Supersymmetric spin operators

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We develop a supersymmetric representation of spin operators which unifies the Schwinger and Abrikosov representations of SU(N) spin operators, allowing a second-quantized treatment of representations of the SU(N) group with both symmetric and antisymmetric character. By applying this to the SU(N) Kondo model, we show that it is possible to develop a controlled treatment of both magnetism and the Kondo effect within a single large-N expansion.

I. MOTIVATION FOR A NEW SPIN REPRESENTATION

Recent experiments on quantum phase transitions in heavy fermion materials have led to a debate about how magnetism condenses out of the metallic state at absolute zero. Certain heavy fermion materials can can be tuned between the magnetic and the paramagnetic state through the use of pressure[1,2] or chemical doping.[3,4] The quantum critical point which separates these two phases is of great current interest, in part because materials in its vicinity may become fundamentally new kinds of metal.[5–7] Heavy fermion materials contain a dense lattice of magnetic moments; conventional wisdom assumes that the spins of the local moments are magnetically screened and of no importance to the magnetic quantum critical point.[8–10] Recent neutron data contradict this viewpoint, by showing that the spin correlations at the quantum critical point are critical in time, but local on an atomic scale,[4,11–13] suggesting that unscreened local moments emerge from the metallic state at the quantum critical point.

If it is indeed true that the magnetic quantum critical points involve local moment physics, then a new theoretical approach is required. Traditionally, heavy fermion physics is modeled using a Kondo lattice Hamiltonian,[14] describing the interaction between a bath of conduction electrons and an array of local moments. One of the well-developed theoretical methods for approaching this model is the large-N expansion,[15–19] where the idea is to use a generalization of the quantum-mechanical spin operators, in which the underlying spin rotation group is generalized from SU(2) to SU(N). The utility of this method derives from the fact that in the limit $N\rightarrow\infty$, it provides an essentially exact, analytic treatment of the Kondo lattice problem. Unfortunately, the way this procedure is carried out at present, magnetic interactions are suppressed as a $1/N^2$ correction, beyond the horizon for a controlled computation. In this paper, we show how we can overcome this shortcoming by the use of a supersymmetric spin representation for local moments.

The theoretical description of interacting local moments poses a fundamental problem: the Pauli spin operator **S** does not satisfy Wick's decomposition theorem, which preempts its use in a Feynman diagram approach. The traditional solution is to represent the spin in terms of either bosons or fermions. In the ''Schwinger boson'' approach,[20,21] the spin operator is represented in terms of an N-component boson b_α [Fig. 1(b)]; in the alternative ''Abrikosov pseudofermion'' representation,[22] the spin is represented by an N-component fermion, f_α, in Fig. 1(a).

Here, $\mathbf{\Gamma}\equiv(\Gamma^1,\ldots,\Gamma^M)$ represents the $M=(N^2-1)/2$ independent SU(N) generators. By combining $2S$ ''Schwinger bosons'' together, one generates a *symmetric* representation of SU(N), denoted by a horizontal Young tableau with $2S$ boxes [Fig. 1(b)]. Conversely, in the pseudofermion approach, $Q\leqslant N$ spin fermions are combined to generate an *antisymmetric* representation of SU(N), denoted by a column Young tableau with Q boxes [Fig. 1(a)].

Most cases of physical interest correspond to an elementary spin or a single box in the Young tableau. Unfortunately, to develop a controlled Feynman diagram expansion, we are obliged to consider a large number of such boxes, letting $N\rightarrow\infty$ keeping either $q=Q/N$ or $m=2S/N$ fixed. In the process of letting $N\rightarrow\infty$, some essential physics is lost. Symmetric representations are ideal for treating magnetism, where the ordered moment involves a highly symmetric condensation of spin bosons, but they lose all information about the Fermi liquid fixed point. Antisymmetric representations capture the development of the Kondo effect and heavy fermion bands, but magnetism is suppressed.

Various authors have tried to develop alternative representations of the spin operator[24] and in this context one interesting idea is to use supersymmetry to simultaneously express the bosonic and fermionic character of local moments.[25–27]

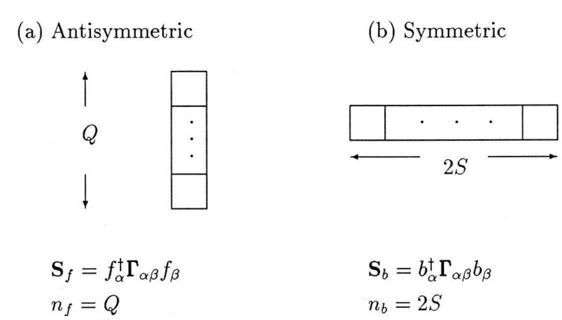


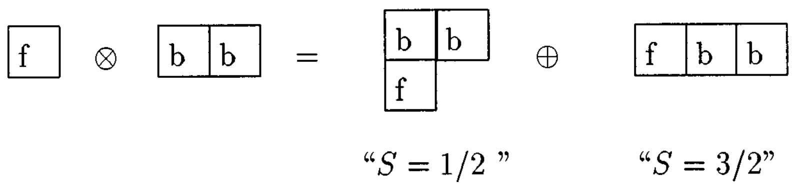
FIG. 1. Young tableaux (Ref. 23) for (a) antisymmetric and (b) symmetric representations generated by the Abrikosov fermion and Schwinger boson representations, respectively.

II. SUPER SPINS

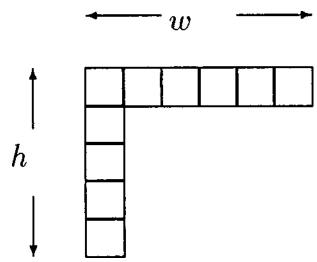
We now examine a class of spin representations which preserve both symmetric and antisymmetric correlations. Consider the spin operator that is a sum of n_f fermions and n_b bosons, given by

$$\mathbf{S}=\mathbf{S}_f+\mathbf{S}_b . \tag{1}$$

By combining $Q=n_f+n_b$ bosons and fermions together, we generate ‘‘L-shaped’’ representations of SU(N). For each choice of n_f and n_b, we generate two irreducible representations. For example, we can combine one fermion and two bosons as follows:



For SU(2), these correspond to a spin-$\frac{1}{2}$ and a spin-$\frac{3}{2}$ representation. To uniquely parametrize an *irreducible* representation, we need to fix the Cazimir $\mathbf{S}^2 \equiv \Sigma_a S_a S_a$. Consider an L-shaped representation of SU(N) of width w and height h:



If the generators of the fundamental representation are normalized according to $\mathrm{Tr}[\Gamma^a \Gamma^b]=\delta^{ab}$, then the expression for the Cazimir of an arbitrary irreducible representation is[28,29]

$$\mathbf{S}^2=\frac{Q(N^2-Q)}{N}+\sum_{j=1,h} m_j(m_j+1-2j), \tag{2}$$

where m_j is the number of boxes in the jth row from the top and Q is the total number of boxes. For an L-shaped Young tableau, $(m_1,m_2,\ldots,m_h)=(w,1,1,\ldots,1)$, so that

$$\mathbf{S}^2=\frac{Q(N^2-Q)}{N}+w(w-1)-h(h-1). \tag{3}$$

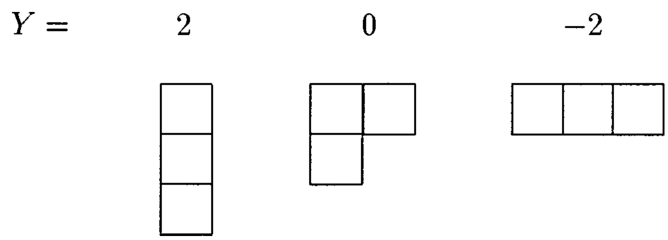
If we substitute $Q=w+h-1$ and $Y=h-w$, we then obtain

$$\mathbf{S}^2=Q(N-Y-Q/N). \tag{4}$$

In this way, each irreducible L-shaped representation of SU(N) is uniquely defined by the two quantities (Q, Y), where Y can assume the values

$$Y=-Q+1,-Q+3,\ldots,Q-1. \tag{5}$$

For example, if $Q=3$, there are three irreducible representations:



We now seek to cast both Q and Y in an operator language. In terms of the boson and fermion operators, the Cazimir can be written

$$\hat{\mathbf{S}}^2=(\hat{\mathbf{S}}_f+\hat{\mathbf{S}}_b)^2. \tag{6}$$

If we expand this expression (Appendix A) by using the completeness relation

$$N\Gamma^a_{\alpha\beta}\Gamma^a_{\delta\gamma}=N\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\beta}\delta_{\delta\gamma}, \tag{7}$$

we are able to express the Cazimir in operator form:

$$\hat{\mathbf{S}}^2=\hat{Q}\left(N-\hat{\mathcal{Y}}-\frac{\hat{Q}}{N}\right), \tag{8}$$

where now

$$\hat{Q}=n_f+n_b \tag{9}$$

fixes the number of boxes and

$$\hat{\mathcal{Y}}=n_f-n_b+\overbrace{\frac{1}{Q}[\theta,\theta^\dagger]}^{P} \tag{10}$$

is the operator measuring the asymmetry $h-w$ of the representation. Here we have introduced the operators

$$\theta^\dagger=f^\dagger_\beta b_\beta, \quad \theta=b^\dagger_\alpha f_\alpha. \tag{11}$$

If we wish to study a spin system described by the (Q, Y) representation, then we must restrict our attention to states $|\psi\rangle$ in the Hilbert space which satisfy

$$\hat{Q}|\psi\rangle = Q|\psi\rangle,$$
$$\hat{\mathcal{Y}}|\psi\rangle = Y|\psi\rangle. \tag{12}$$

Curiously, although this constrains the total number of bosons and fermions, the difference $n_b - n_f$ is only partially constrained, reflecting the fact that bosons and fermions can interconvert without altering the representation.

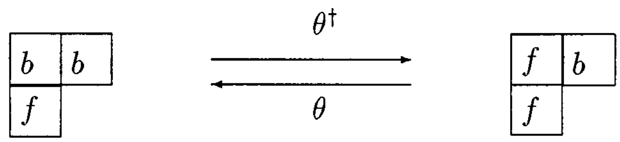
When we represent spins in terms of bosons and fermions, each box in the Young tableau is associated with a fermion or boson, where fermions occupy the column, bosons the row. The corner of the tableau can contain either a boson or a fermion. The operator $\theta^\dagger$ converts the corner box from a boson into a fermion, while θ converts it back again. These operators are the generators of a ''supergroup'' SU(1|1),[30] with the algebra

$$\{\theta^\dagger, \theta\} = Q. \tag{13}$$

The spin operator commutes with these generators,

$$[\mathbf{S}_b + \mathbf{S}_f, \theta] = [\mathbf{S}_b + \mathbf{S}_f, \theta^\dagger] = 0, \tag{14}$$

so that the representation is supersymmetric. Q and $\mathcal{Y}$ also commute with θ and $\theta^\dagger$, and these are the Cazimirs of this group. From Eq. (13), we see that operators $P^b = (1/Q)\theta\theta^\dagger$ and $P^f = (1/Q)\theta^\dagger\theta$ satisfy $P_b + P_f = 1$: they are the projection operators which, respectively, project out states with ''bosons'' or ''fermions'' in the corner of the Young tableau. In this way, we see that $\mathcal{P} = P_b - P_f$ is $+1$ or -1, depending on whether the state has a boson or fermion in the corner of the representation. For example, the representation given by $(Q,Y) = (3,0)$ can be written in two ways:



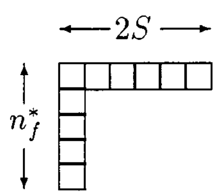
$$\mathcal{P} = 1 \qquad\qquad \mathcal{P} = -1$$
$$n_f - n_b = -1 \qquad\qquad n_f - n_b = 1$$
$$Y = 0 \qquad\qquad Y = 0$$

e.g $\quad \frac{1}{\sqrt{3}} b^\dagger_\sigma (b^\dagger_\uparrow f^\dagger_\downarrow - b^\dagger_\downarrow f^\dagger_\uparrow)|0\rangle \qquad b^\dagger_\sigma (f^\dagger_\uparrow f^\dagger_\downarrow)|0\rangle$

The invariance of the representation under the boson-fermion transformation is a manifestation of the supersymmetry. When a boson is converted into a fermion, the change in $n_f - n_b$ is compensated by the change in $\mathcal{P}$, so that $\mathcal{Y}$ is invariant.

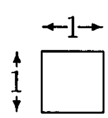
We can alternatively write the constraints in terms of the height $\hat{h} = (\hat{Q} + \hat{\mathcal{Y}} + 1)/2$ and width $\hat{w} = (\hat{Q} - \hat{\mathcal{Y}} + 1)/2$ of the tableau:

$$n_f^* \equiv h = n_f + \frac{1}{Q}\theta\theta^\dagger,$$
$$2S \equiv w = n_b + \frac{1}{Q}\theta^\dagger\theta \tag{15}$$



For the fundamental representation, described by the state

$$|\sigma\rangle = f^\dagger_\sigma|0\rangle \equiv b^\dagger_\sigma|0\rangle \tag{16}$$



$n_f^* = 2S = 1$. Independent of the way we represent the spin, some bosonic and fermionic character is always present, reflecting the fact that a spin can give rise to a ''bosonic'' local moment or it can produce ''fermionic'' singlet bound states. Traditionally, one of the above constraints is dropped: in the approach now adopted, both constraints are simultaneously applied.

There are two ways in which we can use the new constraint. We can work within a ''grand-canonical'' ensemble, where $\hat{Q}$ is fixed, but $\hat{\mathcal{Y}}$ is associated with a chemical potential,

$$H' = H + \zeta\mathcal{Y}. \tag{17}$$

By tuning ζ from negative to positive values, the ensemble is driven from an antisymmetric to a symmetric representation. In fact, since the Cazimir $\mathbf{S}^2 = \hat{Q}[N - \hat{\mathcal{Y}} - (\hat{Q}/N)]$ is linearly related to $\hat{\mathcal{Y}}$, a finite value of ζ is physically equivalent to the introduction of Hund's interaction into the Hamiltonian.

$$H' = H - \frac{1}{Q}\zeta\mathbf{S}^2, \tag{18}$$

where a constant term $\zeta(N - Q/N)$ has been omitted. The supersymmetric spin representation thus enables us to progressively increase the strength of the magnetic interactions by tuning the spin representation.

Alternatively, we may work with a definite representation, where $\hat{\mathcal{Y}}=Y$. The partition function for this model is

$$Z[Q_0,Y]=\mathrm{Tr}[P_{Q_0,Y}e^{-\beta H}], \tag{19}$$

where $P_{Q_0,Y}$ projects out the states with definite $\hat{Q}=Q_0$ and $\hat{\mathcal{Y}}=Y$. By specifying these two constraints, we are still working in an ensemble where the individual number of fermions or bosons are not separately constrained, and in this way we are able to develop a supersymmetric field theory. We can implement these two constraints by carrying out a Fourier transform over the chemical potentials λ and ζ associated with $\hat{Q}$ and $\hat{\mathcal{Y}}$, respectively,

$$Z[Q_0,Y]=\int\frac{d\lambda\, d\zeta}{(2\pi iT)^2}\mathrm{Tr}[e^{-\beta[H+\lambda(\hat{Q}-Q_0)+\zeta(\hat{\mathcal{Y}}-Y)]}], \tag{20}$$

where both $\zeta=\lambda_0+ix$ and $\lambda=\lambda_0+iy$ are integrated along an imaginary axis, $x,y\in[0,2\pi T]$.

III. APPLICATION TO THE UNDERSCREENED KONDO MODEL

A. Formulation of Lagrangian

To illustrate the approach, we develop it for the single-impurity Kondo model, given by

$$H=\overbrace{\sum_{k,\alpha}\epsilon_k c^\dagger_{k\alpha}c_{k\alpha}}^{H_0}+\overbrace{\frac{J}{N}c^\dagger_\alpha\mathbf{\Gamma}_{\alpha\beta}c_\beta\cdot\mathbf{S}}^{H_K}+\overbrace{\lambda\hat{Q}}^{H_Q}+\overbrace{\frac{\zeta}{Q_0}\hat{Q}\hat{\mathcal{Y}}}^{H_Y}. \tag{21}$$

Here, H_0 describes the conduction electron sea, H_K is the interaction between the conduction electron spin density, and the local moment, where $c^\dagger_\alpha=n_s^{-1/2}\Sigma_k c^\dagger_{k\alpha}$, creates an electron at the site of the local moment (n_s=no. of sites). H_Q and H_Y impose the constraints. [Note the way in which H_Y has been written: by multiplying the operator $\hat{\mathcal{Y}}$ by $\hat{Q}$, we cast it in a form which is unchanged upon normal ordering: $\hat{Q}\hat{\mathcal{Y}}=:\hat{Q}\mathcal{Y}:$. By writing it in this way, we can immediately translate the fields to their coherent state representation. Inside a path integral, where $Q=Q_0$ is imposed, we are then able to replace $(1/Q_0)Q\hat{\mathcal{Y}}\to\mathcal{Y}$.]

To date, the one-channel Kondo model has been studied in the large-N approach for completely symmetric[24] and completely antisymmetric[17] representations of **S**. For the latter, the local moment is quenched to form a local Fermi liquid; for symmetric representations the spin S is only partially screened by the Kondo effect to form a spin $S-\frac{1}{2}$. By tuning the representation from the one limit to the other, we are able to examine how the local Fermi liquid interacts with the emergent local moment as the local moment grows. For fully symmetric representations, it is known that the residual Fermi liquid decouples from the partially screened moment.[31] One of the surprising discoveries of this study is that in intermediate representations, the heavy Fermi liquid and the partially screened moment can become antiferromagnetically coupled and the strong-coupling fixed point becomes unstable. We shall see that in this simple model, a two-stage Kondo effect then takes place; richer consequences are likely in a lattice model.

Our first step is to write the constrained partition function as a path integral,

$$Z=\int\mathcal{D}[c,f,b,\lambda,\zeta]e^{-\int_0^\beta[\mathcal{L}_0+H_K+\mathcal{L}_{\rm susy}-(\lambda Q_0+\zeta Y)]d\tau}, \tag{22}$$

where c and f denote Grassman fields, b, λ, and ζ are c-number fields, and we have divided the action into three terms:

$$\mathcal{L}_0=\sum_{\mathbf{k}\sigma}c^\dagger_{\mathbf{k}\sigma}(\partial_\tau+\epsilon_{\mathbf{k}})c_{\mathbf{k}\sigma},$$

$$H_K=-\frac{J}{N}\sum_{\alpha,\beta}[f^\dagger_\alpha c_\alpha c^\dagger_\beta f_\beta+b^\dagger_\alpha c_\alpha c^\dagger_\beta b_\beta], \tag{23}$$

$$\mathcal{L}_{\rm susy}=\sum_\sigma[f^\dagger_\sigma\partial_\tau f_\sigma+b^\dagger_\sigma\partial_\tau b_\sigma]+H_Q+H_Y.$$

The first term describes the conduction electrons, in the second term we have rewritten the local spin in terms of slave fields, and the third term contains the machinery of the supersymmetric representation. We use a single notation for the field operators and the c-number fields that represent them inside the path integral.

Our next step is to formulate the Lagrangian in a form that clearly exhibits the supersymmetry. We shall begin by casting $\mathcal{L}_{\rm susy}$ in a form which is gauge-invariant under time-dependent superrotations. It is convenient to combine the slave fields into a single spinor,

$$\Psi_\sigma=\begin{pmatrix}f_\sigma\\ b_\sigma\end{pmatrix},\quad \Psi^\dagger_\sigma=(f^\dagger_\sigma,b^\dagger_\sigma). \tag{24}$$

Using this notation,

$$\mathcal{L}_{\rm susy}=\sum_\sigma\Psi^\dagger_\sigma[\partial_\tau+\lambda+\zeta\tau_3]\Psi_\sigma-\frac{2\zeta}{Q_0}\theta^\dagger\theta, \tag{25}$$

where τ_3 is a Pauli matrix. Since the starting Hamiltonian and each of the constraints commutes with the supergenerators, the full Lagrangian is invariant (Appendix B) under time-independent superrotations $\Psi_\sigma\to g\Psi_\sigma$, where

$$g=\begin{bmatrix}\sqrt{1-\eta\bar{\eta}} & \eta\\ -\bar{\eta} & \sqrt{1-\bar{\eta}\eta}\end{bmatrix} \tag{26}$$

is an element of the supergroup SU(1|1). The quantities η and $\bar{\eta}$ are conjugate Grassman numbers. If we make this transformation time-dependent, the derivative terms become

$$\Psi^\dagger_\sigma\partial_\tau\Psi_\sigma\to\Psi^\dagger_\sigma[\partial_\tau+(g^\dagger\partial_\tau g)]\Psi_\sigma. \tag{27}$$

Expanding the second term, we obtain

$$\sum_\sigma\Psi^\dagger_\sigma(g^\dagger\partial_\tau g)\Psi_\sigma=\theta^\dagger\partial_\tau\eta+\bar{\eta}\partial_\tau\theta+Q_0\bar{\eta}\partial_\tau\eta, \tag{28}$$

where we have replaced $\sum_\sigma\Psi^\dagger_\sigma\Psi_\sigma\to Q_0$ inside the path integral. Since Z is unchanged by this change of basis, we can integrate over all $g(\tau)$,

$$
\begin{aligned}
Z=&\int D[\bar{\eta},\eta]\int D[c,f,b,\lambda,\zeta]\\
&\times e^{-\int_0^\beta[\mathcal{L}+\theta^\dagger\partial_\tau\eta+\bar{\eta}\partial_\tau\theta+Q_0\bar{\eta}\partial_\tau\eta]d\tau}\\
=&\int D[c,f,b,\lambda,\zeta]\exp\left\{-\int_0^\beta\left(\mathcal{L}-\frac{1}{Q_0}\theta^\dagger\partial_\tau\theta\right)d\tau\right\}.
\end{aligned}
\tag{29}
$$

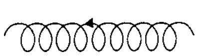
By absorbing the additional term into a redefined

$$
\mathcal{L}^*_{\text{susy}}=\sum_\sigma \Psi^\dagger_\sigma[\partial_\tau+\lambda+\zeta\tau_3]\Psi_\sigma-\frac{1}{Q_0}\theta^\dagger(\partial_\tau+2\zeta)\theta,
\tag{30}
$$

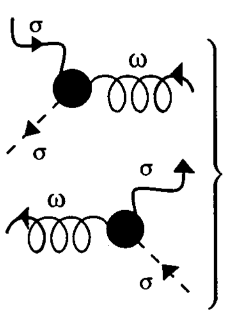
the Lagrangian becomes invariant under time-dependent superrotations. The first term in $\mathcal{L}^*_{\text{susy}}$ describes the level splitting between the bosonic and fermionic components of the spin. The second term describes a residual interaction between the spin and heavy electron fluid. We can factorize this term, to obtain

$$
\begin{aligned}
&-\frac{1}{Q_0}\theta^\dagger(\partial_r+2\zeta)\theta\\
&\quad\to Q_0\alpha^\dagger(\partial_r+2\zeta)\alpha\\
&\qquad+\sum_\sigma \overbrace{[f^\dagger_\sigma b_\sigma(\partial_\tau+2\zeta)\alpha+\alpha^\dagger(\partial_\tau+2\zeta)b^\dagger_\sigma f_\sigma]}^{H_I}.
\end{aligned}
\tag{31}
$$

The first term tells us that the field α represents a *dynamical* fermion with the commutation algebra $\{\alpha,\alpha^\dagger\}=1/Q_0$. This spinless particle mediates the interaction between the spin and the heavy electron fluid; H_I defines the vertex for the decay process $f_\sigma\rightleftharpoons b_\sigma+\alpha$. Notice that since the Schwinger bosons are neutral, the α fermion is a spinless, singly charged excitation. To represent this exchange process, we denote the propagator for the α fermion by the Feynman diagram

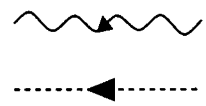
 $= [Q_o(i\omega_n-2\zeta)]^{-1}$ (32)

The vertices which interconvert the heavy electron and spin bosons will be denoted by



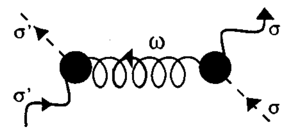
$= i(\partial_\tau+2\zeta)\equiv i(2\zeta-i\omega_n),$ (33)

where the "i" is required to give the correct amplitude for the exchange of the gauge fermion and



$$
= G_b(i\nu_n)=(i\nu_n-\lambda_b)^{-1}
$$
$$
= G_f(i\omega_n)=(i\omega_n-\lambda_f)^{-1}
\tag{34}
$$

represent the propagators for spin bosons and the f electrons. The mediated bare interaction between the spins and the heavy f electrons is then



$= \frac{1}{Q_o}(2\zeta-\omega)$ (35)

It is a rather unique feature of this kind of approach that the spin interactions are carried by fermions rather than bosons. Our final form for $\mathcal{L}^*_{\text{susy}}$ can now be compactly written

$$
\mathcal{L}^*_{\text{susy}}=\sum_\sigma \Psi^\dagger_\sigma\left(\partial_\tau+\lambda+\begin{bmatrix}\zeta & \not{\partial}\alpha\\ \bar{\not{\partial}}\alpha^\dagger & -\zeta\end{bmatrix}\right)\Psi_\sigma+Q_0\alpha^\dagger\not{\partial}\alpha,
\tag{36}
$$

where we have defined $(\partial_\tau+2\zeta)\equiv\not{\partial}$, $(-\partial_\tau+2\zeta)\equiv\bar{\not{\partial}}$.

As our next step, we factorize the Kondo interaction term H_K,

$$
\begin{aligned}
H_K\to H^*_K=&\sum_\sigma [(c^\dagger_\sigma\bar{V}f_\sigma+Vf^\dagger_\sigma c_\sigma)+(c^\dagger_\sigma\bar{\phi}b_\sigma+\phi b^\dagger_\sigma c_\sigma)]\\
&+\frac{N}{J}(\bar{V}V+\bar{\phi}\phi),
\end{aligned}
\tag{37}
$$

where V is a complex c-number field and ϕ is its fermionic partner. If we now introduce

$$
\mathcal{V}=\begin{bmatrix}V\\ \phi\end{bmatrix},\quad \mathcal{V}^\dagger=(\bar{V},\bar{\phi}),
\tag{38}
$$

then the transformed Lagrangian takes the form $\mathcal{L}=\mathcal{L}_0+\mathcal{L}^*_{\text{susy}}+H^*_K$, where

$$
\mathcal{L}_0=\sum_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma}(\partial_\tau+c_{\mathbf{k}})c_{\mathbf{k}\sigma},
$$

$$
H_K=\sum_\sigma [\Psi^\dagger_\sigma\mathcal{V}c_\sigma+c^\dagger_\sigma\mathcal{V}^\dagger\Psi_\sigma]+\frac{N}{J}\mathcal{V}^\dagger\mathcal{V},
\tag{39}
$$

$$
\mathcal{L}^*_{\text{susy}}=\sum_\sigma \Psi^\dagger_\sigma\left(\partial_\tau+\lambda+\begin{bmatrix}\zeta & \not{\partial}\alpha\\ \bar{\not{\partial}}\alpha^\dagger & -\zeta\end{bmatrix}\right)\Psi_\sigma+Q_\sigma\alpha^\dagger\not{\partial}\alpha.
$$

Let us briefly examine the gauge invariance of this Lagrangian. If

$$
h=ge^{i(\theta_Q+\theta_\zeta\tau_3)}
\tag{40}
$$

is a general SU(1|1) matrix, where g takes the form (26), then under the gauge transformation $\Psi_\sigma\to h\Psi_\sigma$, $\mathcal{V}\to h\mathcal{V}$, $\mathcal{L}_0$ and H_K are invariant, but $\mathcal{L}^*_{\text{susy}}$ becomes

$$\mathcal{L}^*_{\text{susy}} \to \sum_\sigma \Psi^\dagger_\sigma h^\dagger \left(\partial_\tau + \lambda + \begin{bmatrix} \zeta & \not{\partial}\alpha \\ \overline{\not{\partial}\alpha^\dagger} & -\zeta \end{bmatrix} \right) h \Psi_\sigma + Q_0 \alpha^\dagger \not{\partial}\alpha. \tag{41}$$

When we expand the first term (Appendix B), we find that

$$\mathcal{L}^*_{\text{susy}}[\lambda,\zeta,\alpha,\alpha^\dagger] \to \mathcal{L}^*_{\text{susy}}[\lambda',\zeta',\alpha',\alpha^{\dagger'}], \tag{42}$$

where $\lambda' = \lambda + i\dot{\theta}_Q$, $\zeta' = \zeta + i\dot{\theta}_\zeta$, and $\alpha' = (\alpha + \eta)e^{-2i\theta_\zeta}$, so $\mathcal{L}$ is invariant under the gauge transformation,

$$\begin{gathered} \Psi_\sigma \to h\Psi_\sigma, \quad \mathcal{V} \to h\mathcal{V}, \\ \lambda \to \lambda - i\dot{\theta}_Q, \quad \zeta \to \zeta - i\dot{\theta}_\zeta, \\ \alpha \to e^{2i\theta_\zeta}\alpha - \eta, \quad \alpha^\dagger \to e^{-2i\theta_\zeta}\alpha^\dagger - \bar{\eta}. \end{gathered} \tag{43}$$

This gauge invariance leads to bosonic and fermionic zero modes. To eliminate them, we must carry out a gauge-fixing procedure. We can always parametrize $\mathcal{V}$ in the form

$$\mathcal{V} = h \begin{bmatrix} V_0 \\ 0 \end{bmatrix} = g \begin{bmatrix} V_0 e^{i(\theta_Q + \theta_\zeta)} \\ 0 \end{bmatrix}, \tag{44}$$

or written out explicitly,

$$\begin{pmatrix} V \\ \phi \end{pmatrix} = V_0 e^{i\theta_f} \begin{pmatrix} \sqrt{1 - \eta\bar{\eta}} \\ -\bar{\eta} \end{pmatrix}, \tag{45}$$

where $\theta_f = \theta_Q + \theta_\zeta$ and $V_0 = \sqrt{\bar{V}V + \bar{\phi}\phi}$ is real. This transformation uniquely specifies both $\bar{\eta} = -(V_0/V)\phi$ and the phase factor and $e^{i\theta_f} = V/|V|$, but does not specify $\theta_b = \theta_Q - \theta_\zeta$. We shall adopt a gauge choice where $\theta_b = 0$. By applying the gauge transformation (43), we can absorb the fermionic fluctuations in $\mathcal{V}$ into a redefinition of the fields. The gauge-fixed hybridization is now

$$H^*_K = V_0 \sum_{\boldsymbol{\sigma}} [c^\dagger_\sigma f_\sigma + f^\dagger_\sigma c_\sigma] + \frac{N V_0^2}{J}. \tag{46}$$

With our gauge choice $\theta_b = 0$, the variable $\lambda_f(\tau) = \lambda + \zeta + i\dot{\theta}_f$ becomes *dynamical*, but the variable $\lambda_b = \lambda - \zeta + i\dot{\theta}_b = \lambda - \zeta$ remains a time-independent integration variable. In this gauge-fixed form of the Hamiltonian, the interaction between the Fermi and spin fluids is entirely contained within $\mathcal{L}^*_{\text{susy}}$, and it is here where we should look if we are to obtain new physics.

In the existing large-N approach to the Kondo model, magnetic interactions are a $1/N^2$ correction to the mean-field theory, an order of accuracy that is beyond current theoretical approaches. A supersymmetric approach enhances the magnetic interactions by a factor of N^2, bringing them within the realm of Gaussian fluctuations about a new large-N mean-field theory. To carry out concrete calculations, we expand around a large-N saddle point of the path integral obtained by taking $N \to \infty$, maintaining $Q/N = q$ and Y/N fixed. By allowing the number of bosons $n_b = N\tilde{n}_b$ to become large, the Bose field is able to condense and form a magnetic moment:

$$\langle b_\sigma \rangle = \sqrt{M} z_\sigma \sim O(\sqrt{N}), \tag{47}$$

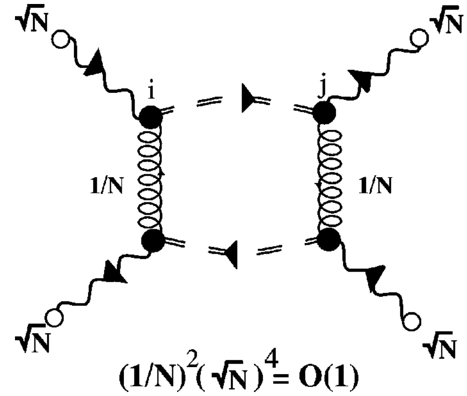


FIG. 2. Magnetic interaction between two spins at site i and j within the current approach. (Note that when $V_0 > 0$, the f fermion can propagate from site to site.) The factor $(\sqrt{N})^4$ associated with the Bose condensate enhances the magnetic interaction by a factor of N^2, so that it appears as the first in a series of RPA diagrams associated with the Gaussian fluctuations of the α fermion.

where z_σ is a unit spinor. The magnetic interaction between spins at different sites is given by Ruderman-Kittel-Kasuya-Yosida (RKKY) diagrams (Fig. 2). The factors of $\sqrt{N}$ associated with the Bose condensate produce an intersite magnetic interaction of order $O(1)$: a factor of N^2 enhancement. These magnetic corrections appear as part of the Gaussian fluctuations of the α fermion, and by calculating them we are able to carry out a controlled treatment of magnetism and the Kondo effect.

B. Mean-field theory

To illustrate this kind of calculation, we develop the machinery for the single-impurity model. Although there are no intersite magnetic interactions, the machinery of the supersymmetry is needed to compute the magnetic interaction between the partially screened local moment and the Fermi liquid in the single-impurity model. The techniques that we now illustrate can be generalized to the lattice.

Our first step is to formally integrate out the conduction (c) and the slave fields (f,b),

$$\begin{gathered} Z = \int D[\alpha, V, \lambda, \zeta] e^{-S_{\text{eff}}}, \\ e^{-S_{\text{eff}}} = \int D[c,f,b] e^{-\int_0^\beta (\mathcal{L}_0 + \mathcal{L}^*_{\text{susy}} + H_K) d\tau}. \end{gathered} \tag{48}$$

Since the second integral is bilinear in the fields, it can be carried out to yield

$$S_{\text{eff}}[\alpha,\lambda,\zeta] = -S\,\text{Tr}\ln[\partial_\tau + \lambda + \zeta\tau_3 + \underline{\Sigma}] + \int_0^\beta [Q_0 \alpha^\dagger \not{\partial}\alpha - \lambda Q_0 - \zeta Y] d\tau \tag{49}$$

and

$$\underline{\Sigma} \equiv \underline{\Sigma}(\tau - \tau') = V_0(\tau) V_0(\tau') \begin{bmatrix} G_e(\tau - \tau') & 0 \\ 0 & 0 \end{bmatrix} + \delta(\tau - \tau') \begin{bmatrix} 0 & \not{\partial}\alpha(\tau) \\ \overline{\not{\partial}\alpha^\dagger}(\tau) & 0 \end{bmatrix} \tag{50}$$

is the self-energy correction induced by the coupling of the f electrons to the conduction electrons (first term) and the mixing between the f and b fields induced by the α fermion (second term), where

$$G_0(\tau-\tau')=\sum_{i\omega_n,\mathbf{k}}\frac{1}{i\omega_n-\epsilon_{\mathbf{k}}}e^{-i\omega_n(\tau-\tau')} \tag{51}$$

is the local conduction electron propagator. $S\,\mathrm{Tr}[\;]$ denotes the ''supertrace,'' defined as the difference between the fermionic and the bosonic trace:

$$S\,\mathrm{Tr}\begin{bmatrix} F & \alpha \\ \beta & B \end{bmatrix}=\mathrm{Tr}[F]-\mathrm{Tr}[B]. \tag{52}$$

Our procedure is then to expand the effective action to quadratic order around the saddle point, where λ, ζ, and $V(\tau)=V_0$ are static, and $\alpha=0$,

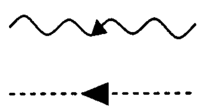
$$S_{\mathrm{eff}}[V,\lambda,\zeta,\alpha]=S_{\mathrm{MF}}+O(\delta\Lambda^2), \tag{53}$$

where $\delta\Lambda$ denotes the fluctuations. $S_{\mathrm{MF}}=\beta F_{\mathrm{MF}}$ determines the leading $O(N)$ mean-field contribution to the free energy. By carrying out the Gaussian integral over the fluctuations in $\delta\Lambda$, we can then determine the $O(1)$ correction to the mean-field theory and the magnetic interactions within the medium.

We begin by computing the saddle point, described by the the mean-field Hamiltonian

$$H_{\mathrm{mft}}=\sum_{\mathbf{k}\sigma}\epsilon_{\mathbf{k}}c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+V_0\sum_{\sigma}[c^{\dagger}_{\sigma}f_{\sigma}+f^{\dagger}_{\sigma}c_{\sigma}]+(\lambda-\zeta)n_b+(\lambda+\zeta)n_f-\lambda Q_0, \tag{54}$$

where V_0 and λ are to be determined self-consistently. This mean-field theory describes a Kondo resonance formed between the conduction electrons and the antisymmetric part of the spin. The residual symmetric part of the spin is unquenched and described by a sharp bosonic level at energy $\lambda_b=\lambda-\zeta$. For the moment, we shall work in the ensemble of definite ζ, examining how the mean field evolves as we increase ζ to favor representations that are increasingly symmetric. If we define $\lambda_f=\lambda+\zeta$, $\lambda_b=\lambda-\zeta$, then in the presence of the finite hybridization, the Green's functions for the slave fields are now given by



$$= G_b(i\nu_n)=(i\nu_n-\lambda_b)^{-1}$$
$$= G_f(i\omega_n)=(i\omega_n-\lambda_f+i\Delta\,\mathrm{sign}(\omega_n))^{-1} \tag{55}$$

where $\Delta=\pi V_0^2\rho$ is the hybridization width of the conduction electron and ρ is the conduction electron density of states. The mean-field free energy is then given by

$$F_{\mathrm{mft}}=NT\sum_n\{\ln[-G_b^{-1}(i\nu_n)]-\ln[-G_f^{-1}(i\omega_n)]\}e^{i\omega_n 0^+}+N\frac{V_0^2}{J}-\lambda Nq. \tag{56}$$

The first term is just the free energy of a free boson. Carrying out the frequency sum on the second term (Appendix C), we obtain

$$\frac{F_{\mathrm{mft}}}{N}=\Phi_f(\xi)+\Phi_b(\lambda_b)+\frac{V_0^2}{J}-\lambda q, \tag{57}$$

where

$$\Phi_f(z)=-2T\,\mathrm{Re}\ln\left[\frac{\Gamma\left(\frac{z+D}{2\pi iT}\right)}{\Gamma\left(\frac{z}{2\pi iT}+\frac{1}{2}\right)}\right]+D/2,$$
$$\Phi_b(z)=T\ln(1-e^{-\beta z}), \tag{58}$$

are the fermionic and bosonic contributions to the energy, D is the conduction-band half-width, and $\xi=\lambda_f+i\Delta$. If we differentiate this result with respect to Δ and λ, we obtain

$$\frac{\pi}{N}\left(\frac{\partial F}{\partial\Delta}+i\frac{\partial F}{\partial\lambda}\right)=\psi\left(\frac{\xi}{2\pi iT}+\frac{1}{2}\right)-\ln\left(\frac{D}{2\pi iT}\right)+\frac{1}{J\rho}-i\pi(q-\tilde{n}_b)=\psi\left(\frac{\xi}{2\pi iT}+\frac{1}{2}\right)-\ln\left(\frac{T_K e^{i\pi(q-\tilde{n}_b)}}{2\pi iT}\right)=0, \tag{59}$$

where $\psi(z)=\partial_z\ln\Gamma(z)$ is the digamma function, $\tilde{n}_b=n_b/N=[\exp(\lambda_b/T)-1]^{-1}$, and $T_K=D\exp[-1/J\rho_0]$ is the Kondo temperature. At zero temperature we may replace $\psi(z)\to\ln z$, so the $T=0$ mean-field equations are then

$$\xi=T_K e^{i\pi(q-\tilde{n}_b)}. \tag{60}$$

If the Bose field condenses, then $\lambda_b=0$, so $\lambda=\zeta$, $\lambda_f=2\zeta$, and $\xi=2\zeta+i\Delta$. In this case, we can solve for the size of the unquenched moment, $M=Nm=N\tilde{n}_b$, and the width Δ of the Abrikosov-Suhl resonance with which it coexists:

$$m=q-\frac{1}{\pi}\cos^{-1}\left(\frac{2\zeta}{T_K}\right)\quad(\zeta>\zeta_c),$$
$$\Delta=\sqrt{(T_K)^2-(2\zeta)^2}, \tag{61}$$

where

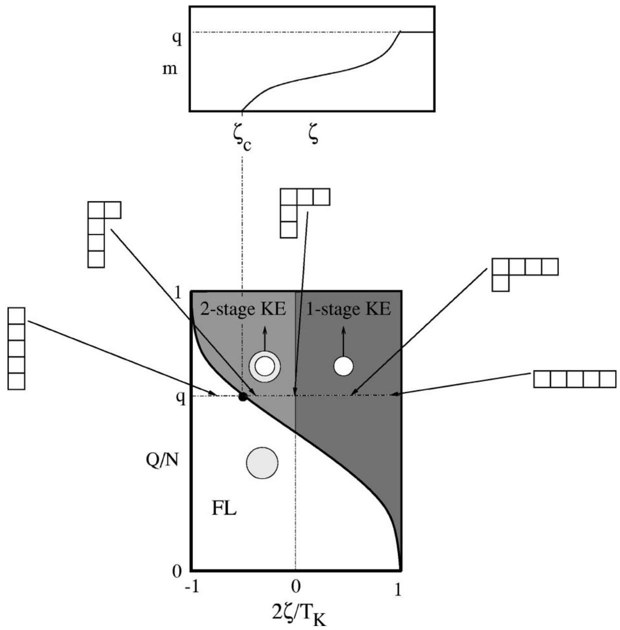


FIG. 3. Top: magnetization of partially screened local moment as a function of ζ. Bottom: Phase diagram for the supersymmetric impurity Kondo model, showing how the representation of the local moment evolves as ζ is increased. Shaded area indicates underscreened region. For $\zeta<0$, the underscreened region involves a two-stage Kondo effect.

$$\zeta_c=(T_K/2)\cos(\pi q) \tag{62}$$

corresponds to a critical value of Hund's interaction beyond which the local moment develops an unscreened component. There are three regions:

$$m=\begin{cases} q, & 2\zeta/T_K>1 \\ 1-\dfrac{1}{\pi}\cos^{-1}\left(\dfrac{2\zeta}{T_K}\right), & 1>2\zeta/T_K>\cos(\pi q) \\ 0, & \cos(\pi q)>2\zeta/T_K. \end{cases} \tag{63}$$

corresponding to an unscreened, partially screened, and fully screened local moment. Figure 3 shows the mean-field phase diagram.

Next, let us consider the residual interactions between the unquenched local moment and the Fermi liquid. The constraint term $\xi\mathcal{Y}$ generates the residual interaction

$$H_I=-\frac{2\zeta}{Q_0}\sum_{\sigma,\sigma'}(f^\dagger_\sigma b_\sigma)(b^\dagger_{\sigma'}f_{\sigma'}) \tag{64}$$

between the heavy-f electron and the unscreened moment. Conventional wisdom, based on the spin-S Kondo model, supposes that such residual interactions are always ferromagnetic. For $\zeta>0$, this is indeed the case, and the fixed point described by the mean-field theory is thus stable. By contrast, if $\zeta<0$, then the residual interaction is *antiferromagnetic*. The presence of such terms is unexpected. We shall see that for the impurity, this leads to a two-stage Kondo effect. In the single-impurity model, the condition $\zeta<0$ corresponds to the requirement that $n_f^*>N/2$, in other words, the requirement that the antisymmetric component of the spin representation is more than half-filled.

C. Calculation of the magnetization using Gaussian fluctuations

One way to examine the consequences of this residual coupling on the single-ion Kondo effect is to compute the field-dependent magnetization of the ground state. The application of a field provides a controlled way of examining the crossovers associated with screening processes. To compute the magnetization, we need to introduce a magnetic field and calculate the field-dependent ground-state energy, including the effect of the Gaussian fluctuations around the mean-field theory. For SU(N), there are $N-1$ ways of introducing the magnetic field. We shall use the form

$$H_B=-B\sum_\sigma m_\sigma(n_{f\sigma}+n_{b\sigma}), \tag{65}$$

where $m_1=-1$, $m_2=1$. With this choice, the field splits off two bosonic and two fermionic states from the other $(N-2)$ levels.

The mean-field free energy in a field is then given by

$$F_{\rm mft}[B]=\sum_\sigma[\Phi_f(\xi_\sigma)+\Phi_b(\lambda_{b\sigma})]+\frac{NV_0^2}{J}-\lambda Q_0-\zeta Y, \tag{66}$$

where $\xi_\sigma=\xi-Bm_\sigma$, $\lambda_{b\sigma}=\lambda_b-Bm_\sigma$. To calculate the magnetization, we must differentiate the free energy with respect to B. (Since the Free energy is stationary with respect to changes in λ and ζ, we do not have to worry about how these fields change with respect to B.) The magnetization is then

$$M=n_{b1}+(n_{f1}-n_{f2})$$
$$=2S+\overbrace{\frac{1}{\pi}\left[\tan^{-1}\left(\frac{\Delta}{-B-\lambda_f}\right)-\tan^{-1}\left(\frac{\Delta}{B-\lambda_f}\right)\right]}^{m_f(B)}. \tag{67}$$

The first term is the residual unquenched moment, the second term represents the spin-polarization of the Kondo singlet. Technically, we should lump this term with the other $O(1)$ corrections that we need to calculate from the Gaussian fluctuations. The mean field only reliably predicts the terms of order N, and thus to this order,

$$M=2S+O(1). \tag{68}$$

To calculate the $O(1)$ term, we need to include the zero-point corrections to the ground-state energy.

There are two types of Gaussian fluctuation around mean-field theory: bosonic fluctuations in V, λ, and ζ, plus the fluctuations of the α field. Fluctuations in V and $\lambda_f=\lambda+\zeta$ are associated with the interactions in the Fermi liquid. These terms renormalize $m_f(B)$ and produce an order $O(1/N)$ correction to the magnetization. Fluctuations in $\lambda_b=\lambda-\zeta$ renormalize the entropy of the free moment, and do not produce any correction to the unscreened moment. The only $O(1)$ corrections to the magnetization are those associated with the fermionic fluctuations. The Lagrangian for these fluctuations is given by

$$\mathcal{L}_\alpha = Q_0 \alpha^\dagger(\partial_\tau + 2\zeta)\alpha + \overbrace{[f^\dagger_\sigma b_\sigma(\partial_\tau + 2\zeta)\alpha + \alpha^\dagger(\partial_\tau + 2\zeta) b^\dagger_\sigma f_\sigma]}^{H_T}. \tag{69}$$

In a field, the number one boson condenses, so that $\lambda_{b1} = 0$. To fulfill the constraint on the Fermi fields, $n_f^* = \text{const}$, we require that $\lambda_f = \lambda_{b1} + 2\zeta + B$ is field-independent, which implies that in a field $2\zeta = 2\zeta_0 - B$.

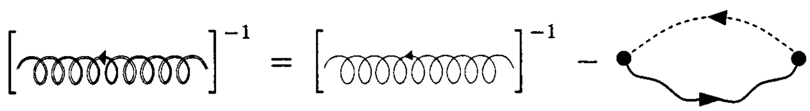
When we expand the effective action S_{eff} to Gaussian order in the α field, we obtain the correction

$$S_\alpha = -\int d1\, d2\, \alpha^\dagger(1) D^{-1}(1-2)\alpha(2), \tag{70}$$

where

$$\begin{aligned} -D^{-1}(1-2) &= Q_0(\partial_\tau + 2\zeta) - (\partial_\tau + 2\zeta)^2 \langle T\theta(1)\theta^\dagger(2)\rangle \\ &= Q_0(\partial_\tau + 2\zeta) - (\partial_\tau + 2\zeta)^2 T \sum G_{f\sigma}(1-2) G_{b\sigma}(2-1) \end{aligned} \tag{71}$$

is the inverse propagator for the α fermion. Written out both diagrammatically and in the frequency domain, this is



$$D^{-1}(\omega) = Q_o(\omega - 2\zeta) - N[i(2\zeta - \omega)]^2 \Phi(\omega) \tag{72}$$

where

$$\Phi(\omega) = \frac{T}{N} \sum_{\sigma\nu} G_{f\sigma}(\omega + \nu) G_{0\sigma}(\nu). \tag{73}$$

It proves convenient to factorize $D^{-1}(\omega)$ as follows:

$$D^{-1}(\omega) = Q_0(\omega - 2\zeta) P(\omega), \tag{74}$$

where

$$P(\omega) = 1 + \frac{\omega - 2\zeta}{q} \Phi(\omega). \tag{75}$$

A detailed calculation of $P(\omega)$ at zero temperature (Appendix D) yields

$$P(\omega - i\delta) = P_0(\omega + B - i\delta), \tag{76}$$

where

$$P_0(\omega) = A_f(\omega)\left[\Delta - \frac{\omega - 2\zeta_0}{\pi q}\left(\ln\frac{T_K}{\omega} + i\pi\tilde{n}_b\right)\right] \tag{77}$$

is the zero-field expression for $P(\omega)$ and $A_f(\omega) = \text{Im}\, G_f(\omega - i\delta)$ is the zero-field f-spectral function.

Now the free energy associated with the Gaussian fluctuations in α is given by

$$\begin{aligned} F_\alpha &= -T\sum_n \{\ln[-D^{-1}(i\omega_n)] - \ln[Q_0(2\zeta - i\omega_n)]\} e^{i\omega_n 0^+} \\ &= -T\sum_n \ln[P(i\omega_n)] e^{i\omega_n 0^+}, \end{aligned} \tag{78}$$

where we subtract the free energy of the auxiliary α field in the absence of interactions to avoid overcounting. Carrying this sum out by contour integration, and taking the zero-temperature limit, the zero-point energy is

$$E_\alpha = -\int_{-D}^{0} \frac{d\omega}{\pi} \text{Im}\ln[P_0(\omega + B - i\delta)], \tag{79}$$

where we have inserted the field dependence by replacing $P(\omega) \to P_0(\omega + B)$. Differentiating this with respect to the applied magnetic field, the screening contribution to the magnetization due to the interaction between the spin and Fermi fluid is given by

$$\begin{aligned} M_\alpha &= -\frac{\partial E_\alpha}{\partial B} \\ &= \int_{-D}^{0} \frac{d\omega}{\pi} \partial_\omega \text{Im}\ln[P_0(\omega + B - i\delta)] \\ &= \left[\frac{1}{\pi}\text{Im}\ln[P_0(\omega + B - i\delta)]\right]_{-D}^{0}. \end{aligned} \tag{80}$$

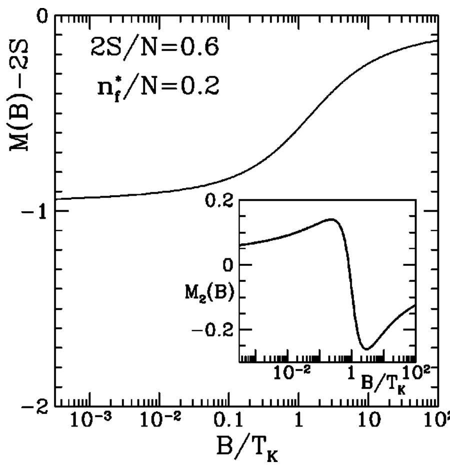


FIG. 4. Showing the field-dependent magnetization for a Kondo model with $\zeta_0>0$. In this case, the Kondo effect is a single-stage process. Inset shows the corrections to the magnetization derived from the residual interaction between the heavy fermion and spin fluid. These corrections are positive (ferromagnetic) at low temperatures and remain small at all temperatures

The lower limit of this sum gives a contribution $-(1/\pi)\mathrm{Im}\ln\{-D[\ln(D/T_K)-i\pi(1+\tilde{n}_b)]\}=-1$. The final result for the fluctuation contribution to the magnetization is then

$$M_\alpha(B)=-1+\frac{1}{\pi}\mathrm{Im}\ln\left[\Delta+\frac{2\zeta_0-B}{\pi q}\ln\left(\frac{T_K e^{i\pi\tilde{n}_b}}{B}\right)\right]. \tag{81}$$

To obtain the total magnetization, we must add this to the result $M_{\mathrm{mft}}=2S+m_f(B)$ obtained from the mean-field theory. The total magnetization, evaluated to order $O(1)$, is then

$$M(B)=\overbrace{[2S-1+m_f(B)]}^{M_1(B)} + \overbrace{\frac{1}{\pi}\mathrm{Im}\ln\left[\Delta+\frac{2\zeta_0-B}{\pi q}\ln\left(\frac{T_K e^{i\pi\tilde{n}_b}}{B}\right)\right]}^{M_2(B)}+O\left(\frac{1}{N}\right), \tag{82}$$

where

$$m_f(B)=\frac{1}{\pi}\left\{\arctan\left[\frac{2\zeta_0+B}{\Delta}\right]-\arctan\left[\frac{2\zeta_0-B}{\Delta}\right]\right\}. \tag{83}$$

We can check this result by completely removing the fermionic contribution. In the limit $\Delta\to 0, 2\zeta_0\to T_K$, $M_1(B)$ and $M_2(B)$ develop discontinuities which precisely cancel, to yield

$$M(B)=2S-\frac{1}{\pi}\left\{\arctan\left[\frac{N\ln\left[\frac{T_K}{B}\right]}{2S\pi}\right]+\frac{\pi}{2}\right\}, \tag{84}$$

which is the residual magnetization of the spin-S Kondo model (see Fig. 4).

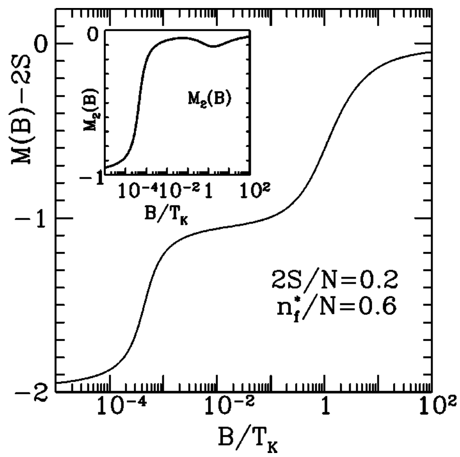


FIG. 5. Field-dependent magnetization for a Kondo model with $\zeta<0$. In this case, the Kondo effect is a two-stage process. The corrections to the magnetization derived from the residual antiferromagnetic interaction between the heavy fermion and spin fluid. These interactions grow to -1 at low temperatures, corresponding to a second screening process at the renormalized Kondo temperature T_K^*.

Let us now restore a finite Δ. The first term $M_1(B)$ represents the screening of the local moment by the Kondo effect. $M_1(B)$ has the limiting values

$$M_1(B)=\begin{cases}2S-1 & (B\ll T_K),\\ 2S & (B\gg T_K).\end{cases} \tag{85}$$

The second term $M_2(B)$ derives from the screening effect produced by the residual interaction between the Fermi liquid and the magnetic moment. For $\zeta_0>0$, this can be rewritten

$$M_2(B)=\frac{1}{\pi}\arctan\left[\frac{2S\pi}{\frac{Q\Delta\pi}{2\zeta_0-B}+N\ln\left[\frac{T_K}{B}\right]}\right]. \tag{86}$$

At low fields, this contribution is small and positive, corresponding to an irrelevant residual ferromagnetic interaction. At $2\zeta_0=B$, this contribution passes continuously through zero, due to a change of sign in the residual interaction. At still higher fields, this correction remains small, and asymptotes to zero.

By contrast, if $\zeta_0<0$, in this case the residual interaction with the Fermi fluid is *antiferromagnetic*. Since $2\zeta_0-B$ is always negative, we can now write

$$M_2(B)=-\frac{1}{2}+\frac{1}{\pi}\arctan\left[\frac{\frac{Q\Delta\pi}{B-2\zeta_0}+N\ln\left[\frac{B}{T_K}\right]}{2\pi S}\right]. \tag{87}$$

For small fields, $M_2(B)\to -1$, so that this term constitutes an *additional* screening contribution to the magnetization. For small negative ζ, we can approximate this expression by

$$M_2(B)=-\frac{1}{2}-\frac{1}{\pi}\arctan\left[\frac{+N\ln\left[\frac{T_K^*}{B}\right]}{2\pi S}\right], \tag{88}$$

where

$$T_K^* = T_K e^{-\pi q \Delta/2\zeta_0} = T_K e^{-\pi q \cot\{\pi[\tilde{n}_f^* - 1/2]\}} \quad (n_f^* > \tfrac{1}{2}), \tag{89}$$

which corresponds to a second screening process, governed by the second-stage Kondo temperature T_K^* (Fig. 5).

An alternative way to derive the same result is to consider how the Schwinger boson field condenses in an applied field. The constraint associated with the bosonic part of the spin is written (16)

$$2S = n_b + \frac{1}{Q}\theta^\dagger \theta. \tag{90}$$

In zero field, the Fermi fluid is unpolarized, and the magnetization is given by the condensed part of the Schwinger Bose field. Suppose we apply a small field that condenses the b_1 component, so that

$$b_\sigma = \sqrt{M}\,\delta_{\sigma 1} + \delta b_\sigma, \tag{91}$$

then the constraint can be rewritten as

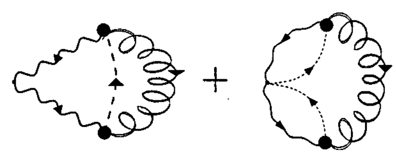
$$2S = M + \sum_\sigma \langle \delta b_\sigma^\dagger \delta b_\sigma \rangle + \frac{1}{Q}\langle \theta^\dagger \theta \rangle \tag{92}$$

or

$$M = 2S - \sum_\sigma \langle \delta b_\sigma^\dagger \delta b_\sigma \rangle - \frac{1}{Q}\langle \theta^\dagger \theta \rangle. \tag{93}$$

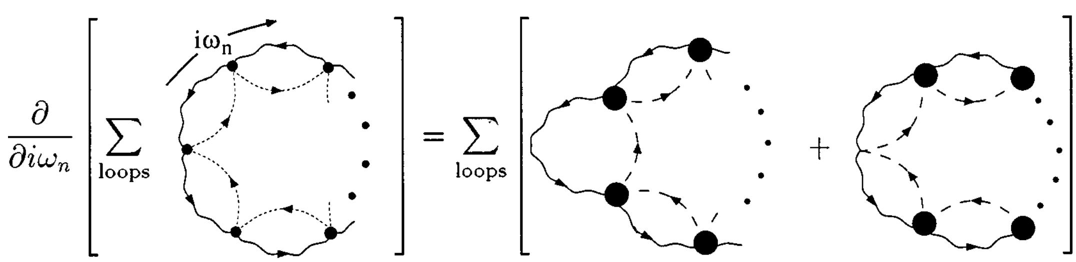
Diagrammatically, these two contributions to moment reduction are given by

$$\sum_\sigma \langle \delta b_\sigma^\dagger \delta b_\sigma \rangle + \frac{1}{Q}\langle \theta^\dagger \theta \rangle =$$



(94)

Of course, only the combination of the two terms is gauge-invariant, but by fixing the gauge, we can assign them each physical meaning. The first corresponds to fluctuations in the direction of the local moment. The second represents the reduction in the amplitude of the moment derived from the interconversion of spins into heavy fermions. We can compute the sum of these diagrams by noting that they are generated by differentiating the RPA diagrams contributing to the fermionic zero-point energy with respect to the frequency:



(95)

so that

$$\sum_\sigma \langle \delta b_\sigma^\dagger \delta b_\sigma \rangle + \frac{1}{Q}\langle \theta^\dagger \theta \rangle = -T \sum_{\omega = i\omega_n} \frac{\partial}{\partial \omega} \ln[P(\omega)] = -M_\alpha, \tag{96}$$

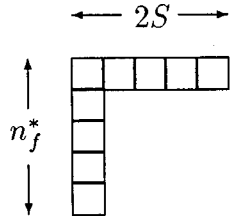
which enables us to identify the reduction in the magnetization with the fluctuations in direction and magnitude of the local moment. In this way, we see that the fluctuations which screen the moment are given by

$$\sum_\sigma \langle \delta b_\sigma^\dagger \delta b_\sigma \rangle + \frac{1}{Q}\langle \theta^\dagger \theta \rangle = \begin{cases} 1 & (n_f^* < N/2), \\ 2 & (n_f^* > N/2), \end{cases} \tag{97}$$

depending on whether a one- or two-stage screening process takes place. Although these results are only calculated to leading order in the large-N expansion, we expect the appearance of integer values for the screening is exact for a local moment.

IV. STRONG-COUPLING PICTURE OF THE TWO-STAGE KONDO EFFECT

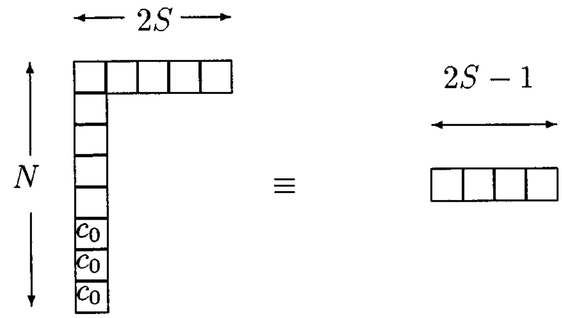
To gain a complimentary insight into the two-stage Kondo effect, it is useful to examine this phenomenon in the strong-coupling limit. Imagine a local moment, described by an L-shaped representation of SU(N), denoted by the Young tableau

$\mathbf{S} =$  (98)

In the ground state of the strong-coupling Hamiltonian

$$H_K = \frac{J}{N} c^{\dagger}_{0\alpha} \mathbf{\Gamma}_{\alpha\beta} c_{0\beta} \cdot \mathbf{S}, \tag{99}$$

electrons form a singlet with the fermionic part of the spin creating a partially screened moment, denoted by a Young tableau with a completely filled first row.

$\mathbf{S}^* = (\mathbf{\Gamma}_{e_0} + \mathbf{S}) =$  (100)

where in this example we have taken $N = 8$. Since the first column of the tableau is a singlet (with N boxes), it can be removed from the tableau, leaving behind a partially screened spin $S - \frac{1}{2}$, described by a row Young tableau with $2S - 1$ boxes. If we now couple the electron at the origin with electrons at site ''1'' via a small hopping matrix element $t \ll J$, then the virtual charge fluctuations of electrons in and out of the singlet at the origin will lead to a residual coupling between the partially screened moment and the electrons at the neighboring site ''1,''

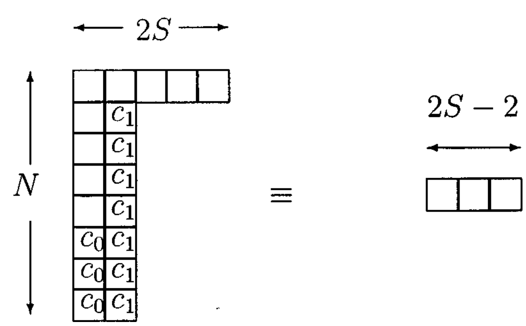
$$H_{(1)} = \frac{J^*}{N} \mathbf{S}^* \cdot c^{\dagger}_{1\alpha} \mathbf{\Gamma}_{\alpha\beta} c_{1\beta}, \tag{101}$$

where $J^* \sim t^2/J$. In the SU(2) Kondo model, only electrons parallel to the residual moment $\mathbf{S}^*$ can hop onto the origin, which gives rise to a ferromagnetic coupling $J^* < 0$. In the SU(N) case, electrons can hop provided they are not in the same spin state as electrons at the origin. The sign of the coupling J^* depends on the number of conduction electrons $n_c = n - n_f^*$, bound at the origin. If the $n_c = N - 1$, electrons hopping onto the origin will have to be parallel to the residual spin, so in this case the coupling is ferromagnetic, $J^* < 0$. By contrast, if $n_c \ll N$, there are many ways for the electron to hop onto the origin with a spin component that is different to the residual moment, so the residual interaction will be antiferromagnetic, $J^* > 0$. By carrying out a large-N calculation in the strong-coupling limit or by making a detailed strong-coupling calculation for SU(N), we are able to confirm that for $N > 2$, J^* changes sign when the number of bound-conduction electrons is less than $N/2$, and in the large-N limit, it is given by[32]

$$J^* = -\frac{t^2}{J(1-\tilde{n}_f^*)\tilde{n}_f^*} \left[\frac{\frac{1}{2} - \tilde{n}_f^*}{1 - \tilde{n}_f^* + \tilde{n}_b} \right], \tag{102}$$

where $\tilde{n}_f^* = n_f^*/N$ and $\tilde{n}_b = 2S/N$.

When $n_f^* > N/2$, $4J^* > 0$, the strong-coupling fixed point becomes unstable, and a second-stage Kondo effect occurs, binding a further $N - 1$ electron at site ''1'' to form a state denoted by the tableau

$\mathbf{S}^{**} = (\mathbf{\Gamma}_{e_0} + \mathbf{\Gamma}_{e_1} + \mathbf{S}) =$  (103)

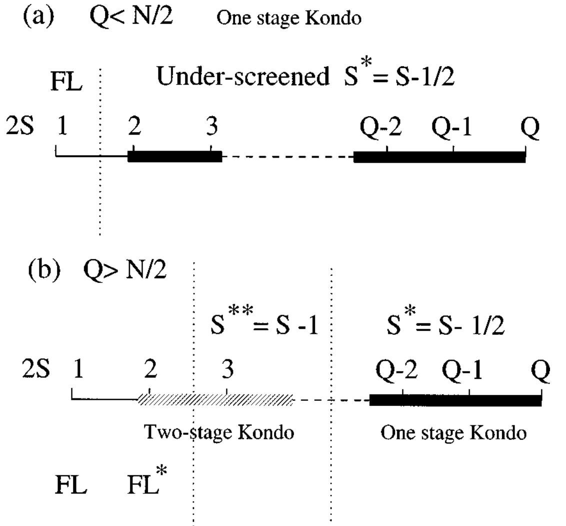


FIG. 6. Two scenarios for the emergence of magnetism as the size of S is progressively increased. (a) Single stage Kondo effect, where for $S>\frac{1}{2}$, a partially quenched moment with spin $S^*=S=\frac{1}{2}$. (b) For $Q>N/2$, the initial emergence of a local moment is accompanied by a two-stage Kondo effect, where, provided $n_f^*>N/2$, the spin is screened from S to $S^{**}=S-1$.

corresponding to a residual spin $S^{**}=S-1$, $M=2S-2$. This final configuration is stable, because an electron at site ''2'' can only hop onto site ''1'' if it is parallel to the unquenched moment.

We see that our supersymmetric approach permits us to examine the consequences of this two-stage Kondo effect, starting from weak coupling. Translated into the weak-coupling language, the two vertical columns of the Young tableau will correspond to two separate screening clouds, of very different radii,

$$l=\frac{v_F}{T_K} \quad \text{and} \quad l^*=\frac{v_F}{T_K^*}, \tag{104}$$

respectively. It is remarkable that a pointlike complex impurity can give rise to two separate length scales in this way.

V. DISCUSSION

In this paper we have developed a spin representation that interpolates between the Schwinger boson and Abrikosov pseudofermion representations, and which exhibits the property of supersymmetry. As an exploratory exercise, we have applied the method to a class of single-impurity Kondo models, where we have been able to examine how local moment behavior emerges as the strength of Hund's interactions between the spins is systematically increased. Suppose we consider a spin representation with Q boxes and examine how the ground state evolves as we progressively increase the moment from $S=\frac{1}{2}$ to $S=Q/2$. One of the surprising discoveries is that there are in fact two routes by which the magnetic moment emerges from the Fermi liquid (Fig. 6), as follows.

(i) One-stage Kondo effect, where $Q<N/2$. Once the spin S exceeds one-half, a partially screened moment is generated. The low-temperature fixed point is described by the coexistence of a Fermi liquid and a moment of spin $S^*=S-1/2$, with a slowly vanishing ferromagnetic coupling between the two degrees of freedom.

(ii) Two-stage Kondo effect, where $Q>N/2$. At intermediate scales, the moment quenches to a spin $S^*=S-\frac{1}{2}$, but the residual coupling to the conduction sea is now antiferromagnetic, and a second-state quenching occurs to a spin $S^{**}=S-1$. When the starting spin is $S=1$, a new singlet phase is formed, with one additional fermionic bound state. (We label this phase FL* in Fig. 6, but do not know at this stage if this state is a Fermi liquid.)

We would now like to discuss the future extension of this approach to a Kondo lattice. Two decades ago, Doniach[14] argued that the properties of a Kondo lattice should depend critically on the ratio of the Kondo temperature to the RKKY interaction $\kappa=T_K/J_{\text{RKKY}}$. Heuristically, the Kondo and RKKY scales are related to the Kondo coupling constant according to

$$T_K \sim De^{-1/J\rho}, \quad J_{\text{RKKY}} \sim J^2/D, \tag{105}$$

where D is the conduction-band width. Doniach pointed out that κ grows with the size of J, arguing that for small J, the system is expected to be antiferromagnetically ordered, but for large J, the magnetism melts to form a ''heavy'' Fermi liquid. Unfortunately, our theoretical understanding is at present limited only to a discussion of energy scales, and little is known about the nature of the transition between these two limiting cases.

Can we shed light on these issues by extending the supersymmetric approach to the lattice? The mean-field solution will in general give rise to a heavy Fermi liquid which coexists with a lattice of underscreened moments. The Fermi surface volume V_{FS} will depend on the size M of the underscreened moments,

$$\frac{V_{\text{FS}}}{(2\pi)^3}=(n_c+n_f^*)/N=(n_c+Q-M)/N. \tag{106}$$

At the mean-field level, these moments can point in any direction, but once we include the effects of the Gaussian fluctuations of the α fermions, two effects will take place.

(i) The local moments will be partially screened by the Kondo effect with the heavy electron fluid.

(ii) The fluctuation free energy will be sensitive to the direction in which the spins condense.

Typically, we expect the fluctuation free energy will be lowest in an antiferromagnetic spin configuration, where, for instance,

$$\langle b_\sigma(\mathbf{x})\rangle=\sqrt{2M}[\cos^2(\mathbf{Q}\cdot\mathbf{x}/2)\delta_{\sigma 1}+\sin^2(\mathbf{Q}\cdot\mathbf{x}/2)\delta_{\sigma 2}]. \tag{107}$$

This dependence on the relative orientation of the moments defines a renormalized ''RKKY'' interaction J^*_{RKKY}. By tuning J, we will be able to examine how the RKKY interaction is renormalized by the Kondo effect and how the staggered magnetization depends on the screening process,

$$M=2S-m(J),$$
$$m(J)=\sum_\sigma \langle \delta b_\sigma^\dagger \delta b_\sigma\rangle+\frac{1}{Q}\langle\theta^\dagger\theta\rangle, \tag{108}$$

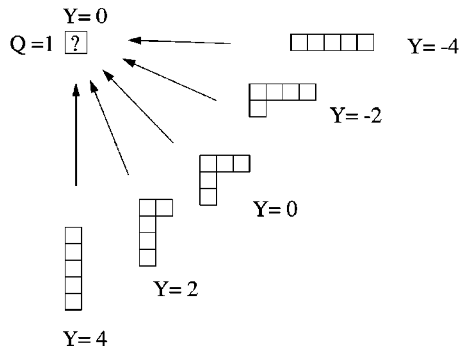


FIG. 7. Illustrating the idea that the properties of a class of L-shaped representations will enable us to triangulate on the properties of a small-N Kondo lattice.

where $m(J)$ is a continuous function of J. The critical value where M vanishes (at small S) defines the point where the magnetism is eradicated by the Kondo effect. In this way, we hope to be able to carry a model calculation of the phase diagram first envisaged by Doniach.

An open question is whether this new approach can shed light on the nature of the quantum critical point separating the magnetic and the paramagnetic phases of the Kondo lattice. Ultimately we are interested in the properties of a Kondo lattice of elementary spins corresponding to one box in Young's tableau. This large-N approach can provide information about the properties of a class of L-shaped representations. If there is any universality associated with the emergence of magnetism at absolute zero, then perhaps a large-N approach will enable us to triangulate on the properties of a real Kondo lattice (Fig. 7).

One of the interesting aspects of the supersymmetric approach is the appearance of fermionic ''phase fluctuations'' between the spin and the heavy electron fluid: it is these fluctuations, described by the gauge fermion α, which mediate the interaction between the magnetic condensate and the heavy electron fluid. In the case where the ''mass'' of this excitation, ζ is positive, the gauge fermion α can be integrated out of the problem, and the interaction between the magnetic and electron fluid could be treated as a point interaction. However, when ζ is negative, the α fermion gives rise to a new bound-state. Will the same phenomenon occur in the lattice? This could lead to the possibility of two different kinds of fixed point.

(i) A Millis Hertz fixed point,[8,9] where the weak ferromagnetic interaction between the magnetic and electron fluid can be treated as a point vertex. In this situation, the transition will be described by the interaction between a Gaussian magnetic fluid and a well-defined Fermi surface.

(ii) A non-Fermi-liquid fixed point, where the dynamical fermion mediating the magnetic interaction becomes an active participant in the physics. If the gauge fermion developed gapless excitations, then the decay of heavy fermions into unquenched spins, described by the process

$$f_\sigma^- \rightleftharpoons b_\sigma + \alpha^-, \tag{109}$$

would lead to a phase with a novel kind of spin-charge separation.

These points will be examined in greater detail in a future publication.

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APPENDIX A: OPERATOR EXPRESSION FOR THE CAZIMIR

In this section, we prove that the Cazimir

$$\mathbf{S}^2 = (b^\dagger \Gamma b + f^\dagger \Gamma f)^2 \tag{A1}$$

can be written

$$\mathbf{S}^2 = Q\left(N - \hat{\mathcal{Y}} - \frac{Q}{N}\right). \tag{A2}$$

To show this relationship, we use the completeness result. Using the normalization

$$\mathrm{Tr}[\Gamma^a \Gamma^b] = \delta^{ab}, \tag{A3}$$

this is

$$\Gamma^\lambda_{ab}\Gamma^\lambda_{cd} + \frac{1}{N}\delta_{ab}\delta_{cd} = \delta_{ad}\delta_{bc}. \tag{A4}$$

By using this to expand

$$\mathbf{S}^2 = (b^\dagger \Gamma b + f^\dagger \Gamma f)^2, \tag{A5}$$

we obtain

$$\mathbf{S}^2 = -\frac{1}{N}Q^2 + (b_a^\dagger b_b + f_a^\dagger f_b)(b_b^\dagger b_a + f_b^\dagger f_a). \tag{A6}$$

We can expand each term in this expansion as follows:

$$\begin{aligned} b_a^\dagger b_b b_b^\dagger b_a + f_a^\dagger f_b f_b^\dagger f_a &= n_b(n_b + N - 1) - n_f[n_f - (N+1)] \\ &= n_b^2 - n_f^2 + NQ + (n_f - n_b). \end{aligned} \tag{A7}$$

Also

$$f_a^\dagger f_b b_b^\dagger b_a = -b_b^\dagger f_b f_a^\dagger b_a + n_b. \tag{A8}$$

Combining these results, we obtain

$$\begin{aligned} \mathbf{S}^2 &= -\frac{1}{N}Q^2 + (n_b^2 - n_f^2) + (N+1)Q - 2b_b^\dagger f_b f_a^\dagger b_a \\ &= -\frac{1}{N}Q^2 + (N+1)Q + Q\left(n_b - n_f - \frac{2}{Q}\theta\theta^\dagger\right) \\ &= -\frac{1}{N}Q^2 + NQ - Q\left(n_f - n_b + \frac{1}{Q}[\theta,\theta^\dagger]\right) \\ &= Q\left(N - \hat{\mathcal{Y}} - \frac{Q}{N}\right), \end{aligned} \tag{A9}$$

where we have used the result $2\theta\theta^\dagger = [\theta,\theta^\dagger] + Q$ to carry out the last step but one.

APPENDIX B: SUPERSYMMETRY OF LAGRANGIAN

In this appendix, we examine the transformation of the Lagrangian

$$\mathcal{L}^*_{\rm susy}=\sum_\sigma \Psi^\dagger_\sigma\left(\partial_\tau+\lambda+\begin{bmatrix}\zeta & \bar{\partial}\alpha\\ \overline{\bar{\partial}\alpha^\dagger} & -\zeta\end{bmatrix}\right)\Psi_\sigma+Q_0\alpha^\dagger\bar{\partial}\alpha \tag{B1}$$

under the transformation $\Psi_\sigma\to h\Psi_\sigma$, where

$$h=e^{i\theta_Q}\begin{bmatrix}\sqrt{1-\eta\bar{\eta}} & \eta\\ -\bar{\eta} & \sqrt{1-\bar{\eta}\eta}\end{bmatrix}\begin{bmatrix}e^{i\theta_\zeta} & 0\\ 0 & e^{-i\theta_\zeta}\end{bmatrix} \tag{B2}$$

is a general member of the group SU(1|1). Under this transformation,

$$\mathcal{L}^*_{\rm susy}\to\mathcal{L}^*_{\rm susy}+\sum_\sigma \Psi^\dagger_\sigma\left((h^\dagger\partial_\tau h)+h^\dagger\left[\begin{bmatrix}\zeta & \bar{\partial}\alpha\\ \overline{\bar{\partial}\alpha^\dagger} & -\zeta\end{bmatrix},h\right]\right)\Psi_\sigma. \tag{B3}$$

When we expand the correction, we obtain

$$\sum_\sigma \Psi^\dagger_\sigma(h^\dagger\partial_\tau h)\Psi_\sigma=\sum_\sigma \Psi^\dagger_\sigma\left(i\dot{\theta}_Q+i\dot{\theta}_\zeta\tau_3+\bar{\eta}\partial_\tau\eta+\begin{bmatrix} & e^{-2i\theta_\zeta}\partial_\tau\eta\\ -e^{2i\theta_\zeta}\partial_\tau\bar{\eta} & \end{bmatrix}\right)\Psi_\sigma, \tag{B4}$$

$$\sum_\sigma \Psi^\dagger_\sigma h^\dagger\left[\begin{bmatrix}\zeta & \\ & -\zeta\end{bmatrix},h\right]\Psi_\sigma=\sum_\sigma \Psi^\dagger_\sigma\begin{bmatrix} & 2\zeta\eta e^{-2i\theta_\zeta}\\ 2\zeta\bar{\eta}e^{2i\theta_\zeta} & \end{bmatrix}\Psi_\sigma+Q(2\zeta\bar{\eta}\eta) \tag{B5}$$

and

$$\sum_\sigma \Psi^\dagger_\sigma h^\dagger\left[\begin{bmatrix} & \bar{\partial}\alpha\\ \bar{\partial}\alpha^\dagger & \end{bmatrix},h\right]\psi_\sigma=\sum_\sigma \Psi^\dagger_\sigma\begin{bmatrix}\bar{\eta}\bar{\partial}\alpha+(\overline{\bar{\partial}\alpha^\dagger})\eta & (e^{-2i\theta_\zeta}-1)\bar{\partial}\alpha\\ (e^{2i\theta_\zeta}-1)\overline{\bar{\partial}\alpha^\dagger} & \bar{\eta}\bar{\partial}\alpha+(\overline{\bar{\partial}\alpha^\dagger})\eta\end{bmatrix}\Psi_\sigma. \tag{B6}$$

Combining Eqs. (B4), (B5), and (B6), we obtain

$$\mathcal{L}^*_{\rm susy}\to\sum_\sigma \Psi^\dagger_\sigma\left(\partial_\tau+\lambda'+\begin{bmatrix}\zeta' & \bar{\partial}'\alpha'\\ \overline{\bar{\partial}'\alpha'^\dagger} & -\zeta'\end{bmatrix}\right)\Psi_\sigma+Q_0\alpha'^\dagger\bar{\partial}'\alpha', \tag{B7}$$

where $\lambda'=\lambda+i\dot{\theta}_Q$, $\zeta'=\zeta+i\dot{\theta}_\zeta$, $\alpha'=e^{-2i\theta_\zeta}(\alpha+\eta)$, and $\alpha'^\dagger=e^{2i\theta_\zeta}(\alpha^\dagger+\bar{\eta})$. The primes on the gauged derivatives denote $\bar{\partial}'=(\partial+2\zeta')$ and $\overline{\bar{\partial}'}=(-\partial+2\zeta')$. The Lagrangian is thus gauge invariant under the transformation

$$\Psi_\sigma\to h\Psi_\sigma,\quad \mathcal{V}\to h\mathcal{V},$$

$$\lambda\to\lambda-i\dot{\theta}_Q,\quad \zeta\to\zeta-i\dot{\theta}_\zeta,\quad \alpha\to e^{2i\theta_\zeta}\alpha-\eta,\quad \alpha^\dagger\to e^{-2i\theta_\zeta}\alpha^\dagger-\bar{\eta}. \tag{B8}$$

APPENDIX C: EVALUATION OF FERMIONIC MEAN-FIELD FREE ENERGY

We wish to calculate

$$F_f=-NT\sum_n \ln[\lambda_f+i\Delta_n-i\omega_n]e^{i\omega_n0^+}, \tag{C1}$$

where $\Delta_n=-\Delta\,\mathrm{sgn}\,\omega_n$. We shall regulate this sum by calculating

$$F_f=-NT\sum_n (\ln[\lambda_f+i\Delta_n-i\omega_n]-[\lambda_f\to\lambda_f+D])e^{i\omega_n0^+}, \tag{C2}$$

which we rewrite as

$$\begin{aligned}F_f&=-2NT\,\mathrm{Re}\sum_{n\geq0}(\ln[\xi+i\omega_n]-\ln[\xi+D+i\omega_n])e^{in0^+}\\&=-2NT\,\mathrm{Re}\sum_{n\geq0}\left(\ln\left[\frac{\xi}{2\pi iT}+\frac{1}{2}+n\right]-\ln\left[\frac{\xi+D}{2\pi iT}+\frac{1}{2}+n\right]\right)e^{in0^+},\end{aligned} \tag{C3}$$

where $\xi=\lambda_f+i\Delta$. Next, using the result

$$\sum_{n\geq0}(\ln[b+n]-\ln[a+n])e^{in0^+}=\ln\left(\frac{\Gamma[a]}{\Gamma[b]}\right)+\frac{i\pi}{2}(b-a), \tag{C4}$$

this becomes

$$\frac{F_f}{N}=-2T\,\mathrm{Re}\ln\left(\frac{\Gamma\left[\frac{\xi+D}{2\pi iT}+\frac{1}{2}\right]}{\Gamma\left[\frac{\xi}{2\pi iT}+\frac{1}{2}\right]}\right)+\frac{D}{2}. \tag{C5}$$

APPENDIX D: CALCULATION OF $P(\omega)$

We begin by writing

$$P(\omega)=1+\left(\frac{\omega-2\zeta}{q}\right)\Phi(\omega), \tag{D1}$$

where

$$\Phi(\omega)=\frac{T}{N}\sum_{\sigma\nu}G_{f\sigma}(\omega+\nu)G_{b\sigma}(\nu). \tag{D2}$$

To calculate $\Phi(\omega)$, we replace the discrete Matsubara sum by a Contour integral, to obtain

$$\Phi(\omega)=\frac{1}{N}\sum_\sigma\int\frac{dz}{2\pi i}n(z)G_{b\sigma}(z)G_{f\sigma}(z+\omega), \tag{D3}$$

where the integral runs counterclockwise around the poles in the Green's functions. Using the spectral decomposition,

$$G_{f\sigma}(z)=\int\frac{d\epsilon}{\pi}A_{f\sigma}(\epsilon)\frac{1}{z-\epsilon}, \tag{D4}$$

this becomes

$$\Phi(\omega)=-\frac{1}{N}\sum_{\sigma}\int\frac{d\epsilon}{\pi}[n_{b\sigma}+f(\epsilon)]\frac{1}{\omega-\epsilon+\lambda_{b\sigma}}A_{f\sigma}(\epsilon), \tag{D5}$$

where $n_{b\sigma}=n(\lambda_{b\sigma})$ is the Bose occupancy. Now

$$\begin{aligned}\sum_{\sigma}\int\frac{d\epsilon}{\pi}\frac{n_{b\sigma}}{\omega-\epsilon+\lambda_{b\sigma}}A_{f\sigma}(\epsilon)&=\sum_{\sigma}\frac{n_{b\sigma}}{\omega-(\lambda_{f\sigma}-\lambda_{b\sigma})+i\Delta_n}\\&=\frac{2S}{\omega-2\zeta+i\Delta_n}=2SG_f(\omega+B),\end{aligned} \tag{D6}$$

where we have replaced $2\zeta\rightarrow 2\zeta_0-B$, and $G_f(i\omega_n)=(i\omega_n-2\zeta_0+i\Delta_n)^{-1}$ is the f propagator in the absence of a field, so that

$$P(\omega)=1-\frac{\omega+B-2\zeta_0}{q}[G_f(\omega+B)\tilde{\eta}_b+I], \tag{D7}$$

where $\tilde{n}_b=2S/N$ and

$$\begin{aligned}I&=-\frac{1}{N}\sum_{\sigma}\int\frac{d\epsilon}{\pi}f(\epsilon)\frac{1}{\omega-\epsilon+\lambda_{b\sigma}}A_{f\sigma}(\epsilon)\\&=-\frac{1}{N}\sum_{\sigma}\int\frac{d\epsilon}{\pi}f(\epsilon+m_\sigma B)\frac{1}{\omega-\epsilon+\lambda_0}A_f(\epsilon),\end{aligned} \tag{D8}$$

where $A_f(\omega)=\mathrm{Im}\,G_f(\omega-i\delta)$. Now since $N-2$ of the levels are unshifted, to leading order in the large-N expansion, we can set $m_\sigma=0$ in this expression. Also, since $\lambda_b=B$ in a magnetic field, we can write $I=I(\omega+B)$, where, at $T=0$,

$$I(\omega)=\int_{-D}^{0}d\epsilon\frac{A_f(\epsilon)}{\pi}\frac{1}{\omega-\epsilon}. \tag{D9}$$

Combining these results together, we can write

$$P(\omega,B)=P_0(\omega+B), \tag{D10}$$

where

$$P_0(\omega)=1-\frac{\omega-2\zeta_0}{q}[G_f(\omega)\tilde{n}_b+I(\omega)] \tag{D11}$$

is the zero-field form of $P(\omega)$. Going on to evaluate $I(\omega)$, we obtain

$$\begin{aligned}I(\omega)&=\int_{-D}^{0}d\epsilon\frac{A_f(\epsilon)}{\pi}\frac{1}{\omega-\epsilon}\\&=\int_{-D}^{0}\frac{d\epsilon}{2\pi i}\left(\frac{1}{\epsilon-\xi}-\frac{1}{\epsilon-\xi^*}\right)\frac{1}{\omega-\epsilon}\\&=-\int_{-D}^{0}\frac{d\epsilon}{2\pi i}\left\{\left(\frac{1}{\epsilon-\xi}-\frac{1}{\epsilon-\omega}\right)\frac{1}{\xi-\omega}-[\xi\rightarrow\xi^*]\right\}\\&=\frac{1}{2\pi i}\left\{\frac{1}{\omega-\xi}\ln\left(\frac{\xi}{\omega}\right)-\frac{1}{\omega-\xi^*}\ln\left(\frac{\xi^*}{\omega}\right)\right\},\end{aligned} \tag{D12}$$

where $\xi=T_K e^{i\pi\tilde{n}_f}$, so that

$$P_0(\omega)=1-\frac{(\omega-2\zeta_0)}{q}\left[\frac{n_b}{\omega-\zeta}+\frac{1}{2\pi i}\left\{\frac{1}{\omega-\xi}\ln\left(\frac{\xi}{\omega}\right)-\frac{1}{\omega-\xi^*}\ln\left(\frac{\xi^*}{\omega}\right)\right\}\right]. \tag{D13}$$

Now by writing $\xi^* e^{i2\pi\tilde{n}_f}$, where $\tilde{n}_f=n_f^*/N$, we can put this in the form

$$P_0(\omega)=1-\frac{(\omega-2\zeta_0)}{q}\left[\frac{\tilde{n}_b+\tilde{n}_f}{\omega-\xi}+\frac{1}{2\pi i}\ln\left(\frac{\xi^*}{\omega}\right)\times\left\{\frac{1}{\omega-\xi}-\frac{1}{\omega-\xi^*}\right\}\right]. \tag{D14}$$

Since $\tilde{n}_b+\tilde{n}_f=q$, there are cancellations between the first two terms which give

$$P_0(\omega)=\left[\left(\frac{-i\Delta}{\omega-\xi}\right)-\frac{(\omega-2\zeta_0)}{\pi q}A_f(\omega)\ln\left(\frac{\xi^*}{\omega}\right)\right]. \tag{D15}$$

Another useful way to rewrite this expression is

$$P_0(\omega)=\left[\left(\frac{-i\Delta}{\omega-\xi}\right)-\frac{\Delta}{q\pi}\mathrm{Re}\left(\frac{1}{\omega-\xi}\right)\ln\left(\frac{\xi^*}{\omega}\right)\right]. \tag{D16}$$

To make contact with the bosonic Kondo model, it is useful to split the first term into a real and an imaginary part, so that

$$\frac{-i\Delta}{\omega-\xi}=\Delta A_f(\omega)-i\Delta\,\mathrm{Re}\left(\frac{1}{\omega-\xi}\right). \tag{D17}$$

One can then move the second term above into the logarithm, writing

$$\begin{aligned}\frac{-i\Delta}{q\pi}\mathrm{Re}\left(\frac{1}{\omega-\xi}\right)\ln\left(\frac{\xi^*}{\omega}\right)-i\Delta\,\mathrm{Re}\left(\frac{1}{\omega-\xi}\right)\\=\frac{-i\Delta}{q\pi}\mathrm{Re}\left(\frac{1}{\omega-\xi}\right)\left[\ln\frac{T_K}{\omega}+i\pi\tilde{n}_b\right]\end{aligned} \tag{D18}$$

to obtain

$$P_0(\omega-i\delta)=A_f(\omega)\left[\Delta-\frac{\omega-2\zeta_0}{\pi q}\left(\ln\frac{T_K}{\omega}+i\pi\tilde{n}_b\right)\right]. \tag{D19}$$

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