

## Dimensional crossover in quasi-one-dimensional and high- $T_c$ superconductors

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The one-dimensional electron gas exhibits spin-charge separation and power-law spectral responses to many experimentally relevant probes. Ordering in a quasi-one-dimensional system is necessarily associated with a dimensional crossover, at which sharp quasiparticle peaks, with small spectral weight, emerge from the incoherent background. Using methods of Abelian bosonization, we derive asymptotically correct expressions for the spectral changes induced by this crossover. Comparison is made with experiments on the high-temperature superconductors, which are electronically quasi-one-dimensional on a local scale.

In this paper, we consider the spectral signatures of dimensional crossover in the continuum theory of a quasi-one-dimensional superconductor. This problem is of interest in its own right and for application to materials which are structurally quasi-one-dimensional, such as the Bechgaard salts (organic superconductors). We believe that it is also interesting as a contribution to the theory of the high-temperature superconductors. Although structurally these materials are quasi-two-dimensional, there is both theoretical and experimental evidence<sup>1</sup> of a substantial range of temperatures in which “stripe” correlations make the electronic structure locally quasi-one-dimensional, a phenomenon we have labeled “dynamical dimension reduction.” Similarly, the  $(\text{ET})_{2x}$  organic superconductors are two-dimensional doped antiferromagnets, which we expect to show similar behavior. More generally, the high-temperature superconducting state emerges from a non-Fermi-liquid normal state, often with a normal-state pseudogap. The quasi-one-dimensional superconductor is the only solvable case in which such an evolution can be traced, theoretically.

A quasi-one-dimensional system can be thought of as an array of “chains,” in which the electron dynamics within a chain is characterized by energy scales large compared to the electronic couplings between chains. Since a one-dimensional system cannot undergo a finite temperature phase transition, any ordering transition with a finite critical temperature  $T_c$  is necessarily associated with a dimensional crossover. The electronic properties at temperatures (or energies) large compared to  $T_c$  can be understood by ignoring the interchain coupling, while at lower temperatures or energies, the behavior is that of a three-dimensional system.

In the one-dimensional electron gas<sup>2</sup> (IDEG), as a consequence of spin-charge separation, the elementary excitations are collective modes with unusual quantum numbers and topological properties: The charge excitations are best understood as soundlike density-wave phasons (or, in dual representation, superconducting quasi-Goldstone modes) when

the system is gapless, and as charge solitons with charge  $\pm 1$  and spin 0 when a charge gap is induced. [The precise meaning of the soliton “charge” is a quantized unit of chirality; see Eq. (33) and the subsequent discussion.] Similarly, the spin excitations of a spin-gapped system are spin solitons with charge 0 and spin 1/2. When the elementary excitations do not have the quantum numbers of the experimentally accessible excited states, spectral functions do not exhibit sharp peaks corresponding to a well defined mode with a definite dispersion relation,  $\omega = \epsilon(k)$ . The single hole spectral function  $G^<(k, \omega)$ , which is measured in angle-resolved photoemission spectroscopy (ARPES), involves excited states with charge  $e$  and spin 1/2, which thus consist of at least one charge soliton and one spin soliton. The dynamic spin structure factor  $S(k, \omega)$ , measured by neutron scattering, involves excited states with spin 1, which thus consist of two spin solitons. (We will see in Sec. IV C that, in fact, the relevant excited states contain two spin solitons and at least two charge antisolitons.)

Below  $T_c$ , where the system is three dimensional, we will show that the solitonic excitations of the IDEG are confined in multiplets with quantum numbers that are simply related to those of the electron. For the case of three-dimensional charge-density wave ordering, this has been known for some time. For the case of the superconductor, it is related to the fact, noted recently by Salkola and Schrieffer,<sup>3</sup> that either a spin soliton or a charge soliton induces a  $\pi$  kink in the superconducting correlations. As a consequence of confinement, there is a finite probability of creating a final state consisting of a single bound spin and charge soliton pair in an ARPES experiment. This will show up as a coherent (delta-function) piece in the zero temperature  $G^<(k, \omega)$ .

In this paper, we show that the coherent piece of the single particle spectral functions has a weight which vanishes in the neighborhood of  $T_c$  in proportion to a positive power of the interchain Josephson energy. It is this fact, that the *spectral weight* of the coherent piece is strongly tempera-

ture dependent below  $T_c$ , rather than either the energy or the lifetime of the normal mode, which is the most notable feature that emerges from our analysis. It is highly reminiscent of behavior observed in ARPES (Refs. 4 and 5) and inelastic neutron-scattering<sup>6</sup> measurements on the high-temperature superconductors. We have also identified a resonant feature in the spin spectrum of a quasi-one-dimensional superconductor that emerges at temperatures well below  $T_c$ .

If the IDEG remains gapless down to  $T_c$ , the superconducting transition is BCS-like, in the sense that both pairing and phase coherence occur at the same time. In this case both are induced by the interchain Josephson tunneling. We will mainly be concerned with the case in which a sort of ‘‘pairing,’’ i.e., the opening of a spin gap  $\Delta_s > 0$ , occurs in the one-dimensional (1D) regime well above  $T_c$ . In this case  $T_c$  is primarily associated with phase ordering, and its scale is set by the superfluid density,<sup>7,8</sup> rather than by the zero-temperature single-particle gap scale  $\Delta_0/2$ . In such circumstances the superconducting state, even at very low temperatures, maintains a memory of the separation of charge and spin which is a feature of the 1D normal state. The unique ‘‘coherence length’’ of a BCS superconductor is replaced by two distinct correlation lengths:<sup>9</sup> a spin length,  $\xi_s = v_s/\Delta_s$ , where  $v_s$  is the spin velocity, and a charge length,  $\xi_c = v_c/\Delta_c$ , where  $\Delta_c \sim 2T_c$ .

The remainder of the paper is divided into two self-contained parts; in Secs. I–IV we derive asymptotically exact results for the spectral properties of a quasi-one-dimensional superconductor in the limit of weak interchain coupling. In Sec. V, we summarize the principal results and discuss their application to experiment, especially in the high-temperature superconductors. The reader who is interested only in results, not their derivation, can skip the intervening sections.

The model we study is defined by the Hamiltonian

$$H = \sum_j \int dx \mathcal{H}_j + H_J, \quad (1)$$

where the sum runs over chains,  $H_j$  is the Hamiltonian of the IDEG on chain  $j$ , and  $H_J$  is the Josephson coupling between chains. In Secs. I–III, we consider the single chain problem ( $H_J=0$ ). The problem is formulated using Abelian bosonization in Sec. I. Next, we discuss the spectral functions for the IDEG without (Sec. II) and with (Sec. III) a spin gap—explicit expressions for various quantities in the presence of a spin gap are reported here. In Sec. IV, we extend these results to the case in which the most relevant interchain coupling is the Josephson tunneling. An adiabatic approximation, which is exact in the limit where  $\Delta_s \gg \Delta_c$ , replaces the spin-charge separation of the purely 1D problem as the central feature of the spectrum—this section contains our principal results. Applications to high-temperature superconductors are described in Sec. V. Various appendices expand upon the derivations in Sec. IV.

## I. ABELIAN BOSONIZATION AND THE SPECTRAL FUNCTIONS

We begin by considering the properties of a single chain in the absence of any interchain coupling; we treat this prob-

lem using Abelian bosonization, which is based on the fact that the properties of an interacting IDEG at low energies and long wavelength are asymptotically equal to those of a set of two independent bosonic fields, one representing the charge and the other the spin degrees of freedom in the system. The widely discussed separation of charge and spin<sup>10,11</sup> in this problem is formally the statement that the Hamiltonian density  $\mathcal{H}_j$  can be expressed as

$$\mathcal{H} = \mathcal{H}_c + \mathcal{H}_s, \quad (2)$$

where the chain index is implicit, and the charge and spin pieces of the Hamiltonian are each of the sine Gordon variety,

$$\mathcal{H}_\alpha = \frac{v_\alpha}{2} \left[ K_\alpha (\partial_x \theta_\alpha)^2 + \frac{(\partial_x \phi_\alpha)^2}{K_\alpha} \right] + V_\alpha \cos(\sqrt{8\pi} \phi_\alpha), \quad (3)$$

where  $\alpha = c, s$  for the charge and spin fields, respectively,  $\theta_\alpha$  is the dual field to  $\phi_\alpha$ , or equivalently,  $\partial_x \theta_\alpha$  is the momentum conjugate to  $\phi_\alpha$ . We consider a sufficiently incommensurate IDEG and therefore set  $V_c = 0$  since it arises from umklapp scattering. Of course, if the umklapp scattering is crucial to explain doped insulator behavior, its role cannot be neglected. Where there is no spin gap, or at temperatures large compared to  $\Delta_s$ , we can likewise set  $V_s = 0$ .

When  $V_s$  is relevant (perturbatively, this means  $K_s < 1$ ), the spin gap is dynamically generated, i.e., it depends both on  $V_s$  and the ultraviolet cutoff in the problem,  $\Lambda$ , according to the scaling relation  $\Delta_s \sim v_s \Lambda [V_s/v_s \Lambda^2]^{1/(2-2K_s)}$ . At the gapless fixed point, spin-rotational invariance requires  $K_s = 1$ , at which point  $V_s$  is perturbatively marginal. It is marginally irrelevant for repulsive interactions ( $K_s > 1$ ) and marginally relevant for attractive interactions ( $K_s < 1$ ). Thus the long distance spin physics is described by  $\mathcal{H}_s$  with  $V_s = 0$  and  $K_s = 1$  for a gapless spin-rotationally invariant phase. Where there is a spin gap in a spin rotationally invariant system, it is exponentially small for weak interactions,  $\Delta_s \sim \sqrt{V_s v_s} \exp[-v_s \Lambda^2/2\pi V_s]$ .

In order to compute correlation functions, we use the Mandelstam representation<sup>12</sup> of the fermion field operators

$$\psi_{\lambda,\sigma}(x) = \mathcal{N}_\sigma \exp[i\lambda k_F x - i\Phi_{\lambda,\sigma}(x)], \quad (4)$$

where  $\mathcal{N}_\sigma$  contains both a normalization factor (which depends on the ultraviolet cutoff) and a ‘‘Klein’’ factor (which can be implemented in many ways) so that  $\mathcal{N}_\sigma$  anticommutes with  $\mathcal{N}_{\sigma'}$  for  $\sigma \neq \sigma'$  and commutes with it for  $\sigma = \sigma'$ . In addition,

$$\Phi_{\lambda,\sigma} = \sqrt{\pi/2} [(\theta_c - \lambda \phi_c) + \sigma(\theta_s - \lambda \phi_s)], \quad (5)$$

where  $\lambda = -1$  for left moving electrons,  $\lambda = +1$  for right moving electrons, and  $\sigma = \pm 1$  refers to spin polarization. From Eq. (4), it is a straightforward (and standard<sup>2</sup>) exercise to obtain the boson representations of all interesting electron bilinear and quartic operators. Physically,  $\phi_c$  and  $\phi_s$  are, respectively, the phases of the  $2k_F$  charge-density wave (CDW) and spin-density wave (SDW) fluctuations, and  $\theta_c$  is the superconducting phase. The long-wavelength components of the charge ( $\rho$ ) and spin ( $S_z$ ) densities are given by

$$\rho(x) = \sum_{\lambda, \sigma} \psi_{\lambda, \sigma}^\dagger \psi_{\lambda, \sigma} - \frac{2k_F}{\pi} = \sqrt{\frac{2}{\pi}} \partial_x \phi_c,$$

$$S_z(x) = \frac{1}{2} \sum_{\lambda, \sigma} \sigma \psi_{\lambda, \sigma}^\dagger \psi_{\lambda, \sigma} = \sqrt{\frac{1}{2\pi}} \partial_x \phi_s.$$

When analyzing results for this model, it is always important to remember that the parameters which enter the field theory are renormalized, and are related to the microscopic interactions in a very complicated manner. For instance, although for a single-component IDEG with repulsive interactions  $V_s$  is always irrelevant, for multicomponent IDEG's, and for the ‘‘IDEG in an active environment,’’ it is common to find a dynamically generated spin gap, even when the microscopic interactions are uniformly repulsive.<sup>13–16</sup>

The bosonized expressions for all electron operators are readily extended to an array of chains by adding a chain index to the Bose fields and to the Klein factors; the Klein factors on different chains must now anticommute with each other. Where single-particle interchain hopping is relevant, the Klein factors appear explicitly in the bosonized Hamiltonian. Where only pair hopping and collective interactions between neighboring chains need be included in the low-energy physics, the Klein factors cancel in  $H$ .

While it is generally simpler to derive results concerning the spectrum, it is important for comparison with experiment to compute actual correlation functions. Specifically, we will consider the transverse spin dynamic structure factor

$$\tilde{\mathcal{S}}(x, t; T) \equiv \langle S_{2k_F}^x \dagger(x, t) S_{2k_F}^x(0, 0) \rangle + \langle S_{2k_F}^y \dagger(x, t) S_{2k_F}^y(0, 0) \rangle, \quad (6)$$

where

$$S_{2k_F}^x = \frac{1}{2} \sum_{\sigma, \sigma'} \psi_{1, \sigma}^\dagger \tau_{\sigma \sigma'} \psi_{-1, \sigma'}, \quad (7)$$

and the  $\tau$  are Pauli matrices. We will also consider the one-hole Green's function,

$$\tilde{G}^<(x, t) \equiv \langle \psi_{-1, \uparrow}^\dagger(x, t) \psi_{-1, \uparrow}(0, 0) \rangle, \quad (8)$$

the singlet-pair correlator,

$$\tilde{\chi}(x, t) \equiv \langle \psi_{1, \uparrow}^\dagger(x, t) \psi_{-1, \downarrow}^\dagger(x, t) \psi_{-1, \downarrow}(0, 0) \psi_{1, \uparrow}(0, 0) \rangle, \quad (9)$$

and the various spectral functions,  $\mathcal{S}$ ,  $G^<$ , and  $\chi$ , obtained by Fourier transforming these correlators. As a consequence of separation of charge and spin,  $\tilde{\mathcal{S}}$ ,  $\tilde{G}^<$ , and  $\tilde{\chi}$  are expressible as a product of spin and charge contributions, and therefore,  $\mathcal{S}$ ,  $G^<$ , and  $\chi$  are convolutions. For instance,

$$G^<(k, \omega) = \int \frac{dq}{2\pi} \frac{d\nu}{2\pi} G_s(k-q, \omega-\nu) G_c(q, \nu). \quad (10)$$

## II. HIGH TEMPERATURE: LUTTINGER LIQUID BEHAVIOR

At temperatures large compared to  $T_c$  and the spin gap,  $\Delta_s$  (or at all temperatures in systems in which  $T_c = \Delta_s = 0$ ), the IDEG exhibits ‘‘Luttinger liquid’’ behavior. Because the Luttinger liquid is a quantum critical system, the response

functions have a scaling form. Specifically, this implies<sup>2</sup> that

$$G^<(k, \omega; T) = T^{2\gamma_c + 2\gamma_s - 1} G^<(k/T, \omega/T; 1), \quad (11)$$

where we define for  $\alpha = c$  or  $s$

$$\gamma_\alpha = \frac{1}{8} (K_\alpha + K_\alpha^{-1} - 2), \quad (12)$$

and that so long as  $K_s \geq 1$ ,

$$\mathcal{S}(k, \omega; T) = T^{(K_s^{-1} + K_c - 2)} \mathcal{S}(k/T, \omega/T; 1). \quad (13)$$

Note that here, and henceforth, we will measure  $k$  relative to  $k_F$  and  $2k_F$ , respectively, when computing the scaling functions  $G^<$  and  $\mathcal{S}$ . If the system is spin-rotationally invariant,  $K_s = 1$  in the above expressions.

The form of these scaling functions can be computed analytically in many cases; this has recently been accomplished in Ref. 17. They may or may not have a peak at energies small compared to the bandwidth, depending on certain essential inequalities. Where there is a peak, it occurs at positive energies  $\omega = \pm v_\alpha k + (\text{const})T$ , but the peak width, however defined, does not narrow in proportion to  $T$  at low temperatures; such a peak does *not* correspond to a quasiparticle.

## III. INTERMEDIATE TEMPERATURE: THE LUTHER-EMERY LIQUID

When  $V_s$  is relevant, the spin sine-Gordon theory scales to a strong-coupling fixed point, and the excitations are massive solitons, in which  $\phi_s$  changes by  $\pm \sqrt{\pi}/2$  (i.e.,  $S_z = \pm 1/2$ ). This problem is most simply treated in terms of spin fermion fields,

$$\Psi_{s, \lambda}^\dagger = \mathcal{N}_s \exp[i\sqrt{\pi}/2(\theta_s - 2\lambda\phi_s)]. \quad (14)$$

The refermionized form of the Hamiltonian is then

$$\mathcal{H}_s = i\tilde{v}_s [\Psi_{s, -1}^\dagger \partial_x \Psi_{s, -1} - \Psi_{s, 1}^\dagger \partial_x \Psi_{s, 1}] + \tilde{\Delta}_s [\Psi_{s, 1}^\dagger \Psi_{s, -1} + \text{H.c.}] + g_s \Psi_{s, 1}^\dagger \Psi_{s, -1} \Psi_{s, -1} \Psi_{s, 1}, \quad (15)$$

where

$$\tilde{v}_s = v_s \left( \frac{1}{4K_s} + K_s \right),$$

$$\tilde{\Delta}_s = \frac{\pi V_s}{\Lambda},$$

$$g_s = 2\pi v_s \left( \frac{1}{4K_s} - K_s \right). \quad (16)$$

For  $K_s = 1/2$ , which is known as the Luther-Emery point,<sup>10</sup> the refermionized model is noninteracting and massive, with a gap  $\Delta_s = \tilde{\Delta}_s$ . Assuming there is a single massive phase of the sine-Gordon theory, the Luther-Emery model<sup>10</sup> will exhibit the same asymptotic behavior as any other model in this phase. Formally, the Luther-Emery point can be thought of as a strong-coupling fixed point Hamiltonian, and  $g_s$ , which vanishes at the fixed point, is the amplitude of a leading irrelevant operator.<sup>18</sup> We will henceforth compute

correlation functions at the Luther-Emery point, and then comment on the effects of deviations from this point.

Now, in computing the various spectral properties of the system, we can distinguish two regimes of temperature: at temperatures large compared to  $\Delta_s$ , the spin gap is negligible, and the results for the Luttinger liquid apply. If the temperature is small compared to the spin gap, then we can compute the spin contributions to the various correlation functions in the zero-temperature limit, and only make exponentially small errors of order  $\exp(-\Delta_s/T)$ . The spin piece of the transverse spin response function can be expressed in terms of the spin fermion fields

$$\tilde{S}_s(x,t) = \langle \Psi_{s,1}^\dagger(x,t) \Psi_{s,-1}^\dagger(x,t) \Psi_{s,-1}(0,0) \Psi_{s,1}(0,0) \rangle. \quad (17)$$

Since the theory reduces, at the Luther-Emery point, to a theory of free massive fermions, the corresponding spectral function can be readily computed with the result, for  $T=0$ ,

$$S_s(k,\omega) = \frac{\omega^2 - 4E_s^2(k/2)}{4v_s^2[q_1 E_s(q_2) - q_2 E_s(q_1)]} \Theta[\omega - 2E_s(k/2)], \quad (18)$$

where the spin soliton spectrum is

$$E_s(k) = \sqrt{v_s^2 k^2 + \Delta_s^2}, \quad (19)$$

and  $q_{1,2}$  are the solutions of the quadratic equation  $\omega + E_s(q) + E_s(k+q) = 0$ . Explicitly,

$$q_{1,2} = \frac{k}{2} \pm \frac{\omega}{2v_s} \sqrt{1 + \frac{4\Delta_s^2}{v_s^2 k^2 - \omega^2}}. \quad (20)$$

The spin piece of the one hole Green's function is more complicated, since it involves nonlocal operators in the re-fermionized form:

$$\tilde{G}_s(x,t) = \langle U_s^\dagger(x,t) \Psi_{s,-1}^\dagger(x,t) \Psi_{s,-1}(0,0) U_s(0,0) \rangle, \quad (21)$$

where the vertex operator  $U_s(x) = e^{i\sqrt{\pi/2}\phi_s(x)}$  with  $\phi_s(x) = \sqrt{\pi/2} \sum_\lambda \int^x dy \Psi_{s,\lambda}^\dagger \Psi_{s,\lambda}$ . From kinematics, it follows that this Green's function consists of a coherent one spin soliton piece and an incoherent multisoliton piece:

$$G_s(k,\omega) = Z_s(k) \delta[\omega - E_s(k)] + G_s^{(multi)}(k,\omega), \quad (22)$$

where the multisoliton piece is proportional to  $\Theta[\omega - 3E_s(k/3)]$ . (Deviations from the Luther-Emery point in the case  $g_s > 0$  will result in the formation of a spin soliton-antisoliton bound state, a "breather," which can shift the threshold energy for multisoliton excitations somewhat.)

At the Luther-Emery point it is possible to obtain closed-form expressions<sup>19</sup> for the matrix elements of the vertex operator between the vacuum and various multisoliton states and from that to compute  $Z_s$  explicitly. We will report this calculation in a forthcoming paper, Ref. 17. Here, we use a simple scaling argument, which can be generalized to the case of nonzero interchain coupling, to derive the principal features of this result, especially the dependence of  $Z_s$  on  $\Delta_s$ . In the absence of a spin gap, and at  $T=0$ ,  $G_s$  can be readily evaluated to give the scaling form

$$G_s = \frac{\pi (v_s \Lambda)^{1/2 - 2\gamma_s}}{\Gamma(\gamma_s) \Gamma(\gamma_s + \frac{1}{2})} (\omega + v_s k)^{\gamma_s - 1} (\omega - v_s k)^{\gamma_s - 1/2} \times \Theta(\omega - v_s |k|). \quad (23)$$

Because the sine-Gordon field theory is asymptotically free, the high energy spectrum, and hence the dependence of  $G_s$  on  $\Lambda$ , is unaffected by the opening of a spin gap. With this observation, it is simply a matter of dimensional analysis to see that

$$Z_s(k) = (\Lambda \xi_s)^{1/2 - 2\gamma_s} f_s(k \xi_s), \quad (24)$$

where  $\xi_s = v_s / \Delta_s$  is the spin correlation length.  $f_s$  is a scaling function which is independent of  $K_s$ . It can be calculated<sup>17</sup> using the exact matrix elements available for  $K_s = 1/2$ , with the result

$$f_s(x) = c \left( 1 - \frac{x}{\sqrt{1+x^2}} \right), \quad (25)$$

where  $c$  is a numerical constant.

The above extends the earlier results of Voit<sup>20</sup> and Wiegmann.<sup>21</sup> In particular, the analytic structure (as a function of  $k$  and  $\omega$ ) of the one soliton contribution to Eq. (22) reproduces that found in earlier work. Those works did not discuss the nonanalyticity at the three soliton threshold, although these are fairly obvious; more muted singularities occur at the five and higher multisoliton thresholds, which we will not discuss explicitly. The specific expression in Eq. (24) is the most important feature of this result for the purposes of the present paper.

The charge pieces of both response functions are unaffected by the opening of the spin gap. Consequently,  $S$  and  $G^<$  have power-law features (which can be a peak or a shoulder depending on  $K_c$ ) at  $\omega = 2E_s(k/2) + \mathcal{O}(T)$  and  $\omega = E_s(k) + \mathcal{O}(T)$ , respectively, with a shape and temperature dependence, both readily computed, determined by the still gapless charge-density fluctuations. For example, we can evaluate the spectral function explicitly<sup>17</sup> at  $T=0$  in the limit  $v_s/v_c \rightarrow 0$  [and for arbitrary  $\omega < 3E_s(k/3)$ ], or when  $|\omega - \Delta_s| \ll \Delta_s$  (for arbitrary  $v_s/v_c$ ):

$$G^<(k,\omega) = \frac{1}{4} \frac{B(\gamma_c, \gamma_c + \frac{1}{2})}{\Gamma(\gamma_c) \Gamma(\gamma_c + \frac{1}{2})} \left( \frac{\Lambda v_c}{2} \right)^{-1/2 - 2\gamma_c} \times Z_s(k) [\omega - E_s(k)]^{2\gamma_c - 1/2} \Theta[\omega - E_s(k)]. \quad (26)$$

Here  $B(x,y)$  is the beta function. Again, the fact that these excitations are not quasiparticles is reflected in the fact that even where peaks in the spectral function occur, they do not narrow indefinitely as  $T \rightarrow 0$ .

In the presence of a spin gap, the spin contribution to the long-distance behavior of the superconducting susceptibility is a constant;

$$\tilde{\chi}(x,t) \sim |\langle U_s^2 \rangle|^2 \langle e^{i\sqrt{2\pi}\theta_c(x,t)} e^{-i\sqrt{2\pi}\theta_c(0,0)} \rangle \sim (\Lambda \xi_s)^{-K_s} \langle e^{i\sqrt{2\pi}\theta_c(x,t)} e^{-i\sqrt{2\pi}\theta_c(0,0)} \rangle. \quad (27)$$



From this, one sees that, within a chain, one can identify  $(\Lambda \xi_s)^{-K_s/2}$  as the ‘‘amplitude’’ and  $\sqrt{2\pi}\theta_c$  as the ‘‘phase’’ of the order parameter.<sup>13</sup>

#### IV. LOW TEMPERATURE: THE 3D SUPERCONDUCTING STATE

For temperatures of order  $T_c$  and below, interchain couplings cannot be ignored. Single-particle hopping and all magnetic couplings are irrelevant by virtue of the pre-existing spin gap. For  $K_c > 1/2$ , the Josephson coupling is perturbatively relevant, but for  $K_c < 1$ , the  $2k_F$  CDW coupling is more relevant. For the simplest realizations of the IDEG,  $K_c < 1$  corresponds to repulsive interactions between charges. However, we have shown<sup>22,13</sup> that for fluctuating or meandering stripes, such as occur in the high-temperature superconductors, the CDW coupling gets dephased, so that the Josephson coupling is the most relevant, even when  $1/2 < K_c < 1$ .

Since we are interested in the onset of superconductivity, we consider the case in which the Josephson coupling between chains is more relevant. The pair tunneling interaction between chains, which appeared in Eq. (1), can be simply bosonized:

$$H_J = -J_{SC} \sum_{(i,j)} \int dx [\Delta_i^\dagger \Delta_j + \text{H.c.}], \quad (28)$$

where the pair-creation operator on chain number  $j$  is

$$\begin{aligned} \Delta^\dagger(x, t) &= \psi_{1,\uparrow}^\dagger \psi_{-1,\downarrow}^\dagger + \psi_{-1,\uparrow}^\dagger \psi_{1,\downarrow}^\dagger \\ &\propto \cos(\sqrt{2\pi}\phi_s) \exp(i\sqrt{2\pi}\theta_c), \end{aligned} \quad (29)$$

and we have left the chain index implicit.

Since the state below  $T_c$  has long-range order, and since we assume that the coupling between chains is weak, it is reasonable to treat it in mean-field approximation,<sup>23</sup> although we continue to treat the one-dimensional fluctuations exactly. Thus, rather than considering a full three-dimensional problem, we consider the effective single chain problem defined by the Hamiltonian

$$\mathcal{H} = \mathcal{H}_s + \mathcal{H}_c - \mathcal{J} \cos(\sqrt{2\pi}\phi_s) \cos(\sqrt{2\pi}\theta_c), \quad (30)$$

where  $\mathcal{J}$  is related to the pair tunneling amplitude by the mean-field relation

$$\mathcal{J} = z J_{SC} (\Lambda/\pi)^2 \langle \cos(\sqrt{2\pi}\phi_s) \cos(\sqrt{2\pi}\theta_c) \rangle, \quad (31)$$

where  $z$  is the number of nearest-neighbor chains. [Since the average of  $\cos(\sqrt{2\pi}\phi_s) \sin(\sqrt{2\pi}\theta_c)$  vanishes, no sine term appears in the effective Hamiltonian (30).] Note that the pair hopping term in Eq. (30) couples charge and spin, as is characteristic of higher dimensional couplings.

The mean-field approximation is exact in the limit of large  $z$  and small  $zJ_{SC}$ . In three dimensions, this mean-field approximation will produce some errors in the critical regime in the vicinity of  $T_c$ , but because of the long correlation length along the chain just above  $T_c$ , the critical region is always small for small  $J_{SC}$ , and well below  $T_c$ , this approximation is safe.<sup>24</sup>

Because of the presence of relevant cosine terms, there are superselection rules which divide Hilbert space into various soliton sectors. The soliton sectors are specified by two integrals:

$$\begin{aligned} N_s &= \sqrt{2/\pi} \int_{-\infty}^{\infty} dx \partial_x \phi_s = \sqrt{2/\pi} [\phi_s(\infty) - \phi_s(-\infty)] \\ &= 2 \int dx S_z, \end{aligned} \quad (32)$$

and

$$N_c = \sqrt{2/\pi} \int_{-\infty}^{\infty} dx \partial_x \theta_c = \sqrt{2/\pi} [\theta_c(\infty) - \theta_c(-\infty)]. \quad (33)$$

$N_s$  is simply the number of spin solitons minus the number of antisolitons or the total value of  $S_z$  in units of  $\hbar/2$ . The interpretation of  $N_c$  is a bit more subtle. Since we are looking at a superconducting state, the electrostatic charge of a quasiparticle is not defined.<sup>25–27</sup> However,  $N_c$  is a conserved ‘‘chirality’’ equal to the number of right moving minus the number of left moving electrons, so that we can still interpret  $eN_c$  as a sort of quasiparticle ‘‘charge’’; it represents the coupling of the quasiparticles to a magnetic flux.<sup>28,25,26</sup>

The presence of the  $\cos(\sqrt{8\pi}\phi_s)$  term in the single chain Hamiltonian results in the quantization of  $N_s$  in integer units. The presence of the  $\cos(\sqrt{2\pi}\phi_s) \cos(\sqrt{2\pi}\theta_c)$  term in  $\mathcal{H}$  results in the quantization condition that  $N_s + N_c$  be an even integer! Physically, this means that excitations can have spin  $\hbar$  and charge 0 ( $N_s = 2$  and  $N_c = 0$ ), spin 0 and charge 2 ( $N_s = 0$  and  $N_c = 2$ ), spin  $\hbar/2$  and charge 1 ( $N_s = 1$  and  $N_c = 1$ ), etc., but that all the exotic quantum numbers of the soliton excitations of the isolated IDEG are killed. Formally, the addition of the pair hopping term to the Hamiltonian of the IDEG leads to a confinement phenomenon. Along the entire segment of chain between two spatially separated  $\pm \sqrt{\pi/2}$  solitons, there is a change in sign of the pair hopping term [see Eq. (29)]. This leads to an energy which grows linearly with the separation  $x$  between solitons,  $\sim \mathcal{J}|x|$ , regardless of whether they are charge or spin solitons or antisolitons.

The importance of this observation becomes clear when we study the operators in whose correlation functions we are interested. Since

$$e^{i\sqrt{\pi/2}\theta_s(x)} \phi_s(y) e^{-i\sqrt{\pi/2}\theta_s(x)} = \phi_s(y) - \sqrt{\pi/2} \Theta(y-x), \quad (34)$$

and

$$e^{i\sqrt{\pi/2}\phi_c(x)} \theta_c(y) e^{-i\sqrt{\pi/2}\phi_c(x)} = \theta_c(y) + \sqrt{\pi/2} \Theta(x-y), \quad (35)$$

it is clear that the fermion annihilation operator  $\Psi_{-1,\uparrow}$  creates a spin antisoliton and a charge antisoliton, while the  $2k_F$  piece of the spin-raising operator,  $S_{2k_F}^+$ , creates a pair of spin solitons and a pair of charge antisolitons. Both these combinations decay into a set of free solitons in the absence of the interchain coupling, but in its presence, the former becomes a bound state, and the latter a resonant state. Thus  $G^<$  develops a coherent piece with a well defined dispersion rela-

tion as superconducting phase coherence between chains occurs.  $\mathcal{S}$  develops a resonant peak at a temperature well below  $T_c$ .

### A. Zero-spin soliton sector

For the case in which the spin gap  $\Delta_s$  of the isolated chain is large compared to the interchain coupling, the fluctuations of the spin field are high energy (fast) compared to any charge fluctuations, and indeed only slightly affected by the onset of superconducting order. In this limit, the eigenstates can be treated in the adiabatic approximation.

In the ground state ( $N_s=0$ ) sector, the spin field fluctuations are little affected by  $\mathcal{H}_J$ ; all spin correlations can thus be computed as in the previous section. Moreover, because of the spin gap, so long as  $T \ll \Delta_s$ , the spin fields can be approximated by their ground state. For computing the charge part of the wave function, we can replace the operator  $\cos(\sqrt{2\pi}\phi_s)$  in  $\mathcal{H}_J$  by its expectation value at zero temperature in the decoupled ground state,

$$\cos(\sqrt{2\pi}\phi_s) \rightarrow \langle \cos(\sqrt{2\pi}\phi_s) \rangle_o \equiv \mathcal{C}_s \sim (\Lambda \xi_s)^{-K_s/2}, \quad (36)$$

where the subscript ‘‘o’’ refers to the expectation value in the ensemble with  $J_{SC}$  set equal to zero (see also Ref. 29). This leaves us with a sine-Gordon equation for the charge degrees of freedom, with potential

$$\mathcal{H}_c \cos(\sqrt{2\pi}\theta_c). \quad (37)$$

Again, we solve this problem by refermionizing

$$\Psi_{c,\lambda}^\dagger = \mathcal{N}_c \exp[i\sqrt{\pi/2}(\theta_c - 2\lambda\phi_c)]. \quad (38)$$

The refermionized form of the Hamiltonian is

$$\begin{aligned} \mathcal{H}_c = & i\tilde{v}_c [\Psi_{c,-1}^\dagger \partial_x \Psi_{c,-1} - \Psi_{c,1}^\dagger \partial_x \Psi_{c,1}] \\ & - \tilde{\Delta}_c [\Psi_{c,1}^\dagger \Psi_{c,-1} + \text{H.c.}] \\ & + g_c \Psi_{c,1}^\dagger \Psi_{c,-1}^\dagger \Psi_{c,-1} \Psi_{c,1}, \end{aligned} \quad (39)$$

where

$$\begin{aligned} \tilde{v}_c = & v_c \left( \frac{1}{4K_c} + K_c \right), \\ \tilde{\Delta}_c = & \frac{\pi \mathcal{C}_s}{\Lambda}, \\ g_c = & 2\pi v_c \left( \frac{1}{4K_c} - K_c \right). \end{aligned} \quad (40)$$

Since  $N_s=0$ , the superselection rule implies  $N_c=2m$ , which upon refermionization is simply the condition

$$-\sum_\lambda \lambda \int dx [\Psi_{c,\lambda}^\dagger \Psi_{c,\lambda}] = N_c/2 = m. \quad (41)$$

It is also interesting to note that the superconducting pair-creation operator can be expressed in an intuitively appealing form in terms of charge soliton creation operators

$$\hat{\Delta}^\dagger \propto \cos(\sqrt{2\pi}\phi_s) \Psi_{c,1}^\dagger \Psi_{c,-1}^\dagger. \quad (42)$$

Recall that here the charge solitons are spinless fermions. This expression emphasizes<sup>13</sup> the fact that spin gap formation, which is associated with the quenching of the fluctuations of the spin-density phase,  $\phi_s$ , can also be identified with the growth of the *amplitude* of the superconducting order parameter. While the charge solitons clearly also make a contribution to the amplitude of the order parameter, the phase of the order parameter comes entirely from the charge.

For  $K_c=1/2$ , just as for the Luther-Emery point for the spin fields, the refermionized Hamiltonian for the charged excitations is noninteracting and massive (gapped), and  $\Delta_c = \tilde{\Delta}_c$ . In computing the asymptotic form of correlations we will set  $K_c=1/2$ . We can now readily compute the expectation value of the pair hopping term so as to relate two physically important quantities: the excitation energy scale  $\Delta_c$  and the interchain portion of the internal energy

$$\begin{aligned} \Delta_c \langle \Psi_{c,1}^\dagger \Psi_{c,-1}^\dagger + \text{H.c.} \rangle &= \mathcal{J} \langle \cos(\sqrt{2\pi}\phi_s) \cos(\sqrt{2\pi}\theta_c) \rangle \\ &= (\Delta_c / \pi \xi_c) u_0(\Delta_c, T), \end{aligned} \quad (43)$$

where  $\xi_c = v_c / \Delta_c$  is the charge correlation length. Equation (43) has the form of a BCS gap equation with

$$u_0(\Delta_c, T) = \int_0^{v_c \Lambda} dx \frac{1}{\sqrt{x^2 + \Delta_c^2}} \tanh\left(\frac{1}{2T} \sqrt{x^2 + \Delta_c^2}\right), \quad (44)$$

where the mean-field relation for  $\Delta_c(T)$  is

$$u_0(\Delta_c, T) = \frac{\pi v_c}{z J_{SC} \mathcal{C}_s^2}. \quad (45)$$

Consequently, we find the familiar BCS relations

$$T_c = 0.57 \Delta_c(0), \quad (46)$$

$$\Delta_c(0) = 2v_c \Lambda \exp[-\pi v_c / z J_{SC} \mathcal{C}_s^2], \quad (47)$$

$$\Delta_c(T) \approx 1.74 \Delta_c(0) \sqrt{1 - T/T_c} \text{ for } T \approx T_c. \quad (48)$$

In general, the actual form of  $\Delta_c(0)$  in terms of  $J_{SC}$  and  $\mathcal{C}_s$  is modified according to the microscopic value of  $K_c$ .

The transverse superconducting phase stiffness  $\kappa_\perp$  (proportional to the superfluid density) is

$$\kappa_\perp = 2\pi a \langle H_J \rangle, \quad (49)$$

where  $d$  is the spacing between chains and  $\langle H_J \rangle$  is given in Eq. (43). Thus at zero temperature  $\kappa_\perp \sim T_c^2 / v_c$ . As is shown in Appendix B, for a system with equal areas of domains in which the stripes run along the  $x$  and  $y$  directions, the macroscopic phase stiffness is equal to the geometric mean of the superfluid density in the directions parallel and perpendicular to the chains,  $\bar{\kappa} = \sqrt{\kappa_\parallel \kappa_\perp}$ . Since the phase stiffness along the chains is simply  $\kappa_\parallel = v_c K_c$ , it follows that  $\bar{\kappa}(T=0)$  is (up to logarithmic corrections coming from  $u_0$ ) simply proportional to  $T_c$ . This is a microscopic realization of a more general phenomenon which occurs in systems with low superfluid density;<sup>8</sup> it is phase ordering, as opposed to pairing, which determines  $T_c$ . In a future publication,<sup>30</sup> we will study the effects of quantum and thermal phase fluctuations on the evolution of the superfluid density of a quasi-one-dimensional superconductor.

With little additional effort, we can study the pair field susceptibility  $\chi$  at energies small compared to  $2\Delta_s$ . In this low-energy limit, as in Eq. (27), we can replace the spin operators in  $\chi$  by their ground-state expectation values.

The charge part of  $\tilde{\chi}$  can be expressed in terms of the charge fermion fields:

$$\tilde{\chi}_c(x,t) = \langle \Psi_{c,1}^\dagger(x,t) \Psi_{c,-1}^\dagger(x,t) \Psi_{c,-1}(0,0) \Psi_{c,1}(0,0) \rangle. \quad (50)$$

At the free charge fermion point ( $K_c = 1/2$ ) the corresponding spectral function is readily evaluated, for  $T=0$  and  $\omega \ll 2\Delta_s$ , with the result

$$\begin{aligned} \chi(k, \omega) = & \left( \frac{C_s u_0}{\xi_c} \right)^2 \delta(k) \delta(\omega) \\ & + \frac{C_s^2 [\omega^2 - 4E_c^2(k/2) + 2\Delta_c^2]}{4v_c^2 |q_1 E_c(q_2) - q_2 E_c(q_1)|} \Theta[\omega - 2E_c(k/2)], \end{aligned} \quad (51)$$

where  $E_c(k)$  and  $q_{1,2}$  are the analogs of Eqs. (19) and (20) with  $\Delta_c$  substituted for  $\Delta_s$  and  $v_c$  for  $v_s$ .

Away from the Luther-Emery point, if  $g_c > 0$  ( $K_c < 1/2$ ), the two solitons repel, and hence the effect of  $g_c$  can be ignored, but for  $g_c < 0$  ( $K_c > 1/2$ ), there is an attractive interaction between the two solitons and hence, this being after all a one-dimensional problem, they form a bound state. This will slightly modify the expression for  $\chi$ .

### B. The one hole sector

In the one soliton sector of the spin Hamiltonian, the adiabatic approximation requires reexamination. While for the most part, the spin modes are fast compared to the charge modes, the Goldstone mode (translation mode of the spin soliton) is slow compared to all other modes, and so must be treated in the inverse adiabatic approximation. Thus we consider the charge Hamiltonian with a spin antisoliton at fixed position  $R_s$ . The pair tunneling term is then

$$\mathcal{H}_s \operatorname{sgn}(x - R_s) \cos(\sqrt{2\pi}\theta_c), \quad (52)$$

where we have used the fact that  $\xi_s = v_s/\Delta_s$  (which characterizes the width of the spin soliton) is small compared to the charge correlation length,  $\xi_c = v_c/\Delta_c$ , to approximate the profile of the spin soliton by a step function. Upon refermionization, the charge Hamiltonian is still of the same form as Eq. (39) with the term proportional to  $\tilde{\Delta}_c$  replaced by

$$\tilde{\Delta}_c \rightarrow -\tilde{\Delta}_c \operatorname{sgn}(x - R_s). \quad (53)$$

For  $K_c = 1/2$ , upon the canonical transformation,

$$\psi_{c,-1} = \Psi_{c,-1}^\dagger, \quad \psi_{c,1} = \Psi_{c,1}, \quad (54)$$

the charge soliton Hamiltonian is of the same form<sup>31</sup> as the fermionic Hamiltonian of a commensurability two Peierls insulator, ‘‘polyacetylene,’’ in the presence of a topological soliton. As is well known,<sup>31</sup> there is an index theorem that implies the existence of a zero energy bound state associated with the soliton, the famous ‘‘midgap state’’ or ‘‘zero mode.’’ All other fermionic states have energies greater than

or equal to  $\Delta_c$ . Importantly, since in this sector  $N_s = -1$ , the superselection rule  $N_c = 2m + 1$ , requires that the fermion number is half integer!

$$-\sum_\lambda \int dx: [\psi_{c,\lambda}^\dagger \psi_{c,\lambda}]: = N_c/2 = m + 1/2. \quad (55)$$

This is essential, since with the midgap state occupied the fermion number is<sup>31</sup>  $+1/2$ , while with it empty the fermion number is  $-1/2$ . The midgap state is associated with the bound state of the spin and charge antisolitons.

To compute the charge contribution to the soliton creation energy we need to evaluate the difference between the ground-state energies of the charge Hamiltonian in the presence and absence of a kink. We have done this by taking the limit of vanishing soliton width of a general expression of Takayama, Lin-Liu, and Maki,<sup>32</sup> (and dividing by 2 for the spinless case). The resulting soliton creation energy is just  $\Delta_c/2$ ; in other words, the rest energy of the electron, i.e., the bound state of a spin soliton and a charge soliton, is

$$\Delta_0 = \Delta_s + \Delta_c/2 \approx \Delta_s. \quad (56)$$

From this discussion, we can immediately conclude that for  $T \ll T_c \ll \Delta_s$ , the one hole spectral function has a coherent piece and a multiparticle incoherent piece,

$$G^<(k, \omega) = Z(k) \delta[\omega - \mathcal{E}(k)] + G^{(multi)}(k, \omega), \quad (57)$$

where

$$\mathcal{E}(k) = \sqrt{v_s^2 k^2 + \Delta_0^2}. \quad (58)$$

This follows from the fact that the bound state of a spin soliton and a charge soliton has the same quantum numbers as a hole. The multiparticle piece<sup>33</sup> has a threshold slightly above the single hole threshold at  $\omega = \mathcal{E}(k) + 2\Delta_c$ .

The overlap factor  $Z(k)$  contains factors from both the spin and the charge parts of the wave function; so long as  $k\xi_s \ll 1$ ,  $Z(k) = Z_c(k)Z_s(0)$  where  $Z_s(0)$  depends on the spin correlation length as in Eq. (24), and  $Z_c(k)$  contains all remaining contributions. We can obtain a scaling form for  $Z_c$  using the same method of analysis employed previously for  $Z_s$ . Specifically, at  $T=0$  in the absence of interchain coupling, and for  $\omega \ll 3\Delta_s$  and  $|k\xi_s| \ll 1$ ,  $G^<$  is given by the expression in Eq. (26). Since the opening of a charge gap does not affect the high-energy physics, the dependence of  $G^<$  on  $\Lambda$  is unaffected by the interchain coupling. Indeed, so long as  $\Delta_c \ll \Delta_s$ , the dependence of  $G^<$  on  $\Delta_s$  is likewise unchanged. Thus, by dimensional analysis, it follows that

$$Z(k) = Z_s(0) (\Lambda \xi_c)^{-1/2-2\gamma_c} A_{\gamma_c} \tilde{f}(k\xi_c), \quad (59)$$

where  $\tilde{f}$  is a scaling function and

$$A_{\gamma_c} = \frac{B(\gamma_c, \gamma_c + 1/2)}{\Gamma(\gamma_c)\Gamma(\gamma_c + 1/2)}. \quad (60)$$

Unfortunately, we do not have exact results from which to compute  $\tilde{f}(x)$  explicitly, but there is no reason to expect it to have any very interesting behavior for small  $x$ .

At temperatures between  $T=0$  and  $T=T_c$ , the same arguments lead to a simple approximate expression for the

spectral function. Specifically, the principal temperature dependence comes from  $\Delta_c$  which is a decreasing function of  $T$ . At mean-field level, the temperature dependence of  $\Delta_c$  can be computed from Eq. (44). In particular, it vanishes at  $T_c$  according to Eq. (48). Since fluctuation effects produce superconducting correlations between neighboring chains at temperatures above  $T_c$ , this simple mean-field behavior will be somewhat rounded, but the qualitative point that  $\Delta_c$  becomes small at temperatures above  $T_c$  is quite robust.

Consequently, the quasiparticle weight  $Z$ , which is proportional to  $\Delta_c^{2\gamma_c+1/2}$ , is a strongly decreasing function of  $T$  which vanishes in the neighborhood of  $T_c$ . The quasiparticle gap,  $\Delta_0$ , on the other hand, is only weakly temperature dependent, dropping from its maximum value  $\Delta_0 = \Delta_s + \frac{1}{2}\Delta_c(0)$  at  $T=0$  to  $\Delta_0 = \Delta_s$  in the neighborhood of  $T_c$ . Scattering off thermal excitations will, of course, induce a finite lifetime for the quasiparticle at finite temperatures.

Neither a charge soliton nor a spin soliton can hop from one chain to the next, but a hole can. The problem of the transverse dispersion of the coherent peak in the single hole spectral function is addressed in Appendix A. Not surprisingly, we find that the effective interchain hopping matrix element  $t_\perp$  is replaced by an effective interchain hopping matrix element,

$$t_\perp^{eff} = Z(k)t_\perp. \quad (61)$$

Thus the dispersion of the coherent peak transverse to the chain direction is an independent measure of the degree of interchain coherence.

### C. The two spin soliton sector

To compute  $\mathcal{S}$ , we need to study states in the  $N_s = 2$  sector. Interestingly (in contrast to the case of an ordered CDW), in a quasi-one-dimensional superconductor, the  $2k_F$  spin-density wave operator also creates two charge antisolitons:  $N_c = -2$ . Again, for the most part, the spin fluctuations are fast and high energy compared to the scale of the charge fluctuations, and can thus be treated in the adiabatic approximation—indeed, they are little affected by the presence of the interchain Josephson coupling. However, there are two low-frequency modes associated with the soliton translational degrees of freedom, which must be treated in the antiadiabatic approximation. Consequently, we obtain an effective Schrödinger equation governing the center of mass motion of the two spin solitons:<sup>34</sup>

$$H^{eff} \approx 2\Delta_s - \frac{1}{2M^*} \sum_{j=1}^2 \frac{\partial^2}{\partial x_j^2} + V(x_1 - x_2), \quad (62)$$

where  $x_j$  is the position of soliton  $j$ ,

$$M^* = \Delta_s / v_s^2, \quad (63)$$

and  $V$  is the adiabatic spin soliton potential, obtained by integrating out the (relatively fast) fluctuations of the charge degrees of freedom.

To compute  $V(R)$ , we again rely on the analogy between the refermionized version of the charge part of the Hamiltonian and solitons in polyacetylene,<sup>31,32,35</sup> In this case,  $V(R)$  is recognized as the difference in the ground-state energy of

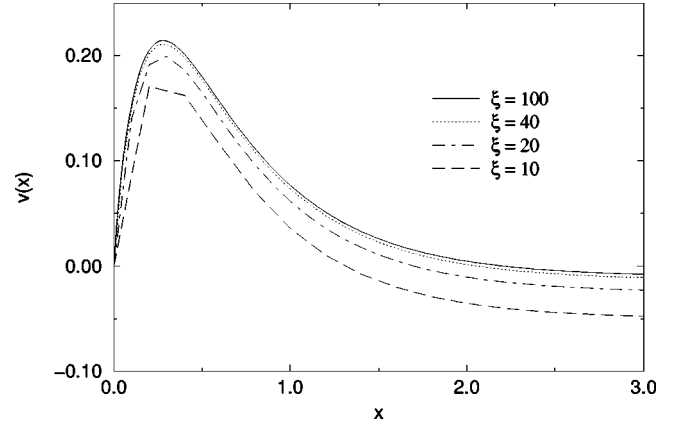


FIG. 1. The scaled version of the adiabatic potential  $v(x)$ , defined in Eq. (67), computed from the SSH model in a system of size 3000 sites with open boundary conditions. The different curves are for different magnitudes of dimerization corresponding to a coherence length of the indicated magnitude in units of the lattice constant.

a massive Dirac fermion in the presence and absence of a pair of zero width solitons separated by a distance  $R$ , i.e., the Hamiltonian in Eq. (39) with

$$\tilde{\Delta}_c \rightarrow \tilde{\Delta}_c \operatorname{sgn}(4x^2 - R^2). \quad (64)$$

Since  $2N_c = -2$ , this energy difference is to be computed in the fermion number  $-1$  sector.

From the results in the previous section, it follows that

$$V(R) \rightarrow \Delta_c \text{ as } R \rightarrow \infty, \quad (65)$$

since in this limit, the two solitons are noninteracting, and reduce to the solution discussed in the previous section. Similarly, since for  $R=0$ , the energy approaches that of the uniform system with fermion number  $-1$ ,

$$V(R) \rightarrow \Delta_c \text{ as } R \rightarrow 0. \quad (66)$$

Moreover, from simple scaling, it is clear that

$$V(R) = \Delta_c [1 + v(R/\xi_c)], \quad (67)$$

where  $v(x)$  is independent of the magnitude of  $\Delta_c$  and  $v(x) \rightarrow 0$  for  $x \rightarrow 0$  and  $x \rightarrow \infty$ . For intermediate  $R/\xi_c$ , we have been unable to obtain an analytic expression for  $v$ , although it is easily derived numerically, as described in Appendix C, with the result shown in Fig. 1. As can be seen,  $v(x)$  rises from 0 to a gentle maximum at  $x \approx 0.3$  where  $v(0.3) \approx 0.2$ , and then drops exponentially back to zero at large separation.

What this means is that there is no true bound state in the spin-1 excitation spectrum. The spin-1 excitations, even in the superconducting state, are always unstable to decay into a pair of far separated spin-1/2 quasiparticles. However, near the threshold energy,  $\omega = 2\Delta_s + \Delta_c$ , there is a nearly bound (resonant) state with a lifetime which is exponentially long. Treating Eq. (62) in the WKB approximation, we see that the decay rate of the resonant state is

$$\Gamma \sim \exp[-B(v_c/v_s)\sqrt{\Delta_s/\Delta_c}], \quad (68)$$

where  $B$  is a constant of order 1.



Using the fact that  $S \sim \Lambda^{-1/K_c}$  in the absence of a charge gap and utilizing the same scaling arguments applied previously to the coherent piece of  $G^<$ , it is easy to see that the weight associated with this resonant state is proportional to  $\Delta_c^{1/K_c-1}$ . However, because the barrier height is small compared to  $\Delta_c$ , the thermal decay of the resonant bound state will become large, due to activation over the barrier, at a temperature well below  $T_c$ .

#### D. The ‘‘BCS-like case’’: No pre-existing spin gap

When there is no spin gap on the isolated chain and there are repulsive interactions in the charge sector, the interchain Josephson coupling is perturbatively irrelevant. Thus the usual case for a quasi-one-dimensional superconductor is the already analyzed case with a pre-existing spin gap. However, it is worthwhile considering the case (with  $K_c < 1$ ) in which both the spin gap and the superconducting coherence are induced by a relevant interchain Josephson coupling. This case, even though quasi-one-dimensional, is much more akin to the usual BCS limit, in that there is a single gap scale in the problem, and pairing (gap formation) and superconducting coherence occur at the same temperature and with roughly the same energy scale. This case has been analyzed extensively in the literature.<sup>36–38</sup> It should be noted, however, that here, too, since the ‘‘normal’’ state is a non-Fermi liquid, the coherent piece of all spectral functions will be strongly temperature dependent below  $T_c$ , and vanish in the neighborhood of  $T_c$ .

### V. SUMMARY AND IMPLICATIONS FOR EXPERIMENT

In this paper, we have obtained explicit and detailed results for the properties of the superconducting state of a quasi-one-dimensional superconductor. We have studied this problem as a quantum critical phenomenon, in which the quantum critical point is reached in the 1D limit of no interchain coupling, and hence we have treated the interchain Josephson coupling as a small parameter. In particular, we expect (as discussed below) that the results are pertinent to underdoped and optimally doped high-temperature superconductors, where self-organized stripe structures render the system locally quasi-one-dimensional.

It is often argued that, even in fairly exotic circumstances, and even when the normal state is a non-Fermi liquid, the superconducting state itself is fairly conventional and BCS-like. We have shown that there are a number of ways in which this expectation is violated. In the first place, there are two ‘‘gap’’ scales,  $\Delta_s \gg \Delta_c$ , whereas in a BCS superconductor there is one,  $\Delta_0$ , and correspondingly two correlation lengths,  $\xi_s$  and  $\xi_c$ , in place of the one,  $\xi_0$ , of BCS theory. However, both gaps are, in a very real sense, superconducting gaps:  $\Delta_s$  is associated with spin pairing (i.e., a nonzero value of  $\langle \Psi_{s,1}^\dagger \Psi_{s,-1} \rangle$ ) and the existence of a local amplitude of the order parameter.  $\Delta_c$  is a measure of interchain phase coherence. In the case in which there is a pre-existing spin gap on the isolated chain,  $\xi_s$  remains finite at the quantum critical point, whereas  $\xi_c$  diverges. The same holds true in the superconducting phase, a bit away from the quantum critical point, where  $\xi_c$  diverges at  $T_c$ , while  $\xi_s$  remains finite.<sup>33</sup>

It is perhaps worth noting that many of these aspects of the superconducting state are considerably more general than the particular model we have solved. Indeed, recently, Lee<sup>39</sup> derived similar results from the gauge theory formulation of a flux phase to superconductor transition. While this derivation presupposes rather different seeming microscopic physics, it does build in the doped insulator character of the superconducting state, which is the essential feature of the results. Likewise, many features we have discussed here bear a close resemblance to the dimensional crossover from a conjectured 2D non-Fermi liquid to a 3D superconductor envisaged in the context of the interlayer tunneling mechanism of high-temperature superconductivity.<sup>40</sup>

#### A. Summary of results

For the benefit of the reader who skipped the technical exposition, we begin by summarizing our most important results. We consider here the case in which there is a pre-existing spin gap,  $\Delta_s/2 \gg T_c$ , on an isolated chain, and we focus on the effects of the interchain Josephson coupling between stripes at lower energies.

##### 1. Thermodynamic effects

The effect of the interchain Josephson coupling is to produce an interchain coherence scale  $\Delta_c(T)$ . At mean-field level,  $\Delta_c(T)$  vanishes for any  $T$  above  $T_c$ , and while fluctuation effects will produce a small amount of rounding to this behavior, because of the large coherence lengths along the chain the degree of rounding will always be small in the quasi-1D limit. It is the coherence scale that determines  $T_c$ , in the sense that

$$T_c \approx \Delta_c(0)/2 \ll \Delta_s/2. \quad (69)$$

[ $\Delta_c$  is expressed in terms of the strength of the Josephson tunneling matrix elements in Eq. (47).] The superfluid densities in the directions transverse and parallel to the chain direction are, respectively,

$$\kappa_\perp = 2au_0\Delta_c^2/v_c, \quad \kappa_\parallel = v_c K_c, \quad (70)$$

where  $u_0$  is a constant [see Eq. (44)] which depends weakly on parameters,  $d$  is the spacing between chains,  $v_c$  is the charge velocity, and  $K_c$  is the charge Luttinger parameter. In two dimensions, if, on average, there is a fourfold rotationally invariant mixture of domains in which the chains run along the  $x$  and  $y$  directions, respectively, the macroscopic superfluid density is isotropic and given by

$$\bar{\kappa}(T) = \sqrt{\kappa_\perp \kappa_\parallel} \sim \Delta_c(T). \quad (71)$$

##### 2. Single hole spectral function

The common theme in the spectral functions is that all dependence on the interchain coupling (and hence all important temperature dependences in the neighborhood of  $T_c$ ) are expressible in terms of the single coherence scale  $\Delta_c$ . Moreover, it is the spectral weight of the coherent features in the spectrum, rather than their energies, which are strongly tem-

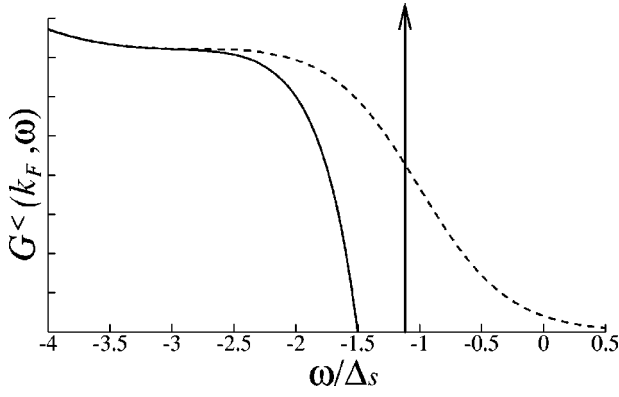


FIG. 2. The temperature evolution of the spectral function. The dashed line depicts  $G^<(k_F, \omega)$  at temperature  $T = \Delta_s/3 > T_c$ , and is calculated using the parameters  $\gamma_c = 0.3$ ,  $K_s = 1/2$ , and  $v_s/v_c = 0.2$ . The solid line represents the spectral function at zero temperature. A coherent  $\delta$ -function peak onsets near  $T_c$  at energy  $\Delta_0 = \Delta_s + \Delta_c(0)/2$ . Here we assume  $\Delta_s/\Delta_c(0) = 5$ . The multiparticle piece starts at a threshold  $2\Delta_c(0)$  away from the coherent peak. The exact shape of the incoherent piece at  $T=0$  is not calculated in the present work and is meant to be schematic.

perature dependent! This is very different from the behavior of the spectral functions near  $T_c$  in a three-dimensional BCS superconductor.

Characteristic shapes of the single hole spectral function above and below  $T_c$  are shown in Fig. 2. Above  $T_c$ , the single hole spectral function is a broad incoherent peak. Below  $T_c$ , there is a coherent delta-function piece and a multiparticle continuum at higher energy,

$$G^<(\vec{k}, \omega) = Z(k_{\parallel}) \delta[\omega - \mathcal{E}(\vec{k})] + G^{multi}, \quad (72)$$

where

$$\mathcal{E}(\vec{k}) = \sqrt{v_s^2 k_{\parallel}^2 + \Delta_0^2} + 2t_{\perp} Z(k_{\parallel}) \cos(k_{\perp} a) + \dots \quad (73)$$

Here  $k_F + k_{\parallel}$  and  $k_{\perp}$  are, respectively, the components of the crystal momentum parallel and perpendicular to the chain direction.

The energy gap for the coherent peak is

$$\Delta_0(T) = \Delta_s + \frac{1}{2} \Delta_c(T), \quad (74)$$

and its spectral weight is given by

$$Z(k) \sim [\Delta_c(T)]^{2\gamma_c + 1/2}. \quad (75)$$

Thus  $Z(k)$  (and with it the transverse bandwidth) is the most strongly temperature-dependent feature of the spectral function.

The multiparticle incoherent piece  $G^{multi}$  starts at a threshold energy  $\mathcal{E}(\vec{k}) + 2\Delta_c(T)$ . This is the origin of the gap between the coherent peak and the incoherent shoulder in Fig. 2. Various forms of damping, including phase fluctuations transverse to the stripes, will broaden this structure, leading to a peak-dip-shoulder form of the spectral function. However, the distance from the coherent peak to the dip should be proportional to  $\Delta_c(T)$  and hence, at  $T=0$ , to  $T_c$ .

### 3. The spin response function

The spin response function is entirely a multiparticle continuum; even below  $T_c$ , we find that any spin-1 mode is unstable to decay into two spin-1/2 ‘‘quasiparticles.’’ However, at low temperature, we find that there is a spin-1 resonant state with an exponentially long lifetime near the threshold energy  $2\Delta_s + \Delta_c = 2\Delta_0$ , with momentum  $2k_F$ , where  $k_F$  is the Fermi momentum on a stripe. Even here, because the barrier to decay is quantitatively small compared to  $T_c$ , we expect that no sharp resonant state will appear in the spectrum in the neighborhood of  $T_c$ . Rather, it will appear as the temperature falls below  $T \approx 0.2\Delta_c(0) \approx 0.4T_c$ .

### B. Implications of two scales

The existence of two scales in the superconducting state appears in different experiments in fairly obvious ways:

(i) Since an electron has spin and charge, the gap measured in single particle spectroscopies, such as ARPES or tunnelling, is  $\Delta_0 = \Delta_s + (1/2)\Delta_c(T)$  [see Eq. (56)]. Manifestly, this gap scale decreases slightly with increasing temperature, but remains large, roughly  $\Delta_s$ , above  $T_c$ . The gap scale  $\Delta_s$  is unrelated to  $T_c$ , and moreover  $\Delta_s(T=0) \gg 2T_c$ , which physically is the statement that the onset of phase coherence, *not* pairing, is what determines  $T_c$ . Consequently, the zero-temperature superfluid density is a better predictor<sup>7,8</sup> of  $T_c$  than  $\Delta_0(T=0)$  [see Eq. (49) and subsequent discussion]. Similarly, pure spin probes, such as NMR or neutron scattering, see a gap which is approximately  $\Delta_s$  per spin 1/2 [see Eqs. (62)].

(ii) Experiments involving singlet pairs of electrons, such as Andreev tunneling, could exhibit<sup>33</sup> an energy scale  $\Delta_c$ ; a scale, moreover, which vanishes at (or near)  $T_c$ , and is related in magnitude to  $T_c$  in a more or less familiar manner,  $\Delta_c(0)/2 \sim T_c$ . More complicated spectroscopies, such as SIS tunneling (e.g., tunneling across a break junction) should reveal gap-like features with both energy scales,  $\Delta_s$  and  $\Delta_c$ .

(iii) The existence of two correlation lengths implies that different measurements will find the order-parameter magnitude depressed over distinct distances: If an impurity destroys the superconducting gap locally, the single-particle density of states, as determined, for instance, with a scanning tunneling microscope, will basically recover over a length scale  $\xi_s$  (although, subtle effects will persist out to a scale  $\xi_c$ ). By contrast, the magnetic-field strength near the core of a vortex, which otherwise would diverge logarithmically at short distances, is reduced inside a ‘‘core radius’’ due to the fact that the superfluid density is depressed (i.e., there is a lower current density per unit phase gradient). Since this latter effect involves only charge motion, the vortex core radius is of order  $\xi_c$ . This ‘‘magnetic’’ core radius is measured, in principle, in muon spin rotation ( $\mu$ SR).<sup>41</sup>

(iv) The superconducting state reflects the non-Fermi-liquid character of the normal state in many ways, but it has a complex scalar order parameter as in a conventional (BCS) superconducting state. This means that we *might* expect well-defined elementary excitations with the quantum numbers<sup>25–27</sup> of the electron quasiparticle, as indeed we have found. However, in a conventional superconductor, the quasiparticle energy is shifted by the opening of the gap, and the lifetimes of all elementary excitations (as observed, e.g., in

ultrasonic attenuation) are strongly temperature dependent below  $T_c$ . In the present case, it is the *spectral weight* associated with the elementary excitations which is strongly temperature dependent, not the lifetime or the energy. Moreover, even as  $T \rightarrow 0$ , the quasiparticle weight remains small, in proportion to a positive power of the distance from the quantum critical point; see Eq. (75).

### C. Two scales in the high-temperature superconductors

It has been noted,<sup>42,43</sup> in the so-called ‘‘Yamada plot,’’ that  $T_c$  in underdoped high-temperature superconductors is proportional to the observed incommensurability<sup>44,45</sup> in the low-energy spin structure factor. The magnetic incommensurability  $\pi/d$  is inversely related<sup>46,47</sup> to the mean separation between charge ‘‘stripes’’  $d$ . Thus the Yamada plot implies that  $T_c$  is inversely proportional to the mean spacing between stripes; as the stripes become more separated, and the electronic structure becomes more one dimensional,  $T_c \rightarrow 0$ . This observation strongly supports the idea that the anomalous electronic properties of these materials reflect the properties of nearby phases of the IDEG. Indeed, many of the spectral features listed above have been observed, with various levels of confidence, in experiments on the high-temperature superconductors:

(i) The best single particle spectra (ARPES and tunneling) exist for  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8$  because it cleaves easily. For underdoped and optimally doped materials, the single-particle gap as measured by tunneling and ARPES (Ref. 48) is found to be large:  $\Delta_0 \geq 35$  meV, in the ‘‘flat-band’’ region near the  $\bar{M}$ , or  $(\pi, 0)$  and  $(0, \pi)$ , points of the Brillouin zone. ( $\Delta_0/2T_c$  lies in the range 2 to 6. The  $\bar{M}$  region is where the maximum of a  $d$ -wave superconducting gap is expected.) The gap persists<sup>49</sup> in some form or other to temperatures well above  $T_c$ . Moreover,  $\Delta_0$  increases with underdoping while  $T_c$  decreases. This gap is quite clearly a superconducting gap in that it has (at low  $T$ ) the characteristic  $d$ -wave form<sup>50</sup> expected of the superconducting gap, and it evolves<sup>51</sup> smoothly with overdoping into a gap of only slightly smaller magnitude which opens, in a more conventional manner, in the neighborhood of  $T_c$ . (We focus on the gap near the  $\bar{M}$  point, especially, because there are both theoretical<sup>52</sup> and experimental<sup>53</sup> reasons to think that the ‘‘flat-band’’ region is associated with states in ‘‘stripes’’ or ‘‘fluctuating stripes.’’) Moreover, as discovered by Uemura and co-workers,<sup>7</sup>  $T_c$  is roughly proportional to the zero-temperature superfluid density for underdoped materials, consistent with the notion<sup>8</sup> that it is phase ordering, not pairing, which determines  $T_c$ .

(ii) Deutscher<sup>54</sup> has argued that the gap scale determined by low-temperature Andreev tunneling spectroscopy is considerably smaller than that determined from single-particle tunneling measurements in underdoped materials, while the two gap scales approach each other in overdoped materials. This issue is well worth revisiting in more detail. The single-particle gap scale is strongly apparent<sup>55</sup> in SIS tunneling spectra—we do not know of any convincing analysis which reveals the smaller charge gap scale in such experiments.

(iii) The vortex core radius has been measured with both scanning tunneling microscopy<sup>56,57</sup> (STM) and  $\mu\text{SR}$ .<sup>41</sup> The  $\mu\text{SR}$  study measures the magnetic-field distribution in the material, and infers the core radius from the high-field cutoff

of the distribution. For large applied fields ( $B \sim 6T$ ), both methods are in rough agreement that the core radius is about 15 Å. However, the core radius deduced from the  $\mu\text{SR}$  measurements is strongly field dependent, so that at low fields ( $B \sim 0.5T$ ) it yields a core radius around 120 Å. By contrast, preliminary evidence<sup>58</sup> from STM experiments suggests that the core radius measured by that method is not strongly field dependent, so that, in low fields, the results of the two methods differ by almost an order of magnitude. However, there appear to be differences in the STM results of different groups.<sup>57</sup> Certainly, the core radius inferred from STM studies<sup>59,60</sup> of the gap suppression in the vicinity of an impurity at zero magnetic field are suggestive of a rather short coherence length. While the experimental results are, by no means, definitive, we would tentatively like to explain the discrepancy between the STM and  $\mu\text{SR}$  results at low field as evidence of the existence of two coherence lengths in the superconducting state.

(iv) It has been realized for a long time that there are no sharp quasiparticle features in the ARPES spectrum near the superconducting gap maximum (near the  $\bar{M}$  point of the Brillouin zone) in the normal state, and it has been argued<sup>61</sup> that they disappear due to a lifetime catastrophe which occurs as the temperature is raised above  $T_c$ . Recent<sup>4,5</sup> high-resolution ARPES measurements in optimally doped  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8$  have revealed an interesting picture of the emergence of these peaks. Within experimental resolution, neither the energy nor the width of the peak changes as the temperature is raised from well below  $T_c$  to slightly above  $T_c$ ; rather, it is the intensity of the peak that is strongly temperature dependent in the neighborhood of  $T_c$ . The intensity vanishes slightly above  $T_c$ , without any apparent change in the shape of the peak itself. Indeed, the sharp temperature dependence of this intensity in the neighborhood of  $T_c$  is consistent with its being proportional to a fractional power of the (local) superfluid density [see Eq. (75)] or from a different perspective<sup>62</sup> to a fractional power of the condensate fraction [see Eq. (51)]. Additional evidence for this comes from an old observation of Harris *et al.*<sup>63</sup> that, as a function of underdoping, the weight in the peak at low temperatures decreases with decreasing superfluid density. Moreover, Shen and Balatsky<sup>64</sup> have argued that a small dispersion of the ARPES peak in the direction perpendicular to the putative stripe direction scales more or less with  $T_c$ , consistent with our Eq. (61). The distance between the coherent peak and the dip feature in ARPES curves near the  $\bar{M}$  point<sup>65</sup> decreases with underdoping, consistent with the zero-temperature distance being proportional to  $T_c$ .

A similar *temperature* evolution has been observed for the so called ‘‘resonant peak’’ in neutron scattering<sup>6</sup> in  $\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$  and  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8$  (although no such feature has been seen<sup>66</sup> in  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ ). We would like to identify this phenomenon, as well, with a dimensional cross-over of the sort discussed here. However, the spin resonance we have found in the present model is clearly not directly related to the observed resonant peak. In particular, all features we have found are peaked at a momentum  $2k_F$ , while the resonant peak is centered on the antiferromagnetic wave vector  $(\pi, \pi)$ . Moreover, the peak we have found disappears through a lifetime catastrophe well below  $T_c$ , while the resonant peak is sharp immediately below  $T_c$ . Clearly, at the very least, to have a theory with anything more than a very



rough caricature of this observed magnetic behavior, we need to expand the considered model to include the effects of the antiferromagnetic “strips” between the “stripes.”

#### D. Further implications for high-temperature superconductors

Finally, we end with a few additional observations concerning insights into the behavior of the high-temperature superconductors that can be obtained from the analysis of this paper:

In the superconducting state of a structurally quasi-1D superconductor, we have found there are two emergent length scales,  $\xi_s$  and  $\xi_c$ . If the quasi-1D electronic structure is self-organized, as it is in the high-temperature superconductors, there are potentially two additional emergent length scales:<sup>67</sup> the mean spacing between stripes,  $d$ , and the persistence length of the stripes,  $\xi_{\text{stripe}}$ .  $d$  can be determined directly from the charge incommensurability<sup>46,68</sup> (or indirectly<sup>46,47</sup> from the spin) structure factor.  $\xi_{\text{stripe}}$  is much harder to determine experimentally, although it is bounded below by the correlation length of the magnetic order.<sup>69,70</sup> So long as  $\xi_{\text{stripe}}$  is the longest length scale in the problem, i.e., so long as  $\xi_{\text{stripe}} \gg \xi_c$ , it is possible to assume, as we have here, that the superconducting properties of the system are quasi-one-dimensional. Where this inequality is violated, the correct theory of the superconducting state needs to be significantly modified. As was pointed out previously,<sup>13</sup> so long as the weaker condition,  $\xi_{\text{stripe}} \gg \xi_s$ , is satisfied, it is possible to have a one-dimensional theory of spin gap formation. At present experiments are unclear about the range of doping and materials for which either of these inequalities is satisfied, which is the most important source of uncertainty in the application of these ideas to the high-temperature superconductors. Certainly, with sufficient overdoping, the stripes lose their integrity and the application of these ideas becomes suspect.

To get a feeling for magnitudes, we can make rough quantitative estimates of the remaining length scales from well-established experimental data in the high-temperature superconductors, although numbers vary from material to material, and as a function of doping concentration  $x$ . The spin velocity in the undoped antiferromagnet is around  $v_s \approx 0.8 \text{ eV \AA}$ , and the superconducting gap is  $\Delta_0 \approx 35 \text{ meV}$ , so  $\xi_s \sim 20 \text{ \AA}$ . The charge coherence length  $\xi_c = \xi_s (v_c / v_s) (\Delta_s / \Delta_c)$ , so if we estimate  $v_c / v_s \sim t / J \approx 2-3$  and  $\Delta_c \approx 2T_c \approx 16 \text{ meV}$ , we find that characteristically  $\xi_c \sim 100-150 \text{ \AA}$ . (This is in good agreement with the  $\mu\text{SR}$  measurement<sup>41</sup> of the vortex core radius cited above.) The spacing between stripes is in the range of four or more lattice constants,  $d \sim 16 \text{ \AA}$ .

A crossover magnetic field, which can be identified as a mean field<sup>71</sup>  $B_{c2}$ , can be estimated as the field at which there is one vortex per coherence length  $\xi_c$  between each pair of neighboring stripes; this leads to an estimate

$$B_{c2} \sim \phi_0 / \xi_c d, \quad (76)$$

where  $\phi_0 = hc/2e$  is the superconducting flux quantum. While  $B_{c2}$  estimated in this fashion is quite large ( $\phi_0 / \xi_c d = 80T$  for  $d = 16 \text{ \AA}$  and  $\xi_c = 100 \text{ \AA}$ ) it is small compared to the characteristic magnetic field,

$$B_s \sim 2\phi_0 / \xi_s w, \quad (77)$$

at which orbital effects lead to the destruction of the spin gap. Here,  $w$  is the “width” of a stripe—i.e., the width of the 1d region involved in spin gap formation. In the “spin gap proximity effect” mechanism<sup>13</sup> proposed previously this would imply that  $w$  is one to two times the crystalline lattice constant. The extremely large value of  $B_s$  rationalizes the lack of any observable reduction of the spin gap temperature in the recent NMR experiments of Gorny *et al.*<sup>72</sup> up to fields as high as 12 T in  $\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$ . (See, also, the discussion in Ref. 73.)

One important difference between a stripe phase and the array of chains studied here, is that in the stripe phase there are additional electronic degrees of freedom which live in the antiferromagnetic strips between the stripes. The two-component nature of the electronic structure of doped antiferromagnets,<sup>74</sup> is characteristic of the microphase separation physics that gives rise to this state. Of course, the antiferromagnetic strips are themselves quasi-one-dimensional magnets,<sup>75</sup> so that any magnetic ordering must be viewed, in similar spirit to that considered here, as resulting from a dimensional crossover. Indeed, it is certainly the spins in the insulating strips that make the dominant contribution to the “resonant peak” observed in neutron scattering. A detailed theory of this peak is beyond the scope of the present model, but is embodied in the spin gap proximity effect.<sup>13</sup>

However, we have found an additional neutron scattering resonance for a quasi-one-dimensional superconductor. While the dimensional crossover causes no bound state in the spin-1 excitations, we find a resonant state of two spin- $\frac{1}{2}$  quasiparticles appearing below  $T \approx 0.4T_c$ . The mode appears at an energy  $2\Delta_s + \Delta_c = 2\Delta_0$ , or twice the single-particle gap as measured by ARPES or tunneling, and at momentum  $2k_F$ , where  $k_F$  defines the Fermi surface associated with a stripe. Since this is a four soliton resonance, it may be qualitatively sensitive to deviations from the limit  $\Delta_c \ll \Delta_s$ , so that the resonance is likely to be most well defined in the underdoped region where  $T_c \ll \Delta_0$ .

Finally, we remark that the ARPES spectrum along the symmetry direction from  $(0,0)$  to  $(\pi, \pi)$ , i.e., along the ray which is expected to pass through the node of a  $d$ -wave gap function, is very different in character from that in the  $\bar{M}$  that we have discussed. In clean samples of optimally doped  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8$ , there is a peak<sup>76,77</sup> in the spectral function both above and below  $T_c$ , and the peak reaches the Fermi surface at a well defined “nodal point,”  $\vec{k}_n = (0.44\pi, 0.44\pi)$ . This peak does not exhibit the characteristics of a quasiparticle peak,<sup>77</sup> in that its width is always larger than its energy; indeed, it seems to exhibit quantum critical behavior reminiscent of a Luttinger liquid. Moreover, there is no qualitative change<sup>77</sup> in the temperature evolution of this peak as the temperature is lowered from two or three times  $T_c$  down to temperatures as low as at least  $1/2 T_c$ ; the character of the nodal excitation seems to be remarkably insensitive to the onset of superconductivity. By contrast, in optimally doped  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$  there is apparently<sup>78</sup> no observable peak along the nodal direction, and indeed little or no spectral weight within about 0.5 eV of the Fermi energy. Indeed, recent neutron-scattering studies<sup>79</sup> of the low-energy



magnetic scattering in the neighborhood of  $2\vec{k}_n$  have revealed the existence of a clean spin gap at low temperatures, which is apparently inconsistent with the existence of any gapless nodal quasiparticle excitations.

It is clear that whatever spectral response is observed near  $\vec{k}_n$  is not associated with the vertical and horizontal stripes studied here, because a stripe wave vector does not span the ‘Fermi surface’ along this direction. It could be associated with diagonal stripes, which have been observed recently in various insulating materials,<sup>80,81</sup> in which case the observed quantum critical behavior might truly be that of a Luttinger liquid. An alternative picture is backflow associated with holes that have not condensed into vertical or horizontal stripes. Both explanations are conceivable as there are strong reasons to expect that the orienting potential, which locks the stripes along a particular (vertical or horizontal) crystallographic direction to be stronger in  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$  than in  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8$ . However, other sources of quantum critical behavior are certainly possible.<sup>82</sup> We will defer further discussion of these classes of excitations to a future study.

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#### APPENDIX A: THE EFFECT OF INTERCHAIN SINGLE-PARTICLE HOPPING

Until now, we have ignored single-particle hopping between chains. This is because, especially in the presence of a spin gap, it is irrelevant in the renormalization-group sense. However, in the superconducting state, we expect the quasiparticles to be able to propagate coherently between chains. Because these terms are irrelevant, their effects on the spectrum can be computed in ordinary degenerate perturbation theory. It is easy to see that to first order in the interchain hopping, the quasiparticle energy is

$$\mathcal{E}(\vec{k}) = \sqrt{v_s^2 k_{\parallel}^2 + \Delta_0^2} + Z(k_{\parallel})\epsilon^{(\perp)}(\vec{k}_{\perp}) + \mathcal{O}(\epsilon^{(\perp)})^2, \quad (\text{A1})$$

where  $\epsilon^{(\perp)}(\vec{k}_{\perp}) = 2t_{\perp} \cos(k_{\perp}a)$  is the interchain contribution to the quasiparticle dispersion, and  $k_{\parallel} + k_F$  and  $\vec{k}_{\perp}$  are, respectively, the components of the crystal momentum parallel and perpendicular to the chain direction.

This is highly reminiscent of the spectrum we would have obtained were we to compute the spectrum of a quasi-one-dimensional superconductor using BCS mean-field theory

$$\begin{aligned} \mathcal{E}_k^{(BCS)} &= \sqrt{[v_F k + \epsilon^{(\perp)}(\vec{k}_{\perp})]^2 + \Delta^2} \\ &= \mathcal{E}_k^{(BCS)} + [v_s k / \mathcal{E}_k^{(BCS)}] \epsilon^{(\perp)}(\vec{k}_{\perp}) + \mathcal{O}(\epsilon^{(\perp)})^2, \end{aligned} \quad (\text{A2})$$

with the differences that the Fermi velocity is replaced by the (slower) spin velocity, the superconducting gap is the sum of the (single-chain) spin gap and the (interchain) charge gap, and the interchain bandwidth is reduced by the quasiparticle weight factor  $Z$ .

#### APPENDIX B: MACROSCOPIC SUPERFLUID DENSITY

In this appendix, we compute the macroscopic phase stiffness (superfluid density) tensor  $K_{ab}[\kappa]$  in two dimensions ( $a=x,y$ ) given a microscopic distribution of the (in general anisotropic) local phase stiffness tensor  $\kappa_{ab}(\vec{r})$ . We include the derivation here for pedagogical purposes, although the results exist elsewhere in the literature.<sup>83</sup>

$\kappa$  determines the relation between the local current density,  $\vec{j}(\vec{r})$  and the gradient of the phase according to

$$j_a(\vec{r}) = \kappa_{ab} \partial_b \theta(\vec{r}). \quad (\text{B1})$$

From the equation of continuity, it follows that  $\vec{\nabla} \cdot \vec{j} = 0$ , so we can express  $\vec{j}$  in terms of a potential,  $j_a(\vec{r}) = \epsilon_{ab} \partial_b \phi(\vec{r})$ , so that

$$\epsilon_{ab} \partial_b \phi(\vec{r}) = \kappa_{ab} \partial_b \theta(\vec{r}). \quad (\text{B2})$$

To compute  $K_{xx}$  in a rectangular geometry, this equation is to be solved subject to the boundary conditions that  $\theta=0$  for  $x=0$  and  $\theta=\Delta\theta$  for  $x=L_x$  (independent of  $y$ ) and (from the condition that no current can flow out of the sample in the  $y$  direction)  $\phi=0$  for  $y=0$  and  $\phi=\Delta\Phi$  for  $y=L_y$ . For a given distribution of  $\kappa$ , we solve this equation for given  $\Delta\theta$  to determine  $\Delta\phi$ , from which we determine  $K$  according to

$$K_{xx}[\kappa] = \Delta\phi / \Delta\theta. \quad (\text{B3})$$

The key observation is the same potential and phase that satisfy Eq. (B3), also satisfy the dual equation

$$\epsilon_{ab} \partial_b \theta(\vec{r}) = \kappa_{ab}^D \partial_b \phi(\vec{r}), \quad (\text{B4})$$

where

$$\kappa_{ab}^D(\vec{r}) \equiv \epsilon_{ac} \kappa_{cd}^{-1}(\vec{r}) \epsilon_{db}. \quad (\text{B5})$$

Therefore

$$K_{xx}[\kappa] K_{yy}[\kappa^D] = 1. \quad (\text{B6})$$

We can apply this general result to the problem of interest here. Consider the case of a square geometry in which, because of some assumed domain structure, the system is macroscopically isotropic ( $\bar{\kappa} \equiv K_{xx} = K_{yy}$ ) despite the existence of microscopic anisotropy in each ‘‘stripe’’ domain. It follows that

$$\bar{\kappa}(\kappa_{\perp}, \kappa_{\parallel}) \bar{\kappa}(1/\kappa_{\parallel}, 1/\kappa_{\perp}) = 1. \quad (\text{B7})$$

It follows that  $\bar{\kappa}(\kappa_{\perp}, \kappa_{\parallel}) = \sqrt{\kappa_{\perp} \kappa_{\parallel}}$ . Other solutions to Eq. (B7) exist, but are not homogeneous functions.

### APPENDIX C: THE EFFECTIVE POTENTIAL $v(x)$

To compute the effective potential which appears in Eq. (62), we consider the discrete version of the refermionized Hamiltonian,<sup>84</sup>

$$H = - \sum_n [t_0 + (-1)^n \Delta(n)/2][c_n^\dagger c_{n+1} + \text{H.c.}], \quad (\text{C1})$$

where  $v_c = 2t_0$  and

$$\Delta(n) = \Delta_c \operatorname{sgn}(R^2 - 4n^2), \quad (\text{C2})$$

corresponding to a pair of solitons separated by a distance  $R$ . (We have set the lattice constant equal to 1.)

We compute the ground-state energy on a system of  $2N$  sites by computing the single-particle eigenvalues, and then summing over the lowest lying  $N-1$  of them to get the total energy as a function of  $R$ . This is precisely the program

carried out previously to study various solitonic states in the Su-Schrieffer-Heeger model of polyacetylene.<sup>31,32,34</sup> With open boundary conditions, the Hamiltonian matrix is tridiagonal, and so particularly simple to study numerically for large system sizes. We have carried out this program numerically for system sizes up to  $2N=3000$ , and for  $\Delta_c=0.2, 0.1, 0.05$ , and  $0.02$ ; the continuum limit is obtained when  $\Delta_c \rightarrow 0$ . Even for the smallest values of  $\Delta_c$ , we find no significant finite-size effects at these large system sizes. The results for  $v(x)$  computed in this way are summarized in Fig. 1. The fact that the asymptotic value of  $v$  is always slightly negative is a reflection of the fact that in the limit of small soliton width (which is equal to 1, the lattice constant, in the present calculation), the soliton creation energy is a very strongly varying function of the width, as found previously by Takayama, Lin-Liu, and Maki,<sup>32</sup> and only approaches its true asymptotic limit  $\Delta_c/2$ , when  $1/\xi_c$  is extremely small.

<sup>1</sup>For a recent perspective on the theoretical and experimental evidence of the importance of stripe correlations in the high-temperature superconductors, see V. J. Emery, S. A. Kivelson, and J. M. Tranquada, *Proc. Natl. Acad. Sci. USA* **96**, 8814 (1999).

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