# Sign of the $\cos\varphi$ conductance term in Josephson tunneling

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Unlike the Josephson supercurrent, $j_c \sin\phi$, the quasiparticle conductance current $(\sigma_0+\sigma_1\cos\varphi)V$, of a Josephson junction is dissipative. For many years the theoretical consensus has been that $\sigma_1$ must be positive. The discrepancy in sign between belief and observation is resolved herein by showing that the theoretical sign of $\sigma_1$ is, in fact, negative.

The purpose of this study is to determine the theoretical sign of the $\cos\varphi$ conductance term of a Josephson junction. The tunneling current across a superconductor-insulator-superconductor junction has both a supercurrent and a dissipative current:[1]

$$j=j_c\sin\varphi+(\sigma_0+\sigma_1\cos\varphi)V, \tag{1}$$

when $\varphi$ is the phase difference. ($\sigma_0$ and $\sigma_1$ depend on $T$ and the voltage $V$.) $j_c(T)$ is the critical supercurrent. The theoretical sign of $\sigma_1$ has long been controversial.[2] Twenty-one years later the question was still viewed as unresolved.[3]

Microscopic theories of $\sigma_1$ have been reviewed, for example, by Harris,[4] and by Langenberg.[5] The consensus is that $\sigma_1$ is positive and of magnitude similar to $\sigma_0$. However, $\sigma_1$ was measured by studying the resonance width of the Josephson plasmon, and $\sigma_1$ was found to be negative.[6] This result has been confirmed in many subsequent experiments.[7] (For $T>0.98T_c$ the sign becomes positive[8]). Some have attempted to change the predicted sign from positive to negative by postulating Lorentzian broadening of the density-of-states singularity at $\pm\Delta$, caused by small-scale inhomogenities.[9] References to other work invoking lifetime broadening are given by Barone and Paternò.[7] Antithetical to such effort is the theorem of Lewis,[10] who showed that scattering by impurities and other static lattice imperfections does not cause lifetime broadening.

The dissipative current can be calculated by summing all golden-rule transitions for quasiparticle tunneling. The tunneling Hamiltonian is

$$H_t=\sum_{k,q}\,(T_{kq}a^{\dagger}_{q\uparrow}a_{k\uparrow}+T^{*}_{kq}a^{\dagger}_{-q\downarrow}a_{-k\downarrow}+T^{*}_{kq}a^{\dagger}_{k\uparrow}a_{q\uparrow}+T_{kq}a^{\dagger}_{-k\downarrow}a_{-q\downarrow}), \tag{2}$$

where electron $k$ states are on the left side of the junction and $q$ states are on the right. (The tunneling is taken to be spin independent. All $k$'s and $q$'s are vectors.) The first two terms take an electron from left to right; and the last two take an electron from right to left.

Now, each BCS state is determined by selecting one of the four operators of each quartet of possibilities for $k$ on the left and $q$ on the right side of the junction:

$$A_k:u_k+v_ka^{\dagger}_{-k\downarrow}a^{\dagger}_{k\uparrow},\quad A_q:u_q+v_qa^{\dagger}_{-q\downarrow}a^{\dagger}_{q\uparrow},$$
$$B_k:a^{\dagger}_{k\uparrow},\quad B_q:a^{\dagger}_{q\uparrow},$$
$$C_k:a^{\dagger}_{-k\downarrow},\quad C_q:a^{\dagger}_{-q\downarrow},$$
$$D_k:v_k-u_ka^{\dagger}_{-k\downarrow}a^{\dagger}_{k\uparrow},\quad D_q:v_q-u_qa^{\dagger}_{-q\downarrow}a^{\dagger}_{q\uparrow}. \tag{3}$$

The $A$'s are the ground pairs; the $D$'s are the excited pairs; and the $B$'s and $C$'s are the fragments of broken pairs (in BCS notation). When the phase difference, $\varphi$, between right and left is zero, all $u_k, v_k, u_q, v_q$ can be taken to be real and positive.

Consider now an elastic (spin-up) quasiparticle transition between an initial state,

$$\Phi_i=a^{\dagger}_{k\uparrow}(u_q+v_qa^{\dagger}_{-q\downarrow}a^{\dagger}_{q\uparrow})\Phi_w, \tag{4}$$

and a final state,

$$\Phi_f=(u_k+v_ka^{\dagger}_{-k\downarrow}a^{\dagger}_{k\uparrow})a^{\dagger}_{q\uparrow}\Phi_w, \tag{5}$$

where $\Phi_w$ is the product of all other selected options from Eq. (3) acting on the vacuum state. (We need not consider the alternative final state, involving the excited pair $D_k$, since we assume the voltage, $V<2\Delta/e$.) The matrix element of the transition is,

$$\mathcal{M}_{kq}=\langle\Phi_f|H_t|\Phi_i\rangle. \tag{6}$$

The first term of $H_t$ contributes $T_{kq}u_ku_q$. The second and third terms each give zero. The fourth term contributes $-T_{kq}v_kv_q$. Accordingly,

$$\mathcal{M}_{kq}=T_{kq}(u_ku_q-v_kv_q). \tag{7}$$

The negative of this result is obtained if $\Phi_i$ and $\Phi_f$ in Eqs. (4) and (5) employ the excited-pair states of Eq. (3) instead of the ground-pair states.

The minus sign in Eq. (7) is the same as the minus sign first derived by BCS (Ref. 11) for quasiparticle transitions that do not flip the electron spin. The sign would be positive, on the other hand, for spin-flip processes that cause nuclear-spin relaxation. The Hebel-Slichter peak[12] in the NMR relaxation rate of Al (just below $T_c$) established the existence of the singularity in the quasiparticle density of states at $\pm\Delta$. Were it not for the minus sign in the coherence factor of Eq. (7), ultrasonic attenuation would also exhibit a Hebel-Slichter peak. ($T_{kq}$ in this context is that for electron-phonon interactions, which are spin independent. Also, $k$ and $q$ states are not then spatially separated.)

An abrupt decrease in ultrasonic attenuation below $T_c$ was first observed by Bömmel in Pb.[13] Similar behavior was soon found in Sn (Ref. 14) and In,[15] and ultimately also in Al.[16] The juxtaposition of the temperature dependence of NMR relaxation (having a Hebel-Slichter peak) with that for ultrasonic attenuation (which falls precipitously) provided the earliest extraordinary evidence for the fundamental validity of BCS theory. The minus sign in the coherence factor of Eq. (7) is, therefore, both historic and far reaching.

The (near) isolation of the two sides of a Josephson junction allows a relative phase difference, $\varphi$, to occur. This phase difference is embodied by letting

$$v_q \to v_q e^{i\varphi}. \tag{8}$$

The square magnitude of the transition matrix element Eq. (7) is then

$$|\mathcal{M}_{kq}|^2 = |T_{kq}|^2 (u_k^2 u_q^2 + v_k^2 v_q^2 - 2u_k v_k u_q v_q \cos\varphi). \tag{9}$$

For quasiparticles near the Fermi energy, $u_k, v_k, u_q, v_q \sim 1/\sqrt{2}$, so

$$|\mathcal{M}_{kq}|^2 \sim \frac{1}{2} |T_{kq}|^2 (1 - \cos\varphi). \tag{10}$$

The $\cos\varphi$ term is negative and has a magnitude comparable to the constant term (as is observed experimentally[6].) The conductance term of Eq. (1) involves (multiplicatively) $|\mathcal{M}_{kq}|^2$, the thermal factors (which depend on $V$), the density-of-states factors for golden-rule transitions, and the quasiparticle effective charge. Only $|\mathcal{M}_{kq}|^2$ depends on $\varphi$.[17]

The foregoing observations are sufficient for the purpose of this paper. The sign of the $\cos\varphi$ conductance term is negative for the same reason that ultrasonic attenuation does not display a Hebel-Slichter peak.

I am grateful to Nigel Goldenfeld for calling my attention to the ‘‘sign of the $\cos\varphi$ conductance’’ problem.

[1] B.D. Josephson, Adv. Phys. **14**, 419 (1965), Eq. (3.10).

[2] Michael Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1975), p. 195. Tinkham observes that the $\cos\varphi$ conductance is the Kramers-Kronig conjugate of the $\sin\varphi$ (supercurrent) term. Moreover, in the low voltage limit, the $\cos\varphi$ term is shown to have the observed magnitude and negative sign.

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[16] R. David and N.J. Poulis, in *Proceedings of the Eighth International Conference on Low Temperature Physics, London, 1962*, edited by R.O. Davies (Butterworths, London, 1963), p. 193.

[17] Fermi's golden rule can be applied to quasiparticle tunneling as long as the frequency components of the phase difference, $\varphi$, satisfy $\hbar\omega \ll 2\Delta$, the superconducting energy gap. The frequency of the Josephson plasmon measured in Ref. 6 was 9 GHz, 60 times smaller than $2\Delta/h$.