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Quantum nutation of the neutron spin

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The possibility to prepare the spin particle in a state, which is a superposition of two states with different energies (in contrast to the classical particle, the energy of which is definite in any time and position) gives a purely quantum condition for the nutation behavior of its spin.

The behavior of the neutron polarization **P** is often analyzed (e.g., Refs. 1 and 2) with the Bloch equation

$$\frac{d\mathbf{P}}{dt}=\gamma[\mathbf{P}\times\mathbf{B}]. \tag{1}$$

According to this equation, the spin inclined to a homogeneous field **B** precesses about it with the classical Larmor frequency

$$\omega_L=|\gamma B|=2|\mu_n B|/\hbar. \tag{2}$$

According to the exact quantum mechanics (QM) approach, the origin of the spin precessions is a change in the phase difference between the states with the spin up $(+)$ and down $(-)$ the field **B** (e.g., Refs. 3–6). In a rf flipper the two states may acquire different total energies. When the neutron kinetic energy exceeds its magnetic Zeeman energy, the QM approach leads[5,6] to the classical Larmor precession picture. A more general case, when the neutron kinetic energy is arbitrary and two states with different energies are coherent states with the spins noncollinear not only to the static field direction but also to each other, is considered in the present paper. It is shown that quantum nutations of the spin not described by the Bloch equation may become essential for ultracold neutrons. Moreover, when not the total momentum but its component less by orders of magnitude is effective, as is the case in reflectometry, quantum nutations may come into play even for thermal neutrons.

We do not consider any consequences of averaging the neutron polarization over neutron wavelengths. This is conveniently made by treating the propagation of an initial (before the apparatus at t_0) neutron wave packet

$$\psi=\int d\mathbf{k}A(\mathbf{k})\exp[i\{\mathbf{k}\cdot\mathbf{r}-E(k)t_0/\hbar\}] \tag{3}$$

through the apparatus. The initial wave packet can be represented as a superposition of plane waves, i.e., of neutron states with a definite energy and momentum. We shall consider the propagation of neutrons in a sharp energy and momentum initial state (at t_0) through the apparatus. The neutron polarization along the beam may then be found by averaging over neutrons with different initial energies and momenta (trajectories). As it follows from the superposition principle of quantum mechanics, the wave-packet description and the plane-wave description are, in principle, equivalent. The plane-wave description is widely used in papers dealing with neutron beams, because it is simpler for analysis. The Bloch equation (1) is formulated in the neutron rest frame, i.e. also for neutrons in a sharp energy and momentum state.

The incident-plane-wave solutions Ψ of the Schrödinger equation may play a heuristic role. The assumptions about perfect monochromatization and polarization of the initial beam, perfect field homogeneity, and perfect resonance parameters of the rf flipper may be relaxed without losing conclusions obtained from analysis of these solutions. The quantity $|\Psi(\mathbf{r},t)|^2$ is proportional to the probability of finding a neutron in a volume $dxdydz$ at a point **r** in the time interval $(t,t+dt)$. The neutron spin orientation in this point and time is fully defined by $\Psi(\mathbf{r},t)$. Therefore, we can describe how the spin behaves as we move along the beam. Different velocities of motion along the beam (different reference frames) may be used. It would be natural to move with the neutron velocity. However, the neutron state may be a superposition of two or more coherent states with different velocities. Therefore, we do not try to define a ''neutron velocity'' but look for a reference frame in which the description of the spin behavior is simpler.

In textbooks on classical mechanics, we find that nutation requires at least that the symmetry axis of a gyroscope be noncollinear to the axis of rotation of the gyroscope. For a 1/2 spin particle such an asymmetry cannot exist, because higher moments would be necessary. However, the possibility to prepare the spin particle in a state, which is a superposition of two or more states with different energies (in contrast to the classical particle where the energy is definite in any time and position), gives a purely quantum condition for the nutation behavior of its spin. A detailed analysis is given below. Now we only mention that quantum nutations of the spin vanish in the classical limit $(\lambda\rightarrow 0)$, as one could expect.

There are different possibilities to prepare neutrons in a state, which is a superposition of two coherent states with different energies and with spins noncollinear not only to the static field direction but also to each other. One of possible experimental schemes is given in Fig. 1. We assume that the boundary between media 1 and 2 is sharp and that the two fields are not collinear. Designate the nuclear potentials and the fields in medium j as $U_{n,j}$ and $\mathbf{B}_j$ $(j=1,2)$. In order to find the wave function in medium 2, one should solve the problem of reflection and transmission of neutrons through the boundary of partition of magnetically noncollinear media (see a detailed consideration in Ref. 7). The superposition of the solutions for the spin components with total energies $E_\uparrow$ and $E_\downarrow$ yields the wave function in medium 2:

$$\Psi(\mathbf{r},t)=\begin{pmatrix} A_\uparrow^+\exp(i\mathbf{K}_\uparrow^+\mathbf{r}-itE_\uparrow/\hbar)+A_\downarrow^+\exp(i\mathbf{K}_\downarrow^+\mathbf{r}-itE_\downarrow/\hbar)\\ A_\uparrow^-\exp(i\mathbf{K}_\uparrow^-\mathbf{r}-itE_\uparrow/\hbar)+A_\downarrow^-\exp(i\mathbf{K}_\downarrow^-\mathbf{r}-itE_\downarrow/\hbar)\end{pmatrix}_{\mathbf{B}_2} \tag{4}$$

(the subscript $\mathbf{B}_2$ shows that a representation with the quantization axis $Z\|\mathbf{B}_2$ is used), where

$$K_{\uparrow\downarrow}^{\pm}=\sqrt{\frac{2m_n}{\hbar^2}[E_{\uparrow\downarrow}-(U_{n,2}\pm|\mu_n\mathbf{B}_2|)]}. \tag{5}$$

The quantities related to the neutron states with the spin parallel to $\pm\mathbf{B}_2$ are designated by the subscripts/superscripts $(\pm)$. The quantities related to the neutron states with different energies are designated by the subscripts/superscripts $(\uparrow)$ and $(\downarrow)$. The squared moduli of the amplitudes $A_{\uparrow\downarrow}^{\pm}$ yield the four probabilities to find the neutron in either $(+)$ or $(-)$ spin states with the energy either $E_\uparrow$ or $E_\downarrow$. Designate the states with different energies as the $(\uparrow)$ and $(\downarrow)$ states. Certain spin orientations ($\uparrow$ spin and $\downarrow$ spin) correspond to these states at any instant and point. The neutron spin orientation is found from superposition of the $(\uparrow)$ and $(\downarrow)$ states. Generally, the $(\uparrow)$ and $(\downarrow)$ states are not orthogonal (the $\uparrow$ spin and the $\downarrow$ spin are not antiparallel).

To describe the neutron spin behavior, choose V for the velocity of the reference frame (it moves along the neutron beam) and rewrite Eq. (4) as

$$\Psi(t)=\begin{pmatrix} A_\uparrow^+\exp(-it\omega_\uparrow/2)\\ A_\uparrow^-\exp(it\omega_\uparrow/2)\end{pmatrix}_{\mathbf{B}_2}\exp\left[\frac{it}{\hbar}(m_n v_{av}^{\uparrow}V-E_\uparrow)\right]$$
$$+\begin{pmatrix} A_\downarrow^+\exp(-it\omega_\downarrow/2)\\ A_\downarrow^-\exp(it\omega_\downarrow/2)\end{pmatrix}_{\mathbf{B}_2}\exp\left[\frac{it}{\hbar}(m_n v_{av}^{\downarrow}V-E_\downarrow)\right], \tag{6}$$

$$\omega_\uparrow=\frac{v}{v_{av}^{\uparrow}}\omega_L,\quad \omega_\downarrow=\frac{v}{v_{av}^{\downarrow}}\omega_L, \tag{7}$$

$$v_{av}^{\uparrow\downarrow}=(v_{\uparrow\downarrow}^{+}+v_{\uparrow\downarrow}^{-})/2,\quad v_{\uparrow\downarrow}^{\pm}=(\hbar/m_n)K_{\uparrow\downarrow}^{\pm}. \tag{8}$$

It follows from Eq. (6) that the $\uparrow$ spin and the $\downarrow$ spin rotate about $\mathbf{B}_2$ with the frequencies, respectively, $\omega_\uparrow$ and $\omega_\downarrow$. If we choose the velocity

$$v_p=(E_\uparrow-E_\downarrow)[m_n(v_{av}^{\uparrow}-v_{av}^{\downarrow})]^{-1} \tag{9}$$

for V, the spin behavior is determined by the function

$$\psi(t)=\begin{pmatrix} A_\uparrow^+\exp\left(-it\frac{\omega_\uparrow}{2}\right)\\ A_\uparrow^-\exp\left(it\frac{\omega_\uparrow}{2}\right)\end{pmatrix}_{\mathbf{B}_2}+\begin{pmatrix} A_\downarrow^+\exp\left(-it\frac{\omega_\downarrow}{2}\right)\\ A_\downarrow^-\exp\left(it\frac{\omega_\downarrow}{2}\right)\end{pmatrix}_{\mathbf{B}_2}$$
$$=\begin{pmatrix}\left[A_\uparrow^+\exp\left(-it\frac{\omega_\uparrow-\omega_\downarrow}{4}\right)+A_\downarrow^+\exp\left(it\frac{\omega_\uparrow-\omega_\downarrow}{4}\right)\right]\exp\left(-it\frac{\omega_\uparrow+\omega_\downarrow}{4}\right)\\ \left[A_\uparrow^-\exp\left(it\frac{\omega_\uparrow-\omega_\downarrow}{4}\right)+A_\downarrow^-\exp\left(-it\frac{\omega_\uparrow-\omega_\downarrow}{4}\right)\right]\exp\left(it\frac{\omega_\uparrow+\omega_\downarrow}{4}\right)\end{pmatrix}_{\mathbf{B}_2}. \tag{10}$$

In the reference frame rotating about $\mathbf{B}_2$ with a frequency

$$\omega_p=-\frac{\omega_\uparrow+\omega_\downarrow}{2}=-\frac{v_p v_{av}}{v_{av}^{\uparrow}v_{av}^{\downarrow}}\omega_L, \tag{11}$$

where

$$v_{av}=(v_{av}^{\uparrow}+v_{av}^{\downarrow})/2, \tag{12}$$

the spin motion is described by

$$\psi(t)=\begin{pmatrix}A_\uparrow^+\\ A_\downarrow^-\end{pmatrix}_{\mathbf{B}_2}\exp\left(-it\frac{\omega_\uparrow-\omega_\downarrow}{4}\right)+\begin{pmatrix}A_\downarrow^+\\ A_\uparrow^-\end{pmatrix}_{\mathbf{B}_2}\exp\left(it\frac{\omega_\uparrow-\omega_\downarrow}{4}\right). \tag{13}$$

One can see from Eq. (11) that the difference between $|\omega_p|$ and ω_L (by definition, $\omega_L>0$) is usually very small.

The ''first spin'' and the ''second spin'' in Eq. (13) do not coincide with the $\uparrow$ spin and the $\downarrow$ spin. It may be shown[8] that Eq. (13) with arbitrary magnitudes and phases of $A_{\uparrow\downarrow}^{\pm}$ describes the neutron spin rotation about an axis (for reasons

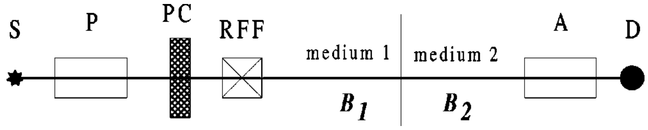

FIG. 1. The neutron beam from the source (S) is polarized with the polarizer (P). The precession coil (PC) makes the neutron polarization vector noncollinear to the guide field. Neutrons in the coherent states with the spin up and down the static field in the rf flipper (RFF) acquire different total energies. The boundary between magnetic media 1 ($\mathbf{B}_1$) and 2 ($\mathbf{B}_2$) is sharp. The behavior of the neutron spin in medium 2 is analyzed. The registering system includes the analyzer (A) and the detector (D).

stated below, we shall call it ''the nutation axis''). The orientation of the nutation axis can be found analytically.[8] The period of the neutron spin rotation about the nutation axis is $2\pi/\omega_n$, where

$$\omega_n = -\frac{\omega_\uparrow - \omega_\downarrow}{2} = -\frac{1}{2}\left(\frac{1}{v_{av}^\uparrow} - \frac{1}{v_{av}^\downarrow}\right) v_p \omega_L = \frac{1}{2}\frac{E_\uparrow - E_\downarrow}{m_n v_{av}^\uparrow v_{av}^\downarrow}\omega_L, \tag{14}$$

but the angular velocity of rotation changes during one period, when the ''first spin'' and the ''second spin'' are not antiparallel.

We may conclude that the neutron spin motion is a superposition of two rotations: the neutron spin rotates with a period $2\pi/\omega_n$ about the nutation axis, which, in its turn, rotates about $\mathbf{B}_2$ with a frequency ω_p. As it follows from Eq. (14), the origin of quantum nutation is a difference in the frequencies of rotation of the ↑ spin ($\omega_\uparrow$) and the ↓ spin ($\omega_\downarrow$) about $\mathbf{B}_2$. The mutual orientation of the ↑ spin and the ↓ spin changes with time. Consequently, the angle between the neutron spin and $\mathbf{B}_2$ changes, leading to quantum nutation.

The rotation about the nutation axis actually looks like ''nutation'' known from classical mechanics. Of course, this analogy with classical mechanics cannot be stretched too far, the more so that the nature of nutations is different. So the rotation of the neutron spin about the nutation axis is more correctly to be called ''quantum nutation.'' The use of the word ''nutation'' may be justified only by the apparent likelihood of the motion of the neutron spin with that of a classical top when it precesses and nutates. It allows being brief in explanations and gives a helpful ''visualization'' of the effect. It is noteworthy that precession is also treated differently in quantum mechanics and in classical mechanics (CM). Nevertheless, the notion ''precession'' is not rejected in QM just because it has an exact definition in CM.

Usually, the kinetic energy of neutrons is much greater than the neutron potential energy and $E_\downarrow \cong E_\uparrow \cong E$. Then the frequency of quantum nutations of the neutron spin is

$$\omega_n \cong \frac{E_\uparrow - E_\downarrow}{4E}\omega_L \cong 3.69\times 10^{-7} H_0^{(RF)} \lambda^2 \omega_L, \tag{15}$$

where ω_n (Hz), ω_L (Hz), λ (Å) is the neutron wavelength, and $H_0^{(RF)}$ (T) is the static field of the rf flipper. Therefore, the frequency of the spin nutations is usually by orders of magnitude lower than ω_L. The nutation frequency is proportional to λ^2, so the nutations in the limit $\lambda \to 0$ just vanish. The length at which the spin makes one nutation is

$$L_n = v_p \frac{2\pi}{\omega_n} = \frac{4\pi}{\omega_L}\left(\frac{1}{v_{av}^\downarrow} - \frac{1}{v_{av}^\uparrow}\right)^{-1} \cong \frac{369}{H_0^{(RF)} B_2 \lambda^3}, \tag{16}$$

where L_n (m), if λ(Å), $H_0^{(RF)}$ (T), B_2 (T). E.g., if $H_0^{(RF)} = 0.01$ T, $B_2 = 0.1$ T, then $L_n = 369$ km for $\lambda = 1$ Å and $L_n = 0.369$ mm for $\lambda = 1000$ Å. In the overwhelming majority of cases the nutations may certainly be ignored. The strong dependence of L_n on the neutron wavelength λ provides that the quantum nutations can be observed only for very large wavelengths.

The length at which the spin makes one precession may also be found:

$$L_p = v_p \frac{2\pi}{|\omega_p|} = \frac{v_{av}^\uparrow v_{av}^\downarrow}{v_{av}} \frac{2\pi}{\omega_L} \cong \frac{1.356\times 10^{-4}}{B_2 \lambda}, \tag{17}$$

where L_p (m), if λ (Å) and B_2 (T). E.g., if $B_2 = 0.1$ T, then $L_p = 1.356$ mm for $\lambda = 1$ Å and $L_p = 1.356$ μm for $\lambda = 1000$ Å.

What is the resultant behavior of the neutron polarization vector $\mathbf{P}$ in a cross section perpendicular to the beam (at L)? It can be found for each L by substitution of $V = 0$ into Eq. (6). The respective analysis shows that $\mathbf{P}$ rotates about a fixed axis (''local dynamic axis'') with a frequency

$$\omega_{RF} = (E_\uparrow - E_\downarrow)/\hbar \tag{18}$$

defined by the rf flipper. When the ↑ spin and the ↓ spin are not antiparallel, the rotation of $\mathbf{P}$ is not uniform (the angular velocity changes during one period $2\pi/\omega_{RF}$). The orientation of the local dynamic axis in each beam cross section can be found analytically.[8] When we move along the beam, the local dynamic axis uniformly rotates about $\mathbf{B}_2$ with a period L_p, and the angle between this axis and $\mathbf{B}_2$ changes with a period L_n. However, the change in this angle is usually negligible, the reason being that the angles of rotation of the the ↑ spin and the ↓ spin [see the two terms in Eq. (6)] about $\mathbf{B}_2$ due to the change in L are practically the same, so their mutual orientation does not change [see also evaluation (16) for L_n].

The calculations from Eq. (4) confirm the results obtained in the analysis given above. When $V = v_p$, the neutron spin motion is, generally, a superposition of two rotations, nutation and precession. It is to be emphasized that no reference frame then exists in which the spin behavior in space and time looks like a mere precession about the field direction. Nor such reference frame exists among noninertial reference frames (V is changing).

Of course, when nutations are negligible (and usually they are), we come to the picture of the classical Larmor precession. The Bloch equation yields a uniform spin precession about the magnetic induction vector in a homogeneous field. On the other hand, we concluded that the neutron polarization vector $\mathbf{P}$ in a cross section perpendicular to the beam rotates about a local dynamic axis that is, generally, not collinear to the field. Yet, such a behavior does not contradict the Bloch equation. Indeed, solving the Bloch equation with the time of the neutron entry into the apparatus as a parameter, we obtain the same behavior of $\mathbf{P}$. However, the Bloch equation may fail even when quantum nutations are negligible. It is derived on the assumption that the neutron experiences the field along its trajectory point by point. This condition is not satisfied when neutron scattering is essential. This condition is not satisfied in the interferometer where the existence of two paths for a neutron is essential. A state with arbitrary orientations of the ↑ spin and the ↓ spin may be prepared in the interferometer. When the ↑ spin and the ↓ spin are not antiparallel, the rotation of **P** about the local dynamic axis is not uniform and can by no means be derived from the Bloch equation. Of course, one may postulate the unusual behavior of **P** in a beam cross section after the interferometer, and describe the further evolution of **P** by the Bloch equation.

In conclusion, note that the spin behavior in a homogeneous field may be even more intricate when the neutron state is a superposition of more than two coherent states with different energies. The analysis of such states is out of the scope of the present paper (more details will be given elsewhere[8]). Here, we only mention that, when the kinetic energy of neutrons greatly exceeds the potential of interaction with magnetic media, the neutron spin evolution in the magnetic media is described by the Bloch equation. However, the study of the neutron spin behavior in a beam cross section is complementary to the conventional three-dimensional (3D) polarization analysis. The corresponding technique may be called ''4D polarization analysis'' (time is the fourth dimension). In principle, it may allow synchronization of processes in a sample under a periodic external force (e.g., magnetization processes in oscillating magnetic fields) to be studied.

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