Wave spectrum of multilayers with finite thicknesses of interfaces

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To describe a multilayer structure with arbitrary thicknesses of the interfaces between layers, we introduce a model in which the dependence of a material parameter along the axis of such a superlattice is described by a Jacobian elliptic sine function. Depending on the value of the modulus κ of the elliptic function, the model describes the limiting cases of multilayers with sharp interfaces ($\kappa=1$, $d/l=0$, where d is the thickness of the interface, l is the period of the superlattice) and of sinusoidal superlattices ($\kappa=0$, $d/l=1/4$), as well as all intermediate situations. We investigate the wave spectrum in such a superlattice. The dependences of the widths of the gaps in the spectrum at the boundaries of all odd Brillouin zones on the ratio d/l are found. It is shown that the thicknesses of the interfaces can be determined if the experimental value of the relation between the widths of the first gap $\Delta\nu_1$ and any other gap $\Delta\nu_n$ is known.

I. INTRODUCTION

The spectrum of waves of different nature (spin, elastic, electromagnetic, etc.) is well-studied theoretically for two types of superlattices, which correspond to the limiting cases of the relation between the thickness of the interfaces d and the period l of the multilayer structure. For the first type $d/l=0$, and the dependence of a material parameter along the superlattice axis z has the form of rectangular spatial pulses. This model is the simplest from a mathematical point of view, because the solution in each layer can be written as a superposition of plane waves, the relation between whose amplitudes can be found from the matching conditions at the boundaries of the layers and from the periodicity of the structure. A transcendental equation is obtained as a result,[1] from which the wave spectrum $\omega=\omega(\mathbf{k})$ can be found, where $\mathbf{k}$ is the wave vector. The transfer-matrix method is effectively used in obtaining this transcendental equation.[2] The model corresponding to infinitely thin interfaces has been widely used in studies of electromagnetic,[3,4] elastic,[5–7] and spin[8–12] waves in superlattices.

To the second limiting case corresponds the situation where the thicknesses of both the interface and the layer are equal to each other, i.e., $d/l=1/4$. The simplest model for this situation is a sinusoidal superlattice. The wave equation in this case is the Mathieu equation, whose wave spectrum is well known. Different types of waves in solids were studied for this case in Refs. 13–15. In Ref. 16 spin waves were considered for both limiting cases, namely for a superlattice with infinitely thin interfaces and for a sinusoidal superlattice.

In real multilayers the ratio d/l can have an arbitrary value between these limiting cases. A model is introduced in the present paper which permits considering this general situation and obtaining the dependence of characteristics of the wave spectrum on the thicknesses of the interfaces in multilayer media.

II. THE MODEL AND METHOD OF CALCULATION

For definitenes we consider here a ferromagnetic superlattice with a magnetic anisotropy $\beta(z)$ varying along the z axis. The direction of the anisotropy is taken to be constant and to coincide with the z axis. We represent $\beta(z)$ in the form

$$\beta(z)=\beta[1+\gamma\rho(z)], \tag{1}$$

where β is the average value of the anisotropy, γ is its relative rms variation, and $\rho(z)$ is a centralized ($\langle\rho\rangle=0$) and normalized ($\langle\rho^2\rangle=1$) function (the angular brackets denote averaging over the period l). To model the general situation we express $\rho(z)$ in the form of a Jacobian elliptic function:

$$\rho(z)=\kappa\left(\frac{\mathbf{K}}{\mathbf{K}-\mathbf{E}}\right)^{1/2}\mathrm{sn}\left(\frac{\pi z}{2d}\right). \tag{2}$$

Here $d=\pi l/8\mathbf{K}$ is the effective thickness of the interfaces. In introducing the effective thickness we chose the numerical coefficient to be $\pi/2$, so that $d/l=1/4$ for the limiting case of the sinusoidal superlattice; in so doing the main variation of a material parameter occurs over the length d for all values of d/l (Fig. 1). $\mathbf{K}$ and $\mathbf{E}$ are the complete elliptic integrals of the first and second kind, respectively, and κ is the modulus of these integrals. The coefficient multiplying $\mathrm{sn}(\pi z/2d)$ is

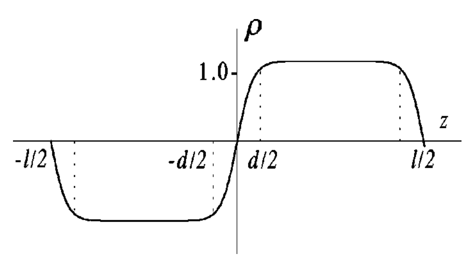

FIG. 1. A typical form of the function (2) for the case where the thickness of the interfaces is much smaller than the thickness of the layers ($d/l=1/8$; $\kappa'=10^{-2}$).

the normalization constant, which follows from the condition $\langle\rho^2\rangle=1$. Equation (2) describes both limiting cases: $d/l=0$ at $\kappa=1$ ($\mathbf{K}=\infty$), $d/l=1/4$ at $\kappa=0$ ($\mathbf{K}=\pi/2$), and all intermediate situations. We describe the dynamics of a ferromagnet by the Landau-Lifshits equation,

$$\dot{\mathbf{M}}=-g\left[\mathbf{M}\times\left(-\frac{\partial\mathcal{H}_m}{\partial\mathbf{M}}+\frac{\partial}{\partial\mathbf{x}}\frac{\partial\mathcal{H}_m}{\partial(\partial\mathbf{M}/\partial\mathbf{x})}\right)\right], \tag{3}$$

with the energy density

$$\mathcal{H}_m=\frac{1}{2}\alpha\left(\frac{\partial\mathbf{M}}{\partial\mathbf{x}}\right)^2-\frac{1}{2}\beta(z)(\mathbf{M}\cdot\mathbf{b})^2-\mathbf{M}\cdot\mathbf{H}. \tag{4}$$

Here $\mathbf{M}$ is the magnetization, $\mathbf{H}$ is the magnetic field, g is the gyromagnetic ratio, α is the exchange parameter, and $\mathbf{b}$ is the direction of the magnetic anisotropy axis, which coincides with the z axis.

The external magnetic field $\mathbf{H}$ and the static part of the magnetization $\mathbf{M}_0$ are also directed along this axis. Representing the magnetization in the form

$$\mathbf{M}(\mathbf{x},t)=\mathbf{M}_0+\mathbf{m}(\mathbf{x},t), \tag{5}$$

performing the usual linearization of Eq. (3) under the condition $|\mathbf{m}|\ll|\mathbf{M}_0|$, and taking $\mathbf{m}\propto\exp(i\omega t)$, we obtain the following equation for the circular projection $\mu=m_x+im_y$:

$$\nabla^2\mu+[\nu-\varepsilon\rho(z)]\mu=0. \tag{6}$$

In writing Eq. (6) we have introduced the notations

$$\nu=\frac{\omega-\omega_0}{\alpha gM}, \quad \varepsilon=\frac{\gamma\beta}{\alpha}, \tag{7}$$

where $\omega_0=g(H+\beta M)$.

In the scalar approximation both the spectrum of elastic waves in a medium with an inhomogeneous density and the spectrum of electromagnetic waves in a medium with an inhomogeneous dielectric permeability are also described by this equation with redefinitions of the parameters. For elastic waves we have

$$\nu=(\omega/v)^2, \quad \varepsilon=\nu\gamma_u, \tag{8}$$

where γ_u is the rms variation of the density of the material and v is the wave velocity. For electromagnetic waves we have

$$\nu=\varepsilon_e(\omega/c)^2, \quad \varepsilon=\nu\gamma_e, \tag{9}$$

where ε_e is the average value of the dielectric permeability, γ_e is its rms deviation, and c is the speed of light.

One might expect that the problem of investigating the wave equation (6) with the function $\rho(z)$ in the form of Eq. (2) is analogous to the problem of spin waves in a ferromagnet with a domain structure. Indeed, the ground state in the latter case is periodically inhomogeneous because of the inhomogeneous orientation of the static part of the magnetization, which is also described by Jacobian elliptic functions:[17]

$$M_x=M_0\,\mathrm{sn}\left(\frac{z}{\kappa}\sqrt{\frac{\beta}{\alpha}}\right), \quad M_y=M_0\,\mathrm{cn}\left(\frac{z}{\kappa}\sqrt{\frac{\beta}{\alpha}}\right), \quad M_z=0. \tag{10}$$

In an investigation of the spin waves in such a system, the magnetization is represented in the form

$$\mathbf{M}(\mathbf{x},t)=\mathbf{M}(z)+\mathbf{m}(\mathbf{x},t), \tag{11}$$

where $\mathbf{M}(z)$ describes the inhomogeneous ground state (10). Linearizing Eq. (3), we obtain the wave equation

$$\nabla^2\mu+\left[\frac{\omega^2}{\alpha(4\pi gM_0)^2}-\frac{\beta}{\alpha M_0^2}(M_x^2-M_y^2)\right]\mu=0. \tag{12}$$

One can see that the coefficients in this equation contain the functions (10) quadratically. Because of this, Eq. (12) transforms into the Lamé equation, and it is this well-studied equation that describes the spin waves in such systems.[18]

Another situation where Jacobian elliptic functions appear is a ferromagnetic film on an antiferromagnetic substrate. In this case the magnetic moment is fixed on one surface of the film, and under the action of the external magnetic field $\mathbf{H}$, which is oriented in the direction opposite to the direction of $\mathbf{M}$, an inhomogeneous rotation of the magnetization occurs along the z axis. The oscillations of the magnetic moment on the background of this inhomogeneous ground state are described by a wave equation that has the form[19]

$$\nabla^2\mu+\left[\frac{\omega^2}{\alpha(4\pi gM_0)^2}-\frac{HM_x(z)}{\alpha M_0^2}\right]\mu=0. \tag{13}$$

Here, in contrast to Eq. (12), the function $M_x(z)$ appears linearly. But this ground state is described by an equation containing the square of the Jacobian elliptic sine function[20]:

$$M_x/M_0=-1+2\kappa^2\,\mathrm{sn}^2\left(z\sqrt{\frac{H}{\alpha M_0}}\right). \tag{14}$$

Thus, the wave equation is the Lamé equation in this case, too. In contrast to the situations described above, the wave equation in our case does not reduce to the Lamé equation. That is why we can use an approximate approach to the solution of an equation with an arbitrary periodic potential. According to the Floquet theorem we seek the solution of Eq. (6) in the form

$$\mu(\mathbf{x})=e^{i\mathbf{kx}}\sum_{p=-\infty}^{\infty}\mu_p e^{ipqz}, \tag{15}$$

where $\mathbf{k}$ is the wave vector, $\mathbf{q}$ is the vector of the reciprocal superlattice ($|\mathbf{q}|=2\pi/l$), and we represent the function $\rho(z)$ by the Fourier series

$$\rho(z)=\sum_{p=-\infty}^{\infty}\rho_p e^{ipqz}. \tag{16}$$

Substituting Eqs. (15) and (16) in Eq. (6), we obtain the equation for μ_p:

$$[\nu-k_x^2+k_y^2-(k_z-pq)^2]\mu_p=\varepsilon\sum_{p_1=-\infty}^{\infty}\mu_{p_1}\rho_{p-p_1}. \tag{17}$$

In the following, we will consider waves propagating along the z axis ($k_z=k$). The equation for μ_{p_1} then becomes

$$[\nu-(k-p_1q)^2]\mu_{p_1}=\varepsilon\sum_{p_2=-\infty}^{\infty}\mu_{p_2}\rho_{p_1-p_2}. \tag{18}$$

Expressing μ_{p_1} from this equation, and substituting it into Eq. (17), we obtain

$$[\nu-(k-pq)^2]\mu_p=\varepsilon^2\sum_{p_1}\sum_{p_2}\frac{\mu_{p_2}\rho_{p_1-p_2}\rho_{p-p_1}}{\nu-(k-p_1q)^2}. \tag{19}$$

Taking into account only the term $p_2=p$ in the second sum of this equation corresponds to the first order of perturbation theory. The dispersion law in this approximation corresponding to the main branch of the solution $(p=0)$ has the form

$$\nu-k^2=\varepsilon^2\sum_{n=-\infty}^{\infty}\frac{\rho_n\rho_{-n}}{\nu-(k-nq)^2}, \tag{20}$$

where we have replaced the summation index p_1 by n.

III. THE WAVE SPECTRUM

Equation (20) is valid for any periodic function $\rho(z)$. Using the well-known coefficients of the representation of the Jacobian elliptic sine function by a Fourier series,[21] we obtain for our case

$$\rho_n=\frac{\pi}{i\sqrt{\mathbf{K}(\mathbf{K}-\mathbf{E})}}\frac{Q^{|n|/2}}{1-Q^{|n|}}\,\mathrm{sgn}(n), \tag{21}$$

where

$$Q=\exp\left(-\frac{\pi\mathbf{K}'}{\mathbf{K}}\right),\quad \mathbf{K}'(\kappa)=\mathbf{K}(\kappa'),\quad \kappa'=\sqrt{1-\kappa^2}. \tag{22}$$

Substituting Eq. (21) into Eq. (20) we obtain the general equation for the wave spectrum $\omega=\omega(\mathbf{k})$ in our case:

$$\nu-k^2=\frac{\pi^2\varepsilon^2}{\mathbf{K}(\mathbf{K}-\mathbf{E})}\sum_n\frac{Q^{|n|}}{(1-Q^{|n|})^2}\frac{1}{\nu-(k-nq)^2}, \tag{23}$$

where $n=\pm1,\pm3\ \ldots$. It is well known that a periodic potential induces the strongest modification of the spectrum in the vicinities of the Brillouin zone boundaries $k=k_{rn}=nq/2$, where n can have both odd and even integer values. Equation (20) determines the modifications of the wave spectrum only in the vicinities of the odd boundaries of the Brillouin zones. The modifications of the spectrum in the vicinities of the boundaries of the even zones are higher-order quantities, which cannot be described by Eqs. (20) and (23). With the proviso that $\varepsilon/\nu\ll1$, the resonances in the sum in Eq. (23) influence one another only slightly. That is why we can restrict ourselves to the two-wave approximation in the vicinity of each odd Brillouin zone boundary, keeping in the sum only the term corresponding to the Brillouin zone n considered:

$$(\nu-k^2)[\nu-(k-nq)^2]=\frac{\pi^2\varepsilon^2}{\mathbf{K}(\mathbf{K}-\mathbf{E})}\frac{Q^{|n|}}{(1-Q^{|n|})^2}. \tag{24}$$

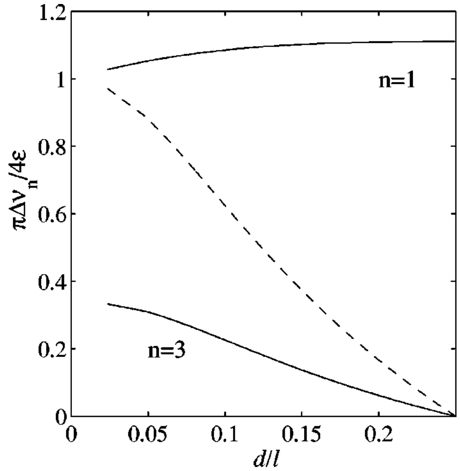

FIG. 2. The dependences of the gap widths in the spectrum $\Delta\nu_n$ on d/l for the first $(n=1)$ and third $(n=3)$ Brillouin zones. The relation $3\Delta\nu_3/\Delta\nu_1$ is also shown by the dashed curve.

We obtain from this equation the general law determining the width of the gap in the spectrum at the boundary of the nth odd Brillouin zone for the superlattice with an arbitrary value of d/l:

$$\Delta\nu_n=\nu_n^+-\nu_n^-=\frac{2\pi\varepsilon}{\sqrt{\mathbf{K}(\mathbf{K}-\mathbf{E})}}\frac{Q^{|n|/2}}{1-Q^{|n|}}, \tag{25}$$

where ν_n^+ and ν_n^- are the solutions of Eq. (24) at $k=k_{rn}$.

In Fig. 2 the dependences of $\Delta\nu_n$ on d/l are depicted for the first $(n=1)$ and third $(n=3)$ Brillouin zones. The ratio $3\Delta\nu_3/\Delta\nu_1$ is shown also by the dashed curve. One can see that the width of the gap for the first Brillouin zone depends only slightly on d/l (it increases when d/l increases), whereas for the third zone the gap width goes to zero when d/l increases. Analysis of Eq. (25) shows that the decrease of the gap width with the increase of d/l occurs for all Brillouin zones except the first. Analytical dependences of $\Delta\nu_n$ on d/l and n can be obtained from Eq. (25) for the limiting cases of small $d/l(\kappa\to1)$,

$$\Delta\nu_n=\frac{4\varepsilon}{\pi|n|}\left[1+\frac{4d}{\pi l}-\frac{1}{2}\left(\frac{\pi^4n^2}{12}-3\right)\left(\frac{4d}{\pi l}\right)^2\right], \tag{26}$$

and for $d/l\to1/4\ \ (\kappa\to0)$,

$$\Delta\nu_n=\sqrt{2}\varepsilon\left(1-\frac{l}{4d}\right)^{(|n|-1)/2}. \tag{27}$$

It follows from Eq. (27) that the widths of all gaps for $n\neq1$ vanish for the sinusoidal superlattice when $d/l=1/4$. This means that in this case the first order of the perturbation theory does not give a contribution to the gap widths. The latter are determined by terms of higher orders which were not taken into account in our analysis: it is known that for a sinusoidal superlattice $\Delta\nu_n\propto\varepsilon^n$.

IV. CONCLUSION

The model introduced in this paper permits describing the dependence of characteristics of the wave spectrum on the thickness of the interfaces d. We have carried out this description in the first order of perturbation theory for the wave spectrum in the vicinities of the boundaries of all odd Brillouin zones. It is shown that the dependence of the widths of the gaps in the spectrum at the boundaries of the Brillouin zones on d/l differs significantly for the different zones. Whereas the gap width of the first zone $\Delta\nu_1$ increases slightly with increasing d/l, the gap widths for the other zones $\Delta\nu_n$, $n\neq 1$, decrease with increasing d/l. Experimental measurement of the ratio between the widths of the gaps at the boundaries of the first and, for example, third Brillouin zones offers the possibility of determining the thickness of the interfaces in a multilayered medium.

We considered an ideal periodic superlattice in this paper. It is known that partial randomization of the superlattice leads to a decrease of the widths of the gaps in the wave spectrum in such a system, and this decrease is different for different Brillouin zones.[22,23] This phenomenon has to be taken into account when one analyzes experimental values of the gap widths. The decrease of the gap induced by the finite thickness of interfaces can be separated, in principle, from the decrease induced by the randomization, because the relation between the gap widths of different Brillouin zones changes in different ways with increasing d/l and increasing randomization.

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