Liouvillian approach to the integer quantum Hall effect transition

Jairo Sinova,[1,2] V. Meden,[1,3] and S. M. Girvin[1]
[1]*Department of Physics, Indiana University, Bloomington, Indiana 47405-7105*
[2]*Department of Physics and Astronomy, University of Tennessee, Knoxville, Tennessee 37996-1200*
[3]*Institut für Theoretische Physik, Universität Göttingen, Bunsenstrasse 9, D-37073 Göttingen, Germany*
(Received 14 February 2000)

We present an approach to the localization-delocalization transition in the integer quantum Hall effect. The Hamiltonian projected onto the lowest Landau level can be written in terms of the projected density operators alone. This and the closed set of commutation relations between the projected densities leads to simple equations for the time evolution of the density operators. These equations can be used to map the problem of calculating the disorder-averaged and energetically unconstrained density-density correlation function to the problem of calculating the one-particle density of states of a dynamical system. At the self-consistent mean-field level, this approach yields normal diffusion and a finite longitudinal conductivity. While we have not been able to go beyond the saddle point approximation analytically, we show numerically that the critical localization exponent can be extracted from the energetically integrated correlation function, yielding $\nu = 2.33 \pm 0.05$, in excellent agreement with previous finite-size scaling studies.

I. INTRODUCTION

The metal-insulator transition in the integer quantum Hall effect (IQHE) is a reentrant zero-temperature quantum phase transition in which the sample goes from an insulating phase with longitudinal conductivity $\sigma_{xx}=0$ to another insulating phase by crossing a conducting critical point ($\sigma_{xx}\neq 0$) as the magnetic field is varied. The critical point occurs between the plateaus of the Hall conductivity σ_{xy} and corresponds to the instance when the Fermi energy is at a critical energy located in the middle of one of the disorder-broadened Landau levels.[1,2]

In general, the disorder-induced metal-insulator transition is a transition in the nature of the states (whether they are localized or delocalized) at the Fermi energy and it does not manifest itself in the density of states which remains smooth across the mobility edge. According to the one-parameter theory of scaling, the states of a two-dimensional noninteracting electron gas are all localized in the presence of arbitrary weak disorder.[3] In the IQHE, however, the presence of a strong magnetic field pointing perpendicular to the plane drastically changes the nature of the states near the middle of the Landau bands. In the noninteracting picture of the IQHE these states are characterized by a localization length

$$\xi(E) \sim \xi_0 \left| \frac{E - E_c^i}{E_0} \right|^{-\nu}, \tag{1}$$

which determines the extent to which the eigenstates of energy E are delocalized. Here ξ_0 denotes a characteristic length scale of the system, e.g., the magnetic length l (see below) and E_0 a characteristic energy scale, e.g., the bandwidth or disorder strength. The critical energy E_c^i is located in the middle of the ith Landau band and, in an infinite-size system, it is the only energy at which the one-particle eigenstates are delocalized within this Landau band. As the Fermi energy (or magnetic field) is varied, the conductivity σ_{xx} will change according to the nature of the states at that energy and sharp peaks in the longitudinal conductivity will be observed.

When studying the IQHE the interaction between the electrons is usually ignored and only the disorder is considered to be responsible for the localization of the single-particle states. This assumption must be checked by comparing the predictions of the noninteracting theory to experimental results[1,2,4] and the outcome of numerical calculations which include the interactions.[5–7] The universal localization exponent $\nu = 2.34 \pm 0.04$ numerically obtained within a noninteracting theory[8–10] is in excellent agreement with experimental measurements of ν,[1,2,4] but it remains a mystery why the strong interactions, which do affect the dynamical exponent z, do not seem to affect ν.[5–7,11] Here we adopt the noninteracting picture. We furthermore assume a strong magnetic field and a Zeeman splitting, which is much larger than the width of each disorder-broadened Landau level. We can then focus on the transition within the lowest Landau level (LLL) and neglect the spin degree of freedom of the electrons.

It has been shown numerically that for a finite system delocalized one-particle wave functions near E_c show multifractal properties characterized by a set of generalized fractal dimensions D_q.[1,12,13] Also, dynamical studies have shown anomalous slow diffusion of wave packets constructed from these multifractal states.[14] Diffusion can be studied using the spectral function of the disorder-averaged retarded density-density correlation function.[15] For the problem considered here the spectral function is given by (after dividing by $\pi\hbar\omega$) (Ref. 12)

$$\overline{S}(r,\omega;E) \equiv \left\langle \left\langle \sum_{i,j} \delta(E - \hbar\omega/2 - E_i)\, \delta(E + \hbar\omega/2 - E_j) \times \psi_i(0)\, \psi_i^*(\mathbf{r})\, \psi_j(\mathbf{r})\, \psi_j^*(0) \right\rangle \right\rangle. \tag{2}$$

Here the $\psi_i(\mathbf{r})$ denote one-particle eigenfunctions and E_i the respective eigenenergies for an electron of a two-dimensional spinless electron gas which is subject to a perpendicular magnetic field and a disorder potential. $\langle\langle\cdots\rangle\rangle$ indicates the ensemble average over the disorder. After taking the disorder average, translational invariance is restored and $\overline{S}$ only depends on the distance $r\equiv|\mathbf{r}|$ from the origin of the plane. Assuming that the eigenstates which contribute in Eq. (2) for $E\approx E_c$ are of multifractal character, it has been argued that $\overline{S}$ decays algebraically,[1,16]

$$\overline{S}(r,\omega\to 0;E\to E_c)\sim\left(\frac{r}{\xi(E)}\right)^{-\eta}, \tag{3}$$

for $\xi_0\ll r\ll\xi(E)$. The anomalous diffusion exponent η is related to the generalized fractal dimension via $D_2=2-\eta$.[1,13]

Assuming a generalized nonlocal (in time and space) relation between the current and the gradient of the density and using the continuity equation, the spectral function in momentum space $S(q,\omega;E)$ at small $q\equiv|\mathbf{q}|$ and ω can be rewritten in terms of a generalized diffusion ''coefficient'' $D(q,\omega)$ for $E\approx E_c$,[15]

$$S(q,\omega;E)=\frac{\rho(E)}{\pi}\,\frac{\hbar q^2 D(q,\omega)}{[\hbar\omega]^2+[\hbar q^2 D(q,\omega)]^2}, \tag{4}$$

where $\rho(E)$ is the density of states per unit area. In the limit of $\omega,q\to 0$ and for large enough system sizes, $D(q,\omega)$ is only a function of qL_ω, where[12]

$$L_\omega\equiv[\rho(E_c)\hbar\omega]^{-1/2}. \tag{5}$$

Through numerical diagonalization and using Eq. (4), Chalker and Daniell[12] have shown that $D(q,\omega)$ approaches a constant D_0 for small qL_ω. The precise value of D_0 is important since the longitudinal conductivity at the critical point is given by the Einstein relation $\sigma_{xx}=e^2\rho(E_c)D_0$ and is expected to be universal. The $qL_\omega\to 0$ limit of $D(q,\omega)$ was later reinvestigated in an extended numerical study.[17] For $qL_\omega\gg 1$, but still in the limit of $q,\omega\to 0$, $D(q,\omega)$ decays as[12]

$$D(q,\omega)\propto D_0(qL_\omega)^{-\eta}. \tag{6}$$

For the anomalous diffusion exponent η, Chalker and Daniell obtain the numerical value $\eta=0.38\pm 0.04$, indicating that the delocalized states near the critical energy indeed have multifractal properties. This value for η was later confirmed in other numerical studies.[14,18] For energies E away from E_c, $S(q,\omega;E)$ vanishes in the small q and ω limit independent of the order in which the limits are taken due to exponential localization of the states.

Most of the progress in the theoretical understanding of the localization-delocalization transition considered here has been through numerical calculations.[2,12,14] Although a field theory was proposed some time ago by Pruisken,[19] up to now no quantitative results such as the critical exponents of the transition have been obtained within this description. More recent studies[20,21] have introduced alternative field theories. Within the framework of these theories it might in the future be possible to analytically determine critical exponents as has been recently successfully achieved for the SU(2) version of the network model.[22] In this paper we present an approach to the transition which may prove more tractable. Although thus far we have not been able to analytically calculate the spectral function, Eq. (2), beyond the self-consistent Born approximation, we have numerically verified the possibility of obtaining the critical exponent ν using this approach.

We start by defining the density correlation function at zero temperature as

$$\tilde{\Pi}(q,t;E)\equiv-\frac{i\theta(t)}{N\hbar l^2}\langle\langle\mathrm{Tr}\{\overline{\rho}_{\mathbf{q}}(t)\overline{\rho}_{-\mathbf{q}}(0)\,\delta(E-H)\}\rangle\rangle, \tag{7}$$

with the one-particle Hamiltonian $H=H_0+H_D$. Here H_0 denotes the kinetic energy of a spinless electron moving in the plane in the presence of a perpendicular magnetic field and H_D is the potential energy for a fixed realization of the disorder potential $V(\mathbf{r})$. l is the magnetic length given by $l^2=\hbar c/(eB)$, where B is the strength of the magnetic field, and $N=L^2/(2\pi l^2)$ is the number of states in the LLL. We consider a square sample of area L^2. By projecting the one-particle density operator $\rho_{\mathbf{q}}\equiv\exp(-i\mathbf{q}\cdot\mathbf{r})$ onto the LLL, denoting the projected density by $\overline{\rho}_{\mathbf{q}}$ (see Sec. II), and taking the one-particle trace Tr over the states in the LLL, we restrict our considerations to the transition in the LLL. It will turn out that the equation of motion for the density operators restricted to the LLL can be solved formally in this case. In the small ω limit we have

$$S(q,\omega;E)=-\frac{1}{2\pi^2}\,\mathrm{Im}\,\Pi(q,\omega;E). \tag{8}$$

Instead of dealing with $\tilde{\Pi}(q,t;E\approx E_c)$ directly, we will integrate $\tilde{\Pi}(q,t;E)$ over *all* energies E and focus our attention on

$$\tilde{\Pi}(q,t)\equiv-i\frac{\theta(t)}{N\hbar l^2}\langle\langle\mathrm{Tr}\{\overline{\rho}_{\mathbf{q}}(t)\overline{\rho}_{-\mathbf{q}}(0)\}\rangle\rangle. \tag{9}$$

Since the localization length only diverges at E_c, the energetically unconstrained diffusion problem considered by investigating $\mathrm{Im}\,\Pi(q,\omega)$ still contains useful information about critical exponents. For instance, let us suppose that at time $t=0$ we create a wave packet localized at the origin constructed from *all* the states of the system (localized as well as delocalized states). For large t only ''delocalized'' states with $\xi(E)/r>1$ can contribute to the probability amplitude of the wave packet at a distance $r/\xi_0\gg 1$ far away from the origin. This implies that in the limit of small $q\xi_0$ only states with $q\xi(E)>1$ and thus $|E-E_c|<E_0(q\xi_0)^{1/\nu}$ contribute to the right hand side of Eq. (9). Hence for $q\xi_0\to 0$ only a fraction $\sim(q\xi_0)^{1/\nu}$ of the states in the LLL contributes and we expect from Eq. (4) that for small qL_ω

$$-\hbar\omega l^2\,\mathrm{Im}\,\Pi(q,\omega)\propto(q\xi_0)^{1/\nu}\frac{(D_0q^2/\omega)}{1+(D_0q^2/\omega)^2}, \tag{10}$$

where the diffusion parameter is a constant D_0. The above argument, which we confirm numerically in Sec. V, gives strong indication that some useful information about the

quantum phase transition can be extracted from $\Pi(q,\omega)$. We again emphasize that this is so because delocalization only occurs at a single critical energy, a characteristic unique to the IQHE where the extended states have zero measure in the energy spectrum. We also point out the importance of the order of limits in obtaining Eq. (10). The limit of $q,\omega\to 0$ is taken by having q approach zero faster than ω so as to obtain a finite diffusion constant. In contrast to the usual approach,[12] in which information about the *anomalous diffusion exponent* η is extracted from the spectral function $S(q,\omega;E\approx E_c)$, we will be able to extract information about the *localization exponent* ν using the same spectral function but integrated over all energies E.

We will show that $\Pi(q,\omega)$, which is an inherent fermionic-disorder-averaged two-particle correlation function, can be reexpressed as the *single-particle* correlation function of an interacting (after the disorder average has been performed) dynamical system with an unusual action. Therefore, in order to extract the dynamical behavior of the original problem, one simply has to study the disorder-averaged density of states of this new action.

The rest of this paper is organized as follows. In Sec. II we introduce the model and mapping of the problem to the new ''Hamiltonian.'' In Sec. III we calculate $\Pi(q,\omega)$ within the self-consistent Born approximation. It displays normal diffusion at this level of approximation. In Sec. IV we introduce the field theoretical approach to the disorder averaging. In Sec. V we will demonstrate numerically the validity of the scaling hypothesis stated in Eq. (10), and finally in Sec. VI we present our conclusions.

II. MODEL AND MAPPING

We consider the two-dimensional spinless electron gas lying in the x-y plane which is subject to a perpendicular magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$ and an external potential $V(\mathbf{r})$. $\hat{\mathbf{z}}$ denotes the unit vector in the z direction. In the symmetric gauge the vector potential is given by $\mathbf{A}=-\frac{1}{2}\mathbf{r}\times\mathbf{B}$ and the one-particle Hamiltonian reads

$$H=H_0+H_D=\frac{1}{2m}\left[\mathbf{p}+\frac{e}{c}\mathbf{A}\right]^2+V(\mathbf{r}). \tag{11}$$

We restrict our investigations to the LLL and thus project the Hamiltonian onto the states in the LLL. The kinetic energy of all the LLL states is the same and after projecting leads to a constant which we will neglect in what follows. Writing the potential energy in Fourier space the Hamiltonian simplifies to

$$H=\sum_{\mathbf{q}} v(-\mathbf{q})\overline{\rho}_{\mathbf{q}}, \tag{12}$$

where $v(\mathbf{q})$ is the Fourier transform of the disorder potential. The projected density operator is given by

$$\overline{\rho}_{\mathbf{q}}\equiv e^{-l^2q^2/4}\tau_{\mathbf{q}}, \tag{13}$$

with $\tau_{\mathbf{q}}$ being the unitary magnetic translation operator which translates the electron a distance $(\mathbf{q}\times\hat{\mathbf{z}})l^2$. The formalism needed to project the density operator $\rho_{\mathbf{q}}=e^{-i\mathbf{q}\cdot\mathbf{r}}$ onto the LLL was developed elsewhere.[23]

The magnetic translation operators have the following special property:

$$\tau_{\mathbf{q}}\tau_{\mathbf{p}}=\exp\left(\frac{il^2}{2}q\wedge p\right)\tau_{\mathbf{q}+\mathbf{p}}, \tag{14}$$

where $q\wedge p\equiv(\mathbf{q}\times\mathbf{p})\cdot\hat{\mathbf{z}}$. Hence their commutation relation defines a closed Lie algebra:

$$[\tau_{\mathbf{q}},\tau_{\mathbf{p}}]=2i\sin\left(\frac{l^2}{2}q\wedge p\right)\tau_{\mathbf{q}+\mathbf{p}}. \tag{15}$$

Also we have

$$\mathrm{Tr}\{\tau_{\mathbf{q}}\}=N\delta_{\mathbf{q},0}. \tag{16}$$

The latter can be proved by noting that the left hand side is proportional to the one-particle trace of $\overline{\rho}_{\mathbf{q}}$. Since the trace is taken over states in the LLL, the projection is unnecessary and we have

$$\mathrm{Tr}\{\overline{\rho}_{\mathbf{q}}\}=\mathrm{Tr}\{e^{-i\mathbf{q}\cdot\mathbf{r}}\}, \tag{17}$$

which vanishes unless $\mathbf{q}=0$.

If there are N states in the Hilbert space, there are N^2 independent operators on the space. However there are exactly N^2 different wave vectors on the torus, so the set of operators $\overline{\rho}_{\mathbf{q}}$ is ''complete''; it spans the set of all operators. The Hamiltonian can be expressed in terms of the $\overline{\rho}_{\mathbf{q}}$ and the Heisenberg equation of motion of the $\overline{\rho}_{\mathbf{q}}$ is closed. This allows us to define the quantum ''Liouvillian'' matrix by

$$\dot{\tau}_{\mathbf{q}}(t)=-i\sum_{\mathbf{q}'}\mathcal{L}_{\mathbf{q}\mathbf{q}'}\tau_{\mathbf{q}'}(t). \tag{18}$$

From the simple commutation properties, Eq. (15), of the $\tau_{\mathbf{q}}$ it readily follows that

$$\mathcal{L}_{\mathbf{q}\mathbf{q}'}\equiv-\frac{2i}{\hbar}v(\mathbf{q}-\mathbf{q}')e^{-l^2|\mathbf{q}'-\mathbf{q}|^2/4}\sin\left(\frac{l^2}{2}q'\wedge q\right). \tag{19}$$

Using the Liouvillian matrix we can immediately write down the formal solution of the equation of motion (18) for $\tau_{\mathbf{q}}(t)$:

$$\tau_{\mathbf{q}}(t)=\sum_{\mathbf{q}'}(e^{-i\mathcal{L}t})_{\mathbf{q}\mathbf{q}'}\tau_{\mathbf{q}'}(0). \tag{20}$$

This leads to a simple expression for the density-density correlation function defined in Eq. (9):

$$\tilde{\Pi}(q,t)=-i\frac{\theta(t)}{\hbar l^2}e^{-l^2q^2/2}\langle\langle(e^{-i\mathcal{L}t})_{\mathbf{q}\mathbf{q}}\rangle\rangle. \tag{21}$$

We can define an N^2 element operator ''superspace'' and view $\mathcal{L}$ as the ''Hamiltonian.'' From this point of view, finding $\mathrm{Im}\,\Pi(q,\omega)$ is the same as finding the *one-particle density of states* for a system with Hamiltonian $\mathcal{L}$:

$$\Pi(q,\omega)=-\frac{i}{\hbar l^2}e^{-l^2q^2/2}\int_0^\infty dt e^{i(\omega+i\delta)t}\langle\langle\,\langle\mathbf{q}|e^{-i\mathcal{L}t}|\mathbf{q}\rangle\,\rangle\rangle$$

$$=\frac{1}{\hbar l^2}e^{-l^2q^2/2}\left\langle\left\langle\,\langle\mathbf{q}|\frac{1}{\omega+i\delta-\mathcal{L}}|\mathbf{q}\rangle\,\right\rangle\right\rangle, \tag{22}$$

where we have introduced states $|\mathbf{q}\rangle$ with $\langle\mathbf{q}|\mathcal{L}|\mathbf{q}'\rangle \equiv \mathcal{L}_{\mathbf{q}\mathbf{q}'}$ and δ is an infinitesimal small positive number.

This remarkable formula is our central result. Let us now try to understand its import. In a crude sense it represents a kind of bosonization of the problem. Ordinarily in an interacting many-body system the equations of motion for the density are not closed but rather involve a hierarchy of additional operators. However, for the special case of one dimension and a linear dispersion relation (the Tomonaga-Luttinger model) the equations of motion *are* closed and the density fluctuations become free bosons[24] even though the underlying particles are interacting. In the present problem (without electron-electron interactions) the equations for the density operators close after projection onto a single Landau level (which for simplicity we have taken to be the lowest). This has several advantages. First we do not have to work separately with retarded and advanced one-particle Green's functions and their products. Second, we note that there are no problems with gauge invariance and conserving approximations. This is because the Liouvillian matrix elements $\mathcal{L}_{\mathbf{q}\mathbf{q}'}$ vanish if either $\mathbf{q}$ or $\mathbf{q}'$ vanish. Thus the total charge in the system is automatically conserved. Finally this representation allows us to establish a hierarchy of length and time scales which should be suitable for renormalization group (RG) analysis. Because the kinetic energy has been quenched, the high momentum of a particle is not associated with high energy. Since the Liouvillian vanishes at small wave vectors, it naturally organizes the decay rates of density fluctuations into short time scales at large wave vectors and long time scales at small wave vectors. As we comment further below, however, there are technical obstacles to be overcome before this RG can be carried out.

We take the disorder to be Gaussian distributed, but not necessarily white noise, i.e., possibly smoothed. We then have

$$\langle\langle v(\mathbf{q})\rangle\rangle = 0 \tag{23}$$

and

$$\langle\langle v(\mathbf{q})v(\mathbf{q}')\rangle\rangle = \frac{2\pi\alpha^2 v^2}{L^2} e^{-l^2q^2(\alpha^2-1)/2}\delta_{\mathbf{q}+\mathbf{q}',0}, \tag{24}$$

which in real space translates into

$$\langle\langle V(\mathbf{r})V(\mathbf{r}')\rangle\rangle = \frac{\alpha^2 v^2}{l^2(\alpha^2-1)}\exp\left[-\frac{|\mathbf{r}-\mathbf{r}'|^2}{2l^2(\alpha^2-1)}\right]. \tag{25}$$

Here v denotes the strength of the disorder potential and α is a dimensionless smoothness parameter. In the limit of a distribution which is extremely smooth ($\alpha\to\infty$), the one-particle electronic density of states approaches a Gaussian[25]

$$\rho_{\alpha=\infty}(E) = \frac{1}{(2\pi)^{3/2}l^2 v}\exp\left[-\frac{1}{2v^2}(E-E_c)^2\right]. \tag{26}$$

An integration over all energies E gives the number of states in the LLL divided by the sample area N/L^2 which is $1/(2\pi l^2)$. For $\alpha=1$ the disorder distribution goes over to the uncorrelated white noise distribution for which Wegner has determined the density of states.[26] At $E=E_c$ it is given by

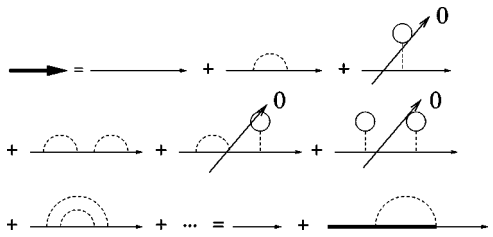

FIG. 1. Partial sum of all noncrossing diagrams of the propagator $\hat{\Pi}(q,\omega)$. For details see the text.

$$\rho_{\alpha=1}(E_c) = \frac{\sqrt{2}}{\pi^2 l^2 v}. \tag{27}$$

III. SELF-CONSISTENT BORN APPROXIMATION

We next calculate $\Pi(q,\omega)$, Eq. (22), in the self-consistent Born approximation. We define the complex self-energy $\Sigma(q,\omega)=\Sigma_R(q,\omega)+i\Sigma_I(q,\omega)$ for the propagator

$$\hat{\Pi}(q,\omega) \equiv \hbar l^2 e^{l^2q^2/2}\Pi(q,\omega) \tag{28}$$

by setting

$$\hat{\Pi}(q,\omega) = \frac{1}{\omega + i\delta - \Sigma(q,\omega)}. \tag{29}$$

Within the self-consistent Born approximation the self-energy is given by the expression

$$\Sigma^B(q,\omega) = \sum_{\mathbf{p}} \langle\langle \mathcal{L}_{\mathbf{q},\mathbf{q}+\mathbf{p}}\mathcal{L}_{\mathbf{q}+\mathbf{p},\mathbf{q}}\rangle\rangle \hat{\Pi}^B(|\mathbf{q}+\mathbf{p}|,\omega). \tag{30}$$

In contrast to standard many-body perturbation theory the right hand side of this expression does not contain an energy sum. In this approximation all noncrossing diagrams for the propagator $\hat{\Pi}(q,\omega)$ are summed, as shown in Fig. 1. In this figure a thick solid line stands for $\hat{\Pi}^B(q,\omega)$ and a thin solid line indicates the ''noninteracting'' propagator $\hat{\Pi}^0(q,\omega)$, which is given by Eq. (29) with $\Sigma(q,\omega)\equiv 0$. $\hat{\Pi}^0$ is independent of q. The consequences of this for a perturbative treatment will be discussed in the next section. In Fig. 1 the vertex with an incoming and an outgoing solid line and a dashed line stands for a matrix element $\mathcal{L}_{\mathbf{q}\mathbf{q}'}$ of the Liouvillian. The disorder average introduces ''contractions,'' i.e., connections, between the dashed lines. In general the Hartree terms are included in the partial sum, but as indicated in Fig. 1, they vanish because of the $q\wedge p$ term in the matrix elements of the Liouvillian.

Using the distribution introduced in the last section [see Eq. (24)] and the definition of the Liouvillian matrix Eq. (19) we obtain the self-consistency equation

$$\Sigma^B(q,\omega) = \frac{2\pi\alpha^2 v^2}{\hbar^2 L^2}\sum_{\mathbf{p}} \frac{e^{-l^2\alpha^2|\mathbf{q}-\mathbf{p}|^2/2}4\sin^2[(l^2/2)q\wedge p]}{\omega + i\delta - \Sigma^B(p,\omega)}. \tag{31}$$

The strength of the disorder v can be scaled out of this equation by replacing $\Sigma^B\to\hbar\Sigma^B/v$ and $\omega\to\hbar\omega/v$.

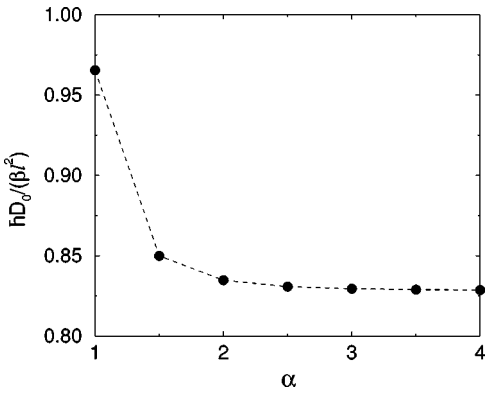

FIG. 2. Diffusion constant D_0 as a function of the smoothness of the disorder α.

As explained in the Introduction the diffusive properties can be read off from the small q and ω limit of the imaginary part of Π. For $q\to 0$ we have $\hat{\Pi}(q,\omega)\to\hbar l^2\Pi(q,\omega)$ and can thus write

$$\hbar l^2\,\mathrm{Im}\,\Pi^B(q,\omega)=\frac{\Sigma_I^B(q,\omega)}{[\omega-\Sigma_R^B(q,\omega)]^2+[\Sigma_I^B(q,\omega)]^2}. \tag{32}$$

Equation (31) can be solved numerically by iteration. Following Eq. (10) the best way to extract the diffusive properties is a ''scaling'' plot in which (for fixed v and α) $-\hbar\omega l^2\,\mathrm{Im}\,\Pi^B(q,\omega)$ is plotted as a function of qL_ω for different small q and ω. Such an evaluation shows that on this level of approximation $-\hbar\omega l^2\,\mathrm{Im}\,\Pi^B(q,\omega)$ is a function of qL_ω only and thus does not display a sign of the prefactor $(q\xi_0)^{1/\nu}$ discussed in connection with Eq. (10). Furthermore, $\mathrm{Im}\,\Pi^B(q,\omega)$ only shows *normal diffusion* with a diffusion constant D_0 which for $q\to 0$ and $\omega\to 0$ is independent of qL_ω. We thus conclude that (as expected) the occurrence of the critical exponents ν and η is a *higher-order fluctuation effect*. For $\omega\to 0$, $\Sigma_R^B(q,\omega)$ goes to zero for all q. Thus D_0 is given by

$$D_0=-\lim_{\omega\to 0}\lim_{q\to 0}\Sigma_I^B(q,\omega)/q^2. \tag{33}$$

Because of the scaling property discussed following Eq. (31), D_0 is proportional to v. As shown in Fig. 2, D_0 also depends on the smoothness α of the disorder. Between $\alpha=1$ (white noise) and $\alpha=2$, D_0 changes by approximately 10%. For $\alpha>2$, the α dependence is extremely weak, and for $\alpha\to\infty$, D_0 saturates at $D_0^{\alpha=\infty}\approx 0.828\,v l^2/\hbar$. For $\alpha=1$ we find $D_0^{\alpha=1}\approx 0.965\,v l^2/\hbar$.

Using the Einstein relation for the conductivity and Eqs. (26) and (27) we obtain

$$\sigma_{xx}^{\alpha=\infty}\approx 0.330\frac{e^2}{h} \tag{34}$$

and

$$\sigma_{xx}^{\alpha=1}\approx 0.869\frac{e^2}{h}. \tag{35}$$

If one is interested in the large α limit, it might be tempting to expand the sine in Eq. (31), as only small p contribute to the sum due to the exponential function. Anticipating that for small q the self-energy is quadratic in q the ansatz $\Sigma^B(q,\omega)=-iq^2\tilde{D}_0$ seems to be plausible. Then the self-consistency equation can be solved analytically, leading to $\tilde{D}_0=(1/\sqrt{2})v l^2/\hbar\approx 0.707 v l^2/\hbar$. A comparison with $D_0^{\alpha=\infty}$ discussed above shows that this procedure *does not* give the correct large α value for D_0. This is due to the fact that in the exact solution of Eq. (31) the range of q values over which $\Sigma^B(q,\omega)$ can be approximated by a purely quadratic function in q shrinks as $1/\alpha$. Thus in the limit $\alpha\to\infty$ it would be necessary to include higher-order terms in the expansion of $\Sigma^B(q,\omega)$ in order to reproduce the numerical result in Eq. (34). Note that $\sigma_{xx}^{\alpha\gg 1}$ obtained above is independent of the correlation parameter α as it should in the limit $\alpha\gg 1$. Since the exact conductivity is universal, the present result is a considerable improvement over the traditional self-consistent Born approximation result for which the conductivity vanishes like α^{-1} in this limit.[27]

In a previous numerical study[17] it was found that $\sigma_{xx}=(0.54\pm 0.04)e^2/h$, independent of the smoothness of the disorder. The results for σ_{xx} obtained within our approach are of the same order of magnitude as the one calculated using purely numerical methods[17] but in contrast to this one our results depend on α. This is due to the fact that we have calculated D_0 within the self-consistent Born approximation but included in the Einstein relation the *exact* density of states at the critical energy.

Using our approach of calculating the disorder-averaged one-particle correlation function for the dynamical system described by the Liouvillian, we observe normal diffusion already at the level of the self-consistent Born approximation. In the usual fermionic picture of noninteracting electrons in the presence of disorder and a magnetic field, much more elaborate techniques, such as, e.g., Borel resummation, instanton methods, the replica trick, and the supersymmetry method, are used to obtain similar results.[1] In particular, in the more traditional approaches, diffusion is not obtained at the saddle point level and it is necessary to include Gaussian fluctuations (i.e., sum ladder diagrams) to obtain diffusion. Because we deal directly with the density itself, we obtain diffusion even at the saddle point level.

IV. FIELD THEORETICAL APPROACH

To go beyond the self-consistent Born approximation it might prove advantageous to bring our approach into a field theoretical framework. This is what we will do in this section. In reformulating $\Pi(q,\omega)$ using field theoretical methods we use the Gaussian integral identity

$$-i\langle\overline{\psi}_{\mathbf{q}}\psi_{\mathbf{q}}\rangle=\langle\mathbf{q}|\frac{1}{\omega+i\delta-\mathcal{L}}|\mathbf{q}\rangle, \tag{36}$$

where

$$\langle\overline{\psi}_{\mathbf{q}}\psi_{\mathbf{q}}\rangle\equiv\frac{1}{Z}\int\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-S_\psi}\overline{\psi}_{\mathbf{q}}\psi_{\mathbf{q}} \tag{37}$$

and

$$S_\psi \equiv -i \sum_{\mathbf{k},\mathbf{k}'} \bar{\psi}_{\mathbf{k}} [\omega + i\delta - \mathcal{L}]_{\mathbf{k},\mathbf{k}'} \psi_{\mathbf{k}'} . \tag{38}$$

The $\psi_{\mathbf{q}}$ denote complex (bosonic) fields and Z is given by

$$Z \equiv \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S_\psi}. \tag{39}$$

In order to ensemble average over the disorder we introduce additional *Grassmann variables* to represent $1/Z$ as a path integral,[1]

$$\frac{1}{Z} = \int \mathcal{D}\bar{\xi}\mathcal{D}\xi\, e^{-S_\xi}, \tag{40}$$

where

$$S_\xi \equiv -i \sum_{\mathbf{k},\mathbf{k}'} \bar{\xi}_{\mathbf{k}} [\omega + i\tilde{\delta} - \mathcal{L}]_{\mathbf{k},\mathbf{k}'} \xi_{\mathbf{k}'} . \tag{41}$$

One can then carry out the ensemble average over the Gaussian-distributed disorder and obtain the generalized functional

$$\bar{Z}(\omega) = \int \mathcal{D}\bar{\xi}\mathcal{D}\xi \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-F(\omega)}, \tag{42}$$

where

$$\begin{aligned} F(\omega) \equiv & \sum_{\mathbf{k}} [(-i\omega + \delta_{\mathbf{k}})\bar{\psi}_{\mathbf{k}}\psi_{\mathbf{k}} + (-i\omega + \tilde{\delta}_{\mathbf{k}})\bar{\xi}_{\mathbf{k}}\xi_{\mathbf{k}}] \\ & + \sum_{\mathbf{k},\mathbf{k}'} \sum_{\mathbf{p},\mathbf{p}'} \langle\langle \mathcal{L}_{\mathbf{k}\mathbf{k}'} \mathcal{L}_{\mathbf{p}\mathbf{p}'} \rangle\rangle [\bar{\psi}_{\mathbf{k}}\bar{\psi}_{\mathbf{p}}\psi_{\mathbf{p}'}\psi_{\mathbf{k}'} \\ & + 2\bar{\psi}_{\mathbf{k}}\psi_{\mathbf{k}'}\bar{\xi}_{\mathbf{p}}\xi_{\mathbf{p}'} + \bar{\xi}_{\mathbf{k}}\bar{\xi}_{\mathbf{p}}\xi_{\mathbf{p}'}\xi_{\mathbf{k}'}]. \end{aligned} \tag{43}$$

Here we have let $\delta \to \delta_{\mathbf{k}}$ so that we can generate the correlation functions by

$$\langle \bar{\xi}_{\mathbf{q}}\xi_{\mathbf{q}} \rangle_\omega = \langle \bar{\psi}_{\mathbf{q}}\psi_{\mathbf{q}} \rangle_\omega = -\frac{\partial \bar{Z}(\omega)}{\partial \tilde{\delta}_{\mathbf{q}}} = -\frac{\partial \bar{Z}(\omega)}{\partial \delta_{\mathbf{q}}}. \tag{44}$$

Once the disorder averaging is done we finally obtain

$$\begin{aligned} F(\omega) = & -i \sum_{\mathbf{q}} [(\omega + i\delta_{\mathbf{q}})\bar{\psi}_{\mathbf{q}}\psi_{\mathbf{q}} + (\omega + i\tilde{\delta}_{\mathbf{q}})\bar{\xi}_{\mathbf{q}}\xi_{\mathbf{q}}] \\ & + \sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4} f(1,2,3,4)[\bar{\psi}_{\mathbf{q}_1}\bar{\psi}_{\mathbf{q}_2}\psi_{\mathbf{q}_3}\psi_{\mathbf{q}_4} \\ & + 2\bar{\psi}_{\mathbf{q}_1}\psi_{\mathbf{q}_4}\bar{\xi}_{\mathbf{q}_2}\xi_{\mathbf{q}_3} + \bar{\xi}_{\mathbf{q}_1}\bar{\xi}_{\mathbf{q}_2}\xi_{\mathbf{q}_3}\xi_{\mathbf{q}_4}], \end{aligned} \tag{45}$$

with

$$\begin{aligned} f(1,2,3,4) = & -\frac{\pi\alpha^2 v^2}{\hbar^2 L^2} e^{-l^2\alpha^2|\mathbf{q}_1 - \mathbf{q}_4|^2/2} 4 \sin\left(\frac{l^2}{2} q_1 \wedge q_4\right) \\ & \times \sin\left(\frac{l^2}{2} q_2 \wedge q_3\right) \delta_{\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_3 + \mathbf{q}_4}. \end{aligned} \tag{46}$$

In contrast to standard many-body theory the action, Eq. (45), does *not* contain a sum over the frequency. ω only enters this equation as an *external parameter*. As already discussed in the last section the noninteracting propagator ($v = 0$) is given by $(\omega + i\delta)^{-1}$ and does not depend on q. Thus a perturbation theory or RG procedure can only be set up after a q-dependent propagator has been generated by self-consistently summing up an *entire class* of diagrams, such as, e.g., the noncrossing diagrams in Sec. III. Furthermore, the interaction f in Eq. (46) has an unusual momentum dependence compared to standard ϕ^4 theory of critical phenomena: It *vanishes* if one of the $\mathbf{q}_i$ goes to zero and is *periodic* in the momenta.

Using the field theoretical approach we can reproduce the approximation discussed in Sec. III, which is usually called self-consistent mean-field or saddle point approximation in the present context. In the absence of symmetry breaking, the middle of the three quartic terms in the action cannot contribute to the saddle point solution since its coefficient vanishes for $\mathbf{q}_1 = \mathbf{q}_4$ and $\mathbf{q}_2 = \mathbf{q}_3$. Hence we can deal separately with the bosonic and the fermionic variables when discussing the saddle point solution. By performing the usual pairing of the fields in the quartic interaction term at the mean-field level we have

$$\begin{aligned} \bar{\psi}_{\mathbf{q}_1}\bar{\psi}_{\mathbf{q}_2}\psi_{\mathbf{q}_3}\psi_{\mathbf{q}_4} = & i\hat{\Pi}^{\mathrm{MF}}(q_1,\omega)\delta_{\mathbf{q}_1,\mathbf{q}_3}\bar{\psi}_{\mathbf{q}_2}\psi_{\mathbf{q}_4} \\ & + i\hat{\Pi}^{\mathrm{MF}}(q_2,\omega)\delta_{\mathbf{q}_2,\mathbf{q}_4}\bar{\psi}_{\mathbf{q}_1}\psi_{\mathbf{q}_3}. \end{aligned} \tag{47}$$

Thus we can write

$$F^{\mathrm{MF}}(\omega) = \sum_{\mathbf{q}} \bar{\psi}_{\mathbf{q}} [-i\omega + \delta + i\Sigma^{\mathrm{MF}}(q,\omega)]\psi_{\mathbf{q}}, \tag{48}$$

and use this in calculating

$$\begin{aligned} i\hat{\Pi}^{\mathrm{MF}}(q,\omega) \equiv \langle \bar{\psi}_{\mathbf{q}}\psi_{\mathbf{q}} \rangle_\omega^{\mathrm{MF}} &= \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-F^{\mathrm{MF}}(\omega)} \bar{\psi}_{\mathbf{q}}\psi_{\mathbf{q}}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-F^{\mathrm{MF}}(\omega)}} \\ &= \frac{i}{\omega + i\delta - \Sigma^{\mathrm{MF}}(q,\omega)} = i\hat{\Pi}^{\mathrm{B}}(q,\omega), \end{aligned} \tag{49}$$

which reproduces the self-consistency, Eq. (31), for the self-energy.

At present we do not know how to evaluate the correlation function beyond the self-consistent mean-field approximation in a controlled way. However, we hope that in the future it will be possible to analytically extend our results.

V. NUMERICAL RESULTS

In this section we will numerically calculate $\mathrm{Im}\,\Pi(q,\omega)$ by exact diagonalization and verify the scaling hypothesis stated in Eq. (10). We closely follow the procedure and notation used by Chalker and Daniell.[12] Motivated by Eq. (8) we define

$$\overline{S}(r,\omega)\equiv-\frac{1}{2\pi^2}\,\mathrm{Im}\,\overline{\Pi}(r,\omega)$$
$$=\left\langle\left\langle\sum_{i,j}\delta(\hbar\omega+E_i-E_j)\psi_i(0)\right.\right.$$
$$\left.\left.\times\psi_i^*(\mathbf{r})\psi_j(\mathbf{r})\psi_j^*(0)\right\rangle\right\rangle. \quad (50)$$

The single-particle wave functions $\psi_i(\mathbf{r})$ can be expanded in the basis of the elliptical θ functions $\phi_m(\mathbf{r})$:

$$\psi_i(\mathbf{r})=\sum_{m=1}^{N}a_i(m)\phi_m(\mathbf{r}), \quad (51)$$

where

$$\phi_m(x,y)=\frac{1}{\sqrt{Ll\pi^{1/2}}}\sum_{s=-\infty}^{\infty}\exp(iX_{m,s}y/l^2)$$
$$\times\exp[-(x-X_{m,s})^2/(2l^2)] \quad (52)$$

and

$$X_{m,s}=m\frac{2\pi}{L}l^2+sL. \quad (53)$$

Then the Fourier transform of Eq. (50) can be written as

$$S(q,\omega)=\frac{1}{2\pi l^2N^2}e^{-l^2q^2/2}$$
$$\times\left\langle\left\langle\sum_{i,j}\delta(\hbar\omega+E_i-E_j)Q_{ij}(k,l)\right\rangle\right\rangle, \quad (54)$$

where

$$Q_{ij}(k,l)=N\left|\sum_{m=1}^{N}a_i(m)a_j^*([m-l])\times\exp\left(i2\pi k\frac{m}{N}\right)\right|^2, \quad (55)$$

and $\mathbf{q}=(2\pi/L)(k,l)=(\sqrt{2\pi/\mathcal{A}N})(k,l)$, with k, l integer. In Eq. (55), $[m+l]$ is defined as being $m+l$ for $1\leqslant m+l\leqslant N$ and $m+l\pm N$, otherwise such that $1\leqslant|m+l\pm N|\leqslant N$. In the numerical calculation we replace the delta function in Eq. (54) by a sharply peaked Gaussian $\delta_\gamma(x)\propto\exp[-x^2/\gamma^2]$ with a broadening[12] $\gamma=0.64\,v/N$ which is of the order of the level spacing. We then have

$$S(q,\omega)=\frac{1}{2\pi l^2N^2}e^{-l^2q^2/2}K(q,\omega), \quad (56)$$

with

$$K(q,\omega)=\frac{\left\langle\left\langle\sum_{i\neq j}\delta_\gamma(\hbar\omega+E_i-E_j)Q_{ij}(k,l)\right\rangle\right\rangle}{\left\langle\left\langle\sum_{i\neq j}\delta_\gamma(\hbar\omega+E_i-E_j)\right\rangle\right\rangle}. \quad (57)$$

This function is suitable for a numerical investigation.[12] We restrict ourselves to a white noise disorder distribution ($\alpha=1$). We calculate $K(q,\omega)$ for values of $2\leqslant k^2+l^2\leqslant25$ and $\hbar\omega=\gamma n$, with $3\leqslant n\leqslant23$, where the limits have been chosen such that $L^{-1}<q<l^{-1}$ and $\hbar\omega\ll v$ but $\hbar\omega$ greater than the level spacing of the finite-size system. The system sizes range from $N=200$ to $N=2000$, and the number of disorder realizations are 500 or 100 depending on the system size. All values of $K(q,\omega)$ were determined to an accuracy better than 1% in the disorder averaging.

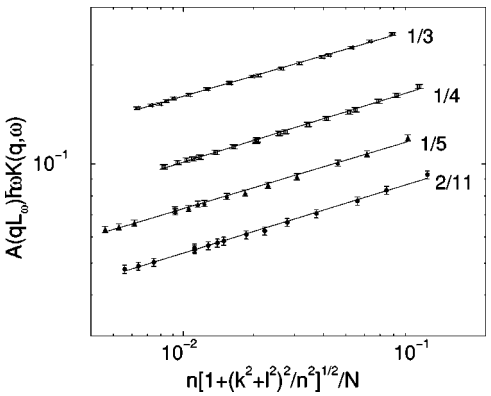

FIG. 3. $\hbar\omega K(q,\omega)$ for different fixed $(qL_\omega)^2\propto(k^2+l^2)/n$ and small q and ω as a function of $n[1+(k^2+l^2)^2/n^2]^{(1/2)}/N$ on a log-log scale. For clear comparison each data set has been multiplied by a constant factor $A(qL_\omega)$ and is labeled by the ratio $(k^2+l^2)/n$.

For a fixed and small value of qL_ω (so that we are in the range of normal diffusion) and $q,\omega\to0$ we expect from Eq. (10) that $\hbar\omega K(q,\omega)$ scales as $(\hbar\omega/v)^{1/2\nu}\propto(ql)^{1/\nu}$. The scaling hypothesis is illustrated in Fig. 3, where we plot $A(qL_\omega)\hbar\omega K(q,\omega)$ for fixed ratios of $(qL_\omega)^2\propto(k^2+l^2)/n$ as a function of $n\sqrt{1+(k^2+l^2)^2/n^2}/N$ on a log-log scale. Here each curve is multiplied by a constant factor $A(qL_\omega)$ [different for each $(k^2+l^2)/n$ ratio] to make the comparison of the different lines easier. Also the factor $\sqrt{1+(k^2+l^2)^2/n^2}$ multiplying $n/N\propto\omega$ is used such that the curves line up horizontally. The fact that data calculated for different system sizes fall onto the same curve [for a fixed ratio of $(k^2+l^2)/n$] indicates that the limits chosen above for k, l, and n do avoid large finite-size effects. On the log-log scale the different data sets fall onto straight lines and can be fitted by power laws (solid lines in Fig. 3).

The localization exponent ν extracted from the slope of the lines in Fig. 3 is shown as a function of $(k^2+l^2)/n$

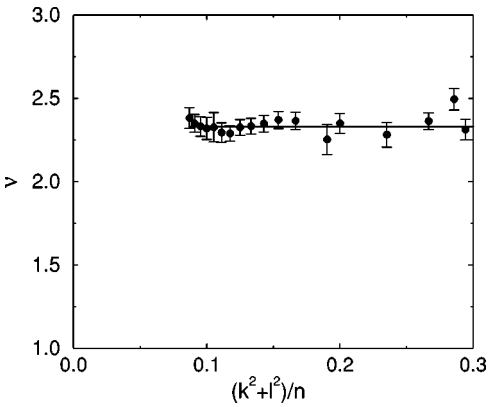

FIG. 4. Localization exponent ν obtained from Fig. 3. The solid line corresponds to the average ν.

$\propto (qL_\omega)^2$ in Fig. 4. Within our error bars and for the qL_ω considered, ν is a constant. Its value $\nu = 2.33 \pm 0.05$ is in excellent agreement with previous finite-size scaling studies[1,2,8–10] and strongly supports the scaling hypothesis, Eq. (10). The fact that the lowest $(k^2+l^2)/n$ points seem to be moving upwards in Fig. 4 is an indication that there are still some finite-size effects for the low values of (k^2+l^2). In contrast to previous numerical studies we are able to obtain information about the critical exponent ν from systems of finite size without doing finite-size scaling.

VI. CONCLUSION

We have presented an analytical and numerical approach to the localization-delocalization transition in the LLL of the IQHE. By using the closed Lie algebra of the density operators in the LLL we are able to write the equation of motion for the densities in a closed form which can be solved formally. Using the solution of the equation of motion for the projected densities we can express the integrated spectral function $\int dE S(q,\omega;E)$ as the disorder-averaged density of states of a dynamical system with a different action. We show analytically that the self-consistent mean-field approximation of the integrated spectral function yields normal diffusion but it misses the critical scaling. However, it is encouraging to note that even at this level of approximation the longitudinal conductivity is in approximate agreement with previous numerical studies.[17] Finally, using exact diagonalization, we are able to extract the localization critical exponent ν from the integrated spectral function by using the scaling hypothesis, Eq. (10), without having to do finite-size scaling. We obtain $\nu = 2.33 \pm 0.05$ in excellent agreement with previous studies.[8–10] We hope that in the future it will be possible to extend our approach beyond the self-consistent mean-field level and analytically extract information about the critical exponent ν.

ACKNOWLEDGMENTS

The authors would like to thank A. Zee, A.H. MacDonald, T. Brandes, K. Schönhammer, and J.T. Chalker for helpful discussions. V.M. is grateful to the Deutsche Forschungsgemeinschaft for financial support during his stay at Indiana University. This work was furthermore supported by NSF Grant No. DMR-9714055 and NSF Grant No. DMR-9820816.

[1] M. Janßen, O. Viehweger, U. Fastenrath, and J. Hajdu, *Introduction to the Theory of the Integer Quantum Hall Effect* (VCH, Weinheim, 1994) and references therein.
[2] For a recent review see B. Huckestein, Rev. Mod. Phys. **67**, 357 (1995).
[3] E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
[4] H.P. Wei, D.C. Tsui, M.A. Paalanen, and A.M.M. Pruisken, Phys. Rev. Lett. **61**, 1294 (1988); H.P. Wei, S.Y. Lin, D.C. Tsui, and A.M.M. Pruisken, Phys. Rev. B **45**, 3926 (1992).
[5] S.-R.E. Yang and A.H. MacDonald, Phys. Rev. Lett. **70**, 4110 (1993); S.-R.E. Yang, A.H. MacDonald, and B. Huckestein, *ibid.* **74**, 3229 (1995).
[6] D.-H. Lee and Z. Wang, Phys. Rev. Lett. **76**, 4014 (1996).
[7] B. Huckestein and M. Backhaus, Phys. Rev. Lett. **82**, 5100 (1999).
[8] B. Huckestein and B. Kramer, Solid State Commun. **71**, 445 (1989); Phys. Rev. Lett. **64**, 1437 (1990).
[9] B. Mieck, Europhys. Lett. **13**, 453 (1990).
[10] Y. Huo and R.N. Bhatt, Phys. Rev. Lett. **68**, 1375 (1992).
[11] Z. Wang, M.P.A. Fisher, S.M. Girvin, and J.T. Chalker, Phys. Rev. B **61**, 8326 (2000).
[12] J.T. Chalker and G.J. Daniell, Phys. Rev. Lett. **61**, 593 (1988).
[13] For a recent review, see M. Janßen, Phys. Rep. **295**, 1 (1998).
[14] B. Huckestein and L. Schweitzer, Phys. Rev. Lett. **72**, 713 (1994).
[15] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin/Cummings, London, 1975).
[16] F. Wegner, Z. Phys. B **25**, 327 (1976).
[17] Y. Huo, R.E. Hetzel, and R.N. Bhatt, Phys. Rev. Lett. **70**, 481 (1993).
[18] T. Brandes, L. Schweitzer, and B. Kramer, Phys. Rev. Lett. **72**, 3582 (1994).
[19] A.M.M. Pruisken, Nucl. Phys. B **235**, 227 (1984).
[20] M.R. Zirnbauer, hep-th/9905054 (unpublished).
[21] M.J. Bhaseen, I.I. Kogan, O.A. Soloviev, N. Taniguchi, and A.M. Tsvelik, cond-mat/9912060 (unpublished).
[22] I.A. Gruzberg, A.W.W. Ludwig, and N. Read, Phys. Rev. Lett. **82**, 4524 (1999).
[23] S.M. Girvin and T. Jach, Phys. Rev. B **29**, 5617 (1984); S.M. Girvin, A.H. MacDonald, and P.M. Platzman, *ibid.* **33**, 2481 (1986); S. M. Girvin, *The Quantum Hall Effect: Novel Excitations and Broken Symmetries*, 1998 Les Houches Summer School Lecture Notes (Springer-Verlag, New York, and Editions de Physique, Les Ulis, France, 2000).
[24] E. Fradkin, *Field Theories of Condensed Matter Systems* (Persus Books, Reading, MA, 1998).
[25] M.E. Raikh and T.V. Shahbazyan, Phys. Rev. B **47**, 1522 (1993).
[26] F. Wegner, Z. Phys. B: Condens. Matter **51**, 279 (1983).
[27] T. Ando, A.B. Fowler, and F. Stern, Rev. Mod. Phys. **54**, 437 (1982).