Scaling properties of one-dimensional Anderson models in an electric field: Exponential versus factorial localization

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We investigate the scaling properties of eigenstates of a one-dimensional Anderson model in the presence of a constant electric field. The states show a transition from exponential to factorial localization. For infinite systems this transition can be described by a simple scaling law based on a single parameter $\lambda_\infty = l_\infty / l_{\rm el}$, the ratio between the Anderson localization length l_∞ and the Stark localization length $l_{\rm el}$. For finite samples, however, the system size N enters the problem as a third parameter. In that case the global structure of eigenstates is uniquely determined by two scaling parameters $\lambda_N = l_\infty / N$ and $\lambda_\infty = l_\infty / l_{\rm el}$.

I. INTRODUCTION

In recent years several studies have investigated the influence of constant uniform electric fields on the localization of electrons in one-dimensional (1D) systems with on-site randomness. In the absence of dc fields, it is by now well known that even small amounts of disorder lead to an exponential localization of all eigenstates.[1,2] On the other hand, in the case of a single-orbital tight-binding model of an electron in a periodic potential, application of a static electric field is known to generate a discrete, uniformly spaced eigenvalue spectrum,[3] known as a Stark ladder, with all eigenfunction localized factorially.[4] For weak fields, the wave functions may be extended over several lattice periods, but with increasing field strength the electron tends to be localized on a specific site. This is known as Stark localization and has been observed experimentally in superlattices[5] which are commonly used for such measurements.[6]

In infinite samples the localization of eigenstates can be characterized in terms of the localization length; the latter is commonly defined from the amplitude decay of eigenstates in the limit $|n| \to \infty$, where n labels the site in the tight-binding picture. The most powerful and informative method available for such studies is the transfer-matrix method. In the presence of a strong electric field, however, it appears to be less efficient due to the factorial nature of the Stark localization. Moreover, for finite systems the structure of eigenvectors cannot be characterized in the same way. One then needs to use other quantities (such as the inverse participation ratio), that are valid both for finite and infinite samples. Through the use of scaling conjectures, one can link then the properties of eigenstates in infinite samples to those of finite samples. Since the scaling approach proved to be extremely useful in describing conductance and its fluctuations (see, e.g., Refs. 7 and 8) in the theory of disordered solids, it also seems natural to use this approach in order to describe localization properties of eigenfunctions of 1D disordered systems in the presence of a constant electric field.

In this paper we study the 1D Anderson model in the presence of a constant electric field in view of scaling properties of its eigenstates. The main question that we want to answer is whether the equivalence between quasi-1D and 1D disordered models[24,13] known up to now can be extended in order to include systems with a constant electric field. For this purpose we analyze the scaling properties of information lengths for infinite and finite samples, which were used successfully in the studies of one- and quasi-one-dimensional systems.[9–15] Contrary to the standard Anderson case, where the ratio of the Anderson localization length l_∞ and the sample size N, i.e., $\lambda_N = l_\infty / N$, is the only relevant scaling parameter, in the present case we find an additional scaling parameter $\lambda_\infty = l_\infty / l_{\rm el}$. Here $l_{\rm el}$ is the Stark localization length, which arises from the applied constant electric field. Hence, the structure of eigenvectors in our model is characterized by two scaling parameters λ_N, and λ_∞.

The structure of the paper is as follows. In Sec. II we describe the mathematical model, and briefly summarize the known facts about the two limiting cases that appear for our model. In Sec. III we discuss different definitions of localization length, which are used in our numerical simulations. In Sec. IV we present numerical data on scaling of localization lengths of eigenstates in infinite and finite systems. Finally, in Sec. V we study the scaling of the whole distribution of eigenvectors. Our conclusions are summarized in Sec. VI.

II. MODEL

Our starting point is the 1D Schrödinger equation in the tight-binding approximation,

$$i\frac{d\psi_n(t)}{dt} = (\epsilon_n + neF)\psi_n(t) + V\psi_{n+1}(t) + V\psi_{n-1}(t), \tag{1}$$

where $\psi_n(t)$ denotes the probability amplitude for an electron to be at site n. Moreover, ϵ_n is the local site energy, V is the hopping element, e is the electron charge, and F the strength of the applied dc field. By applying the transformation $\psi_n(t) = \exp(-iEt)\varphi_n$, one obtains the stationary equation

$$E\varphi_n = V\varphi_{n+1} + (\epsilon_n + neF)\varphi_n + V\varphi_{n-1} \tag{2}$$

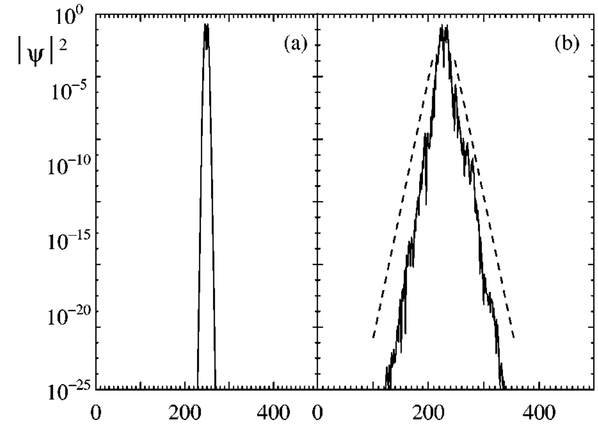

FIG. 1. Two representative eigenfunctions of the 1D tight-binding model [Eq. (1)]. (a) Stark regime with factorial localization ($W=0$, $eF=0.5$, $l_{\rm el}\approx 2$). (b) Anderson regime with exponential localization ($W=5, eF=0$, and $l_\infty \approx 3.4$). The dashed line has a slope $2/l_\infty$.

for the eigenvalues E and the corresponding eigenstates $\varphi_n(E)$. We can distinguish two limiting cases which are relevant for our study: (a) a perfect system (i.e. $\epsilon_n=\epsilon$) with a nonzero electric field[18–20] $F\neq 0$, and (b) a zero field (i.e., $F=0$) with a random on-site potential.

In the case of a perfect system with $F\neq 0$, one can prove that the corresponding eigenstates $\varphi(E)$, known as Wannier-Stark states, show a generic factorial decay, i.e.,[4,16]

$$\varphi_n(E)=J_{m-n}(2/eF)\rightarrow(1/eF)^{|n|}/(|n|!), \quad n\rightarrow\pm\infty, \quad (3)$$

where J_n is a Bessel function of order n. Wannier-Stark states constitute a complete set of energy eigenstates.[17] Their eigenvalues $E_m=meF$ form the so-called Wannier-Stark ladder.[3] A particular Wannier-Stark state φ_n is factorially localized around the nth site, with a localization length of the order of $l_{\rm el}\simeq 1/eF$, i.e., the electric field appears in the denominator of the localization length in Eq. (3). This underscores the fact that F cannot be treated as a small perturbation to the field-free Hamiltonian. An example of such a state is presented in Fig. 1(a).

The other limit of interest corresponds to a zero electric field, with ϵ_n random and δ-correlated, chosen from a distribution $\mathcal{P}_\epsilon$ with mean zero and variance σ^2. Below, in our numerical investigation, we will always assume that $\mathcal{P}_\epsilon$ is a uniform distribution in $[-W/2,W/2]$. Such a model is known in the literature as the Anderson model,[1] and has been studied in great detail in the context of disordered materials. It was shown with mathematical rigor that in the limit of infinite samples this model displays exponentially localized eigenfunctions, no matter how small the disorder is [see Fig. 1(b)]. The rate of decay is measured by the Lyapunov exponent γ which may be evaluated by the transfer-matrix method. To this end, one has to study the asymptotic behavior of the random matrix product ΠM_n, where M_n is defined through the relation

$$\xi_{n+1}=M_n\xi_n, \quad M_n=\begin{pmatrix} v_n & -1 \\ 1 & 0 \end{pmatrix}, \quad v_n=\frac{E-\epsilon_n}{V} \quad (4)$$

for the vector $\xi_n=(x_n,x_{n-1})$, with the matrix M_n known as the transfer matrix. The localization length l_∞ is hence the inverse Lyapunov exponent γ; the latter is evaluated as the exponential rate of increase of an initial vector ξ_1:

$$l_\infty^{-1}=\gamma=\lim_{N\rightarrow\infty}\frac{1}{N}\frac{\ln\left(\prod_{n=1}^{N}\left|M_n\xi_n\right|\right)}{|\xi_1|}. \quad (5)$$

Although the Lyapunov exponent γ for finite N depends on the particular realization of disorder, for $N\rightarrow\infty$ it converges to its mean value.[21] For the calculations below we have used samples of length 5×10^5 for relatively large values of W, and up to 4×10^6 for small values of W.

III. SCALING APPROACH FOR THE EIGENSTATES

Our interest is dedicated to the structure of eigenstates for infinite as well as finite samples, as we tune the disorder and the electric-field strength. Unlike the simpler case of infinite samples, however, the meaning of a localization length for finite samples is not clear. Below we follow the approach developed in the theory of quasi-1D disordered solids which is based on the evaluation of multifractal localization lengths (see, e.g., Ref. 13). The great advantage of this approach is the applicability in both finite and infinite samples.

One of the commonly used quantities in this approach is the so-called entropic localization length, defined through the information entropy $\mathcal{H}_N$ of eigenstates,

$$\mathcal{H}_N=-\sum_{n=1}^{N} w_n \ln w_n, \quad w_n=|\varphi_n^2|, \quad (6)$$

where φ_n is the nth component of an eigenstate in a given basis. For eigenstates normalized as $\Sigma_n w_n=1$, the simplest case of $\varphi_n=N^{-1/2}$ results in an entropy equal to the maximum value: $\mathcal{H}_N=\ln(N)$. We therefore define the localization length L as the number of basis states occupied by the eigenstate φ_n; the latter is equal to $\exp(\mathcal{H}_N)$. We note that, in general, the amplitudes φ_n fluctuate strongly with n, and thus the coefficient of proportionality between L and l_∞ depends on the type of fluctuations.

In order to study the properties of eigenstates in quasi-1D solids, localized on some scale in the finite basis, it was suggested in Ref. 22 to normalize the localization length L in such a way that in the extreme case of fully extended states the quantity L is equal to the size of the basis N. In such an approach, the entropic localization length L_1 is defined as

$$L_1=N\exp(\langle\mathcal{H}_N\rangle-\mathcal{H}_{\rm ref}). \quad (7)$$

In Eq. (7), the ensemble average $\langle\cdots\rangle$ is performed over the number of eigenstates with the same structure, and over realizations of the disorder potential. The normalization factor $\mathcal{H}_{\rm ref}$ has the meaning of an average entropy of the completely extended random eigenstates in a finite basis. For the quasi-1D case the distribution of components φ_n is assumed to correspond to the Gaussian orthogonal ensemble (GOE).[22]

Analogously, a whole set of localization lengths L_q can be defined as[13]

$$L_q = N\left(\frac{\langle P_q\rangle}{P_{\rm ref}^{(q)}}\right)^{1/(1-q)}, \quad P_q = \sum_{n=1}^{N}(w_n)^q, \; q \geqslant 2, \quad (8)$$

where $P_{\rm ref}^{(q)}$ is the average value of P_q for the reference ensemble of completely extended states. One should note that for the particular case $q=2$ the quantity P_2 is known as the participation ratio.[23]

In the context of quasi-1D disordered models in the presence of a constant electric field,[9,15,14] it was shown that all global properties of eigenfunctions are described by the localization parameters

$$\beta_q^{\infty} = \frac{L_q}{l_{\rm el}}, \quad \beta_q^{N} = \frac{L_q}{N}, \quad (9)$$

where the superscripts ∞ and N denote infinite and finite samples, respectively. Moreover it was found that $\beta_q^{\infty,\,N}$ obey some scaling law, i.e., they depend only on the ratio of the characteristic lengths of the system. In the case of infinite samples only two length scales, i.e., l_∞ and $l_{\rm el}$, are relevant. For finite samples, however, a third length N, which is the actual size of the sample, comes into play and has to be taken into account in the scaling theory. According to the scaling conjecture in the modern theory of disordered solids, it was found that for quasi-1D systems in the presence of an electric field,[9,15,14] $\beta_q^{\infty,N}$ follow the scaling laws

$$\beta_q^{\infty} = \beta_q^{\infty}(\lambda_\infty), \quad \beta_q^{N} = \beta_q^{N}(\lambda_\infty, \lambda_N),$$

where

$$\lambda_\infty = \frac{l_\infty}{l_{\rm el}}, \quad \lambda_N = \frac{l_\infty}{N}. \quad (10)$$

Our main question is whether relations of the type of Eq. (10) are also applicable for our 1D Anderson tight-binding model with electric field. In Refs. 13 and 24 it was shown analytically that the eigenstates in 1D and quasi-1D disordered systems, *without* electric field, possess the same gross structure (envelope) on scales comparable with the localization length, while their statistical properties are quite different on a much finer scale of the order of the lattice spacing. That is the reason why many scaling laws, which are dominated by the fluctuations of the envelope, hold both for 1D and quasi-1D systems. The validity of such a statement is however questionable in the presence of electric fields. We will show that such a similarity between quasi-1D and 1D disordered systems also persists in this case.

The first nontrivial question in this context arises about the reference ensemble for the computation of the average entropy $\mathcal{H}_{\rm ref}$. Indeed, in application to 1D Anderson-type models,[10–12] the reference ensemble cannot be chosen as an ensemble of full random matrices, like the GOE. This point is related to the fact that in the 1D tight-binding case fully extended states are not Gaussian random functions but just plane waves which arise for zero disorder. In the presence of an electric field, the situation is even more complicated due to strong dependence of the eigenstates on the electric field. However, and this is our expectation, in spite of the above-mentioned differences, scaling properties of the eigenstates of the 1D model [Eq. (1)] are of the generic type discovered for quasi-1D systems.

For this reason, and in the spirit of Refs. 10 and 12 we define the normalization factors $\mathcal{H}_{\rm ref}$ and $P_{\rm ref}^{(q)}$ from the solution of Eq. (2) for zero disorder and electric field, i.e., $\epsilon_n = 0$ and $F=0$,

$$E^k = 2V\cos\frac{k\pi}{N+1}, \quad \varphi_n^k = \sqrt{\left(\frac{2}{N+1}\right)}\sin\frac{nk\pi}{N+1}, \quad (11)$$

with $k,n = 1,\ldots,N$. The entropy $\mathcal{H}_{\rm ref}$ and the participation ratio $P_{\rm ref}^{(2)}$ of the above eigenstates in the large N limit have the same values for every eigenvalue E^k, i.e.,

$$\mathcal{H}_{\rm ref} = \ln(2N) - 1, \quad P_{\rm ref}^{(2)} = \frac{3}{2N}. \quad (12)$$

IV. SCALING PROPERTIES OF LOCALIZATION LENGTHS

A. Infinite samples

In this section we analyze the scaling properties of eigenstates of infinite systems. In numerical studies the matrices are obviously of finite size N. However, in our analysis below we will always consider the case that $N \gg l_\infty, l_{\rm el}$, and thus the finite (but large) size of the matrix becomes irrelevant. We therefore have used these data to investigate our scaling assumption for β_q^∞ [see Eq. (9)].

As mentioned in Sec. II, the introduction of a nonzero electric field $F \neq 0$ results in an additional length scale $l_{\rm el}$. This length arises when we consider a cross section of the energy band locally tilted by the electric field: $-V/2 + Fn \leqslant E \leqslant V/2 + Fn$ for an energy level E. Therefore the scaling parameter $\lambda_\infty = l_\infty / l_{\rm el}$ enters the problem. Furthermore, if we consider the localization lengths $L^{(q)}$ of Eqs. (7) and (8) as the typical length, which contains most of an eigenvector normalization, we expect that

$$L_q \simeq \begin{matrix} l_\infty & \lambda_\infty \ll 1 \\ l_{el} & \lambda_\infty \gg 1, \end{matrix} \quad (13)$$

i.e., the exponentially localized states progressively become localized factorially as the field increases. This is due to the fact that, for a weak electric field, we have $\lambda_\infty \ll 1$, and thus the dominant localization mechanism, i.e., the one that produces the shortest localization length scale, is the one related to the randomness. From now on we will refer to this as the ''Anderson regime.'' In the opposite limit $\lambda_\infty \gg 1$, the dominant localization mechanism is due to the electric field. We will refer to this regime as the ''Stark regime.'' Based on the previous considerations we expect that the L_q's have the following scaling form (also see Ref. 9 for an equivalent reasoning for quasi-1D systems):

$$L_q = l_\infty f(\lambda_\infty) \quad \text{with} \quad f(\lambda_\infty) \simeq \begin{cases} 1 & \lambda_\infty \ll 1 \\ \dfrac{1}{\lambda_\infty} & \lambda_\infty \gg 1, \end{cases} \quad (14)$$

where $f(\lambda_\infty)$ is related to the scaling function β_q^∞ as $\beta_q^\infty = \lambda_\infty f(\lambda_\infty)$.

Our aim in this paragraph is to support the above-mentioned scaling law based on numerical data, and to extend our knowledge on the structure of the eigenstates in the

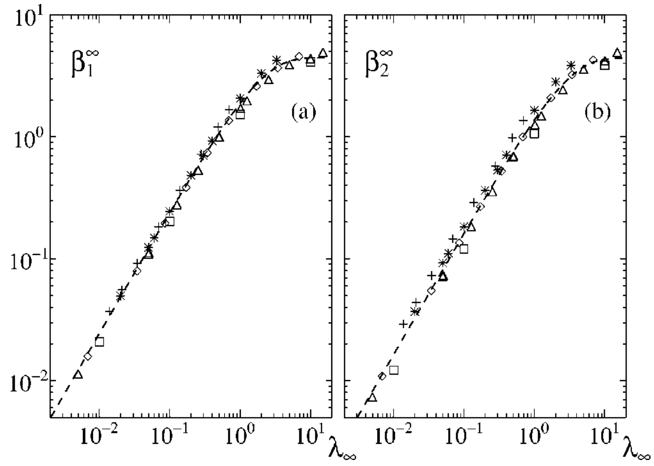

FIG. 2. One-parameter scaling of the (nearly) infinite sample ($N=10^4$) upon variation of λ_∞ in a range $N \gg l_{\rm el}, l_\infty$ ($W=2.62$, 3.87, 5, 7.35, and 10, and $eF \in [10^{-4},2]$). A least-squares fit according to Eq. (15) is shown as dashed line. (a) Scaling of β_1^∞ ($a_1^0=4.45$ and $a_1^1=0.55$). (b) Scaling of β_2^∞ ($a_2^0=4.43$ and $a_2^1=0.37$).

intermediate regime between the two discussed limits. In order to study scaling properties of the localized eigenstates, we have used the transfer-matrix method for the calculation of l_∞ as well as the direct diagonalization of the Hamiltonians that are associated with Eq. (1), for finite but long chains of size $N=10^4$. In all numerical calculations below we used $V=1$. We then varied the disorder strength W as well as the dc field strength in a regime, where always $l_\infty, l_{\rm el} \ll N$. One should stress here that both localization lengths l_∞ and L_q are functions of the energy E. For this reason, in our numerical experiments, we consider ensembles of states specified by the values of the energy E in a small window $E \in [0.95,1.05]$. The size of the energy window was chosen in such a way that the localization length l_∞ is approximately constant inside this window (in all cases, $\Delta l_\infty / l_\infty \leq 0.06$). The values of β_1^∞ and β_2^∞ are then obtained by averaging over all eigenstates which are found inside the energy window for a set of different realizations of disorder. As a result, the total number of eigenstates used for the calculation of β_q^∞ was more than 1500.

A detailed analysis of the numerical data gives evidence of a scaling behavior of the form of Eqs. (10) and (14) with the scaling function

$$\beta_q^\infty \approx a_q^0 [1-\exp(-a_q^1 \lambda_\infty)], \tag{15}$$

where the parameters a_q^0, and a_q^1 are determined from a least-squares fit. We have found that $a_q^0=4.45$ (4.43) and $a_q^1=0.55$ (0.37) for $q=1$ (2). We note here that a similar scaling function was found in the framework of quasi-1D systems for $q=1$.[14] Our data, together with a fit according to Eq. (15), are presented in Fig. 2. We observe a nice agreement with the scaling assumption of Eqs. (10) and (14).

B. Finite samples

In realistic situations one always deals with finite samples. In such cases an understanding of the statistical properties of conductance is of major importance. Since these properties are directly related to the structure of eigenstates, it is important to investigate the statistical properties of eigenstates for finite systems. This is the goal of the present subsection.

For finite N and zero electric field, it was shown in Refs. 10–12 that the statistical properties of 1D Anderson models are characterized by a single scaling parameter $\lambda_N = l_\infty / N$. Moreover, the scaling relation for the eigenvectors was found to be very simple:

$$\beta_q^N = \beta_q^N(\lambda_N) = \frac{c_q \lambda_N}{1+c_q \lambda_N}, \tag{16}$$

where the constants c_q were found to be $c_1 \approx 2.6$ and $c_2 \approx 1.5$. In fact, this scaling relation is exact only for $q=2$. For other cases of small values q, including $q=1$, however, it is very close to the correct one (see details in Ref. 13).

Once the electric field is turned on, however, a new length scale $l_{\rm el}$ (with respect to the standard Anderson models) appears. This can already be seen from the previous paragraph, where on the basis of numerical results we were able to conclude that for infinite 1D Anderson models in the presence of an electric field the scaling properties of the eigenvectors are characterized by the parameter λ_∞. Since the sample size now enters as a third length scale, the second scaling parameter λ_N is likely to show up. Thus we expect that the statistical properties of the eigenstates, and accordingly the β_q^N's, are going to be determined by the two parameters λ_∞ and λ_N which arise due to the competition between the characteristic lengths l_∞ and $l_{\rm el}$ of the corresponding infinite sample and the actual size N of the sample. In the rest of this section we are going to present numerical evidence that for finite 1D Anderson models in the presence of an electric field; the statistical properties of the eigenstates are characterized by the two scaling parameters λ_∞ and λ_N. To this end, we will concentrate on the localization measures β_q^N [see Eq. (9)], which are the finite-size counterparts of β_q^∞.

To find the localization lengths L_q for finite samples of size N, we have used the same approach as in Sec. IV A. The average values of L_q were calculated by choosing only the eigenstates which had eigenvalues within a small energy window $E \in [0.95,1.05]$. Additionally, we performed an ensemble averaging over at least 100 realizations of the disorder potential. For each N the total averaging thus involved several hundreds up to several thousands of eigenstates. In all these calculations the sample size was varied from $N=200$ up to 1000.

To test scaling assumption (10) for finite systems, we first analyze the behavior of the localization measures $\beta_{1,2}^N$ once λ_N is fixed. This is the finite sample counterpart of the scaling analysis presented in Sec. IV A. In Fig. 3 we report our numerical results. Different symbols correspond to various sample sizes N's and disorder strengths W's such that always $\lambda_N=1$. The good overlap confirms the scaling dependence $\beta_q^N = \beta_q^N(\lambda_\infty)$ conjectured in Eq. (10).

Let us now try to gain some insight in the asymptotic form of the scaling law of $\beta_q^N(\lambda_\infty, \lambda_N = \text{const})$. For $\lambda_\infty \ll 1$, the system is in the Anderson regime, where β_q^N is given by

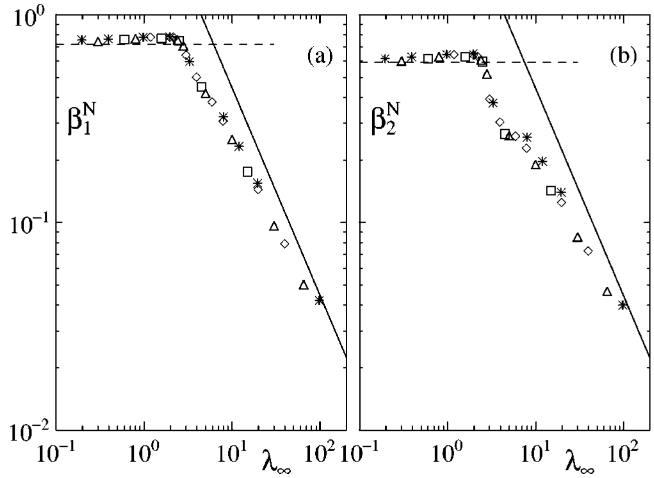

FIG. 3. Finite sample scaling of β_q^N as a function of λ_∞ with $\lambda_N=1$. Different symbols correspond to various sample sizes $N \in [200,1000]$ and disorder strengths $W \in [0.3,0.6]$; the electric field was chosen appropriately $(eF \in [5\times10^{-4}, 5\times10^{-1}])$. Dashed lines correspond to Eq. (17), where the values c_q were taken from a least-squares fit of Fig. 4(a) and 4(b) (see below). Full lines represent the scaling law derived in Eq. (19), where the values for a_q^0 where taken from Fig. 2. (a) Scaling of β_1^N ($a_1^0=4.45$, $c_1=2.59$). (b) Scaling of β_2^N ($a_2^0=4.43$, $c_2=1.45$).

Eq. (16). The latter expression does not depend on λ_∞, and thus we can conclude that β_q^N has to saturate to a constant which is given by

$$\beta_q^N(\lambda_\infty \ll 1,\lambda_N) \approx \frac{c_q\lambda_N}{1+c_q\lambda_N}, \tag{17}$$

where $\lambda_N=$const The dashed lines in Fig. 3 show the expected saturation plateau given by Eq. (17) for $\lambda_N=1$. The agreement with the numerical data is very good.

In the opposite limit $\lambda_\infty \gg 1$, Stark localization sets the dominant length scale $l_{\rm el}$. Since we can always choose the strength of the electric field F such that $N, l_\infty \gg l_{\rm el}$, and assuming continuity in the form of β_q^N, we can approximate the latter with the help of Eq. (15). For $\lambda_\infty \gg 1$ this formula yields $\beta_q^\infty \approx a_q^0$. Next, by changing variables and going from β_q^∞ to β_q^N, we obtain

$$\beta_q^N = \frac{l_{\rm el}\beta_q^\infty}{N} = \frac{\lambda_N a_q^0}{\lambda_\infty}. \tag{18}$$

Displaying β_q^N versus λ_∞ in a double logarithmic fashion, this yields

$$\ln(\beta_q^N) \approx \ln(\lambda_N a_q^0) - \ln(\lambda_\infty). \tag{19}$$

This linear behavior (19) is shown by solid lines in Fig. 3, it approximately describes the numerical data. Deviations are due to the fact that the approximation via the scaling law of Eq. (15) actually requires not only $l_{\rm el} \ll N$ but also $l_\infty \ll N$, which is not fulfilled in our case. Nevertheless it gives a reasonable estimate.

We now turn to the case where λ_∞ is fixed and λ_N varies. Our numerical results, for $\lambda_\infty=0.01$, 1, and 20, corresponding to the Anderson, intermediate, and Stark regimes, respectively, are shown in Fig. 4 where now we refer to the variable

$$Y_q = \frac{\beta_q}{1-\beta_q}. \tag{20}$$

The points corresponding to the same λ_∞ (but different l_∞ and $l_{\rm el}$) fall onto the same smooth curve with a good accuracy, confirming scaling hypothesis (10). From Fig. 4 one can see that the behavior of Y_q is different in the two regimes $\lambda_\infty \gg 1$ ($\lambda_\infty \ll 1$) where localization is due to the Stark (Anderson) mechanism. As a matter of fact, as we are increasing λ_∞ two asymptotic regimes start to build up which have the same slope and differ only by a constant shift.

To understand the behavior of $Y_q(\lambda_N)$ as a function of λ_∞, we first try to illuminate the limiting cases. Let us start with the limit $\lambda_\infty \ll 1$. This condition defines the Anderson regime, where Eq. (16) holds, and thus a behavior

$$\ln(Y_q) = \ln(\lambda_N) + \ln c_q \tag{21}$$

in terms of the variable Y_q is expected over the whole range of λ_N. This expectation is shown in Fig. 4 by solid lines. Since λ_∞ is small but still finite, we can also estimate c_q from Eq. (15). The limit $\lambda_N \ll \lambda_\infty \ll 1$ corresponds to the Anderson regime of a nearly infinite sample. Expanding Eq. (15) to first order yields $\beta_q^\infty \approx a_q^0 a_q^1 \lambda_\infty$. Substituting this expression into Eq. (18), we end up with the following term for β_q^N:

$$\beta_q^N \approx a_q^0 a_q^1 \lambda_N. \tag{22}$$

Inserting Eq. (22) into the definition of Y_q, and assuming $\lambda_N \ll 1$, we obtain

$$\ln(Y_q) \approx \ln(\lambda_N) + \ln(a_q^0 a_q^1), \tag{23}$$

which implies $c_q \approx a_q^0 a_q^1$. A comparison of $c_q = 2.59\,(1.55)$ and $a_q^0 a_q^1 = 2.45\,(1.64)$ shows a very good agreement. Thus we conclude that our scaling function (15) is consistent with Eq. (21), as it should be in the above limit.

In the opposite limit of $\lambda_\infty \gg 1$, we have to distinguish between the following two cases. When $\lambda_N \gg \lambda_\infty$, the sample size N sets the smallest length scale. In this limit the eigenstates extend over the whole sample. Then, by continuity, we expect that the behavior of $Y_q(\lambda_N)$ for $\lambda_N \gg 1$ will be given by Eq. (21). Our numerical data [see Figs. 4(e) and 4(f)] support this hypothesis. The second case, in which $\lambda_N \ll \lambda_\infty$ holds, can be viewed as the extreme Stark regime of an infinite sample. In that limit one obtains Eq. (19) again, which yields

$$\ln(Y_q) \approx \ln(\lambda_N) + \ln\left(\frac{a_q^0}{\lambda_\infty}\right). \tag{24}$$

The asymptotic behavior [Eq. (24)] is reported in Figs. 4(e) and 4(f) by dashed lines and, agrees quite well with our numerical data.

From the above analysis we conclude that, at

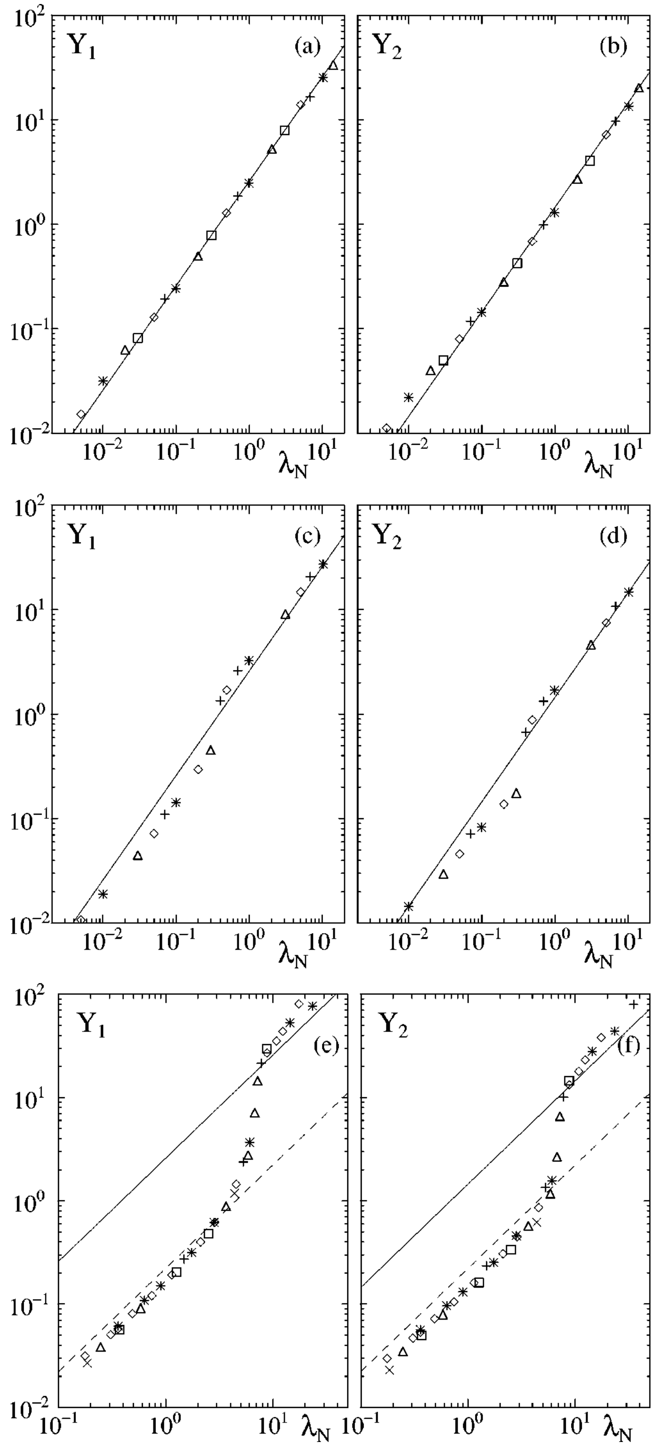

FIG. 4. Scaling of $Y_q=\beta_q/(1-\beta_q)$ in the finite sample upon variation of λ_N for $\lambda_\infty=0.01$, 1, and 20. Different symbols correspond to various disorder strengths $W\in[0.1,6]$ and sample sizes $N\in[200,1000]$. Full and dashed lines correspond to the limiting cases given by Eqs. (16) (with $c_1=2.59, c_2=1.45$) and (24) (with $a_1^0=4.45, a_2^0=4.35$), respectively. (a) and (b) Scaling of Y_1, and Y_2 for $\lambda_\infty=0.01$ (Anderson regime). (c) and (d) Scaling of Y_1, and Y_2 for $\lambda_\infty=1$ (intermediate regime.) (e) nd (f) Scaling of Y_1, and Y_2 for $\lambda_\infty=20$ (Stark regime).

$$\lambda_\infty \simeq \lambda_N, \tag{25}$$

two asymptotic regimes are created due to the interplay between Anderson and Stark localization mechanisms. Although these estimations are made only on a very rough level, they describe our numerical findings rather well.

V. SCALING OF THE DISTRIBUTION OF EIGENVECTORS

As a complement to the above analysis, we show in this section that the distributions of squared components of eigenvectors are parametrized in the same fashion by λ_N and λ_∞. Again we restrict ourselves to a definite energy window $E\in[0.95,1.05]$, where all eigenvectors corresponding to eigenvalues in that window were computed for several realizations of disorder. The total number of eigenstates were in all cases more than 1000.

Before examining the scaling properties of the distribution let us distinguish between the various cases that appear due to the competition between the three characteristic length scales $N, l_{\rm el}$, and l_∞ (see Sec. IV). For simplicity we define these regimes only by their limiting cases, which read as follows:

(1) $l_\infty < l_{\rm el} < N$, $l_{\rm el} < l_\infty < N$ (infinite sample),
(2) $l_\infty < N < l_{\rm el}$, $N < l_\infty < l_{\rm el}$ (Anderson regime),
(3) $N < l_{\rm el} < l_\infty$, $l_{\rm el} < N < l_\infty$ (Stark regime).

The first pair in this list corresponds to the scaling behavior of the infinite sample, since the sample size is always larger than the other two competing lengths. For this case, and $q=1$ and 2, we have shown already in Sec. IV A that β_q^∞ follows a single parameter scaling with respect to λ_∞. We will show here that the whole distribution of eigenvector components is also parametrized according to the same scaling parameter. The first question which arises for the infinite sample is the proper normalization of the squared entries of the eigenstates $w_n=|\varphi_n|^2$. A normalization with respect to the number of sites N does not seem appropriate, since we are interested in the limit $N\rightarrow\infty$. Since w_n has to scale with some length, however, following our previous strategy [see Eq. (9)], we introduce the variable

$$r=\ln(w_n l_{\rm el}), \tag{26}$$

and investigate the distribution $p(r)$. For our calculations we consider matrices of size $N=10^4$, while we let $l_{\rm el}, l_\infty \ll N$ and varied λ_∞. For each λ_∞ we considered two different disorder strengths ($l_\infty\approx 6$ and 10) and adjusted the dc field strength appropriately. The results for $\lambda_\infty=10$, 1, and 0.1 are shown in Fig. 5(a)–5(c) in a semilogarithmic plot. The assumed scaling of $p(r)$ with λ_∞ is clearly visible.

The second and third pairs of conditions always involve the sample size N. Therefore, scaling according to λ_∞ *and* λ_N has to be taken into account. For these cases renormalization with respect to the sample size is meaningful; hence we define the rescaled squared entries of eigenstates as

$$r=\ln(w_n N). \tag{27}$$

Before turning to an analysis of our numerical data, let us first qualitatively analyze the form of $p(r)$. In the limit $N \ll l_\infty, l_{el}$ the system does not ''feel'' any localization due to disorder or electric field. Therefore all eigenstates are given approximately by Eq. (11). The distribution $p(r)$ is then given by

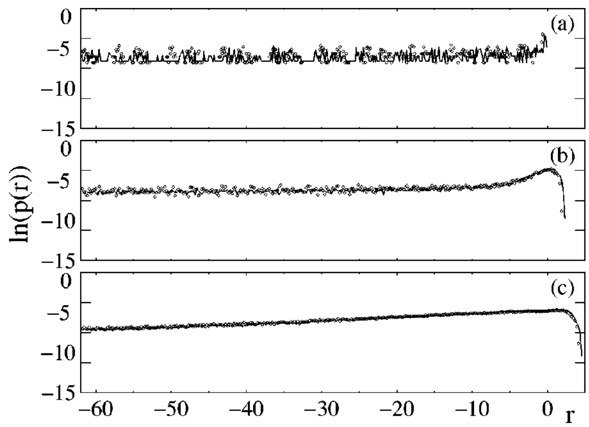

FIG. 5. Scaling of the entire distribution of squared eigenvector components $p(r)$ with λ_∞ in the case of nearly infinite samples ($N=10^4$). Two different pairs of $l_\infty, l_{el} \ll N$ are denoted by full lines ($W=3.49$ and $eF=1.666$, 0.166, and 0.016) and symbols ($W=2.62$, and $eF=1$, 0.1, 0.01), while keeping λ_∞ fixed. (a) $\lambda_\infty=10$. (b) $\lambda_\infty=1$. (c) $\lambda_\infty=0.1$.

$$p(r)=\frac{e^r}{\pi\sqrt{e^r(2-e^r)}}. \tag{28}$$

Plotting $\ln[p(r)]$ versus r for $r \ll 0$ yields a curve with slope 1/2, as can be verified by expanding Eq. (28). Moreover, Eq. (28) shows a sharp peak around $r=0$.

In the case where $l_\infty \ll N, l_{el}$ the disorder sets the relevant length scale, and the system resembles an infinite Anderson model with exponentially localized eigenstates $w_n = \exp(-|n-n_0|/l_\infty)$. For small r this particular form of eigenstates leads to[11]

$$p(r)=l_\infty/N. \tag{29}$$

In a semilogarithmic representation this results in a nearly horizontal curve of height $\ln(l_\infty/N)$ for $r \ll 0$, which drops rapidly for some $r>0$.

For the second pair of conditions, i.e. the Anderson regime, the scaling properties of the distribution were already analyzed in Ref. 11. A good agreement with the limiting equations (28) and (29) was found.

The more interesting case is the pair of conditions with label 3, where the competition between N and l_{el} is dominant. In this case $\lambda_\infty \gg 1$, and thus the localization mechanism is due to the Stark effect. The resulting distribution for some representative values of λ_∞ and λ_N are shown in Fig. 6. One can clearly see that distributions corresponding to different sample sizes N and disorder strengths W, but having the same λ_N and λ_∞, coincide. Moreover, as we move from higher to smaller values of λ_N, the shape of the distributions changes drastically. In the case $\lambda_N \gg 1$ (corresponding to $N \ll l_{el}$) the eigenstates can be considered as extended with respect to the sample size, and thus we again obtain Eq. (28) [see Fig. 6(a)]. The peak of the distribution broadens and moves to the right upon an increase of λ_N, as can be seen from Fig. 6(b). At the same time, for negative values of r the distribution possesses long tails. This becomes more and more apparent as we move toward the Stark regime [Fig. 6(c)]. In the strong-field limit the eigenstates are essentially localized at one site, i.e., $w_n \simeq \delta_{nm}$. In this case, one has $p[r \simeq \ln(N)] \sim 1/N$, while long tails appear due to factorial localization. We conclude this section by noting that the scaling of the distributions of squared eigenvector components with λ_∞ and λ_N implies a scaling of the localization parameters (10) for arbitrary q.

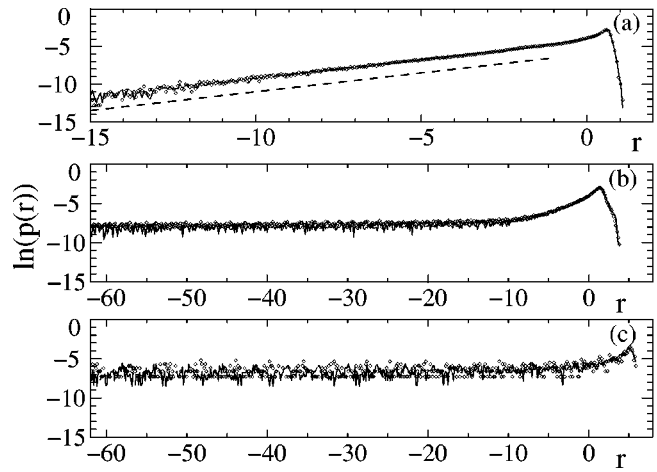

FIG. 6. Scaling of the entire distribution of squared eigenvector components $p(r)$ in the Stark regime ($\lambda_\infty=20$) in the case of finite samples ($N\in[230,1200], W\in[0.01,2], eF\in[10^{-3},1]$). Full lines and symbols denote different sample sizes (and thus different strengths of disorder), while λ_N is kept fixed to a chosen value. (a) $\lambda_N=100$ (the dashed line has a slope 1/2). (b) $\lambda_N=2$. (c) $\lambda_N=0.05$.

VI. SUMMARY

We have studied a 1D tight-binding model in the presence of a constant electric field. For such a model we can distinguish between two regimes where localization is due to totally different mechanisms. The first regime is the Anderson regime, which is defined through the condition $\lambda_\infty \ll 1$. Here localization due to disorder is the dominant mechanism that controls the statistical properties of the eigenstates. In the opposite limit, $\lambda_\infty \gg 1$, the localization is due to the presence of the electric field. This is the Stark regime. Our numerical study deals with the scaling properties of eigenstates both for infinite and finite samples. This study was motivated by the remarkable scaling law that has been found for quasi-1D models with electric field.[9,14,15] Our results indicate that in both infinite and finite samples with disorder and electric field, the eigenstates have generic properties, regardless of the dimensionality of the system, provided that an appropriate renormalization (with respect to the corresponding ''extended'' states) is done. Thus we show that the similarity between 1D and quasi-1D eigenstates should also persist for systems with electric field.

We found that for infinite systems the statistical properties of the eigenstates are characterized by the single parameter λ_∞. This conclusion was based on an extensive numerical analysis of both the localization measures [Eq. (9)] and the whole distribution of squared eigenvector components. Moreover for $\beta^\infty_{q=1,2}$ we have found a simple scaling law [Eq. (15)] that describes our numerical data quite nicely. This expression can be used in order to find the strength of the applied dc electric field once l_∞ is known for the field-free model.

Moreover, we have performed a finite length scaling analysis. Our numerical analysis showed that for finite systems the statistical properties of the eigenstates are characterized by two parameters, namely, λ_∞ (in the case of infinite systems) and λ_N. The latter parameter involves the actual size of the sample which enters into the scaling analysis as a third length scale. We found that the localization measures $\beta^N_{q=1,2}$ show a totally different asymptotic behavior in $\lambda_N \to 0,\infty$ as we increase the parameter λ_∞. Based on some analytical arguments, we estimated that this occurs approximately at $\lambda_N \simeq \lambda_\infty$. This can be used as a criterion to identify which localization mechanism (Anderson or Stark localization) is responsible for the structure of eigenstates. It will be interesting to investigate if the same asymptotic behavior also holds for higher moments q. Finally, we studied the whole distribution of squared eigenvector components, and showed that it also follows the same scaling behavior with respect to the two scaling parameters λ_∞ and λ_N.

The main result of our work is the fact that scaling properties of eigenstates of infinite systems are described by one parameter scaling λ_∞, whereas for finite systems an additional parameter λ_N is also needed. In particular, both localization lengths, the entropy localization length as well as the one defined via the inverse participation ratio, follow the universal scaling law of Eq. (10) after an appropriate normalization. This is in contrast to the standard Anderson models without electric field, where only one parameter (λ_N) is needed to describe the scaling properties of eigenstates.

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