Extension to order β^{23} of the high-temperature expansions for the spin-$\frac{1}{2}$ Ising model on simple cubic and body-centered cubic lattices

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Using a renormalized linked-cluster-expansion method, we have extended to order β^{23} the high-temperature series for the susceptibility χ and the second-moment correlation length ξ of the spin-1/2 Ising models on the sc and the bcc lattices. A study of these expansions yields updated direct estimates of universal parameters, such as exponents and amplitude ratios, which characterize the critical behavior of χ and ξ. Our best estimates for the inverse critical temperatures are $\beta_c^{sc}=0.221\,654(1)$ and $\beta_c^{bcc}=0.157\,372\,5(6)$. For the susceptibility exponent we get $\gamma=1.2375(6)$ and for the correlation length exponent $\nu=0.6302(4)$. The ratio of the critical amplitudes of χ above and below the critical temperature is estimated to be $C_+/C_-=4.762(8)$. The analogous ratio for ξ is estimated to be $f_+/f_-=1.963(8)$. For the correction-to-scaling amplitude ratio we obtain $a_\xi^+/a_\chi^+=0.87(6)$.

I. INTRODUCTION

As a part of an ongoing long-term program of computer-based calculations and analyses of high-temperature (HT) series for two-dimensional[1] and three-dimensional[2,3] lattice spin models, we have extended by two terms the high-temperature series for the spin-1/2 Ising model on the simple cubic (sc) and the body-centered cubic (bcc) lattices. In the first analysis presented here, we shall consider only the HT expansions through β^{23} for the susceptibility χ and the second-moment correlation length ξ, mainly in order to update the *direct* estimates of the corresponding critical indices γ and ν.

For the sc lattice, the longest expansions of these quantities already in the literature reach order β^{21}. They were obtained and analyzed in Refs. 2 and 3 only a few years ago. In the case of the bcc lattice, the published series[4,5] for χ and ξ, also extending through β^{21}, were calculated by Nickel two decades ago. The progress in such computations has been slow due to the exponential growth of their complexity with the order of the expansion, so that even adding only a few terms to the present results is a laborious task. Within the renormalized linked-cluster[6] expansion method, used in our work, one must overcome many problems of combinatorial nature concerning graph generation, classification, and partial resummation, and a special effort must be devoted to keep under strict control the numerous possible sources of error. In our case, a final severe test is provided by having the program reproduce, in three dimensions, established data like the series for the nearest-neighbor spin correlation on the sc lattice, which is already tabulated[7] through β^{27}, and, in two dimensions, the series for χ and μ_2 on the simple square lattice, which are known[4,5] through β^{35} and beyond.[8,9] After the completion of this work, a preprint[10] has been issued which also reports independently extended expansions for χ and μ_2 on both the the sc and the bcc lattice through orders β^{23} and β^{25}, respectively. Our series coefficients agree with those of Ref. 10 as far as the expansions overlap. This adds further confidence about the correctness of the results since our implementation of the linked cluster expansion procedure is rather different from that described in Ref. 10.

Any enrichment of the exact information on the three-dimensional (3D) Ising model is still of general interest. Here we have used these data to improve the knowledge of the non-negligible singular corrections[11] to the leading critical singularities of χ and ξ and, as a consequence, the accuracy of the direct HT series estimates of all critical parameters. As stressed in Refs. 4 and 5, the corrections to scaling first showed up unambiguously when the bcc series were extended to order β^{21}, the last three coefficients being crucial.[5] It is therefore helpful to produce more coefficients, in order to stabilize and possibly refine the quality of the information extracted from the series.

The plan of this note is as follows: after setting our notational conventions in Sec. II, we tabulate the series coefficients for χ and μ_2 through order 23, with respect to the usual HT expansion variable $v=th(\beta)$. In Sec. III we report the results of our extrapolations for the critical temperatures, for the critical exponents γ and ν, for the universal ratio C_+/C_- of the critical amplitudes of the susceptibility above and below the critical point, for the analogous ratio f_+/f_- of the correlation-length amplitudes, and for the ratio a_ξ^+/a_χ^+ of the correction-to-scaling amplitudes.[12] Our estimates are compared with the latest numerical calculations by series, by stochastic methods, and by perturbative renormalization-group (RG) techniques, in the fixed-dimension (FD) approach[13–19] and in the ϵ-expansion approach.[17–22] Less recent studies have been already reviewed in our Refs. 2 and 3.

II. DEFINITIONS AND NOTATIONS

In order to introduce our notation, we shall specify by the Hamiltonian

$$H\{s\}=-\frac{J}{2}\sum_{\langle\vec{x},\vec{x}'\rangle} s(\vec{x})s(\vec{x}') \qquad (1)$$

the nearest-neighbor three-dimensional spin-1/2 Ising model in zero magnetic field. Here $s(\vec{x})=\pm 1$ is the spin variable at the lattice site $\vec{x}$, and the sum extends over all nearest-neighbor pairs of sites. We shall consider expansions in the usual HT variable $\beta = J/k_B T$ called ''inverse temperature'' for brevity. However, for convenience, we shall tabulate the series coefficients with respect to the expansion variable $v = th(\beta)$.

The susceptibility is expressed in terms of the connected two-spin correlation function $\langle s(\vec{x}) s(\vec{y}) \rangle_c$ by

$$\chi(\beta) = \sum_{\vec{x}} \langle s(0) s(\vec{x}) \rangle_c = 1 + \sum_{r=1}^{\infty} a_r \beta^r, \tag{2}$$

and the second moment of the correlation function is defined as

$$\mu_2(\beta) = \sum_{\vec{x}} \vec{x}^2 \langle s(0) s(\vec{x}) \rangle_c = \sum_{r=1}^{\infty} b_r \beta^r. \tag{3}$$

In terms of χ and μ_2 the second-moment correlation length ξ is defined by

$$\xi^2(\beta) = \frac{\mu_2(\beta)}{6\chi(\beta)}. \tag{4}$$

For easy reference we report here the complete expansions of χ and μ_2, rather than only the lastly computed two coefficients. For the susceptibility on the sc lattice we have

$$\begin{aligned}\chi^{sc}(v) = {} & 1 + 6v + 30v^2 + 150v^3 + 726v^4 + 3510v^5 + 16710v^6 + 79494v^7 + 375174v^8 + 1769686v^9 + 8306862v^{10} \\ & + 38975286v^{11} + 182265822v^{12} + 852063558v^{13} + 3973784886v^{14} + 18527532310v^{15} + 86228667894v^{16} \\ & + 401225368086v^{17} + 1864308847838v^{18} + 8660961643254v^{19} + 40190947325670v^{20} + 186475398518726v^{21} \\ & + 864404776466406v^{22} + 4006394107568934v^{23} + \cdots .\end{aligned}$$

For the second moment on the sc lattice we have

$$\begin{aligned}\mu_2^{sc}(v) = {} & 6v + 72v^2 + 582v^3 + 4032v^4 + 25542v^5 + 153000v^6 + 880422v^7 + 4920576v^8 + 26879670v^9 + 144230088v^{10} \\ & + 762587910v^{11} + 3983525952v^{12} + 20595680694v^{13} + 105558845736v^{14} + 536926539990v^{15} + 2713148048256v^{16} \\ & + 13630071574614v^{17} + 68121779384520v^{18} + 338895833104998v^{19} + 1678998083744448v^{20} \\ & + 8287136476787862v^{21} + 40764741656730408v^{22} + 199901334823355526v^{23} + \cdots .\end{aligned}$$

For the susceptibility on the bcc lattice we have

$$\begin{aligned}\chi^{bcc}(v) = {} & 1 + 8v + 56v^2 + 392v^3 + 2648v^4 + 17864v^5 + 118760v^6 + 789032v^7 + 5201048v^8 + 34268104v^9 + 224679864v^{10} \\ & + 1472595144v^{11} + 9619740648v^{12} + 62823141192v^{13} + 409297617672v^{14} + 2665987056200v^{15} \\ & + 17333875251192v^{16} + 112680746646856v^{17} + 731466943653464v^{18} + 4747546469665832v^{19} \\ & + 30779106675700312v^{20} + 199518218638233896v^{21} + 1292141318087690824v^{22} \\ & + 8367300424426139624v^{23} + \cdots .\end{aligned}$$

For the second moment on the bcc lattice we have

$$\begin{aligned}\mu_2^{bcc}(v) = {} & 8v + 128v^2 + 1416v^3 + 13568v^4 + 119240v^5 + 992768v^6 + 7948840v^7 + 61865216v^8 + 470875848v^9 \\ & + 3521954816v^{10} + 25965652936v^{11} + 189180221184v^{12} + 1364489291848v^{13} + 9757802417152v^{14} \\ & + 69262083278152v^{15} + 488463065172736v^{16} + 3425131086090312v^{17} + 23896020585393152v^{18} \\ & + 165958239005454632v^{19} + 1147904794262960384v^{20} + 7910579661767454248v^{21} \\ & + 54332551216709931904v^{22} + 372033905161237212392v^{23} + \cdots .\end{aligned}$$

III. ANALYSIS OF THE SERIES

In terms of the reduced inverse temperature $\tau^{\#} = 1 - \beta/\beta_c^{\#}$, the asymptotic critical behavior of the susceptibility is expected to be[11]

$$\chi^{\#}(\beta) \simeq C_+^{\#} (\tau^{\#})^{-\gamma} (1 + a_\chi^{+\#} (\tau^{\#})^{\theta} + \cdots + e_\chi^{+\#} \tau^{\#} + \ldots) \tag{5}$$

as the critical point $\beta_c^{\#}$ is approached from below. (Here and in what follows, the superscript # stands for either sc or bcc,

TABLE I. The sequences of approximants for β_c and γ defined by Eqs. (7) and (9), respectively, and the sequences of the appropriate extrapolations using alternate pairs, as obtained from χ on the bcc lattice. For the extrapolations we have assumed that $\theta = 0.504$.

n	$(\beta_c)_n$ from Eq. (7)	Extrapol. of $(\beta_c)_n$	γ_n from Eq. (9)	Extrapol. of γ_n
18	0.1573815		1.244335	
19	0.1573806		1.244174	
20	0.1573807	0.1573761	1.244049	1.238519
21	0.1573800	0.1573759	1.243889	1.238114
22	0.1573799	0.1573743	1.243760	1.237595
23	0.1573793	0.1573746	1.243620	1.237599
24	0.1573791	0.1573739	1.243501	1.237475
25	0.1573787	0.1573740	1.243374	1.237421

as appropriate, and will be dropped whenever unnecessary. The index + (-) denotes, as usual, quantities associated with the high (low) temperature side of the critical point.) Similarly, for the correlation length ξ, we expect

$$\xi^{\#}(\beta) \simeq f_{+}^{\#}(\tau^{\#})^{-\nu}(1 + a_{\xi}^{+\#}(\tau^{\#})^{\theta} + \cdots + e_{\xi}^{+\#}\tau^{\#} + \ \dots) \tag{6}$$

as $\tau \to 0^{+}$.

The exponents γ, ν, and θ are universal quantities, whereas the critical amplitudes $C_{+}^{\#}$, $f_{+}^{\#}$, the amplitudes $a_{\chi}^{+\#}$, $a_{\xi}^{+\#}$ of the leading nonanalytic correction-to-scaling terms, and the amplitudes $e_{\chi}^{+\#}$, $e_{\xi}^{+\#}$ of the leading analytic corrections are nonuniversal, as suggested by the superscript #. Experimentally accessible universal combinations can be formed out of the critical amplitudes.[12] Here we shall be concerned with series estimates of the universal ratios C_{+}/C_{-}, f_{+}/f_{-}, and a_{χ}^{+}/a_{ξ}^{+}. Notice that for the critical amplitudes we have adopted the notation of Ref. 18 and of other recent studies rather than that of Ref. 12.

A. Estimates of the critical points

As a first step of the analysis, we shall examine the series for the susceptibility whose coefficients have the smoothest pattern of behavior, so that they are generally used to estimate the critical temperatures. These estimates will also be used to bias the determination of the critical exponents and of the universal amplitude ratios; therefore, their accuracy is crucial. Let us begin by considering the results obtained by a very efficient variant of the ratio method introduced by Zinn-Justin[23] (see also Ref. 8).

We evaluate β_c from the sequence

$$(\beta_c)_n = \left(\frac{a_{n-2}a_{n-3}}{a_n a_{n-1}}\right)^{1/4} \exp\left[\frac{s_n + s_{n-2}}{2s_n(s_n - s_{n-2})}\right] = \beta_c + O\left(\frac{1}{n^{1+\theta}}\right) \tag{7}$$

where

$$s_n = \left[\ln\left(\frac{a_{n-2}^2}{a_n a_{n-4}}\right)^{-1} + \ln\left(\frac{a_{n-3}^2}{a_{n-1}a_{n-5}}\right)^{-1}\right] \Big/ 2. \tag{8}$$

This is an *unbiased* method, in the sense that no additional accurate information must be used together with the series in order to get the estimates of the critical parameters, but we found it useful to improve the procedure by biasing it with the value of θ as follows. For sufficiently large n, the sequence of estimates $(\beta_c)_n$ shows very small regular oscillations due to the loose structure of the lattice. Moreover, the odd and even subsequences of $(\beta_c)_n$ have a residual decreasing trend which is very nearly linear on a $1/n^{1+\theta}$ plot, as suggested by Eq. (7). Therefore, simply taking the highest-order term of the sequence $(\beta_c)_n$ as the final estimate would be an inadequate choice. We have preferred to extrapolate separately to $n \to \infty$ the successive odd and even pairs of estimates $(\beta_c)_n$, assuming that we know the value of θ well enough. The two sequences of extrapolated values need further extrapolation which allows also for the small residual curvature of the plot and leads to the final estimates $\beta_c^{sc} = 0.221\,654(1)$ in the case of the sc lattice, and $\beta_c^{bcc} = 0.157\,372\,5(6)$ in the case of the bcc lattice. The errors we have reported account generously both for the present uncertainty in θ (whose effects in this analysis are very small anyway) and for the uncertainty of the second extrapolation. For the correction-to-scaling exponent we have assumed the value $\theta = 0.504(8)$, obtained by the FD perturbative RG.[17] Also in the rest of this paper the central values of all θ-biased estimates will refer to this value. However, in the calculations of this and Sec. III B, we have also considered a much larger uncertainty, in order to make sure that our results are compatible with somewhat higher central values such as $\theta = 0.52(3)$, proposed in Ref. 5 (as well as in Ref. 24, with a smaller error), or with $\theta = 0.53(1)$ suggested in Ref. 25. An even larger central value $\theta = 0.54(3)$ was indicated in Refs. 26 and 27, while an experimental measure reported in Ref. 28 yields $\theta = 0.57(9)$. In the case of the bcc lattice, as an example of our extrapolation procedure, we have reported in Table I the last eight terms of the sequence $(\beta_c)_n$ and the results of the initial extrapolation of the last six successive alternate pairs of terms. Our final result for the critical inverse temperature of the Ising model on the sc lattice is completely compatible, although much less precise than the value $\beta_c^{sc} = 0.221\,654\,59(10)$ obtained from an extensive Monte Carlo (MC) study by a dedicated cluster processor[24] and generally considered as the best available estimate. Our central value of β_c^{bcc}, obtained similarly, is only slightly smaller, but more precise than the value $\beta_c^{bcc} = 0.157\,373(2)$ suggested in the Nickel and Rehr analysis.[5,23,29] We should finally mention that Professor D. Stauffer[30] kindly informed us that he still tends to favor the somewhat larger central estimate $\beta_c^{sc} = 0.221\,659$, basing on the HT analysis in Ref. 31, as well as on his own recent simulation[30] of the critical dynamics and on analogous work in Ref. 32. We also recall that a similar value $\beta_c^{sc} = 0.221\,659\,5(26)$ was indicated a decade ago in the Monte Carlo simulation of Ref. 33. In the context of our analysis, these values lie approximately halfway between the highest-order approximant $(\beta_c^{sc})_{23} \approx 0.221\,667$ and our final estimate obtained from extrapolation. To close this section, three remarks are in order. First: the reliability of our analysis procedure has been corroborated by repeating it with the recently computed $O(\beta^{26})$ series for the self-avoiding-walk (saw) model on the sc lattice.[34] This is a relevant test because the structure of the corrections to scaling (namely the sign and size of the correction amplitude and the value of the confluent exponent[3]) is expected to be quite similar to the Ising sc case. For the saw model we have observed that the central value for β_c indicated by our procedure is essentially stabilized after reaching the order β^{23} and agrees closely with that indicated in Ref. 34, while the error decreases as higher-order coefficients are included in the analysis. Our procedure has also been tested and confirmed by other arguments in Ref. 34. Second, due to the higher coordination number of the bcc lattice, the corresponding series have a greater ''effective length'' than the sc series, and therefore all estimates obtained for the bcc lattice will be systematically more accurate. Third, as expected, the inclusion in our analysis of the two additional coefficients for the expansion of χ on the bcc lattice, computed in Ref. 10, does not essentially modify our central estimate of β_c^{bcc}, but only reduces its uncertainty to the value reported here.

B. Estimates of the critical exponents

By using a related variant[8,23] of the ratio method and by analogous arguments, fairly good estimates can be obtained also for the exponents γ and ν. We construct the approximation sequence

$$\gamma_n = 1 + \frac{2(s_n + s_{n-2})}{(s_n - s_{n-2})^2} = \gamma + O\left(\frac{1}{n^\theta}\right) \qquad (9)$$

with the same definition as above for s_n. Also in this case, for sufficiently large n, the successive estimates γ_n (as well as the analogous ones ν_n obtained from the series coefficients of ξ^2), appear to be nearly linear on a $1/n^\theta$ plot, and therefore we can follow an extrapolation procedure completely analogous to the one previously described. However, in the exponent calculation, the corrections are *a priori* larger and therefore the procedure involves relative errors larger than in the case of β_c. In order to illustrate this numerical procedure in the case of the bcc lattice, we have reported in Table I the last eight terms of the sequence γ_n and the results of the extrapolation of the last six successive alternate pairs of terms. The estimates inferred from the analysis of these data are $\gamma = 1.2378(10)$ and $\nu = 0.629(2)$ in the case of the sc series and $\gamma = 1.2373(6)$, $\nu = 0.629(1)$ from the bcc series. As expected, the relative uncertainties for the exponent ν are larger because of the slower approach of the second moment series to its asymptotic behavior. The dependence of these estimates on the value of θ used in the extrapolation can be expressed as follows: $\gamma = 1.2378 + 0.016(\theta - 0.504) \pm 0.0010$ and $\nu = 0.629 + 0.02(\theta - 0.504) \pm 0.0020$ in the case of the sc lattice; $\gamma = 1.2373 + 0.012(\theta - 0.504) \pm 0.0006$ and $\nu = 0.629 + 0.016(\theta - 0.504) \pm 0.0010$ in the case of the bcc lattice.

In order to confirm these estimates for the exponents, we shall resort also to (unbiased and biased) analyses by inhomogeneous differential approximants (DA's).[8,35] By unbiased DA's, we obtain somewhat larger estimates both for the critical inverse temperatures and for the exponents, which, however, show a clear decreasing trend. Therefore these data should also be further extrapolated, but, unfortunately, this is not as straightforward as in the case of the Zinn-Justin method. Thus we did not insist on this route and preferred to perform *biased* series analyses, either (i) by the first-order *simplified* differential approximants (SDA) introduced and discussed in Ref. 3, in which both β_c and the correction-to-scaling exponent θ are fixed, or alternatively (ii) by conventional second-order inhomogeneous DA's, in which θ and β_c are varied in a small neighborhood of their expected values, following the method of Ref. 5. Let us also add that in all cases in which we have relied on SDA's, we have also repeated the same calculation, first subjecting the series to the biased variable change introduced by Roskies[36] in order to regularize the leading correction to scaling and then computing simple Padé approximants. In this way we have always obtained completely consistent results, although they are sometimes affected by larger uncertainties.

We have used the procedure (i) to study the residue of the logarithmic derivative of χ or of ξ^2 at the critical singularity. In the case of the sc lattice series, rather than our own estimate of β_c, we have used the more accurate (but otherwise completely consistent) value $\beta_c^{sc} = 0.221\,654\,59(10)$ of Ref. 24. Thus we estimate $\gamma = 1.2378(10)$ and $\nu = 0.6306(8)$.

In the analysis of the bcc lattice series, we have taken as a bias the value suggested by our extended ratio-method analysis $\beta_c^{bcc} = 0.157\,372\,5(6)$. In this case we get the values $\gamma = 1.2375(6)$ and $\nu = 0.6302(4)$. By using the Fisher scaling law,[37] we get $\eta = 0.037(3)$ from the sc series and $\eta = 0.036(2)$ from the bcc series. For both lattices we have used the same value (and uncertainty) of θ as previously discussed and we have easily allowed for the residual decreasing trend of the exponent estimates, because SDA's values show a smaller spread than DA's. We can also mention that in the bcc lattice case, the linearized dependence of the exponent central estimates on the bias values of β_c and θ can be described as follows: $\gamma = 1.2375 + 0.01(\theta - 0.504) + 90(\beta_c - 0.1573725)$ and $\nu = 0.6302 + 0.015(\theta - 0.504) + 40(\beta_c - 0.1573725)$. We shall take as our final estimates for the exponents those obtained by SDA's from the bcc lattice, which are best converged.

The so-called M2 method of Ref. 38 is a very useful extension of the above-mentioned Roskies' procedure.[36] In the case of the bcc lattice it suggests $\beta_c^{bcc} = 0.157\,372\,0(4)$ with $\gamma = 1.2374(4)$ and $\theta = 0.56(3)$, in good consistency with the other approaches. On the other hand, in the case of the sc lattice, the results of the M2 method at order β^{23}, namely, $\beta_c^{sc} = 0.221\,659(2)$, $\gamma = 1.2395(5)$ with $\theta = 0.50(2)$ are not essentially changed with respect to those obtained in Ref. 31 from the analysis of our previous $O(\beta^{21})$ series.

TABLE II. A comparison among recent estimates of the critical exponents γ and ν.

	This work	Series[a]	Series[b]	MC[c]	MC[d]	MC[e]	FD exp.[f]	ϵ exp.[f]
γ	1.2375(6)	1.237(2)	1.2371(4)	1.2372(17)	1.2367(20)	1.2353(25)	1.2396(13)	1.2380(50)
ν	0.6302(4)	0.6300(15)	0.63002(23)	0.6303(6)	0.6296(7)	0.6294(10)	0.6304(13)	0.6305(25)

[a]Reference 5.
[b]Reference 39.
[c]Reference 24.
[d]Reference 25.
[e]Reference 40.
[f]Reference 17.

In conclusion, provided that the sequences of estimates are carefully extrapolated using the independently computed value of θ, the determination of the exponents by the improved ratio method and by biased DA's or SDA's are completely consistent, though the latter method gives slightly more accurate results. At this order of expansion, asymptotic trends seem to be already stabilized and the uncertainties in the HT series estimates are significantly reduced. A sample of recent estimates of the critical exponents is reported in Table II and briefly commented on in the rest of this section. The agreement of our results with the values $\gamma=1.2396(13)$ and $\nu=0.6304(13)$, indicated by the FD perturbative RG,[17] or the values $\gamma=1.2380(50)$ and $\nu=0.6305(25)$, suggested by the ϵ expansion,[17] is still good. However, we should observe that through the years, as the length of the HT series has increased, the exponent estimates have been moving towards the slightly lower central values $\gamma\approx 1.237$ and $\nu\approx 0.630$. Indeed, very similar values had already been suggested some time ago by Chen, Fisher, and Nickel[26] who studied soft spin models of the Ising universality class, chosen so to have negligible amplitudes for the leading corrections to scaling. Within this approach, the analysis[5] of HT series through order β^{21} for the bcc lattice gave $\gamma=1.237(2)$ and $\nu=0.6300(15)$. A study[39] of the HT series through $O(\beta^{20})$ for the sc lattice, along the same lines as in Refs. 5, 26, indicates $\gamma=1.2371(4)$ and $\nu=0.63002(23)$. Recently, this method was adapted also to MC simulations in Ref. 24, which reports $\gamma=1.2372(17)$ and $\nu=0.6303(6)$. Analogously in Ref. 25, the estimates $\nu=0.6296(7)$ and $\eta=0.0358(9)$ are obtained, implying $\gamma=1.2367(20)$. Even lower central estimates of the exponents, namely $\gamma=1.2353(25)$ and $\nu=0.6294(10)$, have been obtained in a MC simulation of the Ising model by a finite-size scaling analysis[40] which allows for the corrections to scaling.

C. Estimates of universal amplitude ratios

By taking advantage also of the low-temperature expansion of χ on the sc lattice, extended in Ref. 7 to order u^{26} [here $u=\exp(-4\beta)$], and of the older series for the bcc lattice computed to order u^{23} in Ref. 41, we can give a new direct estimate of the universal ratio C_+/C_-. Using the low-temperature series for ξ^2, computed for the sc lattice in Ref. 42 through u^{23}, we can also compute the ratio f_+^{sc}/f_-^{sc}. These quantities have been repeatedly evaluated in recent years by various techniques with increasing accuracy.

We have used first-order SDA's to compute $C_\pm^{\#}=\lim_{\tau\to 0^\pm}|\tau|^\gamma\chi^{\#}$. In the sc lattice case, by choosing $\beta_c^{sc}=0.221\,654\,59$, $\gamma=1.2375$, and $\theta=0.5$, we obtain $C_+^{sc}/C_-^{sc}=4.762(8)$. Here the uncertainty refers to the sharp bias values indicated above. In order to compare this result with others obtained by slightly different assumptions, the dependence of our estimate on the bias values γ_b and θ_b for the critical and the correction exponents can be linearly approximated by $C_+^{sc}/C_-^{sc}=4.762+6.06(1.2375-\gamma_b)+0.7(\theta_b-0.5)\pm 0.008$. The ratio is insensitive to the choice of β_c^{sc} within its quoted uncertainty. In the case of the bcc lattice, we have used the same bias values for γ and θ together with $\beta_c^{bcc}=0.157\,372\,5$, obtaining $C_+^{bcc}/C_-^{bcc}=4.76(3)$. In this case the error (mainly coming from the uncertainty of C_-^{bcc}) is larger, but the result is completely consistent with the sc lattice estimate.

The experimental measurements of this ratio range between 4.3 and 5.2 (Refs. 12,17, and 18) and are perfectly compatible with our estimates. Other recent numerical evaluations are summarized in Table III. However, some comments are helpful for understanding these results. The previous evaluations by Fisher and co-workers,[27,43] used shorter series and bias values $\gamma=1.2395$ and $\beta_c^{sc}=0.221\,630$ somewhat different from ours, thus yielding the slightly larger value 4.95(15). The ratio C_+/C_- can also be obtained from approximate parametric representations of the scaling equation of state[17,39,44]: here we quote only the most recent[39] such estimate: 4.77(2).

The MC simulation of Ref. 45 gave the somewhat larger value 5.18(35), while the more recent and higher precision study of Ref. 46 yields 4.75(3) and the work of Ref. 47 reports 4.72(11).

TABLE III. A comparison among recent estimates of the susceptibility universal amplitude ratio C_+/C_-.

This work	Series[a,b]	Eq.State[c]	MC[d]	MC[e]	MC[f]	FD exp.[g]	ϵ exp.[g]
4.762(8)	4.95(15)	4.77(2)	5.18(35)	4.75(3)	4.72(11)	4.79(10)	4.73(16)

[a]Reference 27.
[b]Reference 43.
[c]Reference 39.
[d]Reference 45.
[e]Reference 46.
[f]Reference 47.
[g]Reference 17.

TABLE IV. A comparison among recent estimates of the correlation-length universal amplitude ratio f_+/f_-.

This work	Series[a,b]	Eq. State[c]	MC [d]	MC [e]	FD exp.[f]	ϵ exp.[g]
1.963(8)	1.96(1)	1.961(7)	2.06(1)	1.95(2)	2.013(28)	1.91

[a]Reference 27.
[b]Reference 43.
[c]Reference 39.
[d]Reference 45.
[e]Reference 46.
[f]Reference 48.
[g]Reference 18.

Within the ϵ-expansion approach to the RG, the estimate 4.73(16) is obtained, while the FD expansion gives the result 4.79(10).[17] The value 4.72(17) was obtained in Ref. 48.

In a similar way, we have computed $f_+^{sc}/f_-^{sc}=1.963(8)$, assuming $\nu=0.6302$. The quoted uncertainty allows also for the uncertainties in the estimates of ν and θ. Other estimates appearing in the recent literature are summarized in Table IV. Our result compares well with the estimate 1.96(1) obtained in Refs. 27 and 43 by shorter series as well as with the recent estimate 1.961(7) of Ref. 39. The MC estimate of Ref. 45 was 2.06(1), whereas in Ref. 46 the value 1.95(2) is reported. The latest ϵ-expansion estimate[18] is 1.91 (with no indication of error bars) and the FD estimate[48] is 2.013(28). The recent experimental estimates of this ratio range between 1.9(2) and 2.0(4).

We believe that the close agreement between our series estimates and the latest determinations of these universal ratios is due to the careful allowance of the confluent corrections to scaling by SDA's. Indeed, even using the longer series presently available, simple Padé approximants, notoriously inadequate to describe the singular corrections to scaling, suggest estimates sizably larger, while the conventional DA's lead to a wider spread in the estimates. Further improvements of the direct series determination of these ratios should probably await for an extension of the low-temperature series.

By using only the HT extended series presented here, we can also reevaluate the universal ratio a_ξ^+/a_χ^+. Let us recall that, as observed in Refs. 3,5,26,29 and argued in earlier studies[49] for the spin-1/2 Ising model on the sc, the bcc, and the fcc lattices, the amplitudes of the leading correction-to-scaling terms have a negative sign, both for the susceptibility and the correlation length. The values of these amplitudes can be most simply determined, also in this case, by using the SDA's mentioned above. Our estimate for the universal ratio between these amplitudes: $a_\xi^{+sc}/a_\chi^{+sc}=0.95(15)$ from the sc lattice series, and $a_\xi^{+bcc}/a_\chi^{+bcc}=0.87(6)$ from the bcc lattice series, improves the accuracy of our previous results[3] obtained from the analysis of shorter series. These results have to be compared with the FD result $a_\xi^+/a_\chi^+=0.65(5)$ obtained in Ref. 50 and with the HT result $a_\xi^{+bcc}/a_\chi^{+bcc}\approx 0.85$ of Refs. 5 and 29.

IV. CONCLUSIONS

We have extended through order β^{23} the HT expansions of the susceptibility and of the second correlation moment for the spin-1/2 Ising model, on the sc and the bcc lattices. As a first application of our calculation, we have updated the direct HT estimates of universal critical parameters of the Ising model with some improvement over previous analyses in the accuracy and in the agreement with the latest calculations by approximate RG methods and by various numerical methods.

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