COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in **Physical Review B.** *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on "Effect of long-range interactions on the critical behavior of the continuous Ising model"

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Available mathematical results entail the absence of a proper ordering transition taking place at finite temperature in the case $d=1$, $\sigma=1.1$, where the authors of the article commented on find evidence of such a transition. Moreover, in contrast to the findings of the authors of the article, the case $d=2$, $\sigma=0$ entails a ground-state energy per particle diverging logarithmically with increasing sample size, hence an infinite transition temperature.

THE SETTING

We present here the potential models in a slightly more general way, together with some relevant mathematical results. Let us consider a classical system, consisting of n-component unit vectors $\mathbf{u}_k$ ($n=2,3$) associated with a d-dimensional lattice Z^d, $d=1,2$; let $\mathbf{x}_k$ denote dimensionless coordinates of the lattice sites, and let $u_{k,\alpha}$ denote Cartesian spin components with respect to an orthonormal basis $\mathbf{e}_\alpha$. The interaction potential is assumed to be translationally invariant, ferromagnetic (FM), and, in general, anisotropic in spin space

$$W=W_{jk}=-\epsilon\,\Psi(r)\,\Omega,$$

$$\Omega=\Omega_{jk}=a u_{j,n}u_{k,n}+b\sum_{\alpha<n}u_{j,\alpha}u_{k,\alpha}; \tag{1}$$

$$r=|\mathbf{x}_j-\mathbf{x}_k|,\quad \Psi(r)\geqslant 0,\quad \epsilon>0,\quad a\geqslant 0,\quad b\geqslant 0,$$

$$a^2+b^2>0,\quad \max(a,b)=1. \tag{2}$$

Here ϵ is a positive quantity setting energy and temperature scales (i.e., $T^*=k_BT/\epsilon$) and can be scaled away from the following formulas; Ψ is a dimensionless quantity, with $\Psi(1)=1$; finally, let $\zeta_k=u_{k,3}$. The correlation function resulting from a certain model will be denoted by $G_{sym}[(d,\Psi,T^*);r]$, and its transition temperature (possibly zero) by $\Theta_{sym}(d,\Psi)$; here sym is a short-hand symbol taking into account the angular dependency. Since we shall be mentioning and comparing different potential models, it is useful to define a compact notation for their orientational terms Ω_{jk}; symbols to be used hereafter are

Is for $n=1$, Ising model;
$A2$ for $n=2$, $a>b=0$;
$A3$ for $n=3$, $a>b=0$.

These three models entail a discrete degeneracy of the ground state (DD models for short); in contrast to them, other extensively studied models are defined by $0=a<b$ or $a=b$, i.e., $O(m)$-invariant interactions, $m\geqslant 2$, producing a continuously degenerate ground state (CD models for short); among them, let us just define the symbol pr for $n=2$, $a=b$, planar rotators.

INEQUALITIES

A number of rigorous inequalities have been proven in the literature, which make it possible to compare correlation functions of different interaction models (correlation inequalities,[1,2] and others, based on them, mentioned below); some of them involve the three DD models.

(1) The correlation function decreases with increasing n,[3–6] i.e.,

$$\begin{aligned}0&\leqslant G_{A3}[(d,\Psi,T^*);r]\\&\leqslant G_{A2}[(d,\Psi,T^*);r]\\&\leqslant G_{Is}[(d,\Psi,T^*);r].\end{aligned} \tag{3}$$

(2) By Well's inequality and its generalizations[7–10] there exists a positive number $\gamma>1$ such that

$$\begin{aligned}G_{Is}[(d,\Psi,\gamma T^*);r]&\leqslant G_{A3}[(d,\Psi,T^*);r]\\&\leqslant G_{A2}[(d,\Psi,T^*);r]\\&\leqslant G_{Is}[(d,\Psi,T^*);r].\end{aligned} \tag{4}$$

EXISTENCE OR ABSENCE OF PHASE TRANSITIONS

When $d=1$, or when $d=2$ and for CD models, finite-range interactions produce orientational disorder at all finite temperatures, in the thermodynamic limit;[11,12] in other words $\lim_{N\to\infty}F(T^*,N)=0$, $\forall T^*>0$, where $F(T^*,N)$ denotes the magnetization per spin for a sample consisting of N particles. On the other hand, in two dimensions, nearest-neighbor

(NN) interactions of appropriate anisotropy (DD models and, in general, $a>b\geqslant 0$) produce an ordering transition taking place at finite temperature (FMT for short); $A2$ and $A3$ models in two dimensions and with NN interactions have seldom been studied in the literature; previous simulation estimates of their transition temperatures are $\Theta_{A2}(d=2,\ \mathrm{NN})=1.315\pm 0.015$ (Ref. 13) and $\Theta_{A3}(d=2,\ \mathrm{NN})=0.88\pm 0.01$.[14]

Moreover, FM interactions possessing reflection positivity and of sufficiently long range[11,12,15] can produce a FMT even when $d=1$; the inverse-power case

$$\Psi(r)=\psi_\sigma(r)=r^{-d-\sigma},\quad \sigma>0 \tag{5}$$

has been extensively studied (see, e.g., Refs. 12 and 15); here the condition $\sigma>0$ is needed in order to avoid an infinite ground-state energy per particle,[16–19] and hence an infinite transition temperature. However, the case $-d\leqslant\sigma\leqslant 0$ has been studied in connection with nonextensive thermodynamics.[18–22]

When $d=1,2$ and for CD models, the system exhibits orientational disorder at all finite temperature for $\sigma\geqslant d$, and a FMT for $0<\sigma<d$;[12] when $d=1$, the FMT survives up to $\sigma=1$ in the Ising case.[12]

When $d=1$, the above mentioned results entail absence of a FMT for both $A2$ and $A3$ when $\sigma>1$, and existence of a FMT for both $A2$ and $A3$ when $0<\sigma\leqslant 1$, as well as rigorous bounds on their transition temperatures; previous simulation estimates of transition temperatures are $\Theta_{A2}(d=1,\sigma=1)=1.04\pm 0.02$ and $\Theta_{A3}(d=1,\sigma=1)=0.735\pm 0.015$.[23]

COMPARISONS AND CONCLUSIONS

Comparison between the results summarized above and the article commented on,[29] leads to the conclusion that some aspects of it are questionable:

(1) The authors of the article find evidence of a FMT when $d=1,\sigma=1.1$ [see their Eq. (1) and Tables I and II], whereas, according to the mentioned mathematical results, models defined by $d=1,\sigma>1$ produce orientational disorder at all finite temperatures (in the thermodynamic limit).

In some specific cases where the FMT is absent, it has been possible to show that $F(T^*,N)$ decreases so slowly with increasing sample size that the absence of order in the thermodynamic limit becomes compatible with its existence for a finite but macroscopic sample,[24–27] which exhibits a size-dependent pseudotransition temperature, eventually vanishing in the thermodynamic limit. Such a behavior has been shown to occur for pr in $d=2$ and with NN interaction, where the Berezinskiĭ-Kosterlitz-Thouless transition is known to take place.[25,27,28] This may also be an explanation for the findings of the authors of the article.

(2) When $d=2,\sigma=0$ (see Table III in the article), the model produces an infinite ground-state energy in the thermodynamic limit, and one should expect an infinite transition temperature, or switch to the framework of nonextensive thermodynamics.

More precisely, even if one uses the cutoff of the authors of the article, the ground-state energy per particle diverges with sample size L, as can be seen by considering the sum

$$S(L)=\sum_{p\in Z}\sum_{q\in Z}{}^{*}\ (p^2+q^2)^{-1}, \tag{6}$$

where the star means $0<(p^2+q^2)\leqslant L^2/4$; this becomes asymptotically $2\pi\ln(L/2)$ (see also Refs. 18 and 19). In this case one could expect a sample-size dependent pseudotransition temperature, now slowly diverging in the thermodynamic limit. On the other hand, in the framework of nonextensive thermodynamics[18–22] the interaction energy to be used in simulation for a sample of linear size L and for the power law considered here can be redefined by

$$H=-c(L)\sum_{j<k}\zeta_j\zeta_k r^{-2},\quad c(L)=\frac{1}{\ln L}. \tag{7}$$

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