

String limit of vortex current algebra

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The Poisson structure generating the Hamiltonian dynamics of string vortices is reconstructed within the current algebra picture as a limiting case of the standard brackets associated to fluids with a smooth vorticity field. The approach implemented bypasses the use of Dirac's procedure. The fine structure of the dynamical algebra is derived for planar fluids by implementing an appropriate spatial fragmentation of the vorticity field, and the limit to the point vortex gas is effected. The physical interpretation of the resulting local currents is provided. Nontrivial differences characterizing the canonical quantization of point vortices and the current algebra quantization are also illustrated.

I. INTRODUCTION

The formation of vortices and their interactions in superfluid media¹ such as ⁴He (and closely related systems such as type-II superconductors²) have been detected and thoroughly studied at the classical level since forty years ago. Recent experimental results concerning the Bose-Einstein condensates show the emergence of vortices in the condensates³ thus providing a further scenario in which vortices can be investigated. Despite the large number of physical systems exhibiting excited states with vortices, a quite mild interest has been raised by the study of the quantum aspects inherent in their dynamics that, on the contrary, should be relevant both because vortex formation occurs at very low temperatures, and because vortex interactions take place on microscopic/mesoscopic spatial scales, where quantum effects are important.⁴

Such a situation is probably originated by the great difficulties in formulating, within the quantum field theory of superfluid media (and closely related systems), a quantization scheme which supplies both an effective representation of the fluid topological excitations (vortex states), and a viable approach to investigate the formal aspects of the theory. In particular, the dynamical degrees of freedom activated by the vortex emergence exhibit a structural complexity which renders dramatically difficult any attempt to construct explicitly the Hilbert space for the fluid quantum states.^{5,6} Such a program is further complicated by the fact that, since vortices are extended object equipped with a possibly nontrivial topological structure, a consistent quantum description of the fluid should incorporate as well the vortex topology in terms of constant of motions representing generalized circulations.

Such aspects have been thoroughly studied in a series of paper within the geometric quantization scheme both for fluids with a vorticity field confined on filaments⁶⁻⁸ (gas of line vortices), and for fluids whose state is described by a smooth (extended) vorticity field.^{9,10} A large amount of work has been devoted to such two models of fluids in order to realize the unitary irreducible representations of the field operators and the ensuing construction of the Hilbert space. Despite the recognition within the geometric scheme of several basic,

both group-theoretic and algebraic, features that characterize the fluid structure and its description, almost no attention has been directed to considering how the vortex quantization is influenced by the limiting process which relates the previous models through the squeezing of the extended vorticity field to a set of disjoint lines.

In this paper we start to investigate this limit and the advantages it entails as to the stringlike-vortex description, and try to emphasize some nontrivial aspects concerning the quantization of vorticity fields in the planar case.

One of the first attempts to quantize the vortex dynamics was developed in Ref. 12 for a model of almost parallel line vortices within the canonical approach based on coordinates and momenta. Its natural extension to planar systems of superfluids with point vortices raised a certain interest several years later mainly in relation to the possibility of observing fractional statistics. The difficulties inherent in the quantization process were completely recognized in Ref. 5 where the canonical scheme was employed to construct the quantum field theory of three-dimensional (3D) vortices characterized by a singular vorticity field

$$\mathbf{w}(\mathbf{x}) = k \oint_{\Gamma} d\mathbf{x}(s) \delta^3(\mathbf{x} - \mathbf{x}(s)), \quad (1)$$

with vortex strength k , where Γ is a possibly self-knotted string with any number of components. Such an arbitrarily complex object provided a realistic generalization of the model of parallel vortex lines by introducing explicitly the topological structure of line vortices. The components $x_j(s, t)$ of the 3D vector $\mathbf{x}(s, t)$ representing the smooth curve $\Gamma \in \mathbf{R}^3$ (s is the string parameter on Γ , and $j = 1, 2, 3$) supplied the coordinates at each time t , whereas the canonically conjugate momenta

$$P_i(s, t) = \frac{\delta \mathcal{L}}{\delta (\partial_t x_i)}$$

were obtained from the Lagrangian functional

$$\mathcal{L} := -H + \frac{k\rho}{3} \oint_{\Gamma} d\mathbf{x} \cdot \left(\frac{\partial \mathbf{x}}{\partial t} \wedge \mathbf{x} \right)$$

(the fluid density ρ is assumed to be constant) containing the ideal fluid energy H [see Eq. (5) below]. Momenta P_j entail the dynamical constraints $P_i - (\rho k/3) \epsilon_{ijk} x_j \partial_s x_k = 0$ that revealed the singular character of \mathcal{L} , and showed how the stringlike vortex dynamics actually takes place on a submanifold of the standard phase space $\mathcal{P} := \{[P_j(s,t), x_j(s,t)]\}$. The price of implementing the canonical picture based on the brackets

$$\{x_i(s,t), P_j(s',t)\} = \delta(s-s') \delta_{i,j} \quad (2)$$

was to reconstruct Eq. (2) within the Dirac procedure so as to incorporate the dynamical constraints. The dynamics was thus formulated through the Dirac brackets $\{A, B\}_* := \{A, B\} + \langle A|C|B\rangle$, where

$$\langle A|C|B\rangle \equiv \int ds \int ds' \{A, \Phi_{\alpha}(s)\} C_{\alpha,\beta}(s,s') \{\Phi_{\beta}(s'), B\},$$

$\{A, B\}$ represents the standard Poisson brackets, $C_{\alpha,\beta}$ are elements of the matrix $C := |\{\Phi_{\alpha}, \Phi_{\beta}\}|^{-1}$ and the functionals $\Phi_{\alpha}(s)$, $\alpha=1,2$, of the canonical variables essentially identify with the components of the part of $\mathbf{P} - (\rho k/3) \mathbf{x} \wedge \partial_s \mathbf{x}$ orthogonal to the vector field $\mathbf{x}(s,t)$. The main issue of Dirac's formalism was the unexpected coordinate brackets

$$\{x_i(s), x_j(s')\}_* = \frac{1}{k\rho} \epsilon_{ijk} \delta(s-s') \partial_s x_k(s) \quad (3)$$

showing how coordinates x_j cannot be regarded any longer as independent variables. At the quantum level, Eq. (3) entailed the remarkable effects that the projections of Γ on the planes $x_i - x_j$, $i, j=1,2,3$, are affected manifestly by the quantum uncertainty, and led to imagine Γ as a tubular domain representing the intrinsically approximate position of the vortex core (see Ref. 13).

A further observation is suggested by Eq. (3): in spite of the local canonical character of coordinates, the algebraic structure exhibited by Eq. (3) is actually nonlocal consistent with the fact that Γ is a true three-dimensional object. In Ref. 5, this led the authors to construct the algebra of currents, which we review in the sequel, so as to avoid the dependence on local parametrizations as well as on Dirac's formalism.

The first goal of this paper is to show how the algebraic structure involved by the functional picture based on Γ -dependent currents can be derived in a direct way from the standard Lie-Poisson structure

$$\{F, G\}(\mathbf{w}) = \frac{1}{\rho} \int d^3x \mathbf{w} \cdot \left(\text{curl} \frac{\delta F}{\delta \mathbf{w}} \wedge \text{curl} \frac{\delta G}{\delta \mathbf{w}} \right), \quad (4)$$

without implementing Dirac's procedure. The structure (4), that generates the vortex dynamics¹⁴ when the vorticity field $\mathbf{w}(\mathbf{x})$ is smooth [namely its components $w_j(\mathbf{x}) \in C^{\infty}(\mathbf{R}^3)$], contains as a limiting case the Poisson structure for vorticity fields \mathbf{w} collapsed on an array of strings (singular limit), no matter how complex the underlying topological structure is.

The second purpose of the present paper is to consider the effect of the singular limit on the vortex Lie-Poisson (LP) structure in the two-dimensional case, where the limit con-

sists in squeezing the vorticity fields on (a set of) isolated points of the ambient plane. The resulting point vortex gas is well known to represent the reference model for a number of systems with vortex excitations such as superfluid films¹⁵ of ⁴He (adsorbed both on planar substrates and on porous materials), planar superconductors,² and Josephson junctions' arrays¹⁶ (see also Ref. 17 and references therein). Our interest in analyzing the 2D singular limit comes from the wish of establishing a clear link between the smooth case and the singular case. More specifically, we aim at unveiling the algebraic structure of point vortices within the ampler framework of the current algebra (CA) of the smooth case.

The point of view adopted here is that in the 2D case the CA contains an explicit many-body structure related to the spatial distribution of positive/negative vorticity which deserves to be investigated. Such a fine structure of \mathcal{A} (storing information on the spatial distribution of \mathbf{w}) paves the way to the emergence of the canonical Poisson structure that customarily characterizes the point vortex dynamics. The 3D case can be also studied from this viewpoint even if the fragmentation must be developed based on the complex topological structure of \mathbf{w} . In this respect, however, the Arnold cells¹⁸ should represent the 3D counterpart of 2D fragmentation. The latter enables us to shed light on certain features that characterize, at the quantum level, the construction of the CA of the point vortex model, and unexpectedly prevents it from matching the version of the algebra obtained within the field theory formalism.

The paper is organized as follows: In Sec. II, after introducing the standard derivation of Euler's equations by means of appropriate Lie-Poisson brackets, the CA picture is reviewed for the 3D case and its relevance for the quantization of the vortex dynamics is showed. In Sec. III the CA picture is used to perform in a consistent way the string limit of formula (4) in order to construct the Poisson structure for a gas of stringlike objects (a more formal derivation is furnished in the Appendix based on the Clebsch potential picture). Some applications of the brackets thus obtained are illustrated as well. In Sec. IV the many-body structure of vortex dynamics is investigated in the 2D case and the fine structure of the CA is evidenced via an appropriate fragmentation of the vorticity field. The latter is related to the CA reconstructed for 2D point vortices within the canonical quantization. The CA of the point vortices is compared with the CA of the smooth case and their inequivalence at the quantum level is proven: a suitable semiclassical limit is shown to reconcile such situations.

II. CURRENT ALGEBRA APPROACH TO VORTEX DYNAMICS

The classical motion of a perfect fluid with velocity field \mathbf{v} , vorticity field $\mathbf{w} = \text{curl} \mathbf{v}$, and Hamiltonian

$$H[\mathbf{v}] = \frac{\rho}{2} \int_{\mathbf{R}} d^3x \mathbf{v}^2(\mathbf{x}), \quad (5)$$

is governed by the Euler equation $\dot{\mathbf{v}} = -\mathbf{v} \cdot \nabla \mathbf{v}$. Observing that $\mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{v} \wedge \mathbf{w} - \nabla \mathbf{v}^2/2$, the vorticity equation

$$\dot{\mathbf{w}} = -\text{curl}(\mathbf{v} \wedge \mathbf{w}), \quad (6)$$

based on representing the fluid state through \mathbf{w} , is easily obtained from the Euler equation.¹⁹ The derivation of both equations is easily performed by means of the usual Lie-Poisson (LP) brackets²⁰

$$\{F, G\}[\mathbf{v}] = \frac{1}{\rho} \int d^3x \text{curl } \mathbf{v} \cdot \left(\frac{\delta F}{\delta \mathbf{v}} \wedge \frac{\delta G}{\delta \mathbf{v}} \right), \quad (7)$$

where F and G are functions that depend on \mathbf{v} , and the notation $\int d^n x$ denotes, from now on, the integration on the whole space \mathbf{R}^n . It is important to observe how a consistent use of Eq. (6) requires that \mathbf{v} is expressed as a functional of \mathbf{w} . This is achieved by imposing the divergenceless condition $\text{div } \mathbf{v} = 0$ on \mathbf{v} , namely by identifying \mathbf{v} with $\mathbf{V}(\mathbf{x}) = \text{curl } \mathbf{U}(\mathbf{x})$, where the vector potential $\mathbf{U}(\mathbf{x})$ is defined as

$$\mathbf{U}(\mathbf{x}) = \int d^3y G(\mathbf{x} - \mathbf{y}) \mathbf{w}(\mathbf{y}),$$

and the Green function G in 3 and 2 dimensions reads $G(\mathbf{x} - \mathbf{y}) = 1/(4\pi|\mathbf{x} - \mathbf{y}|)$ and $G(\mathbf{x} - \mathbf{y}) = (1/2\pi)\ln|\mathbf{x} - \mathbf{y}|$, respectively.

In the case when F and G are assumed to depend on the equivalence class $[\mathbf{v}] = \{\mathbf{v}' : \mathbf{v}' = \mathbf{v} + \nabla f\}$, the LP brackets take the form (4) in that $\delta F / \delta \mathbf{v} = \text{curl}(\delta F / \delta \mathbf{w})$. Explicitly, this amounts to assuming that F and G depend on \mathbf{w} namely on the divergence-free field \mathbf{V} .

Replacing \mathbf{v} with \mathbf{V} in $H[\mathbf{v}]$ incorporates explicitly the divergenceless feature in the theory. This leads to the quadratic form

$$H[\mathbf{w}] \equiv \frac{\rho}{2} \int d^3x \int d^3y G(\mathbf{x} - \mathbf{y}) \mathbf{w}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{y}),$$

which generates Eq. (6) via brackets (4).

An alternative formulation⁵ of vortex dynamics can be given in terms of current algebra \mathcal{A} . The latter consists of functionals of $[\mathbf{v}]$ (the currents) defined as

$$J_{\mathbf{a}}[\mathbf{v}] = \rho \int d^3x \mathbf{a} \cdot \mathbf{v} = \rho \int d^3x \mathbf{A} \cdot \mathbf{w} = J_{\mathbf{A}}[\mathbf{w}], \quad (8)$$

where \mathbf{a} belongs to the algebra $\mathcal{G} = \{\mathbf{a} : \mathbf{a} = \text{curl } \mathbf{A}\}$ of divergence-free vector fields. One can easily check that $J_{\mathbf{a}}[\mathbf{v}] = J_{\mathbf{a}}[\mathbf{v} + \nabla f]$. The algebraic structure of \mathcal{A} shows up via the equation²¹

$$\{J_{\mathbf{a}}, J_{\mathbf{b}}\}[\mathbf{v}] = J_{[\mathbf{a}, \mathbf{b}]}[\mathbf{v}], \quad (9)$$

where $[\mathbf{a}, \mathbf{b}] = \text{curl}(\mathbf{a} \wedge \mathbf{b})$, that is fulfilled by two any currents of \mathcal{A} . The structure constants of \mathcal{A} are readily worked out by introducing the subalgebra of the mode currents \mathcal{A}_F . The latter is defined by noting that any current $J_{\mathbf{A}}[\mathbf{w}]$ can be expressed in terms of the Fourier transform $\mathbf{A}(\mathbf{q})$ relative to \mathbf{A} as

$$J_{\mathbf{A}}[\mathbf{w}] = \rho \int d^3x \mathbf{A} \cdot \mathbf{w} = \int d^3q \mathbf{A}(\mathbf{q}) \cdot \mathbf{J}_{\mathbf{q}}[\mathbf{w}],$$

where the mode current $\mathbf{J}_{\mathbf{q}}[\mathbf{w}] := \mathbf{e}_m J_{\mathbf{q}}^m[\mathbf{w}]$ has vector components

$$J_{\mathbf{q}}^m[\mathbf{w}] = \rho \int d^3x e^{i\mathbf{x}\mathbf{q}} w_m(\mathbf{x}). \quad (10)$$

Then one can easily show that the basic brackets of \mathcal{A} are given by

$$\{J_{\mathbf{q}}^m, J_{\mathbf{p}}^n\}[\mathbf{w}] = - \sum_{k=1}^3 C_{m,n}(\mathbf{q}, \mathbf{p}) J_{\mathbf{q}+\mathbf{p}}^k[\mathbf{w}], \quad (11)$$

where $C_{m,n}(\mathbf{q}, \mathbf{p}) := \mathbf{e}_k \cdot [(\mathbf{q} \wedge \mathbf{e}_m) \wedge (\mathbf{p} \wedge \mathbf{e}_n)]$ are the structure constants, and \mathbf{e}_k , $k=1,2,3$ are the unit vectors of the 3D Euclidean basis. The equations of motion of any current $J_{\mathbf{q}}^m[\mathbf{w}]$ can be easily derived via the brackets (11) once $H[\mathbf{w}]$ itself has been rewritten as a functional of currents $J_{\mathbf{q}}^m[\mathbf{w}]$ by inverting formula (10) (see Ref. 21). This proves that \mathcal{A}_F indeed furnishes a set of observables which is complete, namely \mathcal{A}_F represents an alternative scheme in which representing the fluid dynamics. Brackets (11) also shows how the subalgebra of mode currents is the most advantageous set of commutators in which implementing the quantization process. Nevertheless, it important to recall that the parent set of commutators (9) indicate that the quantization process is equivalent to construct the unitary irreducible representations of the group of diffeomorphism which is one of the hardest, unsolved problem of the theory of group representations.

III. STRING LIMIT OF 3D CURRENT ALGEBRA

Upon performing the limit which confines the vorticity field on a stringlike domain (that is, on a vortex filament), namely considering

$$\mathbf{w}(\mathbf{x}) \rightarrow \mathbf{w}_*(\mathbf{x}) \equiv \sum_a k_a \oint_{\Gamma_a} d\mathbf{y}_a(s) \delta^3(\mathbf{x} - \mathbf{y}_a(s)) \quad (12)$$

the current (8) becomes

$$J_{\mathbf{A}}[\mathbf{w}] = \rho \sum_a k_a \oint_{\Gamma_a} d\mathbf{x}_a \cdot \mathbf{A}(\mathbf{x}_a),$$

whereas the current $\{J_{\mathbf{A}}, J_{\mathbf{B}}\}[\mathbf{w}]$ reduces to

$$\{J_{\mathbf{A}}, J_{\mathbf{B}}\}[\mathbf{w}] = \rho \sum_a k_a \oint_{\Gamma_a} d\mathbf{x}_a \cdot [\mathbf{a}(\mathbf{x}_a) \wedge \mathbf{b}(\mathbf{x}_a)].$$

Therefore the form assumed by Poisson brackets (4) when limit (12) is enacted must be consistent with this result. Observing that

$$\frac{\delta \mathcal{F}}{\delta \mathbf{x}(s)} = \frac{\partial f}{\partial \mathbf{x}(s)} - \frac{d}{ds} \frac{\partial f}{\partial \dot{\mathbf{x}}(s)}(s) \quad (13)$$

for any function $\mathcal{F} = \int dq f[\mathbf{x}(q), \dot{\mathbf{x}}(q)]$, where $\dot{\mathbf{x}}(s) := d\mathbf{x}(s)/ds$, one easily obtains

$$\frac{\delta J_{\mathbf{A}}}{\delta \mathbf{x}(s)} = \rho k \dot{\mathbf{x}}(s) \wedge \text{curl } \mathbf{A}.$$

This entails, in turn,

$$\pi(s) \cdot \left[\frac{\delta J_{\mathbf{A}}}{\delta \mathbf{x}(s)} \wedge \frac{\delta J_{\mathbf{B}}}{\delta \mathbf{x}(s)} \right] = k^2 \rho^2 \pi(s) \cdot (\mathbf{a} \wedge \mathbf{b})_{\mathbf{x}(s)},$$

provided $\tau(s) = \dot{\mathbf{x}}(s)$ is a unit vector, i.e., the parameter s is identified with the arc length of Γ . This result suggests the substitution

$$\text{curl} \frac{\delta F}{\delta \mathbf{w}} \rightarrow \dot{\mathbf{x}}_a \wedge \frac{\delta}{\delta \mathbf{x}_a} \frac{\partial F}{\partial k_a} \quad (14)$$

as a consequence of limit (12), where the index a takes into account the possible many-component structure $\Gamma = \{\Gamma_a\}$ of the string model. Hence the string LP brackets for a many-component line vortex turn out to be

$$\{F, G\}[\mathbf{w}] = \sum_a \oint_{\Gamma_a} \frac{k_a d\mathbf{x}_a}{\rho} \cdot \left(\frac{\delta}{\delta \mathbf{x}_a} \frac{\partial F}{\partial k_a} \wedge \frac{\delta}{\delta \mathbf{x}_a} \frac{\partial G}{\partial k_a} \right).$$

A simplified version is also available in the form

$$\{F, G\}[\mathbf{w}] = \sum_a \frac{1}{\rho k_a} \oint_{\Gamma_a} d\mathbf{x}_a \cdot \left(\frac{\delta F}{\delta \mathbf{x}_a} \wedge \frac{\delta G}{\delta \mathbf{x}_a} \right), \quad (15)$$

which can be used in a consistent way provided both F and G have a linear dependence on k_a 's as the currents of \mathcal{A} . Another derivation of Eq. (15) is described in Appendix A, where we reformulate the LP brackets within the Clebsch picture of fluids in such a way that the dependence on the diffeomorphism action is expressed explicitly.

A simple way to test the validity of the brackets just obtained consists in checking whether they reproduce correctly the equation of motion for the vortex filament by calculating explicitly the right hand side of $\partial_t \mathbf{x} = \{\mathbf{x}, H\}$. The effect of limit (12) on H and \mathbf{U} is that of exhibiting them into the form

$$H[\mathbf{w}_*] = \frac{\rho}{2} \sum_a \sum_b k_a k_b \oint_{\Gamma_a} \oint_{\Gamma_b} \frac{d\mathbf{x}_a \cdot d\mathbf{x}_b}{4\pi |\mathbf{x}_a - \mathbf{x}_b|},$$

and

$$\mathbf{U}(\mathbf{x}) = \sum_a k_a \oint_{\Gamma_a} \frac{d\mathbf{x}_a}{4\pi |\mathbf{x} - \mathbf{x}_a|},$$

respectively. Considering the single string case one finds

$$\{\mathbf{x}, H\} \equiv \frac{\delta H}{\delta \mathbf{x}(s)} \wedge \frac{\dot{\mathbf{x}}(s)}{\rho}, \quad (16)$$

where the functional derivative of H is given by

$$\begin{aligned} \frac{\delta H}{\delta \mathbf{x}(s)} &= -\frac{k\rho}{4\pi} \oint_{\Gamma} d\mathbf{y} \cdot \int_0^L dr \frac{\delta}{\delta \mathbf{x}(s)} \frac{\dot{\mathbf{x}}(r)}{|\mathbf{x}(r) - \mathbf{y}|} \\ &= \rho \dot{\mathbf{x}}(s) \wedge k \oint_{\Gamma} \frac{[\mathbf{x}(s) - \mathbf{y}] \wedge d\mathbf{y}}{4\pi |\mathbf{x}(s) - \mathbf{y}|^3}. \end{aligned}$$

Then, the expected equation of motion

$$\partial_t \mathbf{x} = \lambda(\Gamma) \dot{\mathbf{x}} + k \oint_{\Gamma} \frac{(\mathbf{x} - \mathbf{y}) \wedge d\mathbf{y}}{4\pi |\mathbf{x} - \mathbf{y}|^3} = \lambda(\Gamma) \dot{\mathbf{x}} - \mathbf{V}(\mathbf{x}; \Gamma),$$

with $\lambda(\Gamma) := \dot{\mathbf{x}} \cdot \mathbf{V}(\mathbf{x}; \Gamma)$, is achieved by explicitly calculating the wedge product in Eq. (16). Notice that the component $\lambda(s, \Gamma)$ generates displacements of Γ that are parallel to Γ itself due to its longitudinal character. The above result easily extends to the many-component case.

IV. FINE STRUCTURE OF 2D CURRENT ALGEBRA

For planar vortices the notation of CA formalism can be simplified in view of the fact that $\mathbf{w} = w\mathbf{e}_3$, $\mathbf{a} = \text{curl}(\mathcal{A}\mathbf{e}_3) = \nabla \mathcal{A} \wedge \mathbf{e}_3$ with $\nabla = \mathbf{e}_1 \partial_{x_1} + \mathbf{e}_2 \partial_{x_2}$. In particular, brackets (4) reduce to

$$\{F, G\}[w] = \frac{1}{\rho} \int d^2x w \cdot \left(\nabla \frac{\delta F}{\delta w} \wedge \nabla \frac{\delta G}{\delta w} \right), \quad (17)$$

which, upon observing that a generic current is available in the two forms $J_{\mathbf{a}}[\mathbf{v}] = \rho \int d^2x \mathbf{a} \cdot \mathbf{v} = \rho \int d^2x \mathcal{A} w = J_A[w]$, provides the current brackets

$$\{J_A, J_B\}[w] = \rho \int d^2x w \cdot (\nabla A \wedge \nabla B) = J_{\{A, B\}}[w],$$

where $\{A, B\}_x = \mathbf{e}_3 \cdot (\nabla A \wedge \nabla B)$.

The two-dimensional LP structure just worked out can be reformulated in such a way that the partition of the ambient plane \mathbf{R}^2 in many subdomains is accounted for explicitly. This is realized through the representation of the unit constant function

$$1 = \sum_a \Theta_a(\mathbf{x}) \quad (18)$$

in terms of Heaviside functions $\Theta_a(\mathbf{x}) := \Theta(\mathbf{x}; S_a)$ nonvanishing inside the domain S_a . The underlying idea is to show that implementing the fragmentation process within the brackets formalism leads to recognize the single components $w_a(\mathbf{x})$ of \mathbf{w} (associated to plane domains S_a) as independent dynamical degrees of freedom.

The rule for selecting such domains is based on separating the negative islands (where $w < 0$) from the positive ones (where $w > 0$). Such a situation indeed is usual since the condition $\int d^2x w(\mathbf{x}) = 0$ is customarily assumed to exclude unphysical vortex configurations whose energy cost is too high. On the other hand, the stable character of such domains is ensured by the hydrodynamic laws of perfect fluids which state the conservation of (the structure of) space patterns (the partition in positive/negative vorticity domains, in the present case) when the evolution is driven by area preserving diffeomorphisms.

Using Eq. (18) any current $J_A[w]$ can be reexpressed in terms of local currents as

$$J_A[w] = \sum_a J_A^{(a)}[w] = \sum_a \rho \int_{S_a} d^2x \mathcal{A}(\mathbf{x}) w(\mathbf{x}), \quad (19)$$

where $J_A^{(a)}[w] = \rho \int d^2x \mathcal{A}(\mathbf{x}) w(\mathbf{x}) \Theta_a(\mathbf{x})$. In this way the additional information concerning the spatial distribution of vorticity is explicitly taken into account in the current description of the fluid. At the quantum level, the quantity $J_A^{(a)}[w]$ with $A \equiv 1$ are expected to represent the quanta of vorticity located in S_a . A simple calculation shows that the LP brackets of local currents are given by

$$\{J_A^{(a)}, J_B^{(b)}\}[w] = \delta_{ab} J_{\{A, B\}}^a[w] \quad (20)$$

provided $w_a(\mathbf{x})=0$ for $\mathbf{x} \in \partial S_a$. The vanishing of \mathbf{w} on the boundary separating different confining domains is crucial to eliminate the contributions coming from the divergent character of

$$\nabla \Theta_a(\mathbf{x}) = \oint_{\partial S_a} d\mathbf{y} \wedge \mathbf{e}_3 \delta^2(\mathbf{x}-\mathbf{y})$$

on the boundary of S_a . This fact motivates as well the choice of the set of plane domains S_a based on distinguishing positive from negative vorticity domains. Furthermore the LP brackets for any two currents can be rewritten by means of the formula

$$\{J_A, J_B\}[w] = \sum_a \int_{S_a} \frac{d^2x}{\rho} \mathbf{w}_a \cdot \left(\nabla \frac{\delta J_A}{\delta w_a} \wedge \nabla \frac{\delta J_B}{\delta w_a} \right) \quad (21)$$

explicitly exhibiting the fine structure of the vorticity domains. This also implies that $\{w_a := w(\mathbf{x}) : \mathbf{x} \in S_a\}$ can be considered as a set of fluid dynamical variables.

It is worth noting how the fragmentation picture just introduced allows one to enlarge the set of currents so as to include current whose labels A not necessarily vanish for $|\mathbf{x}| \rightarrow \infty$. The case in which $A = x_1$, $B = x_2$ is illustrative of this. From Eq. (20) one readily obtains the canonical coordinate-like brackets

$$\{J_{x_1}^{(a)}, J_{x_2}^{(b)}\}[w] = \delta_{ab} J_1^{(a)}[w] = \delta_{ab} \rho K_a, \quad (22)$$

where $K_a = \int_{S_a} d^2x w(\mathbf{x})$, that relate the present field-theory description to the pointlike vortex gas description of the Helmholtz standard model. The quantum version of Eq. (22) implies that the information related to the (average) position of the vortex domain S_a on the x_1 axis cannot be given together with that concerning the position on the x_2 axis. This clearly mimics the effects of the canonical quantization rule standardly used for point vortices²⁴ as well as the uncertainty affecting the position of the string in the 3D case. A nice magnetic-like interpretation of $J_{x_1}^{(a)}$, $J_{x_2}^{(a)}$ is also available. Rewriting first $x_1 w(\mathbf{x})$ [$x_2 w(\mathbf{x})$] in $J_{x_1}^{(a)}$ ($J_{x_2}^{(a)}$) as

$$x_r w(\mathbf{x}) = \mathbf{v} \wedge \text{curl}(x_r \mathbf{e}_3) - \text{div}(x_r \mathbf{e}_3 \wedge \mathbf{v}), \quad r = 1, 2,$$

and using then formula $\mathbf{e}_3 \text{div}(\mathbf{A} \wedge \mathbf{e}_3) = \text{curl } \mathbf{A}$ one finds $J_{x_2}^{(a)} \equiv \mathbf{e}_1 \cdot \mathbf{P}_{(a)}$ and $-J_{x_1}^{(a)} \equiv \mathbf{e}_2 \cdot \mathbf{P}_{(a)}$, where

$$\mathbf{P}_{(a)} \equiv \mathbf{p}_{(a)} + \oint_{\gamma_a} (\mathbf{e}_3 \wedge \mathbf{x})(\mathbf{v} \cdot d\mathbf{x}),$$

with $\mathbf{p}_{(a)} := \rho \int_{S_a} d^2x \mathbf{v}$ and $\gamma_a = \partial S_a$. Such an expression makes visible the structure of generalized magnetic moments characterizing $\mathbf{P}_{(a)}$ in which $\mathbf{p}_{(a)}$ represents the total momentum pertaining to the domain S_a , while the circulation term can be seen as an effective vector potential $\mathbf{A} = B \mathbf{e}_3 \wedge \mathbf{r}$, where $\mathbf{r} = \langle \mathbf{x} \rangle = \oint (d\mathbf{x} \cdot \mathbf{v}) \mathbf{x} / K_a$ and K_a is the magnetic field B . Such a picture matches the magnetic approach to the point vortex quantization presented in Ref. 17.

Similarly to the 3D case, the Fourier mode algebra is obtained by considering the Fourier decomposition $A(\mathbf{x}) = \int d^2q A(\mathbf{q}) e^{i\mathbf{q}\mathbf{x}}$ and defining the mode currents

$$J_{\mathbf{q}}[w] = \int d^2x w(\mathbf{x}) e^{i\mathbf{q}\mathbf{x}}, \quad (23)$$

that represent the functionals whereby reconstructing any current as illustrated by the formula $J_A[w] = \rho \int d^2x A(\mathbf{x}) w(\mathbf{x}) = \int d^2q A(\mathbf{q}) J_{\mathbf{q}}[w]$. Moreover, the Poisson brackets of the $J_{\mathbf{q}}[w]$'s are readily derived from Eq. (20) which provides the formula

$$\{J_{\mathbf{q}}, J_{\mathbf{p}}\}[w] = -\mathbf{e}_3 \cdot (\mathbf{q} \wedge \mathbf{p}) J_{\mathbf{q}+\mathbf{p}}[w],$$

whereby its quantum mechanical counterpart

$$[J_{\mathbf{q}}, J_{\mathbf{p}}][w] = -i\hbar \mathbf{e}_3 \cdot (\mathbf{q} \wedge \mathbf{p}) J_{\mathbf{q}+\mathbf{p}}(w) \quad (24)$$

is derived. The resulting algebra coincides with the well known algebra $W(\infty)$ (see, e.g., Ref. 22).

Now, going to the case of point vortices, it is interesting to illustrate the diversity characterizing the scheme based on the canonical variables and the procedure relying on the CA. Quantizing a classical 2D vortex gas is usually performed by replacing its classical Poisson brackets²³ $\{x_a, y_b\} = \delta_{ab}/\rho k_a$ with the commutators $[x_a, y_b] = i\hbar \delta_{ab}/\rho k_a$ (see, e.g., Ref. 24). The definition of the \mathbf{q} currents for a pointlike vorticity distribution ensues directly from Eq. (23)

$$J_{\mathbf{q}}(w) = \int d^2x w(\mathbf{x}) e^{i\mathbf{q}\mathbf{x}} = \sum_a k_a e^{i\mathbf{q}\mathbf{x}_a},$$

where $k_a = w(\mathbf{x}_a)$, and local currents are recognized to have the form $J_{\mathbf{q}}^a = e^{i\mathbf{q}\mathbf{x}_a}$. As a consequence of the Baker-Campbell-Hausdorff formula $\exp(-\frac{1}{2}[Q, P]) \exp Q \exp P = \exp(Q+P)$ it is found that

$$e^{i\mathbf{q}\mathbf{x}_a} e^{i\mathbf{p}\mathbf{x}_a} e^{i\Phi_a(q,p)} = e^{i(\mathbf{p}\mathbf{x}_a + \mathbf{q}\mathbf{x}_a)}.$$

The phase $\Phi_a(q,p) = (\hbar/2\rho k_a) \mathbf{e}_3 \cdot (\mathbf{q} \wedge \mathbf{p})$ is the nontrivial effect deriving from the canonical quantization. In fact, while the commutator of two any local currents still generates a current [see Eq. (20)] since

$$[J_{\mathbf{q}}^a, J_{\mathbf{p}}^b] = \delta_{ab} 2i k_a \sin[\Phi_a(q,p)] J_{\mathbf{p}+\mathbf{q}}^a, \quad (25)$$

the attempt to reconstruct the CA, namely the commutators (24), fails due to the nonlinearity of the sine factor arising in Eq. (25) that prevents the superposition of local current $J_{\mathbf{p}+\mathbf{q}}^a$ with different label a . The usual result is recovered however either in the limit $\hbar \rightarrow 0$ or when $k_a \rightarrow \infty$, both entailing a semiclassical picture of vortices.

On the other hand, writing explicitly the current commutator

$$[J_{\mathbf{q}}, J_{\mathbf{p}}] = 2i \sum_a k_a \sin[\Phi_a(q,p)] J_{\mathbf{p}+\mathbf{q}}^a$$

shows the presence of an underlying magnetic-like structure where two generators of planar displacements (magnetic translations) commute provided the area element in the mode space $\mathbf{e}_3 \cdot (\mathbf{q} \wedge \mathbf{p})$ is equal (up to a factor π) to the multiple fluxon $n(2\rho k_a/\hbar)$, $n \in \mathbf{N}$. Also, it is worth noting that the structure (24) is partially recovered, namely

$$[J_{\mathbf{q}}, J_{\mathbf{p}}] = 2ik \sin[\Phi(q,p)] \sum_a J_{\mathbf{p}+\mathbf{q}}^a, \quad (26)$$

when assuming the standard (low temperature) vorticity configuration $|k_d| = k \equiv \hbar/m$ (due to the Feynman-Onsager condition, where m is the Helium atomic mass), for each vortex. In this case local currents share the same sine factor which keeps memory of the pointlike form of vortex cores. A similar portrait has been depicted in Refs. 19 and 25 by considering certain realizations of the algebra $\text{diff}(T^2)$ and $\text{su}(N)$ and their link via contraction. More precisely, $\text{su}(N)$ has been shown to be equipped with commutators whose structure constants have the form $N \sin[\mathbf{e}_3 \cdot (\mathbf{q} \wedge \mathbf{p})/N]$ which reproduces that of $\text{diff}(T^2)$ for $N \rightarrow \infty$.

V. CONCLUSIONS

Based on a heuristic approach, we have shown in Sec. III that the LP structure of string vortices can be evinced by combining the effect of the string limit (12) on currents and the request that the algebraic structure of CA is preserved. Such a heuristic way bypasses the complexity of Dirac's formalism.

A more detailed procedure has been supplied in Appendix A based on the Clebsch potential picture of perfect fluids and the explicit use of diffeomorphisms as the dynamical variables in terms of which reformulating the LP brackets.

The applications of formula (15) are at least twofold. First it is a crucial ingredient in constructing the functional operator form of the currents of \mathcal{A} in the implementation of the geometric quantization scheme^{7,9} for string vortices. Second, one can take advantage from formula (15) to study the algebraic structure of string functionals such as Chen iterated path integrals²⁶ that represent the higher order topological charges of the string.¹¹

The limiting process reveals the possible many-component structure of the string. Such an aspect is fully accounted in the planar vortex case discussed in Sec. IV and is used to make evident the fine structure of the current algebra. Relying on such a formulation of the dynamical algebra, we have reconstructed the CA for (planar) point vortices showing how the canonical quantization process yields a different algebraic structure for the local currents. The difference disappears upon passage to an appropriate semiclassical limit.

This effect is of course explainable as the manifestation of the structural inequivalence between a model with a discrete distribution of the vorticity and a (smooth) vorticity field theory. It might be used in an explicit way to characterize the transition from the (low temperature) rarefied gas of point vortex pairs to a fluid with many interacting vortices, which takes place in planar superfluids when temperature is raised. The many-vortex fluid induces a more intense vortex interaction which possibly requires a fieldlike description capable of describing vortex cores which are no longer reducible to pointlike objects.

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APPENDIX

The use of diffeomorphisms $\mathbf{x} \rightarrow \mathbf{y} = \boldsymbol{\eta}(\mathbf{x})$ as the dynamical variables of the system is based on the Clebsch potentials (CP) picture of the vorticity field \mathbf{w} . Within the CP picture assigning the field \mathbf{v} (or \mathbf{w} , i.e., the state of the fluid) is equivalent to defining the set of CP

$$\{[\mathcal{U}_j, (\alpha_j, \beta_j, \varphi_j)] \cup_j \mathcal{U}_j = \mathbf{R}^3\},$$

on a suitable covering of \mathbf{R}^3 , such that $\mathbf{v} \equiv k(\alpha_j \nabla \beta_j + \nabla \varphi_j)$ and $\mathbf{w} \equiv k \nabla \alpha_j \wedge \nabla \beta_j$ in the chart \mathcal{U}_j (index j referred to local charts is dropped in the sequel to simplify formulas). Triads of CP provide an alternative system of coordinates represented by the map (with its inverse)

$$[\alpha(\mathbf{x}), \beta(\mathbf{x}), \varphi(\mathbf{x})] \rightleftharpoons \mathbf{x} = \mathbf{x}(\alpha, \beta, \varphi),$$

whose definiteness is ensured by the fact that its Jacobian

$$I(\mathbf{x}) = \nabla \varphi \cdot (\nabla \alpha \wedge \nabla \beta) = \frac{\mathbf{v} \cdot \mathbf{w}}{k^2} \quad (\text{A1})$$

is nonvanishing (namely the topological charge is nonzero.¹⁴) The Jacobian furnishes further geometric information. In particular, a set of six equations can be easily worked out from Eq. (A1) two of which read

$$I \frac{\partial \mathbf{x}}{\partial \varphi} = \nabla \alpha \wedge \nabla \beta, \quad \nabla \varphi = I \frac{\partial \mathbf{x}}{\partial \alpha} \wedge \frac{\partial \mathbf{x}}{\partial \beta}, \quad (\text{A2})$$

while the others are obtained by cyclic permutations of α, β, φ . The \mathbf{w} can be thought of as the set of its *fibers* (that is its integral curves) filling the whole ambient space \mathbf{R}^3 . Fibers $\mathbf{x}(\alpha, \beta, \varphi)$, in turn, embody the topological structure of \mathbf{w} and are homotopic to each other (the extended version of such a review can be found in Ref. 9).

The time evolution involves the change driven by the time-dependent diffeomorphisms $\boldsymbol{\eta}$

$$[a(\mathbf{x}), b(\mathbf{x}), f(\mathbf{x})] \rightarrow [\alpha(\mathbf{x}), \beta(\mathbf{x}), \varphi(\mathbf{x})],$$

where $\alpha(\mathbf{x}) := a[\boldsymbol{\eta}(\mathbf{x})]$, $\beta(\mathbf{x}) := b[\boldsymbol{\eta}(\mathbf{x})]$, $\varphi(\mathbf{x}) := f[\boldsymbol{\eta}(\mathbf{x})]$, and $\boldsymbol{\eta}(\mathbf{x}) \equiv \mathbf{x}$ at the initial time. This allows one to regard $\mathbf{y} = \boldsymbol{\eta}(\mathbf{x})$ as a dynamical variable. The kernel of formula (4) for $F = J_A$, $G = J_B$ reduces to $\mathbf{w} \cdot (\mathbf{a} \wedge \mathbf{b})$ which represents the result we must reproduce by introducing \mathbf{y} -dependent functional derivatives. Upon expressing a current as

$$J_A[\mathbf{w}] = k\rho \int d^3x \mathbf{A}(\mathbf{x}) \cdot [\nabla \alpha(\mathbf{x}) \wedge \nabla \beta(\mathbf{x})], \quad (\text{A3})$$

where α, β contain $\boldsymbol{\eta}$, the functional derivative of J_A can be written through the formula

$$\frac{\delta J_A}{\delta y^k} \nabla y^k = \rho \mathbf{w}(\mathbf{x}) \wedge \mathbf{a}(\mathbf{x}) := D_y J_A. \quad (\text{A4})$$

The identity $\mathbf{w} \cdot (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{w} \cdot \mathbf{w}^2) \cdot [D_y J_A \wedge D_y J_B]$, where $\mathbf{w} = \nabla \alpha(\mathbf{x}) \wedge \nabla \beta(\mathbf{x})$ must be considered as dependent on $y^k = \boldsymbol{\eta}^k(\mathbf{x})$, is derived from Eq. (A4). Then the brackets

$$\{J_A, J_B\}(\mathbf{w}) = \frac{1}{\rho} \int d^3x \frac{\mathbf{w}}{w^2} \cdot [D_y J_A \wedge D_y J_B], \quad (\text{A5})$$

can be defined. Since $D_y J_A$ is also available in the form

$$D_y J_A \equiv \rho I(\mathbf{x}) \frac{d\mathbf{x}}{d\phi} \wedge \mathbf{a}(\mathbf{x}), \quad (\text{A6})$$

one finds that the string limit of Eq. (A5) is well defined in that the factor $1/w^2$ generating a divergent contribution is compensated by two factors $I(\mathbf{x})$ coming from $D_y J_A$ and

$D_y J_B$. In the string limit, in fact, one can easily show that the approximation $x_3 \approx \phi$ can be locally implemented around the vortex core as a consequence of the confinement of vorticity inside a thin cylinder. In view of Eq. (A2) this implies that $I(\mathbf{x}) \approx |\mathbf{w}(\mathbf{x})|$. Consequently, one finds

$$\frac{D_y J_A}{\rho |\mathbf{w}|} = \frac{d\mathbf{x}}{d\phi} \wedge \mathbf{a}(\mathbf{x}) \rightarrow \left[\frac{\delta J_A}{\delta \mathbf{x}} \right]_{\Gamma},$$

where the subscript Γ recalls that $\mathbf{x} \in \Gamma$ and ϕ identifies with the arclength in the above expression.

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- ¹R.J. Donnelly, *Quantized Vortices in He II* (Cambridge University Press, Cambridge, England, 1991).
- ²*Superfluidity and Superconductivity*, 2nd ed., edited by D.R. Tilley and J. Tilley (Hilger, London, 1986).
- ³F. Dalfovo, S. Giorgini, Lev P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999).
- ⁴Beyond various references listed below a pair of recent papers on this topic are U.R. Fischer, *Ann. Phys. (N.Y.)* **278**, 62 (1999); J.P. Kottmann and A.M. Schakel, *Phys. Lett. A* **242**, 99 (1999).
- ⁵M. Rasetti and T. Regge, *Physica A* **80**, 217 (1975).
- ⁶G.A. Goldin, R. Menikoff, and D.H. Sharp, *Phys. Rev. Lett.* **58**, 174 (1987).
- ⁷G.A. Goldin, R. Menikoff, and D.H. Sharp, *Phys. Rev. Lett.* **67**, 3499 (1991).
- ⁸V. Penna and M. Spera, *J. Math. Phys.* **30**, 2778 (1989).
- ⁹V. Penna and M. Spera, *J. Math. Phys.* **33**, 901 (1992); V. Penna, M. Rasetti, and M. Spera, *Contemp. Math.* **219**, 173 (1998).
- ¹⁰Y. Wu, *J. Math. Phys.* **34**, 2342 (1993).
- ¹¹V. Penna, M. Rasetti, and M. Spera, *Int. J. Mod. Phys. B* **4**, 1289 (1990); V. Penna and M. Spera, *ibid.* **6**, 2209 (1992).
- ¹²A.L. Fetter, *Phys. Rev.* **162**, 143 (1967).
- ¹³This becomes evident by observing that one of the three coordinates required to describe a 1D curve can be always used to parametrize locally the curve itself. Setting $s \equiv x_k$ enables the reduction of Eq. (3) to the canonical form $\{x_i(s), x_j(s')\}_{**} = \epsilon_{ijk} \delta(s-s')/k\rho$. It is worth noting that such a result makes the set of points representing the intersections of Γ with a given plane $x_k = \text{const}$ equivalent to a planar model of point vortices with off-plane couplings.
- ¹⁴E.A. Kuznetsov and A.V. Mikhailov, *Phys. Lett.* **77A**, 37 (1970).
- ¹⁵*Proceedings of the 20th International Conference on Low Temperature Physics*, edited by R.J. Donnelly, [*Physica B* **194-196** (1994)].
- ¹⁶Vortex dynamics and Quantum effects in Josephson junction arrays are reviewed in the *Proceedings of the ICTP Workshop on Josephson Junction Arrays*, edited by H.A. Cerdeira and S.R. Shenoy [*Physica B* **222**, 336 (1996)].
- ¹⁷V. Penna, *Phys. Rev. B* **59**, 7127 (1999).
- ¹⁸V.I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, Berlin, 1978).
- ¹⁹V.I. Arnold and B.A. Khesin, *Topological Methods in Hydrodynamics* (Springer-Verlag, New York, 1998).
- ²⁰J. Marsden and A. Weinstein, *Physica D* **7**, 305 (1983).
- ²¹M. Rasetti and T. Regge, in *Highlights of Condensed Matter Theory*, edited by F. Bassani (Compositori, Bologna, 1985).
- ²²V. Zeitlin, *Physica D* **49**, 353 (1991).
- ²³P.G. Saffman, *Vortex Dynamics* (Cambridge University Press, Cambridge, England, 1992); H. Aref, *Annu. Rev. Fluid Mech.* **15**, 345 (1983).
- ²⁴R.Y. Chiao, A. Hansen, and A. Moulthrop, *Phys. Rev. Lett.* **55**, 2887 (1985).
- ²⁵J. Hoppe, *Int. J. Mod. Phys. A* **4**, 5235 (1989).
- ²⁶K.-T. Chen, *Adv. Math.* **23**, 181 (1977); *Bull. Am. Math. Soc.* **83**, 831 (1977).