# Fresnel coefficients at an interface with a lamellar composite material

O. Acher, A. L. Adenot, and F. Duverger
*CEA Le Ripault, B.P. 16, F-37260 Monts, France*

(Received 4 August 1999; revised manuscript received 7 February 2000)

Different effective-medium theories (EMT's) are used to describe the high-frequency and optical properties of composite materials. However, these theories reveal not only differences in the evaluation of the effective permeability and permittivity, but also in their definitions. Rytov gave definitions of the effective permeability and permittivity that are clearly incompatible with the extended Bruggeman definitions when the skin effect occurs. An analysis of the exactly solvable case of a lamellar composite is performed using both approaches. Since most experimental determinations of the permeability and permittivity of composites rely on reflection-transmission measurements, it is of foremost importance to determine which definitions of the permeability and permittivity should be used to express the Fresnel coefficients under the conventional form. For that purpose, we derive the reflection and transmission coefficients at an interface between a composite material and the air, without any effective-medium hypothesis for the composite. This derivation is performed on a periodic composite containing conducting inclusions separated by a dielectric plane. We point out that in the interface region, evanescent modes are present and cannot be described by an effective-medium approach. We infer the proper definitions of the permeability and the permittivity of a composite from the expression of the Fresnel coefficients and from the expression of the refractive index of the propagative mode. We show that the extended Bruggeman definitions are basically correct, but that small correction terms due to the modes at the interface should be taken into account in some cases. A numerical example is given to show these interface effects. An experimental result is also presented. It illustrates that the permeability determined from reflection-transmission measurement disagrees with the definitions given by Rytov but agrees with our definitions.

## I. INTRODUCTION

A large effort has been dedicated to the investigation of inhomogeneous materials, in particular in terms of optical or microwave properties. Effective-medium Theories (EMT's) have been extensively used to describe composite materials with an effective permittivity and permeability.[1–7] The definitions of the effective quantities are generally given without justification and act as the starting point of the calculation of these quantities as a function of the permittivity and permeability of the constituents, and of the composite topology. However, different definitions of the effective permeability and permittivity are found. In the case of metal-dielectric mixtures with characteristic dimensions of the inclusions smaller than the wavelength in the effective medium but with conductor characteristic dimensions that are not small compared to the penetration depth in the conductor, it has been pointed out that the different sets of definitions could yield significantly different predictions.[8,9] In particular, the different definitions lead to an effective permeability for a composite with no magnetic constituent that may be either equal to unity or significantly lower. Moreover, if much attention has been paid to the way of obtaining precise computations of the effective permeability and permittivity of composites[6,10–12] and of deriving bounds,[3,13–15] little attention has been paid to establish that these quantities would yield the proper reflection and transmission coefficients. In particular, to our view it has not been established that these effective quantities could be used to compute the Fresnel coefficients[16] of a wave incoming on the composite material except in a particular case treated by Pottel.[17] However, it is of the foremost importance to establish which quantities should be used to account for the Fresnel coefficients and the refractive index of the effective medium, since most experimental results are based on reflection and/or transmission measurements.[4–6,8,12,18] Fresnel coefficients have been derived for rough surfaces,[19] but not to our knowledge for semi-infinite composites with conducting inclusions.

The paper is organized as follows. In Sec. II, we describe a particular topology in which the propagation of an electromagnetic wave inside the infinite composite can be expressed analytically. We recall the definitions proposed by Rytov for the effective permeability and permittivity. We show that this topology can also be described in the frame of the extended Bruggeman theory, which is equivalent to the dynamic Maxwell-Garnett approach[9] in this particular case. We point out that this approach is based on different definitions of the effective properties. In Sec. III, we consider the more general case of a periodic composite. In addition, we introduce an interface between the composite and the air. We derive the expression of the Fresnel coefficients for a wave incident on the interface as a function of the fields in the composite far from the interface. This derivation takes into account the possible presence of evanescent surface modes very near the interface that cannot be described by an effective-medium approach. A formal identification with the conventional expressions of the Fresnel coefficients and of the refractive index leads to our definitions for the effective permeability and permittivity. It is also shown that these definitions can be used to calculate the Fresnel coefficients on slabs of composites and multilayers. Numerical and experimental results are presented in Sec. IV. Section V discusses our results and compares our approach with previous attempts to give precise definitions of the effective perme-

0163-1829/2000/62(20)/13748(9)/$15.00 PRB 62  

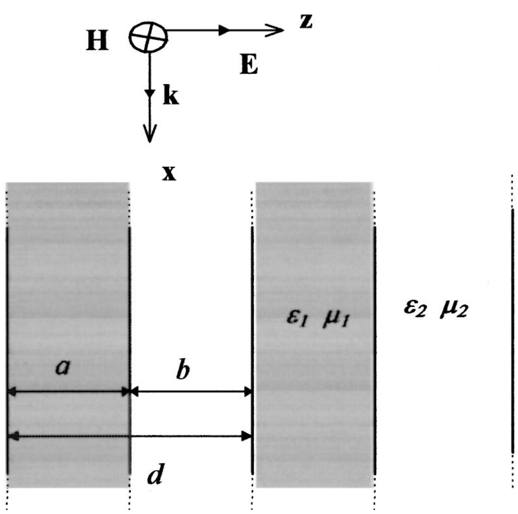


FIG. 1. Sketch of an infinite lamellar composite, consisting of a periodic alternation along the **z** axis of metallic (1) and insulating (2) sheets of thickness $a$ and $b$. The wave propagates in the medium along the **x** direction with the **H** field parallel to the $y$ axis.

ability and permittivity, and Sec. VI concludes this paper.

## II. PROPAGATION OF AN ELECTROMAGNETIC WAVE IN THE INFINITE LAMELLAR INSULATING/METALLIC COMPOSITE MEDIUM

We first investigate a very simple topology: the infinite periodic alternation of dielectric and metallic sheets. Wave propagation in layered media has been studied extensively.[20] Let us consider an electromagnetic plane wave propagating in a finely stratified medium composed of layers of a metallic material 1 of thickness $a$ alternated with layers of a dielectric material 2 of thickness $b$. The complex permittivity and permeability are noted $(\varepsilon_1,\mu_1)$ for material 1 and $(\varepsilon_2,\mu_2)$ for material 2. The microwave magnetic field **H** and the propagation vector are parallel to the lamination plane, as sketched on Fig. 1. In the following, **E** is the microwave electric field, $n$ is the refractive index of the medium deduced from the velocity of the wave in the medium, and $k_0=\omega/c$ the wave vector in vacuum at the wavelength $\lambda_0$. **x** is taken as the axis of propagation and **z** as the normal to the metal and dielectric planes. The metal volume fraction is noted $q=a/(a+b)$. The permittivity of the metal is given by $\varepsilon_1=-j\sigma/(\omega\varepsilon_0)$. For a conventional amorphous ferromagnetic layer with $\sigma=7.1\times10^5\,\mathrm{S\,m^{-1}}$, this yields $\varepsilon_1=-j\times1.3\times10^7$ at 1 GHz. The thickness $a$ is assumed to be small compared with $\lambda_0$, but due to the high permittivity of the metallic material 1, it will not be taken necessarily as small compared to the penetration depth in the metal, which is of the order of $\lambda_0/|(\varepsilon_1\mu_1)|^{1/2}$. The dielectric material 2 has a thickness of $b$ small enough so that $b\ll\lambda_0/|(\varepsilon_2\mu_2)|^{1/2}$. The period $d=a+b$ is supposed to be small compared to the wavelength of the propagative mode, that is $d\ll\lambda_0/|n|$. For the other eigenpolarization (**E** parallel to the layers) the material behaves as a metallic reflector.

### A. Description using the Rytov approach

With the convention for the fields in $\exp(-jk_0nx+j\omega t)$, the Maxwell equations lead to

$$\frac{\partial\mathbf{H}_y}{\partial z}=-j\omega\varepsilon_0\varepsilon\mathbf{E}_x,$$

$$-jk_0n\mathbf{H}_y=j\omega\varepsilon_0\varepsilon\mathbf{E}_z, \tag{1}$$

$$\frac{\partial\mathbf{E}_x}{\partial z}+jk_0n\mathbf{E}_z=-j\omega\mu_0\mu\mathbf{H}_y.$$

Once the equations are solved, the conditions of continuity and periodicity at the boundaries are applied. As shown by Rytov,[10] the dispersion relation is expressed by the following transcendental equations:

$$\frac{\alpha_1}{\varepsilon_1}\tan\left(\frac{\alpha_1 a}{2}\right)=-\frac{\alpha_2}{\varepsilon_2}\tan\left(\frac{\alpha_2 b}{2}\right), \tag{2}$$

or

$$\frac{\varepsilon_1}{\alpha_1}\tan\left(\frac{\alpha_1 a}{2}\right)=-\frac{\varepsilon_2}{\alpha_2}\tan\left(\frac{\alpha_2 b}{2}\right) \tag{3}$$

where $\alpha_1=k_0\sqrt{\varepsilon_1\mu_1-n^2}$ and $\alpha_2=k_0\sqrt{\varepsilon_2\mu_2-n^2}$.

Equation (2) has a unique solution with a low modulus, labeled $n_{l=0}$, given by

$$\alpha_1=k_0\sqrt{n_1^2-n_{l=0}{}^2}\approx k_0n_1,\quad \tan(\alpha_2 b/2)\approx\alpha_2 b/2. \tag{4}$$

High-order modes $n_{l>0}$ exist for $|n_l|>(\lambda_0/b)$. They are given by

$$n_{l>0}\approx jl\frac{\lambda_0}{b}. \tag{5}$$

A similar study of Eq. (3) shows that it has no solution with a small modulus. It yields solutions $n_m$ with $|n_m|>(\lambda_0/a)$, and this set of modes is expressed by

$$n_m\approx jm\frac{\lambda_0}{a}. \tag{6}$$

As a consequence, the only propagative solution has an index $n=n_{l=0}$. Evanescent modes are described by Eqs. (5) and (6) and their propagation length is of the order of the period $d$ or shorter.

In his paper, Rytov mentioned the existence of the high order modes, but he focused only on the propagative mode. He defined the effective properties $\overline{\varepsilon}$ and $\overline{\mu}$ of the medium as $n=\sqrt{\overline{\varepsilon}\overline{\mu}}$ and $\langle E\rangle/\langle H\rangle=\sqrt{\mu_0\overline{\mu}}/\sqrt{\epsilon_0\overline{\epsilon}}$, where the angular brackets indicate the average of the field over the period $d$ for the propagative mode.

It is easy to show that these definitions are equivalent to

$$\langle B\rangle=\mu_0\overline{\mu}\langle H\rangle, \tag{7}$$

$$\langle D\rangle=\varepsilon_0\overline{\varepsilon}\langle E\rangle. \tag{8}$$

Since the thickness $b$ of the dielectric is small compared to the wavelength, and using $|\varepsilon_1|\gg|\varepsilon_2|$, one has

$$\overline{\varepsilon}=\varepsilon_2\left(\frac{q}{1-q}A+1\right), \tag{9}$$

$$\overline{\mu}=\frac{qA\mu_1+(1-q)\mu_2}{qA+(1-q)},\tag{10}$$

$$n^2=\overline{\varepsilon\mu}=\varepsilon_2\left(\frac{qA}{1-q}\mu_1+\mu_2\right),\tag{11}$$

where

$$A=\frac{\tan(k_0\sqrt{\varepsilon_1\mu_1}a/2)}{k_0\sqrt{\varepsilon_1\mu_1}a/2}\tag{12}$$

describes the skin effect in the metal.

If the thickness $a$ of the metal is small enough so that no skin effect occurs at the frequency under investigation, then $A\approx1$. The expression for $\overline{\mu}$ reduces then to $\overline{\mu}=q\mu_1+(1-q)\mu_2$, and $\overline{\varepsilon}=\varepsilon_2/(1-q)$, which corresponds to the Wiener relations. Now, let us examine the same topology using the formalism of EMT's extended to finite frequencies.

### B. Description using the extended effective-medium theory

EMT's have been extensively studied theoretically and experimentally[1–7] and extended for composites with metallic inclusions of various shapes.[8,11,21,22] We briefly remind the reader below the derivation of the effective permeability and permittivity for the particular case of a lamellar composite. A first step in a widespread approach of EMT's is to express the polarizability of a particle in a surrounding medium, in order to describe the interactions in the effective medium. Depending on the effective-medium theory, the nature of the surrounding medium changes: it can be the matrix, for the Maxwell-Garnett approach, or the effective medium, for the Bruggeman approach. For the particular case of a lamellar topology, it can be shown that the different effective-medium approaches lead to the same expressions of the effective parameters. Following the Bruggeman approach for a binary composite made of a material 1 and a material 2, a simple way to compute the effective characteristics is to calculate the polarizability of particles 1 and 2 in the effective medium, and then to write that the effective medium is homogeneous. For an ellipsoid of material $i$, of permeability $\mu_i$ and of polarizability $p_i$ surrounded by the effective medium of permeability $\tilde{\mu}$, polarized by an external field $H_0$, the expression of the polarizability is given by

$$p_i=\frac{\mu_i-\tilde{\mu}}{\tilde{\mu}+L(\mu_i-\tilde{\mu})},$$

where $L$ is the geometric depolarizing coefficient: $L=0$ for an infinite plane polarized along its infinite directions and $L=1$ if the polarizing field is perpendicular to the plane.

According to the effective-medium approach,

$$qp_1+(1-q)p_2=0,\tag{13}$$

where $q$ is the volume fraction of material 1 (metal). Applying this method to the topology described previously (see Fig. 1), the particles are infinite sheets of metal 1 or dielectric 2, and the first step is to evaluate the polarizability of an infinite sheet of metal. The expression of the magnetic polarizability in presence of eddy currents is required to extend the Bruggeman EMT. For an infinite metallic plane polarized along its infinite direction by $H_0$, the polarizability $p_1$ can be related to $\langle B_1\rangle$ the average of the magnetic induction within the plane by

$$p_1=\frac{\langle B_1\rangle-\mu_0\tilde{\mu}H_0}{\mu_0\tilde{\mu}H_0}\tag{14}$$

because of the continuity of $H_0$. One may have $\langle B_1\rangle\neq\mu_0\mu_1H_0$ because of the eddy currents. In the planar conducting particle of width $a=2w$, the internal field $H_1$ is described by

$$\Delta H_1+k_1^2H_1=0,\tag{15}$$

where $k_1=\omega\sqrt{\varepsilon_1\mu_1}/c$. Using the continuity of $H$ (parallel to the **y** axis) at $z=-w$ and $z=+w$, one gets

$$H_1(z)=H_0\frac{\cos(kz)}{\cos(kw)}.\tag{16}$$

Then, the average $\langle B_1\rangle$ of the magnetic induction within the particle is

$$\langle B_1\rangle=\frac{1}{2w}\int_{-w}^{+w}\mu_0\mu_1H_1(z)dz=A\mu_0\mu_1H_0,\tag{17}$$

where the coefficient $A$ is given by Eq. (12) and accounts for the eddy-current effects. The magnetic polarizability of the planar conducting particle in the effective medium of permeability $\tilde{\mu}$ for a polarizing field parallel to its plane is

$$p_1=(A\mu_1-\tilde{\mu})/\tilde{\mu}\tag{18a}$$

for the dielectric sheet, $A=1$, and

$$p_2=(\mu_2-\tilde{\mu})/\tilde{\mu}.\tag{18b}$$

Then, the application of Eq. (13) leads to

$$\tilde{\mu}=qA\mu_1+(1-q)\mu_2.\tag{19}$$

The average of $B$ over particles 1 and 2 is related to the external field $H_0$ by

$$\langle B\rangle=[qA\mu_1+(1-q)\mu_2]\mu_0H_0.\tag{20}$$

As a conclusion, the effective-medium permeability defined by the extended Bruggeman theory is consistent with

$$\langle B\rangle=\mu_0\tilde{\mu}H_0.\tag{21}$$

The same method has to be applied to determine the effective permittivity in presence of eddy currents with the polarizing field $E_0$ perpendicular to the plane. Using the continuity of **D** instead of **H** and computing the average of **E** instead of **B** in the metallic particle, the electric polarizability of the metallic planar particle 1 perpendicular to the polarizing field in the effective medium of permittivity $\tilde{\varepsilon}$ is

$$g_1=\frac{\varepsilon_1/A-\tilde{\varepsilon}}{\varepsilon_1/A}.\tag{22a}$$

For a planar dielectric particle

$$g_2=\frac{\varepsilon_2-\tilde{\varepsilon}}{\varepsilon_2}.\tag{22b}$$

The expression for $\tilde{\varepsilon}$ can be derived from the above equations, and shown to be consistent with

$$D_0=\varepsilon_0\tilde{\varepsilon}\langle E\rangle, \tag{23}$$

where $D_0$ is the electric displacement field (along **z**) in the dielectric plane. Using the assumption $|\varepsilon_1|\gg|\varepsilon_2|$, the effective permittivity can be expressed as

$$\tilde{\varepsilon}=\frac{\varepsilon_2}{1-q}. \tag{24}$$

The following expression of the effective index given by $\tilde{n}=\sqrt{\tilde{\varepsilon}\tilde{\mu}}$ follows from Eqs. (19) and (24):

$$\tilde{n}^2=\varepsilon_2\left(\frac{qA}{1-q}\mu_1+\mu_2\right). \tag{25}$$

An asymmetry between the expressions of $\tilde{\varepsilon}$ and $\tilde{\mu}$ can be noted in Eqs. (21) and (23). This is associated with the polarization of the wave. The magnetic field is parallel to the conducting planes, whereas the incoming electric field is normal to the conducting planes.

### C. Comparison between the two models

It is clear that for the topology under investigation, the Bruggeman approach leads to a set of definitions for the permittivity and the permeability [Eqs. (21) and (23)] that is different from the set of definitions [Eqs. (7) and (8)] associated with the Rytov approach. The corresponding values of the permeability and the permittivity, given by Eqs. (19) and (24) in the former case, and by Eqs. (9) and (10) in the latter case, are different if the skin effect is not negligible, that is, if $A$ differs from unity. Since these values have been derived using an exact approach for the lamellar composite under investigation, the differences should not be attributed to some difficulties in carrying on precise evaluations. It is clear that at least one approach will yield incorrect Fresnel coefficients if the effective quantities are used to compute the reflection and transmission coefficients. However, both approaches yield the same value for the refractive index. It is straightforward from Eqs. (11) and (25) that $\tilde{n}=n$.

This clearly shows the need to find a correct definition for the permeability and permittivity of composites that yield not only the proper refractive index, but also the proper Fresnel coefficients. This approach is presented below.

## III. FRESNEL COEFFICIENTS AT THE INTERFACE BETWEEN A PERIODIC COMPOSITE TOPOLOGY AND AIR

Now, we will derive the expressions of the reflection and transmission coefficients at the interface between a composite medium and air as a function of the fields inside the composite. The difficulty is that there are generally many modes at the vicinity of the interface.[17] These modes are essential to ensure the continuity of the fields through the interface. For example, it can be easily shown in the above example of a finely stratified medium that the propagative mode does not meet the continuity requirements at an interface between such a composite and air. It means that the high-order modes are excited. The key idea in the following

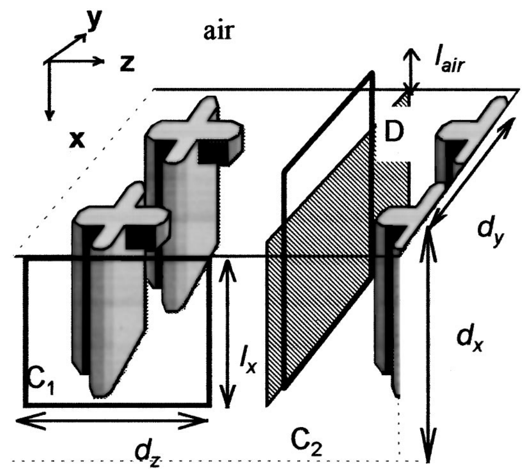


FIG. 2. Sketch of a semi-infinite periodic composite. The integration contours in the vicinity of the interface are represented. The plane labeled $D$ contains no metallic element.

will be to relate the reflected and transmitted waves to the fields far enough from the interface, so that the high-order modes have vanished and only the fundamental mode remains. This will be done under some assumptions, and there may be cases where the reflection and transmission properties should be expressed as a function of the evanescent modes as well as of the propagative mode.

The expression of the refractive index will also be established. Then, we will postulate that the effective permeability and permittivity are defined as the quantities that yield the proper values of the Fresnel coefficients and of the refractive index, using the conventional form of their expressions for an homogeneous medium. This is relevant, to our view, because it is mainly the reflection and transmission coefficients that are available through experimentation.

Some assumptions should be made, though we are able to deal with a far more general case than the topology investigated by Rytov. The topology under investigation is sketched on Fig. 2.

(i) The composite is periodic, with a period $d_x$ along **x**, $d_y$ along **y**, and $d_z$ along **z**. These periods are much smaller than the wavelength $\lambda_0$ in the vacuum. One of its constituents is a dielectric material. The permittivity and the permeability of the dielectric material are noted $\varepsilon_2$ and $\mu_2$, and $d_z|\sqrt{\varepsilon_2\mu_2}|\ll\lambda_0$. It may contain metallic elements with possibly extremely high permittivity, and with thickness larger than the skin depth.

(ii) The incident plane wave is linearly polarized, with the **E** field along **z**, and **H** field along **y**. Only one wavelike mode along **x** propagates deep inside the composite for an incident wave vector along **x**. In particular, this means that the incident polarization is an eigenpolarization. This mode can be written $\mathsf{E}(x,y,z,t)=\mathbf{E}(x,y,z)\exp(j\omega t-jk_0nx)$, and $\mathsf{H}(x,y,z,t)=\mathbf{H}(x,y,z)\exp(j\omega t-jk_0nx)$, with $\mathbf{E}(x,y,z)$ and $\mathbf{H}(x,y,z)$ having the same periodicity as the composite. $k_0$ is the wave vector in the air, and $n$ is the refractive index of the propagative mode.

(iii) High-order modes may be present at the interface, but they vanish at distances $l_x$ from the interface much smaller than $\lambda_0/|n|$. It means that they are evanescent modes. Besides, the permeability of all constituents is sufficiently small, so that $|\mu_i(l_x+d_x)|\ll\lambda_0$ $(i=1,2)$. The permittivity of the dielectric material 2 is also sufficiently small, so that $|\varepsilon_2(l_x+d_x)|\ll\lambda_0$.

(iv) The composite contains a plane $D$ parallel to the **xy** plane, enclosing only dielectric material with limited permittivity, and in particular no conducting element. This does not mean that the case is restricted to the lamellar composites described in Sec. II, as can be shown on Fig. 2.

The incident field is labeled $\mathbf{E}_i e^{j(\omega t-k_0 x)}$. Because of the periodicity (i) along the **y** and **z** axes, the amplitude of the reflected wave can be expressed as a series of Fourier modes by

$$\mathbf{E}_r=\sum_{l=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\mathbf{E}_r^{l,m}e^{-j(2\pi l/d_y)y}e^{-j(2\pi m/d_z)z}\overline{e^{j}k_{l,m}x} \tag{26}$$

associated with the wave vector

$$k_{l,m}^2=k_0^2-\left(l\frac{2\pi}{d_y}\right)^2-\left(m\frac{2\pi}{d_z}\right)^2. \tag{27}$$

According to assumption (i), there is no diffracted ray. Sufficiently far from the interface, the reflected wave is a plane wave of amplitude $\langle\langle E_r^z\rangle\rangle$, where $\langle\langle\ \rangle\rangle$ indicates an average along **y** and **z** over a periodic cell at the interface ($x=0$). The reflection coefficient is then

$$R=\frac{\langle\langle E_r^z\rangle\rangle}{E_i}. \tag{28}$$

It is convenient to introduce the total fields $E_0$ and $H_0$ at the interface. The continuity of the tangential components is written

$$E_i+E_r^z=E_0^z,\quad H_i-H_r^y=H_0^y.$$

Since the quantities $E_r^z$, $E_0^z$, $H_r^y$, and $H_0^y$ vary spatially over a period, it is useful to average these relations over a two-dimensional periodic cell in the plane of the interface. Then

$$R=\frac{\breve{Z}-Z_0}{\breve{Z}+Z_0}, \tag{29}$$

where

$$\breve{Z}=\frac{\langle\langle E_0^z\rangle\rangle}{\langle\langle H_0^y\rangle\rangle} \tag{30}$$

and

$$Z_0=\frac{E_i}{H_i}=\sqrt{\frac{\mu_0}{\varepsilon_0}}$$

is the impedance of the air.

In the same way, we can define the transmission coefficient

$$T=\frac{\langle\langle E_0^z\rangle\rangle}{E_i} \tag{31}$$

and express it by

$$T=\frac{2\breve{Z}}{\breve{Z}+Z_0}. \tag{32}$$

For an homogeneous medium, the quantity $E_0^z/H_0^y$ is its impedance $Z=Z_0\sqrt{\mu/\varepsilon}$. Equations (29) and (32) are then the conventional Fresnel relations at an interface. In contrast, in the case of a composite with metallic particles the evanescent modes contribute to the fields at the interface. Therefore, we have to find the expressions for $\langle\langle E_0^z\rangle\rangle/\langle\langle H_0^y\rangle\rangle$ and for the refractive index $n$ in the medium to define the effective permittivity $\breve{\varepsilon}$ and permeability $\breve{\mu}$ consistent with

$$\breve{Z}=\frac{\langle\langle E_0^z\rangle\rangle}{\langle\langle H_0^y\rangle\rangle}=Z_0\sqrt{\frac{\breve{\mu}}{\breve{\varepsilon}}} \tag{33}$$

and

$$n=\sqrt{\breve{\mu}\breve{\varepsilon}}. \tag{34}$$

The choice of taking Eqs. (33) and (34) as the definitions of the effective permeability and permittivity ensures that the reflection and transmission coefficients at normal incidence can be computed from these quantities, using the conventional Fresnel relations. This holds not only for the reflection and transmission at an interface, but also for a multilayer containing composite materials, since it is clear that the Fresnel coefficients of a multilayer can be expressed as a function of the Fresnel coefficients at the interfaces and of the refractive index.[23]

The first task is to express $\breve{Z}$ as a function of the fields inside the composite, sufficiently far from the interface so that only the propagative mode remains. For this purpose, we apply the Maxwell laws written in the integral formalism on integration contours.

A closed integration contour $C_1$ in the **xz** plane is sketched on Fig. 2. It starts on the interface between air and the composite. It extends over $d_z$ along **z**, and along a small distance $l_x$ along **x**, far enough from the interface so that all the evanescent modes have damped. According to assumption (iii), only the fundamental mode propagates below $l_x$. The relation between the average of **E** on contour $C_1$ and the average of **B** on the surface $S_1$ defined by $C_1$ is found by using the Maxwell-Faraday law

$$\int_{C_1}\mathbf{E}\cdot d\mathbf{l}=\oiint_{S_1}\frac{d\mathbf{B}}{\partial t}\cdot d\mathbf{S}. \tag{35}$$

Since this can be written at any $y$, it follows that

$$\langle\langle E_c^z\rangle\rangle-\langle\langle E_0^z\rangle\rangle=-j2\pi\frac{l_x}{\lambda_0}Z_0\langle\langle\mu H^y\rangle\rangle_{l_x}. \tag{36}$$

$E_c^z$ is the **z** component of the electric field at $l_x$ inside the composite, where the wave is considered as single mode. Since $l_x$ is very small in comparison with $\lambda_0$, and the permeability of all constituents is limited according to assumption (iii), it is clear that the right-hand side of the equation is small compared to $Z_0\langle\langle H_0^y\rangle\rangle$, and that a good approximation of the average along **y** and **z** of the **z** component of the electric field at the interface is $\langle\langle E_0^z\rangle\rangle\approx\langle\langle E_c^z\rangle\rangle$. Since this is true at any $x$ between $l_x$ and $l_x+d_x$ since $l_x+d_x\langle\langle\lambda_0$, one also has $\langle\langle E_0^z\rangle\rangle \approx \langle E_c^z\rangle_V$, where $\langle E_c^z\rangle_V$ is the volume average of the **z** component of the electric field over a period in the composite, far enough from the interface so that only the propagation mode is present. This quantity can be computed from the propagation properties in the infinite composite, without taking into account any interface effect. In the expression of $\breve{Z}$ [Eq. (30)] the numerator can be approximated as $\langle E_c^z\rangle_V$.

A second integration contour $C_2$ is defined in the $D$ plane, containing only dielectric material. It starts at a distance $l_{\text{air}}$ above the surface much smaller than the wavelength but large enough so that the evanescent modes have vanished, and $H = \langle\langle H_0\rangle\rangle$. It has a length $d_y$ along **y**, and drops at a distance $l_x$ under the surface. Then, by applying the Maxwell-Ampère law

$$\int_{C_2} \mathbf{H}\cdot d\mathbf{l} = \oiint_{S_2} \frac{\partial \varepsilon \mathbf{E}}{\partial t}\cdot d\mathbf{S}, \tag{37}$$

we get

$$\langle H^y\rangle_{d_y} - \langle\langle H_0\rangle\rangle = j2\pi \frac{(l_x + l_{\text{air}})}{Z_0\lambda_0}\varepsilon_2 \langle\langle E^z\rangle_{d_y}\rangle_{l_x + l_{\text{air}}}, \tag{38}$$

where $\langle\ \rangle_{d_y}$ corresponds to an average in the $D$ plane over a period along **y** at $x = l_x$. Assumptions (i) and (iii) indicate that $l_x + l_{\text{air}}$ is very small in comparison with $\lambda_0$ and that $\varepsilon_2$ is not large. As a consequence, the second term is small and can be neglected. Then $\langle H^y\rangle_{d_y} \approx \langle\langle H_0\rangle\rangle$. This holds also at any $x$ between $l_x$ and $l_x + d_x$. As a consequence, $\langle H^y\rangle_D \approx \langle\langle H_0\rangle\rangle$, where $\langle H^y\rangle_D$ is the average of $H^y$ in the $D$ plane over a two-dimensional periodic cell of dimension $(d_x, d_y)$. This average is made in a region far enough from the interface so that it can be evaluated from the fields corresponding to the single propagating mode in the composite. It should be mentioned that this does not hold if the contour $C_2$ encloses metallic elements. Since the tangential component of the electric field is continuous, the electric field at the interface between a metallic element of the composite and the air may yield very large $\varepsilon_1 E^z$ products, and the right-hand side of Eq. (38) may no longer be negligible. It should be remembered that for metals $\varepsilon_1$ can be of the order of $10^7$ or more in the microwave range, and the screening by surface currents can develop over a thickness much smaller than the skin depth.[24]

Neglecting the right-hand side of Eqs. (36) and (38) leads to the following expression of the impedance:

$$\breve{Z} \approx \frac{\langle E_c^z\rangle_V}{\langle H^y\rangle_D}. \tag{39}$$

Now, it is necessary to express the refractive index as a function of the fields inside the composite, in the region where only the fundamental mode propagates.

The Maxwell-Ampere equation is written in the $D$ plane at a sufficiently large distance from the interface so that only the propagative mode remains, with fields of the form given in assumption (ii). Integration in the $D$ plane on a two-dimensional unit cell along **x** and **y** yields

$$n\langle H^y\rangle_D = -c\langle D^z\rangle_D, \tag{40}$$

where $\langle D^z\rangle_D$ is the average of the component along the **z** axis of the displacement field in the plane $D$ filled with dielectric material.

Similarly, the Maxwell-Faraday equation is written in the composite medium for the propagative mode at a large distance from the interface. Integrating over the volume of a three-dimensional unit cell and using the fields given in assumption (ii) yield

$$n\langle E_c^z\rangle_V = -c\langle B_c^y\rangle_V. \tag{41}$$

Combining Eqs. (39)–(41), one has

$$\varepsilon_0\mu_0 n^2 = \frac{\langle D^z\rangle_D \langle B_c^y\rangle_V}{\langle E_c^z\rangle_V \langle H^y\rangle_D} \tag{42}$$

and

$$\breve{Z}^2 = \frac{\langle B_c^y\rangle_V \langle E_c^z\rangle_V}{\langle H^y\rangle_D \langle D^z\rangle_D}. \tag{43}$$

It is then straightforward to deduce the expression of the effective permeability and permittivity:

$$\mu_0\breve{\mu} = \frac{\langle B_c^y\rangle_V}{\langle H^y\rangle_D}, \tag{44}$$

$$\varepsilon_0\breve{\varepsilon} = \frac{\langle D^z\rangle_D}{\langle E_c^z\rangle_V}. \tag{45}$$

As a result, it should be emphasized that the definitions for the effective permeability and permittivity given by the Rytov model in Eqs. (7) and (8) do not yield the Fresnel coefficients. In contrast, Eqs. (21) and (23) derived for a particular composite topology from the extended Bruggeman model can be identified with Eqs. (44) and (45) that are obtained in a more general case. They are consistent with the Fresnel coefficients. It is straightforward to establish that the expression of the reflection or transmission coefficients for a multilayer system comprising composite materials can be computed from the conventional Fresnel formulas for multilayers, using definitions (33) and (34) for the permittivity and permeability of the composite. This is because these definitions yield both the reflection and transmission coefficients at the interface and the refractive index.

The effective permeability should be therefore defined as the ratio of the average of **B** over the volume of the composite to the average **H** field in the dielectric plane $D$ (and not to the average of the **H** field in the whole composite as Rytov did). The effective permittivity should be defined as the ratio of the component of the average **D** field along **z** in the dielectric plane to the average of **E** along $z$ over the whole composite. These results have been obtained taking into account the presence of high-order modes at the interface of the semi-infinite composite, and it is remarkable that the expressions depend only on the fields for the fundamental mode propagating in the composite.

## IV. NUMERICAL AND EXPERIMENTAL RESULTS

Our theoretical approach has been compared with both numerical and experimental results.

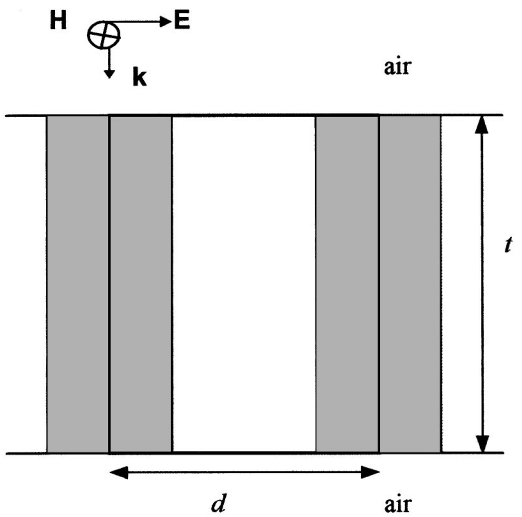


FIG. 3. Sketch of the periodic cell for numerical simulation.

### A. Numerical results

A slab of lamellar material sketched in Fig. 3 is investigated by a finite-element method. It consists of alternating metallic sheets of conductivity $1.1\times 10^7\ \mathrm{S\,m^{-1}}$ and thickness $a=20\ \mu$m and dielectric sheets of permittivity $\varepsilon_2=3$ and thickness $b=10\ \mu$m. At $F_0=10$ GHz, the ratio of the thickness $a$ of the metal sheet to the skin depth $\delta$ is $a/\delta=13.3$, and the skin effect is large. In contrast, the period $d=a+b$ is smaller than the wavelength in air by a factor of $10^3$.

The reflection coefficient at 10 GHz was computed using the Maxwell equations on finite elements with mode matching and a periodicity condition on the boundaries.[25] It was also evaluated from the expressions of the permeability and the permittivity given by Rytov [Eqs. (9) and (10)] and by the extended Bruggeman approach [Eqs. (19) and (24)]. The reflection coefficient $R$ of a wave under normal incidence at frequency $f$ on a slab of material of thickness $t$, permittivity $\varepsilon$, and permeability $\mu$ in air is given by

$$R=\frac{Z_r-1}{Z_r+1} \tag{46}$$

with

$$Z_r=\sqrt{\frac{\mu}{\varepsilon}}\tanh\left[\operatorname{arctanh}\left(\sqrt{\frac{\varepsilon}{\mu}}\right)+j\,\frac{\omega}{c}\sqrt{\varepsilon\mu}\,t\right]. \tag{47}$$

Figure 4 compares the modulus of the reflection coefficient expressed in decibels obtained using the different approaches as a function of the ratio $t/d$ of the slab. For $t$ greater than 50 $\mu$m, the numerical approach is in very good agreement with the reflection calculated using our definitions of the effective permeability and permittivity. The result obtained using Rytov definitions of the effective quantities clearly departs from the numerical approach at any thickness. When $t$ decreases and is of the order of $d$ or smaller, the reflection coefficient derived from the Bruggeman formulas differs from the numerical estimates. This is because the evanescent modes are no longer negligible, and the right-hand side of Eqs. (36) and (38) should no longer be neglected.

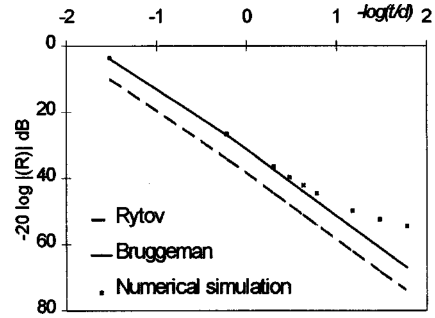


FIG. 4. Modulus of the reflection coefficient on a slab of laminated metal/dielectric composite expressed in decibels vs the logarithm of the ratio of the thickness $t$ to the period $d$. Continuous line, prediction using the Bruggeman approach; dashed line, prediction using the Rytov definitions; squares, numerical simulation.

### B. Experimental results

The effective permeability of a laminated composite was measured using an APC7 coaxial line.[26] 35-$\mu$m-thick copper ribbons have been glued with epoxy resin on 15-$\mu$m-thick polymer ribbons and wound into a torus with inner diameter 3.04 mm, outer diameter 7 mm, and height 3 mm. When illuminated in a coaxial line by the fundamental transverse electric and magnetic mode, this material topology is representative of the lamellar topology of Figs. 1 and 3. The volume fraction $q$ of metal is 0.4, and the conductivity of copper is $5.8\times 10^7\ \mathrm{S\,m^{-1}}$. The effective permeability of the sample was deduced from the measured complex reflection and transmission coefficients using the same procedures as for homogeneous materials. The experimental values are shown in Fig. 5, along with the effective permeability obtained from the Bruggeman and the Rytov definitions. Experimental values agree with the Bruggeman definition, whereas the permeability according to Rytov should remain unity over the whole frequency range, which is clearly not the case.

## V. DISCUSSION

We first discuss the validity of our results and possible extensions. Then, we discuss other definitions that have been proposed for the permeability and the permittivity of composites. We emphasize that the form of our definitions is particularly appropriate for the evaluation of the effective

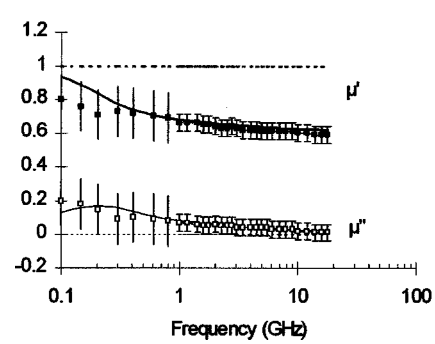


FIG. 5. Real and imaginary permeability of a copper/polymer laminated composite. Squares and dots with errors bars, measured values; thin and thick continuous lines, predicted by the extended Bruggeman model; thin and thick dashed lines, predicted by the Rytov model.

properties of a material using numerical calculations.

In contrast with previous approaches of EMT, we defined the effective permeability and permittivity not as a starting point of further calculations, but as convenient parameters to compute the reflection, transmission, and propagation properties for a composite. The definitions we have derived are confirmed by numerical and experimental results for a particular composite topology. Since the detailed structure of the surface layer plays no role in the proof, it is clear that adding a thin cap layer with metallic elements at the very surface does not affect our approach, provided that the surface elements do not intersect the $D$ plane, filled with the dielectric. In this way, it is possible to extend our definitions to lamellar materials with conducting bridges between some planes.[22] The presence of the dielectric plane $D$ parallel to **H** is clearly a key hypothesis in our approach, though it may be possible to extend our demonstration to a more wavy surface. In Ref. 9, in which the optical and microwave properties of a wire array are studied, this plane does not exist in the composite but an insulating surface can be defined between the metallic inclusions, and the effective permeability including the skin effect derived using an extended Maxwell-Garnett approach yields the correct Fresnel coefficients. Note that if this assumption is invalid, for example, due to the presence of a continuous strip of a few angströms of metal along the **E** field at the surface, the magnitude of the reflected and transmitted waves are related primarily to the properties of the surface layer and not to the properties of the bulk composite.[24]

In the absence of eddy-current effects, our definitions are equivalent to the Rytov definitions. In this case, the continuity of the component of the magnetic field parallel to the incoming field $H_0$ in the whole composite leads to $H_0 = \langle H \rangle_V = \langle H \rangle_D$. With the same hypothesis, the Maxwell-Gauss equation leads to $\langle D \rangle_V = \langle D \rangle_D$. It is therefore inconsequential whether the volume average of the fields or the average in the dielectric constituent is computed, and all the definitions for the effective quantities are equivalent. Although the absence of eddy current is not always explicitly stated, in Ref. 21, for example, the use of the Rytov definitions is valid. The proof that they can be used to compute the Fresnel coefficients has been given by Pottel[17] in the specific case of a lamellar composite, and the present work extends this proof to a more general case.

In the presence of a skin effect, it has long been recognized[8,27,28] that the application of the Rytov definition for the permeability would lead to $\mu = 1$ for a composite made of nonmagnetic inclusions. Since this is clearly not supported by experimental evidence (mainly obtained from reflection-transmission measurements), very few papers use the raw Rytov definitions, with some exceptions. In Ref. 11, the authors make the choice to define the effective permittivity from the propagation constant of a wave in the material, taking the effective permeability of the composite as unity. From the definition $\overline{k}(\omega) = 2\pi\sqrt{\overline{\varepsilon}(\omega)}/\lambda_0$, they find an effective permittivity that is suitable to derive the index of the composite. It yields the velocity and the attenuation length of a wave in the composite material, but it should not be used to compute the reflection and transmission coefficients at the interfaces. It should also be pointed out that the Rytov definitions yield the proper refractive index. Equation (40) also holds for a volume average over a period, and its product with Eq. (41) proves that the square of the index is the product of the permeability and the permittivity as defined by Rytov.

We have shown that the extended Bruggeman approach leads to the proper definitions in the case of a lamellar composite. In a related approach, it has been pointed out[8] in the case of spherical conductive particles dispersed in a dielectric matrix that the Rytov definitions do not hold. Other attempts have been made to give rigorous definitions of the permeability and permittivity in the presence of eddy currents.[27] It has been proposed to renormalize in the Rytov definitions the field $\langle H \rangle$ as $\langle H \rangle - 4\pi\langle M_J \rangle$, where $\langle M_J \rangle$ is the average contribution of the eddy currents to the magnetic moment. This approach has been justified by considering the average of Maxwell equations. It leads to the same expression as the extended Bruggeman approach for the permeability of spherical conducting particles dispersed in an insulating matrix.[8] However, we did not investigate in a general case whether this approach leads to definitions equivalent to ours.

In another work,[28] the lack of clearly established definitions for the effective quantities in the case of composites containing conductive constituents is emphasized. If the permeability should be defined as the ratio of some average of **B** to some average of **H**, a key question is, how do the averages differ to yield a permeability that may differ from unity for nonmagnetic composites? A proposition is made, based on the observation of the Maxwell-Ampère and the Maxwell-Faraday laws under the integral form. The authors suggest that **B** should be averaged over a surface, and **H** over its contour. However, no clue is given where the contour should be drawn. No indication can be obtained of the use of a permeability defined on such aesthetic criteria. Indeed, when dealing with composite materials with known inclusion topologies, the capabilities of the software used for solving Maxwell equations in complex structures[25] make the issue of the definition of the effective quantities the main question. The computation can be performed numerically, provided that one knows what has to be computed. The form of our definitions is very convenient for this purpose, since the result can be evaluated from the solution of Maxwell equations in the structure. In contrast, the form given by Ref. 27 does not allow a direct evaluation once the fields are known in the material.

A possibly puzzling aspect of our definitions is the asymmetry between the expressions of the permeability and permittivity revealed by Eqs. (44) and (45). One might have expected that the permittivity should be a ratio of a volume average of **D** to a surface average of **E**, instead of Eq. (45). When one considers a lamellar composite such as the composite of Fig. 1, this asymmetry can be related to the fact that at the interfaces, the continuity of the tangential component of **H** and the normal component of **D** play a similar role in the determination of the effective quantities. In the case of an isotropic composite, the asymmetry between the magnetic and electric quantities may be attributed to the fact that one medium may have a very large conductivity, whereas the permeability is limited, according to assumption (iii). The right-hand side of Eq. (38) can be neglected only if the contour $C_2$ encloses only dielectric material, whereas the right-hand side of Eq. (36) can be neglected even though the contour $C_1$ encloses both a dielectric and a magnetic conductor.

## VI. CONCLUSION

We pointed out that several definitions have been given for the effective permeability and permittivity of a composite material. We mentioned two that are widely used in the literature. We investigated the case of a laminated composite and showed that the two sets of definitions yield different results for the reflection and transmission coefficients in the case where the skin effect occurs. We showed that this cannot be attributed to an imperfect homogenization procedure, as propagation in this particular composite topology is exactly solvable.

We derived the expression of the Fresnel coefficients for a wave incident on a semi-infinite composite. Among several assumptions concerning the composite and the incoming wave, we made the hypothesis that evanescent modes may be present at the surface, but that only one mode would propagate deep inside the composite. Moreover, we supposed that there was a plane of dielectric material in the composite, parallel to the **H** field and to the propagation vector. Then, we showed that the reflection and transmission coefficients could be written in conventional forms using proper definitions of the effective permeability and permittivity of the composite [Eqs. (44) and (45)]. Our definitions agree with the extended Bruggeman model definitions for a lamellar composite and not with those proposed by Rytov. The validity of the description using an effective permeability and permittivity has been outlined. In particular, for very thin plates of the composite, however, the effective permeability and permittivity according to our definitions may no longer be used to compute the Fresnel coefficients. This is because evanescent modes propagate over distances that are not negligible compared to the plate thickness. It has been checked experimentally and by numerical analysis that for thicker plates of composites, the Fresnel relations based on our definitions are correct. Since most experimental data on the permeability and permittivity of composite materials are obtained from reflection and/or transmission measurements, we find it an important result to give definitions that are consistent with the experimental data.


[1] D. A. G. Bruggeman, Ann. Phys. (Leipzig) **24**, 636 (1935).

[2] W. Lamb, D. M. Wood, and N. W. Ashcroft, Phys. Rev. B **21**, 2248 (1980).

[3] D. J. Bergman, Phys. Rev. Lett. **44**, 1285 (1980).

[4] W. G. Egan and D. E. Aspnes, Phys. Rev. B **26**, 5313 (1982).

[5] G. A. Niklasson and C. G. Granqvist, J. Appl. Phys. **55**, 3382 (1984).

[6] W. M. Robertson, G. Arjavalingam, and S. L. Shinde, J. Appl. Phys. **70**, 7648 (1991).

[7] V. A. Permyakov, Sov. Radiophys. **39**, 558 (1996).

[8] D. Rousselle, A. Berthault, O. Acher, J. P. Bouchaud, and P. G. Zérah, J. Appl. Phys. **74**, 475 (1993).

[9] T. E. Huber, F. Boccuzzi, and L. Silber, Phys. Rev. B **59**, 7446 (1999).

[10] S. M. Rytov, Zh. Eksp. Teor. Fiz. **29**, 605 (1956); [Sov. Phys. JETP **2**, 466 (1956)].

[11] D. Stroud and F. P. Pan, Phys. Rev. B **17**, 1602 (1978).

[12] Y. Yagil, P. Gadenne, C. Julien, and G. Deutscher, Phys. Rev. B **46**, 2503 (1992).

[13] Z. Hashin and S. Shtrikman, J. Appl. Phys. **33**, 3125 (1962).

[14] G. W. Milton, Appl. Phys. Lett. **37**, 300 (1980).

[15] D. E. Aspnes, Phys. Rev. B **25**, 1358 (1982).

[16] P. M. Jacquart, O. Acher, and P. Gadenne, Opt. Commun. **108**, 355 (1994).

[17] R. Pottel, Z. Angew. Phys. **10**, 8 (1958).

[18] O. Acher, P. M. Jacquart, J. M. Fontaine, P. Baclet, and G. Perrin, IEEE Trans. Magn. **30**, 4533 (1994).

[19] D. E. Aspnes, Phys. Rev. B **41**, 10 334 (1990).

[20] L. M. Brekhovskikh, *Waves in Layered Media* (Academic, San Diego, 1980).

[21] E. F. Kuester and C. L. Holloway, IEEE Trans. Electromagn. Compat. **36**, 300 (1994).

[22] O. Acher, A. L. Adenot, F. Lubrano, and F. Duverger, J. Appl. Phys. **85**, 4639 (1999).

[23] M. Born and E. Wolf, *Principle of Optics* (Pergamon, Oxford, 1959).

[24] S. Fahy, C. Kittel, and S. G. Louie, Am. J. Phys. **56**, 989 (1988).

[25] F. Oelhoffen, F. Lubrano, and F. Farail, in Proceedings of the 2nd Workshop on Finite Element Methods in Electromagnetic Wave Problems, Sienna 1994, published in COMPEL 13-A, James and James.

[26] O. Acher, J. L. Vermeulen, P. M. Jacquart, J. M. Fontaine, and P. Baclet, J. Magn. Magn. Mater. **136**, 269 (1994).

[27] A. P. Vinogradov, L. V. Panina, and A. K. Sarychev, Dokl. Akad. Nauk (SSSR) , **306**, 847 (1989); [Sov. Phys. Dokl. **34**, 530 (1989)]; A. N. Lagarkov, A. K. Sarychev, Y. R. Smychkovich, and A. P. Vinogradov, J. Electromagn. Waves Appl. **6**, 1159 (1992).

[28] J. B. Pendry, A. J. Holden, D. J. Robbins, and W. J. Stewart, IEEE Trans. Microwave Theory Tech. **47**, 2075 (1999).