Anisotropy and frequency dependence of the hopping magnetoresistance in the high-frequency limit in three-dimensional samples

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The magnetic-field dependence of the magnetoconductivity due to quantum interferences of isotropic three-dimensional samples in the strongly localized regime is investigated at high frequencies. To this end the three-site model is used. The investigation shows that the magnetoconductivity at high frequencies is anisotropic, that is; it depends on the direction between the electric and the magnetic field. A simple relationship between the transverse part of the magnetoconductivity and the parallel part of the magnetoconductivity for all magnetic fields is obtained. The magnetoconductivity for crossed electric and magnetic fields is always larger than for parallel magnetic and electric fields. As a function of the magnetic field the magnetoconductivity is a quadratic function with respect to the magnetic field for small magnetic fields, a linear function for moderate fields and saturates at high fields. As a function of frequency the magnetoconductivity increases with decreasing frequency at frequencies which are low, but in the range of applicability of the three-site model. At very high frequencies the magnetoconductivity passes into a plateau. On the plateau the magnetoconductivity becomes independent of frequency.

I. INTRODUCTION

The investigation of the influence of quantum interference effects on the transport properties of disordered systems has received considerable attention in recent years. Also in the hopping regime, where quantum interferences are naively not expected to be important on length scales larger than the localization length, such interferences were shown to determine the magnetotransport properties of insulating samples at low magnetic fields. A first explanation for the magnetoresistance of insulating samples was given by Nguyen, Spivak, and Shklovskii,[1] who argued that phase coherence is maintained during a hop, so that elastic scattering of partial electron waves can give rise to quantum interferences, and therefore can lead to magnetoresistance.

So far such interferences have been studied in many papers both theoretically (see, e.g., Refs. 1–8) and experimentally (see, e.g. Refs. 9–13). In all papers it is found that the dc magnetoconductivity increases with decreasing temperature. Furthermore, in most papers the magnetoconductivity is a quadratic function with respect to the magnetic field for small fields, a nearly linear function for moderate fields and saturates at high magnetic fields. In some samples (see, e.g., Ref. 14) quantum oscillations of the magnetoconductivity have been observed, which were investigated further in the Refs. 3 and 4.

The impact of a magnetic field on the quantum interferences is governed by the flux penetrating the area of characteristic configurations. If the system is strongly localized scattering at one intermediate site is most important, since the transition probabilities are exponentially small functions with respect to the ratio between site separation and localization length, so that the scattering center can be anywhere but close to the initial or the final site. If we now consider a hop from an intial site to a final site with scattering of partial waves by one intermediate site, we find that the area penetrated by the magnetic field is always larger for hops transverse to the field than for hops into the direction of the magnetic field. Consequently, the effect is highly anisotropic on the microscopic level.

The current itself is a vector, and since in the presence of an electric and a magnetic field only two vectors are available if the system is isotropic, the symmetric part of the magnetic-field-induced change of the current $\delta \boldsymbol{j}$ has to lie within the vector space spanned by the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$.[16] Consequently, it has the structure

$$\delta \boldsymbol{j} = \delta\sigma^{\perp}(H)\boldsymbol{E} + \frac{\delta\sigma^{\parallel}(H) - \delta\sigma^{\perp}(H)}{H^2}(\boldsymbol{E}\boldsymbol{H})\boldsymbol{H}. \qquad (1)$$

If $\delta\sigma^{\parallel} \neq \delta\sigma^{\perp}$ anisotropy is present. In this case the current is not always parallel to $\boldsymbol{E}$. In general, there is no principle which determines the differences between $\delta\sigma^{\parallel}$ and $\delta\sigma^{\perp}$ in advance. However, according to the theoretical results of Refs. 1–8 and to the experimental results of Refs. 9–13, the magnetoconductivity of a macroscopic sample is isotropic, so that $\delta\sigma^{\parallel}(H) = \delta\sigma^{\perp}(H)$. Only in Ref. 15 was anisotropy observed in a strong electric field. There $\delta\sigma^{\perp} = 1.94\delta\sigma^{\parallel}$ was detected in the quadratic regime with respect to the magnetic field.

To explain the isotropy usually the geometry of the percolation path is invoked. Since in such a sample the percolation path writhes somehow through the sample and the resistance of the whole sample is deduced only from one resistance, the critical resistance, which can have any direction in space, the anisotropy is claimed to be wiped out. On the other hand, if one would try to construct an effective-medium theory for the magnetoconductivity the physical picture would be different. In this case there would be no percolation path. Instead every hop would be of the order of the

Mott length, and into the direction of the electric field. Consequently, also anisotropy could be expected. Then, another reason for the occurence of isotropy could be that the systems in most experiments were close to the metal-insulator transition. In this case the physical picture sketched above would no longer hold, since in this case scattering at many intermediate sites is of importance. If this is true the scattering paths can probably sufficiently diffuse out, so that the area penetrable by flux for hops transverse to the field is as large as for hops into the direction of the field.

At present we are not able to check the latter assumption, since close to the metal-insulator transition it is necessary to take into account also damping and level broadening effects due to the nondiagonal parts of the density matrix and, moreover, to construct a suitable method to find a solution for the configuration-averaged Green's function. First attempts to incorporate such effects in the investigation of the magnetoconductivity have been formulated in Refs. 17–19. However, what can be checked is the argument used to explain the isotropy of the magnetoconductivity. The main point of the argument is that in percolation theory the conductivity of the whole sample is calculated from one resistor only, from the critical resistor. Since the critical resistor can have any direction in space the anisotropy is averaged out in the course of the averaging procedure. The same situation, that is the situation in which the current is calculated from one resistor only, can be modeled with the three-site model, introduced by Holstein.[20] In this model the conductivity of the macroscopic sample is calculated from three sites only. To this end the transport equations are solved for three sites and the current is calculated in this approximation. The configuration-averaged current is obtained by averaging over all possible side lengths of the triangle, and over all orientations of the surface normal of the triangle in space. For three sites the calculation can be performed exactly, so that no percolation theory or effective medium method has to be used.

Clearly, if the transport equations are solved for three sites only, one cannot expect to calculate the resistance for a macroscopic sample for all cases of interest, but only for those range of frequencies, for which the main contributions to the current originate from jumps between nearby sites only. The range of applicability of the three-site model agrees with that of the two-site model (see, e.g., Refs. 21 and 22), introduced by Pollak and Geballe.[23,24] Since in this model the typical hopping length is of the order $\alpha^{-1}\ln(\nu/s)$, where α^{-1} is the localization length, ν is the attempt-to-escape frequency and $s=-i\omega$ is the frequency of the applied electric field, the hopping length increases with decreasing frequency. The requirement that the hopping length does not exceed the mean site separation appreciably, reduces therfore the practical applicability of the the concrete results to rather high frequencies. Clearly, one can not expect that anisotropy is produced just by increasing frequency, so that anisotropy is also present at low frequencies, if present at high frequencies. Below we show that for strongly localized samples anisotropy is present at high frequencies and we derive a simple relationship, which relates the transverse part of the magnetoconductivity to the longitudinal part of the magnetoconductivity. The construction of an effective theory for the description of the magnetoconductivity at low frequencies and in the dc limit, which takes into account anisotropy, is devoted to another paper.

Beside the dependence of the magnetoconductivity on the magnetic field we also investigate the frequency dependence. We find that in the range of applicability of the three-site model the magnetoconductivity decreases with increasing frequency at low frequencies. At high frequencies the magnetoconductivity becomes frequency independent.

II. BASIC EQUATIONS

In the situation of interest the dynamics of the electrons is governed by the Hamilton operator[24]

$$H=\sum_m [\epsilon_m - e(\boldsymbol{E}(t)\boldsymbol{R}_m)]a_m^+ a_m + \sum_{\boldsymbol{q}} \hbar\omega_q\left(b_{\boldsymbol{q}}^+ b_{\boldsymbol{q}}+\frac{1}{2}\right) + \sum_{m\neq m'} J_{mm'}(\boldsymbol{H})\Phi_{mm'}a_m^+ a_{m'}. \tag{2}$$

Here ϵ_m and $\boldsymbol{R}_m$ are the energy and the position vector of the site with index m, a_m^+, and a_m are creation and annihilation operators for electrons at site m, $b_{\boldsymbol{q}}^+$ and $b_{\boldsymbol{q}}$ are creation and annihilation operators for acoustical phonons with wave vector $\boldsymbol{q}$ and frequency $\omega_{\boldsymbol{q}}$, and $\Phi_{m'm}$ is the multiphonon operator. The electric field $\boldsymbol{E}(t)$ is switched on suddenly at $t=0$. The magnetic field is taken into account only in the phase factor of the resonance integral (α inverse localization length)

$$J_{mm'}=J_0\exp\left(-\alpha\left|R_{mm'}\right|+i\frac{e\boldsymbol{H}}{2\hbar c}[\boldsymbol{R}_m\times\boldsymbol{R}_{m'}]\right), \tag{3}$$

so that attention is paid only to the influence of the magnetic field on quantum interferences. Wave-function shrinkage and spin effects are ignored.

If the Hamilton operator (2) is used the transport equation takes the form

$$sC_m(U_m+\boldsymbol{E}R_m)=\sum_{m'}\Gamma_{m'm}(U_{m'}-U_m), \tag{4}$$

in the linear approximation with respect to the electric field. Here $C_m=f_m(1-f_m)$ (f_m is the Fermi distribution with site energy ϵ_m), $s=-i\omega$ (ω is the frequency of the applied electric field), and U_m is the local electrochemical potential at site m. The quantities $\Gamma_{m'm}$, the resistances, are the transition rates. In the absence of the magnetic field their calculation can be restricted to two-site processes, which describe only direct hops between the initial and the final site. As these contributions are independent of the magnetic field, the consideration of the impact of the magnetic field requires also the consideration of higher-order processes, which result from interferences between alternative hopping paths. If the system is strongly localized only scattering at one intermediate site has to be taken into account, so that

$$\Gamma_{mm'}=\Gamma_{mm'}^{(2)}+\Gamma_{mm'}^{(3)}(\boldsymbol{H}) \tag{5}$$

holds, where $\Gamma^{(2)}$ and $\Gamma^{(3)}$ are the two-site and three-site hopping rates, respectively. For weak electron-phonon coupling the two-site rates are given by[24]

$$\Gamma^{(2)}_{mm'}=\nu\exp\Bigg(-2\alpha|R_{mm'}| - \frac{|\epsilon_m-\epsilon_F|+|\epsilon_{m'}-\epsilon_F|+|\epsilon_m-\epsilon'_m|}{2kT}\Bigg), \quad (6)$$

where ν is the attempt-to-escape frequency and ϵ_F is the Fermi energy. Since the three-site contributions describe the interference between the amplitude for the direct hop with the amplitude for a hopping path with scattering at one intermediate site it has the structure

$$\Gamma^{(3)}_{mm'}(\boldsymbol{H})=\sum_{m_1}\Gamma_{mm_1m'}(\boldsymbol{H}). \quad (7)$$

Here m_1 is the index corresponding to the scattering center. Since we are interested only in the magnetoconductivity we take into account only the symmetric part of the rate with respect to the magnetic field, which, for weak electron-phonon coupling, is given by

$$\Gamma_{mm_1m'}(\boldsymbol{H})=\gamma^{m_1}_{mm'}\left[\cos\left(\frac{e\boldsymbol{H}[\boldsymbol{R}_{mm'}\times\boldsymbol{R}_{m_1m'}]}{2\hbar c}\right)-1\right], \quad (8)$$

with

$$\gamma^{m_1}_{mm'}=\nu J_0\left(\frac{1}{\epsilon_{mm_1}}+\frac{1}{\epsilon_{m'm_1}}\right) \times\exp(-\alpha|R_{mm'}|-\alpha|R_{mm_1}|-\alpha|R_{m_1m'}|) \times\exp\left(-\frac{|\epsilon_m-\epsilon_F|+|\epsilon_{m'}-\epsilon_F|+|\epsilon_{mm'}|}{2kT}\right). \quad (9)$$

The rates (8), (9) can be derived using various techniques. They have been derived in Ref. 2 with renormalized perturbation expansion, in Refs. 5 and 25 using the Konstantinov-Perel method, and in Ref. 6 with nonequillibrium Green's functions. Since in the strongly localized regime the mean separation between the sites is large as compared to the localization length, the three-site rates are small as compared to the two-site rates, so that

$$\Gamma^{(2)}_{mm'}\gg\Gamma^{(3)}_{mm'}(\boldsymbol{H}) \quad (10)$$

holds, independent of the strength of the magnetic field.

If the electrochemical potentials are known the configuration-averaged current is calculated according to

$$\boldsymbol{j}(s)=\frac{e\beta s}{\Omega}\left\langle\sum_m \boldsymbol{R}_mC_m[U_m+e(\boldsymbol{E}R_m)]\right\rangle, \quad (11)$$

where the bracket symbolizes the configuration average.

III. CURRENT IN THREE-SITE APPROXIMATION

In order to investigate the magnetoconductivity at high frequencies we use the three-site model. In this model the current is calculated from three sites only. The model was used by Holstein in his investigation of the hopping Hall effect.[20] Here we apply it to the magnetoconductivity. In our opinion, the advantage of the model is in that it can be solved exactly. No percolation theory or effective medium method is needed. Furthermore, the range of validity of the model is well understood (see, e.g., Refs. 21 and 22). It agrees with that of the two-site model introduced by Pollak and Geballe.[23] In the two-site model the characteristic hopping length is of the order $\ln(\nu/s)/(2\alpha)$. Since the hopping length increases with decreasing frequency the current can no longer be calculated from two sites only, if the characteristic hopping length exceeds the mean site separation appreciably, so that the model applies only to high frequencies.

For three sites the transport equations take the form

$$\begin{aligned}(sC_1+\Gamma_{12}+\Gamma_{13})U_1-\Gamma_{12}U_2-\Gamma_{13}U_3&=-sC_1e(\boldsymbol{E}\boldsymbol{R}_1),\\ -\Gamma_{12}U_1+(sC_2+\Gamma_{12}+\Gamma_{23})U_2-\Gamma_{23}U_3&=-sC_2e(\boldsymbol{E}\boldsymbol{R}_2),\\ -\Gamma_{13}U_1-\Gamma_{23}U_2+(sC_3+\Gamma_{13}+\Gamma_{23})U_3&=-sC_3e(\boldsymbol{E}\boldsymbol{R}_3).\end{aligned} \quad (12)$$

Consequently, the problem reduces to the calculation of the inverse of a 3×3 matrix. To calculate the current we first solve Eq. (12) with respect to the electrochemical potentials. The calculation is elementary but lengthy and therefore not presented here in detail. Using the electrochemical potentials we find that every triangle contributes to the configuration averaged current with

$$\left\langle\frac{es\beta}{\Omega}[\boldsymbol{R}_1C_1(U_1+e(\boldsymbol{E}\boldsymbol{R}_1))+\boldsymbol{R}_2C_2(U_2+e(\boldsymbol{E}\boldsymbol{R}_2)) +\boldsymbol{R}_3C_3(U_3+e(\boldsymbol{E}\boldsymbol{R}_3))]\right\rangle.$$

Since we are calculating the average of the current according to the rule

$$\boldsymbol{j}(s)=\int d\boldsymbol{R}_1d\boldsymbol{R}_2d\boldsymbol{R}_3d\epsilon_1d\epsilon_2d\epsilon_3N(\epsilon_1)N(\epsilon_2)N(\epsilon_3)\boldsymbol{j}(1,2,3), \quad (13)$$

we have actually overcounted the number of configurations. Since three bonds contribute to the current, and every bond has two ends we then have to devide the results by 6, so that the configuration averaged current is given by

$$\boldsymbol{j}(s)=\frac{es\beta}{6\Omega}\int d\boldsymbol{R}_1d\boldsymbol{R}_2d\boldsymbol{R}_3d\epsilon_1d\epsilon_2d\epsilon_3N(\epsilon_1)N(\epsilon_2)N(\epsilon_3) \times[\boldsymbol{R}_1C_1(U_1+e(\boldsymbol{E}\boldsymbol{R}_1))+\boldsymbol{R}_2C_2(U_2+e(\boldsymbol{E}\boldsymbol{R}_2)) +\boldsymbol{R}_3C_3(U_3+e(\boldsymbol{E}\boldsymbol{R}_3))]. \quad (14)$$

If we use the solution of the transport equations we find

$$\boldsymbol{j}(s)=\left\langle\frac{e\beta s^2}{6\Omega D(1,2,3)}\{sC_1C_2C_3\Gamma_{12}e(\boldsymbol{E}\boldsymbol{R}_{12})\boldsymbol{R}_{12}+[\Gamma_{12}\Gamma_{23} +\Gamma_{12}\Gamma_{13}+\Gamma_{23}\Gamma_{13}]C_1C_2(e\boldsymbol{E}\boldsymbol{R}_{12})\boldsymbol{R}_{12}+\text{cyclic}\}\right\rangle, \quad (15)$$

where the cyclic denotes the cyclic permutation of the indices 1, 2, and 3, and

$$D(1,2,3)=s[s^2C_1C_2C_3+s[C_1C_2(\Gamma_{13}+\Gamma_{23})+C_1C_3(\Gamma_{12}+\Gamma_{23})+C_2C_3(\Gamma_{12}+\Gamma_{13})]+(C_1+C_2+C_3)(\Gamma_{12}\Gamma_{13}+\Gamma_{12}\Gamma_{23}+\Gamma_{23}\Gamma_{13})] \tag{16}$$

is the determinant of the matrix of the system (12). Since, in the averaging procedure, all sites are equal they yield the same contribution to the configuration-averaged current, so that we can omit the cyclic terms in the following, and multiply the result simply by 3.

To proceed further, we take advantage of the fact that in the course of the averaging procedure the resistances can take on any direction in space. To this end we first put one site, let us say site 3, on the origin of the coordinate system. The z axis is chosen parallel to the magnetic field. Furthermore, we have $\boldsymbol{R}_{12}=\boldsymbol{R}_{13}-\boldsymbol{R}_{23}$. Now we decompose the vectors $\boldsymbol{R}_{13}$ and $\boldsymbol{R}_{23}$ into their longitudinal and their transverse part with respect to the direction of the magnetic field, according to $\boldsymbol{R}_{13}=\boldsymbol{R}_{13}^{||}+\boldsymbol{R}_{13}^{\perp}$ and $\boldsymbol{R}_{23}=\boldsymbol{R}_{23}^{||}+\boldsymbol{R}_{23}^{\perp}$. Since in the averaging procedure all sites are treated equally to every configuration with $\boldsymbol{R}_{13}=\boldsymbol{R}_{13}^{||}+\boldsymbol{R}_{13}^{\perp}$ and $\boldsymbol{R}_{23}=\boldsymbol{R}_{23}^{||}+\boldsymbol{R}_{23}^{\perp}$ there is also a configuration which differs only in that $\boldsymbol{R}_{13}^{\perp}$ and $\boldsymbol{R}_{23}^{\perp}$, are replaced by $-\boldsymbol{R}_{13}^{\perp}$ and $-\boldsymbol{R}_{23}^{\perp}$. The cosine in Eq. (8) is not affected by this operation. Invariant are also the quantities $\Gamma^{(2)}$ and γ_{ij}^{k}, since they depend only on the side lengths of the triangle, which remain unchanged. The scalar product in Eq. (10) projects the correponding vectors onto the direction of the magnetic field. Consequently, in the averaging procedure $(\boldsymbol{E}\boldsymbol{R}_{12})\boldsymbol{R}_{12}$ is replaced by $(\boldsymbol{E}\boldsymbol{R}_{12}^{||})\boldsymbol{R}_{12}^{||}$.

Consider now the situaton $\boldsymbol{E}\perp\boldsymbol{H}$. Again we decompose the vectors $\boldsymbol{R}_{13}$ and $\boldsymbol{R}_{23}$ into their longitudinal and into transverse part with respect to the direction of the magnetic field. If we now apply the same argumentation we find that, in the course of the averaging procedure, $(\boldsymbol{E}\boldsymbol{R}_{12})\boldsymbol{R}_{12}$ is replaced by $(\boldsymbol{E}\boldsymbol{R}_{12}^{\perp})\boldsymbol{R}_{12}^{\perp}$. Consequently, we have

$$\boldsymbol{j}^{||/\perp}(s)=\left\langle \frac{e\beta s^2}{2\Omega D(1,2,3)}\{sC_1C_2C_3\Gamma_{12}e(\boldsymbol{E}\boldsymbol{R}_{12}^{||/\perp})\boldsymbol{R}_{12}^{||/\perp}+[\Gamma_{12}\Gamma_{23}+\Gamma_{12}\Gamma_{13}+\Gamma_{23}\Gamma_{13}]\times C_1C_2(e\boldsymbol{E}\boldsymbol{R}_{12}^{||/\perp})\boldsymbol{R}_{12}^{||/\perp}\}\right\rangle, \tag{17}$$

where $\boldsymbol{j}^{||}$ ($\boldsymbol{j}^{\perp}$) is the current for $\boldsymbol{E}||\boldsymbol{H}$ ($\boldsymbol{E}\perp\boldsymbol{H}$).

To simplify Eq. (17) further we use the inequality (10) to linearize Eq. (17) with respect to the three-site transition probabilities. Then the magnetic-field-induced change of the current is given by

$$\delta\boldsymbol{j}^{||/\perp}(s)=\left\langle \frac{e^2\beta s^4}{2\Omega D^2(1,2,3)}\Gamma_{132}[\boldsymbol{R}_{13}^{||/\perp}a_{21}^3-\boldsymbol{R}_{23}^{||/\perp}a_{12}^3]^2\right\rangle \boldsymbol{E}, \tag{18}$$

where

$$a_{12}^3=\Gamma_{13}^{(2)}C_2(C_1+C_3)+\Gamma_{23}^{(2)}C_1C_2+sC_1C_2C_3, \tag{19}$$

and the determinant $D(1,2,3)$ is given by Eq. (16), with Γ replaced by $\Gamma^{(2)}$.

IV. ANISOTROPIC MAGNETORESISTANCE

Since, in general, the transverse part of a vector with respect to the magnetic field is different from the longitudinal part, it appears that already Eq. (18) indicates, that the magnetoconductivity is anisotropic. To investigate this point further we first perform the angular integrations. To this end we split the integrations over the triangle into two parts, the average over the side lengths of the triangle, and the average over the orientation of the triangle in space. However, since the integrand of Eq. (19) depends only on the side lengths of the triangle $R_1=|\boldsymbol{R}_{23}|$, $R_2=|\boldsymbol{R}_{13}|$, and $R_3=|\boldsymbol{R}_{13}-\boldsymbol{R}_{23}|$ and on the angles of the unit vectors $\boldsymbol{R}_{13}/R_2$ and $\boldsymbol{R}_{23}/R_1$ via the vectors $\boldsymbol{R}_{13}$ and $\boldsymbol{R}_{23}$, we first introduce an additional integration over the third side, using the identity

$$1=2\int_0^{\infty}dR_3R_3\delta(R_3^2-R_1^2-R_2^2+2R_1R_2\cos\psi), \tag{20}$$

where ψ is the angle between the vectors $\boldsymbol{R}_{13}$ and $\boldsymbol{R}_{23}$. ψ satisfies the equation

$$\cos\psi=\cos(\phi_1-\phi_2)\sin\theta_1\sin\theta_2+\cos\theta_1\cos\theta_2. \tag{21}$$

Here θ_1 (θ_2) and ϕ_1 (ϕ_2) are the polar and azimutal angle of the vector $\boldsymbol{R}_{23}$ ($\boldsymbol{R}_{13}$), respectively. Thereafter, the integrations over the angles can be performed exactly (see the remarks in Appendix A). Doing so, we obtain

$$\Delta\boldsymbol{j}^{||/\perp}=\frac{4\pi}{3}e^2\beta s^4\int d\epsilon_1d\epsilon_2d\epsilon_3N(\epsilon_1)N(\epsilon_2)N(\epsilon_3)\times\int_0^{\infty}dR_1dR_2\int_{|R_1-R_2|}^{R_1+R_2}dR_3\frac{R_1R_2R_3\gamma_{12}^3}{D^2(1,2,3)}g^{||/\perp}(h)\times[a_{12}^3a_{21}^3R_3^2+(a_{12}^3-a_{21}^3)(R_1^2a_{12}^3-R_2^2a_{21}^3)]\boldsymbol{E}, \tag{22}$$

where

$$h=\frac{eH\sqrt{4R_1^2R_2^2-(R_1^2+R_2^2-R_3^2)^2}}{4\hbar c}=\frac{eHS}{\hbar c}, \tag{23}$$

the dimensionless magnetic field, is equal to the number of flux quanta penetrating the area S of the triangle. The functions $g^{||}(h)$ and $g^{\perp}(h)$ are different from each other. They are given by

$$g^{||}(h)=\frac{3}{h^2}\left(\frac{\sin(h)}{h}-\cos(h)\right)-1 \tag{24}$$

and

$$g^{\perp}(h)=\frac{1}{2h}\frac{d}{dh}(h^2g^{||}(h)). \tag{25}$$

Consequently, the longitudinal part $\delta\sigma^{||}(H)$ differs from the transverse part $\delta\sigma^{\perp}(H)$. Equation (25) entails that

$$\delta\sigma^{\perp}(H)=\frac{1}{2H}\frac{d}{dH}(H^2\delta\sigma^{||}(H)). \tag{26}$$

Equation (26) is applicable to magnetic fields of any strength. In the quadratic approximation with respect to the magnetic field, this equation results in the relationship

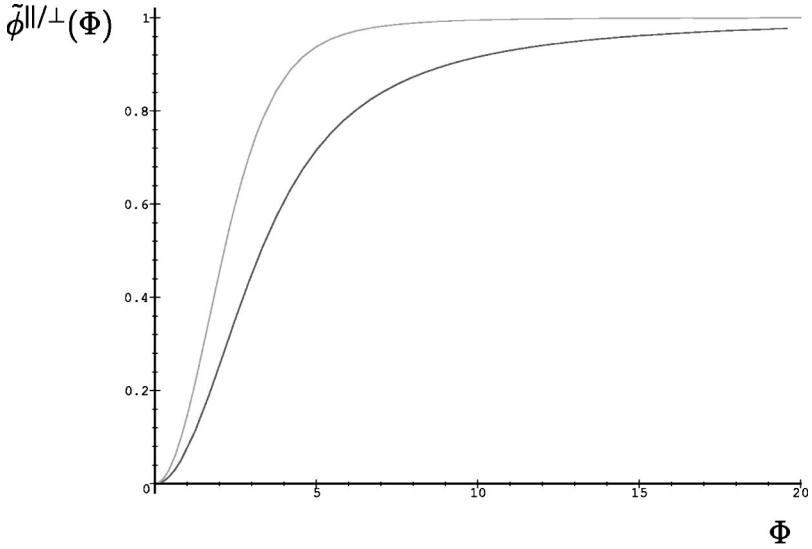

FIG. 1. Longitudinal part (thick line) and transverse part (thin line) of the magnetoconductivity versus Φ.

$$\delta\sigma^{\perp}(H)=2\,\delta\sigma^{||}(H)+o(H^2). \tag{27}$$

Since the transverse part of the magnetoconductivity is related to the parallel part by the relationship (27) we restrict the further investigation to the investigation of $\delta\sigma^{||}$.

V. FREQUENCY DEPENDENCE

While the angular integrations, which have lead to anisotropy, could be performed exactly, the integrations over the side lengths of the triangle and the site energies can only be performed approximately. To see where the most important contributions come from, we first investigate the integrand further. If all site energies were equal the integral would be zero, due to the symmetry of the integrand. Thus, in order to obtain a nonvanishing contribution the site energies should be different. Since the integrand is exponentially small with respect to the distance of the site energies of site 1 and site 2 on the Fermi energy, and the integrand is symmetric with respect to exchange of 1 and 2, we infer that the initial and the final site should be close to the Fermi energy. Since, furthermore, the energy of site 3 has to be different from that of the sites 1 and 2, it has to be situated outside the range of accessible energies for sites 1 and 2. This notion is verified by a closer look on the integrand. If ϵ_3 is outside the range of ϵ_1 and ϵ_2, then the only ϵ_3 dependence which remains is the ϵ_3 dependence of the preexponential factor in γ_{12}^3 Eq. (9). If we use furthermore the Fermi energy as the zero point for the energy axis, we find

$$\begin{aligned}\delta\sigma^{||/\perp}(s)=&-\frac{4\pi^2}{3}e^2\beta N_F^2\Lambda\int d\epsilon_1 d\epsilon_2\int_0^{\infty}dR_1dR_2\\&\times\int_{|R_1-R_2|}^{R_1+R_2}dR_3R_1R_2R_3^3g^{||/\perp}(h)\\&\times\frac{\Gamma_{12}^{(2)}}{\left(1+\frac{1}{s}\Gamma_{12}^{(2)}\frac{C_1+C_2}{C_1C_2}\right)^2}e^{\alpha R_3-\alpha R_1-\alpha R_2},\end{aligned} \tag{28}$$

where N_F is the density of states at the Fermi energy, and

$$\Lambda=2J_0\int_{-\infty}^{\infty}d\epsilon_3\frac{N(\epsilon_3)}{\epsilon_3}. \tag{29}$$

For large $\rho_c=\ln(\nu/s)$ the integrations yield (see Appendix B)

$$\delta\sigma^{||/\perp}=\frac{8\pi^2\ln 2}{9}e^2kT\Lambda\alpha^{-8}N_F^2s(\rho_c/2)^6\tilde{\phi}^{||/\perp}(\Phi), \tag{30}$$

where

$$\begin{aligned}\tilde{\phi}^{||}(\Phi)=&1-3{}_2F_2(1,2,3/2,5/2;-\Phi^2/4)\\&+2{}_2F_2(2,2,5/2,5/2;-\Phi^2/4),\end{aligned} \tag{31}$$

and

$$\tilde{\phi}^{\perp}(\Phi)=1-{}_2F_2(2,2,5/2,5/2;-\Phi^2/4). \tag{32}$$

Here ${}_2F_2$ is the hypergeometric function and

$$\Phi=\frac{eH}{8\hbar c\alpha^2}\rho_c^{3/2}. \tag{33}$$

The functions $\tilde{\phi}^{||}$ and $\tilde{\phi}^{\perp}$ are depicted in Fig. 1. From the figure we see that the magnetoconductivity is a quadratic function with respect to the magnetic field for small fields, a nearly linear function for moderate fields, and saturates for high fields. Furthermore, the transverse part of the magnetoconductivity is always larger than the longitudinal part of the magnetoconductivity.

If we calculate the conductivity in the same approximation we find (see Appendix C)

$$\sigma(s)=\frac{4\pi}{3}\frac{4\ln 2}{5}e^2N_F^2kT(2\alpha)^{-5}s\rho_c^5, \tag{34}$$

so that

$$\frac{\delta\sigma^{||}(H,s)}{\sigma(s)}\propto\alpha^{-3}\Lambda\rho_c\tilde{\phi}(\Phi). \tag{35}$$

Since both Φ and ρ_c depend only on s and H the ratio (35) is temperature independent. In the quadratic regime with respect to the electric field we have

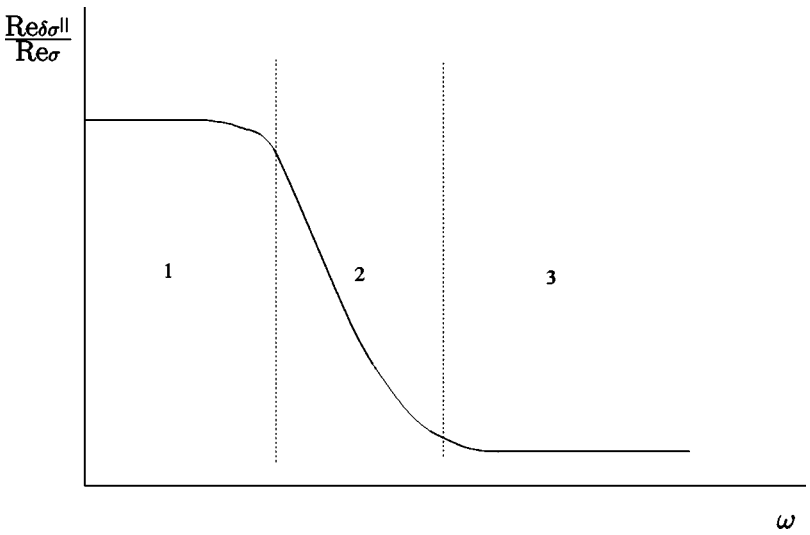

FIG. 2. Sketch of the frequency dependence of the magnetoconductivity. (1) lowest frequencies and multiple hoppping regime, (2) three-site model with $\omega/\nu \ll 1$, (3) three-site model with $\omega/\nu \gg 1$.

$$\tilde{\phi}^{||}(\Phi)=\frac{2}{25}\Phi^2+o(\Phi^2), \tag{36}$$

so that

$$\frac{\mathrm{Re}\delta\sigma^{||}(H,s)}{\mathrm{Re}\sigma(s)}=\frac{3}{3200}\pi\frac{e^2H^2\Lambda}{\hbar^2c^2\alpha^7}\ln^4\frac{\nu}{\omega}. \tag{37}$$

Consequently, the effect increases with decreasing frequency.

The equations derived so far are valid for $\nu/s \gg 1$. Consequently, they do not apply to the frequency range $\nu/s \ll 1$. For such high frequencies the denominator in Eq. (28) can be replaced by 1, so that the magnetoconductivity becomes independent of frequency. On the plateau we obtain

$$\delta\sigma^{||/\perp}=36\pi^2e^2kTN_F^2\Lambda\alpha^{-8}\nu I^{||/\perp}(\Phi), \tag{38}$$

where

$$I^{||/\perp}(\Phi)=-\frac{1}{288}\int_0^\infty dx_1dx_2\int_0^{x_1}dx_3(x_1+x_2-x_3)(x_2+x_3) \times x_1^3g^{||/\perp}[\Phi 2\sqrt{x_2x_3(x_1+x_2)(x_1-x_3)}]e^{-x_1-x_2}. \tag{39}$$

In the quadratic regime with respect to the magnetic field Eq. (39) leads to

$$\delta\sigma^{||}=\frac{1287}{40}\pi^2e^2kTN_F^2\Lambda\alpha^{-8}\nu\frac{e^2H^2}{\hbar^2c^2\alpha^4}+o(H^2). \tag{40}$$

The conductivity also passes into a plateau in this limit. For the conductivity we obtain (see Appendix C)

$$\sigma=3e^2N_F^2kT\alpha^{-5}\nu. \tag{41}$$

Consequently, in the quadratic regime with respect to the electric field we have

$$\frac{\mathrm{Re}\delta\sigma^{||}(\omega)}{\mathrm{Re}\sigma(\omega)}\bigg|_{\omega/\nu\ll 1}\Big/\frac{\delta\sigma^{||}(\omega)}{\sigma(\omega)}\bigg|_{\omega/\nu\gg 1}=\frac{1}{11440}\ln^4\left(\frac{\nu}{\omega}\right). \tag{42}$$

Again we observe, that the magnetoconductivity increases with decreasing frequency. In a further publication we show,[26] that $\mathrm{Re}\delta\sigma^{||}(\omega)/\mathrm{Re}\sigma(\omega)$ depends at most weakly on frequency at very low frequencies and in the multiple hopping regime, so that qualitatively the behavior sketched in Fig. 2 is obtained.

VI. CONCLUSIONS

In the paper we have calculated the magnetoconductivity of highly insulating samples in the high-frequency limit. To this end we have used the three-site model, which could be solved exactly. According to our calculation the magnetoconductivity in the strongly localized regime is anisotropic in the high-frequency limit. We have derived a simple relationship which relates the transverse part of the magnetoconductivity to the parallel part of the magnetoconductivity. It turns out that the magnetoconductivity for magnetic fields applied transverse to the electric field is always larger, as for magnetic fields applied parallel to the electric field. We do not expect that anisotropy is produced merely by increasing frequency, so that we expect that the magnetoconductivity in the strongly localized regime is also anisotropic. In that our results are in contradiction to those of the Refs. 1–8, which predict an isotropic magnetoconductivity for macroscopic samples. They do, however, compare to some extent with the experiments of Ref. 15. There $\delta\sigma^{\perp}/\delta\sigma^{||}=1.94$ was measured in three-dimensional GaAs samples in the quadratic regime with respect to the magnetic field. Our theory predicts $\delta\sigma^{\perp}/\delta\sigma^{||}=2$, also in the quadratic regime. However, since in the experiments the dc conductivity in the non-Ohmic regime was measured, these experiments cannot be taken as direct verification.

From our calculation we deduce that the reason for the absence of anisotropy in previous calculations is in the neglect of correlations between the direction of the electric field, the direction of the magnetic field and the surface normal of the surface spaned by the intermediate sites. In Refs. 1 and 7, in which the magnetoconductivity in the dc limit was investigated using percolation theory, and Refs. 2–5, in which the magnetoconductivity was investigated using effective methods, the focus was on the properties of the critical resistor. The critical resistor is an object $f(\boldsymbol{S},\boldsymbol{H})$ which de-

pends on the oriented area of the surface $\boldsymbol{S}$ and on the magnetic field $\boldsymbol{H}$. Since the calculation of the configuration average amounts to an integration over all directions of the surface normal, in all areas the result of the averaging procedure is isotropic. In our approach we first solve the rate equations, calculate the current and then perform the averaging. Before averaging, the current depends on the direction of the electric field, on the direction of the magnetic field, and on the direction of the surface normal of the triangle. Thus it is an object $\boldsymbol{j}(\boldsymbol{E},\boldsymbol{S},\boldsymbol{H})$ which is a linear function of $|\boldsymbol{E}|$, but depends on the angle between the electric field and the surface normal, the angle between the magnetic field and the surface normal, and the magnitudes of the vectors $\boldsymbol{S}$ and $\boldsymbol{H}$. In the averaging procedure the direction of the surface normal is integrated out, but in doing so it has to be taken into account that the expression is not only a function of the angle between $\boldsymbol{S}$ and $\boldsymbol{H}$, but also a function on the angle between $\boldsymbol{S}$ and $\boldsymbol{E}$. The latter angle is absent in those treatments which focus on the critical resistor only. If the angle is taken into account in the integrations the dependence on the angle between the electric and the magnetic field remains. Consequently, the consideration of the direction of the electric field is an important ingredient in the calculation of the magnetoresistance, in that the situation is entirely analogous to the Hall effect. Therein, averaging also the dependence on the direction between the electric field and the magnetic field has to be taken into account. Otherwise the Hall effect would be zero. The percolation theory was applied to the investigation of the Hall effect in Refs. 28 and 29.

It should, however, also be mentioned, that in most experiments isotropic magnetoresistance is observed. The reason for the occurence might be in that the samples used were close to the metal-insulator transition. The main assumption of our calculation is that the system is strongly localized, so that at most scattering at one intermediate site can be of importance. Consequently, our results are not expected to hold close to the metal-insulator transition, where scattering at many intermediate sites is relevant. This fact already indicates that the results are not universal, but depend on the degree of localization, and therefore on the sample in question.

From our results it follows that, as a function of the magnetic field, the magnetoconductivity at fixed real Laplace frequency s is a quadratic function with respect to the magnetic field for low fields, a linear function for moderate fields and saturates for high fields. As a function of frequency the magnetoconductivity decreases with increasing frequency at low frequencies, that is for frequencies $\omega \ll \nu$. In this regime a decrease of the frequency acts just like an increase of the magnetic-field strength. For $\omega \gg \nu$ the magnetoconductivity becomes independent of frequency, that is it passes into a plateau.

Finally, we note that our expression for the current also exhibits the *p-n* anomaly for symmetric densities of states first discussed in Ref. 5, that is the magnetoconductivity changes sign if electrons are replaced by holes. This anomaly is completely analogous to the *p-n* anomaly of the Hall effect, and results from the fact that the most important contributions to the integrations arise from scattering sites with site energies far from the Fermi energy, so that Λ changes sign if electrons are replaced by holes if the density of states is symmteric, and so does the magnetoconductivity.

APPENDIX A: ANGULAR INTEGRATIONS

Since the integrations are very lengthy we cannot discuss them entirely here, but describe only the general way. In order to perform the angular integrations we first insert the identity (20) into Eq. (18). Then the magnetoconductivity can be written in the form

$$\delta\sigma^{||/\perp} = \frac{e^2\beta s^4}{2}\int d\epsilon_1 d\epsilon_2 d\epsilon_3 N(\epsilon_1)N(\epsilon_2)N(\epsilon_3) \times \int_0^\infty dR_1 dR_2 \times \int_{|R_1-R_2|}^{R_1+R_2} dR_3 R_1 R_2 R_3 \times \frac{\gamma_{12}^3}{D^2(\rho_1,\rho_2,\rho_3)} K^{||/\perp}(\rho_1,\rho_2,\rho_3), \tag{A1}$$

where

$$K^{||/\perp}(\rho_1,\rho_2,\rho_3) = \int_0^{2\pi} d\phi_1 d\phi_2 \int_0^\pi d\theta_1 d\theta_2 \sin\theta_1 \sin\theta_2 \times \left[\cos\left(\frac{eHR_1R_2}{2\hbar c}\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)\right)-1\right] \times \delta\left(\frac{R_3^2-R_1^2-R_2^2}{2R_1R_2}+\cos\psi\right) Q^{||/\perp}, \tag{A2}$$

and

$$Q^{||} = (R_2\cos\theta_2 a_{21}^3 - R_1\cos\theta_1 a_{12}^3)^2, \tag{A3}$$

$$Q^{\perp} = (R_2\sin\theta_2\cos\phi_2 a_{21}^3 - R_1 a_{12}^3 \sin\theta_1\cos\phi_1)^2. \tag{A4}$$

Here $\rho_i = (R_i, \epsilon_i)$, $i = 1,2,3$.

First we consider $K^{||}$. If we perform the integrations over the angles ϕ_1 and ϕ_2 and use the properties of the δ function, we obtain

$$K^{||} = 4\pi\int_0^\pi d\theta_1 d\theta_2 \theta\left(1-\left|\frac{y-\cos\theta_1\cos\theta_2}{\sin\theta_1\sin\theta_2}\right|\right)\frac{\cos\left[\frac{eHR_1R_2}{2\hbar c}\sqrt{\sin^2\theta_1\sin^2\theta_2-(y-\cos\theta_1\cos\theta_2)^2}\right]-1}{\sqrt{\sin^2\theta_1\sin^2\theta_2-(y-\cos\theta_1\cos\theta_2)^2}} \times \sin\theta_1\sin\theta_2(R_2 a_{21}^3\cos\theta_2 - R_1 a_{12}^3\cos\theta_1)^2, \tag{A5}$$

where $\theta(x)$ is the unit step function, and $y = (R_1^2+R_2^2-R_3^2)/(2R_1R_2)$. The remaining integrations in the calculation of the function $K^{||}$ are elementary but lengthy and therefore not presented here in detail. We only would like to point out that in order

to perform these integrations it is convenient to use the identity $\sin(x+\pi/2)=\cos(x)$ to rewrite Eq. (A5) in such a way that the range of integrations is restricted to the interval $(0,\pi/2)$. Then all trigonometric functions in the integrand are either monotonous increasing or decreasing functions within the range of integrations. Thereafter, the integrations reduce to elementary integrals if the standard change of variables is used, that is if trigonometric functions are used as integration variables.

Now we focus on $K^{\perp}$. Since in this case $Q^{\perp}$ also depends on the azimutal angles, the integrations over the angles ϕ_1 and ϕ_2 cannot be performed as before. It turns out to be convenient to perform first the integrations over ϕ_2. If we use the periodicity of the trigonometric functions we obtain

$$K^{\perp}=\pi\int_0^{2\pi}d\phi_1\int_0^{\pi}d\theta_1 d\theta_2 \sin\theta_1\sin\theta_2\left[\cos\left(\frac{eHR_1R_2}{2\hbar c}\sin\theta_1\sin\theta_2\sin\phi_1\right)-1\right](R_2^2(a_{21}^3)^2\sin^2\theta_2+R_1^2(a_{12}^3)^2\sin^2\theta_1$$
$$-2R_1R_2a_{12}^3a_{21}^3\sin\theta_1\sin\theta_2\cos\phi_1)\times\delta(y-\cos\theta_1\cos\theta_2-\cos\phi_1\sin\theta_1\sin\theta_2). \tag{A6}$$

To proceed further we change the integration variables according to $w=\sin\phi_1$. Thereafter, the integration over w can again be performed easily and we obtain

$$K^{\perp}=-\frac{1}{2}K^{\parallel}+2\pi\int_0^{\pi}d\theta_1 d\theta_2\sin\theta_1\sin\theta_2\,\theta\left(1-\left|\frac{y-\cos\theta_1\cos\theta_2}{\sin\theta_1\sin\theta_2}\right|\right)\frac{\left[\cos\frac{eHR_1R_2}{2\hbar c}\sqrt{\sin^2\theta_1\sin^2\theta_2-(y-\cos\theta_1\cos\theta_2)^2}\right]-1}{\sqrt{\sin^2\theta_1\sin^2\theta_2-(y-\cos\theta_1\cos\theta_2)^2}}$$
$$\times[R_2^2(a_{21}^3)^2+R_1^2(a_{12}^3)^2-2R_1R_2ya_{12}^3a_{21}^3]. \tag{A7}$$

The remaining integrations can be performed as before. First the range of integrations is reduced to the interval $(0,\pi/2)$, using the identities of the type $\sin(x+\pi/2)=\cos(x)$. Thereafter, all trigonometric functions are either monotonous increasing or decreasing functions within the range of integrations, so that again the standard substitution can be applied, that is trigonometric functions are used as new intergration variables. The remaining integrations are elementary but lengthy and therefore not presented here in detail.

APPENDIX B: FURTHER INTEGRALS

In order to perform the integrations in Eq. (28) we first introduce dimensionless integration variables, according to $\alpha R_i\to R_i$, $\beta\epsilon_i\to\epsilon_i$. Furthermore, we use the Fermi energy as the zesro point for the energy axis. Then, Eq. (28) can be written in the form

$$\sigma^{\parallel}=\frac{4\pi^2}{3}e^2N_f^2\Lambda\alpha^{-8}kTsI, \tag{B1}$$

where $[\rho_c=\ln(\nu/s)]$

$$I=-\int_0^{\infty}dR_1dR_2\int_{|R_1-R_2|}^{R_1+R_2}dR_3R_1R_2R_3^3$$
$$\times g^{\parallel}[h'(R_1,R_2,R_3)]Q(R_3-\rho_c/2)e^{R_3-R_1-R_2}, \tag{B2}$$

with

$$h'(R_1,R_2,R_3)=\frac{eH}{4\hbar c\alpha^2}\sqrt{4R_1^2R_2^2-(R_1^2+R_2^2-R_3^2)^2}. \tag{B3}$$

The function Q is given by

$$Q(R_3-\rho_c/2)=\frac{1}{s}\int d\epsilon_1 d\epsilon_2\frac{\Gamma_{12}^{(2)}}{\left(1+\frac{1}{s}\Gamma_{12}^{(2)}\frac{C_1+C_2}{C_1C_2}\right)^2}. \tag{B4}$$

For the functions C_i we use the low-temperature approximation $C_i\approx\exp(-|\epsilon_i|)$. Since the application of this approximation requires $|\epsilon_i|\gg 1$, the application of this approximation is, strictly spoken, not valid if the main contributions to the integrations arise from small $|\epsilon_i|$. The consideration of this fact, however, leads only to an additional numerical factor which appears in the logarithm of the dimensionless critical hopping length ρ_c. Since in the region of interest ν/s is large this contribution is negligible.

If we use this approximation the integrations over the energies can be performed exactly. They lead to

$$Q(R_3-\rho_c/2)=4\left[\frac{1}{1+\exp(\rho_c-2R_3)}-\frac{1}{1+2\exp(\rho_c-2R_3)}\right]$$
$$+2\left[e^{2R_3-\rho_c}\ln(1+e^{\rho_c-2R_3})-e^{2R_3-\rho_c}\right.$$
$$\left.\times\ln\left(1+\frac{1}{1+\exp(2R_3-\rho_c)}\right)\right] \tag{B5}$$

The function $Q(R)$ has a maximum close to $R=0$ and a finite support. Now we use the identity

$$\int_0^{\infty}dR_1dR_2\int_{|R_1-R_2|}^{R_1+R_2}dR_3F(R_1,R_2,R_3)$$
$$=4\int_0^{\infty}dR_1dR_2dR_3F(R_1+R_2,R_2+R_3,R_3+R_1), \tag{B6}$$

valid for functions F symmetric with respect to the first two arguments, to cast Eq. (B2) into the form

$$I=\int_0^\infty dR_1 4Q(R_1-\rho_c/2)\phi(R_1), \tag{B7}$$

where

$$\phi(R_1)=-\int_0^\infty dR_2 e^{-2R_2}\int_0^{R_1} dR_3(R_1+R_2-R_3) \times(R_2+R_3)R_1^3 g^{||}[h'(R_1+R_2-R_3,R_2+R_3,R_1)]. \tag{B8}$$

The function $\phi(R)$ is a monotoneous increasing function, which increases like R^6 for $R\to\infty$ (see below). Consequently, we have

$$I\approx\phi(\rho_c/2)4\int_{-\infty}^\infty dRQ(R)=\phi(\rho_c/2)A, \tag{B9}$$

where $A=16\ln 2$. In Eq. (B8) the characteristic R_2 is of order 1. The characteristic R_3, however, is large. Therefore, R_2 can be neglected, when compared with $\rho_c/2$ or R_3. Doing so we obtain

$$\phi(\rho_c/2)=-(\rho_c/2)^3\int_0^\infty dR_2 e^{-2R_2}\int_0^{\rho_c/2} dR_3(\rho_c/2-R_3) \times R_3 g^{||}[h'(\rho_c/2+R_2+R_3,R_2+R_3,\rho_c/2)]. \tag{B10}$$

In the same approximation we find

$$h'(\rho_c/2+R_2+R_3,R_2+R_3,\rho_c/2) \approx\frac{eH}{\hbar c\alpha^2}\sqrt{(\rho_c/2)R_2R_3(\rho_c/2-R_3)}. \tag{B11}$$

To eliminate the root we introduce a new variable z, using the identity

$$1=2\int_0^\infty dz z\,\delta(z^2-R_2R_3(\rho_c/2-R_3)), \tag{B12}$$

and perform the integration over R_3. Then we obtain

$$\phi(\rho_c/2)=\frac{4}{96}(\rho_c/2)\tilde{\phi}^{||}(\Phi), \tag{B13}$$

where Φ is given by Eq. (33), and

$$\tilde{\phi}^{||}(\Phi)=-\frac{3}{4}\int_0^\infty dx\int_0^1 dy e^{-x}\frac{y g^{||}(\Phi\sqrt{xy})}{\sqrt{1-y}}. \tag{B14}$$

To proceed further we use the equality

$$g^{||}(u)=-3\frac{d}{du}\frac{\sin(u)}{u}-1 \tag{B15}$$

and perfom an integration with respect to parts. Then the remaining integrations can be performed with the formulas given in Ref. 27. They yield Eq. (31). The function $\tilde{\phi}^{\perp}$ is obtained from $\tilde{\phi}^{||}$ using Eq. (26).

APPENDIX C: TWO-SITE MODEL

To calculate the conductivity we use the two-site model. In this model the transport equations reduce to

$$sC_1[U_1+e(\boldsymbol{E}\boldsymbol{R}_1)]=\Gamma_{12}^{(2)}(U_2-U_1),$$
$$sC_2[U_2+e(\boldsymbol{E}\boldsymbol{R}_2)]=\Gamma_{12}^{(2)}(U_1-U_2). \tag{C1}$$

The configuration averaged current is calculated according to

$$\boldsymbol{j}(s)=\frac{\beta e s}{2\Omega}\int d\boldsymbol{R}_1 d\boldsymbol{R}_2 d\epsilon_1 d\epsilon_2 N(\epsilon_1)N(\epsilon_2) \times[\boldsymbol{R}_1C_1(U_1+e(\boldsymbol{E}\boldsymbol{R}_1))+\boldsymbol{R}_2C_2(U_2+e(\boldsymbol{E}\boldsymbol{R}_2))]. \tag{C2}$$

The factor 1/2 in front of the current again removes overcounted points. If we solve Eq. (C1), insert the solution into the expression for the current and perform the angular integrations we obtain

$$\sigma(s)=\frac{2\pi}{3}e^2N_F^2\beta\int_0^\infty dRR^4\int d\epsilon_1 d\epsilon_2\frac{\Gamma_{12}^{(2)}}{1+\frac{1}{s}\Gamma_{12}^{(2)}\frac{C_1+C_2}{C_1C_2}}. \tag{C3}$$

To perform the intergations we again use the low-temperature approximation for the functions C_i, that is we replace $C_i\approx\exp(-|\epsilon_i|/kT)$. Here the Fermi energy was used as the zero point for the energy axis. Again we point out that strictly speaking, the use of this approximation requires that the main contributions arise from large $|\epsilon_i|/kT$, which is in principle not the case here. The consideration of this fact leads, however, only to a change of the numerical factor in the logarithm of the characteristic hopping length,[24] which is negligible.

First we consider the situation $s/\nu\ll 1$. In the approximation chosen the integrations over the energies can be performed exactly. Doing so we obtain

$$\sigma(s)=\frac{2\pi}{3}kTe^2N_F^2(2\alpha)^{-5}sI(\nu/s), \tag{C4}$$

where

$$I(\nu/s)=\int_{-\rho_c}^\infty dx(x+\rho_c)^4 \times\left[8\ln\frac{1+2e^{-x}}{1+e^{-x}}+2e^x\ln\frac{1+1/(1+e^x)}{1+e^{-x}}\right], \tag{C5}$$

and $\rho_c=\ln(\nu/s)$. The second term in the bracket is a function with a sharp maximum close to $x=0$. The first term approaches $8\ln 2$ for $x\to-\infty$, and zero for $x\to\infty$. Consequently, the main contribution to the integration arises from the first term, from which we obtain

$$I(\nu/s)=\frac{8\ln 2}{5}\rho_c^5. \tag{C6}$$

If we combine Eqs. (C4) and (C6) we obtain Eq. (34).

For $s\gg\nu$ the denominator of Eq. (C3) can be replaced by 1. Then the integrations can readily be performed and yield immediately Eq. (41).

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