Third-order renormalization group applied to the attractive one-dimensional Fermi gas

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(Received 23 May 2000)

We consider a Callan-Symanzik and Wilson renormalization-group (RG) approach to the infrared problem for interacting fermions in one dimension with backscattering. We compute the third-order (two-loop) approximation of the β function using both methods, and compare it with the well-known multiplicative Gell–Mann Low approach. We point out a previously unnoticed strong instability of the third-order fixed point with respect to an arbitrary dimensionless parameter, which suggests a RG flow toward a strong-coupling phase.

I. INTRODUCTION

The problem of the one-dimensional Fermi-gas model of a metallic conductor, in the low-energy approximation, has been approached using three methods: conventional many-body techniques[1] and, mainly, bosonization[2–4] and renormalization-group (RG) methods.[5–10] In this paper we will be concerned with the latter approach. A formulation of the Gell–Mann Low multiplicative RG for this problem was introduced in Ref. 5. The model considered was the *g-ological model*, which describes a weakly interacting one-dimensional fermion system with Tomonaga-type (g_2, g_4) and backscattering (g_1) interactions. Phonons are neglected. That method provided a satisfactory understanding of the infrared behavior in the case of a weak *repulsive* (effective) interaction. A short list of the most relevant results in this case may be the following (for extensive reviews see, e.g., Refs. 6 and 11): (i) the RG flows toward the *Luttinger liquid*[12] fixed point;[13,14] (ii) there is a line of nontrivial fixed points; and (iii) in the infrared limit the system is not asymptotically free, as in the *Fermi-liquid* case, but is described by anomalous exponents. These results also were recovered and rigorously proved in the case of a periodic potential using the Wilson RG.[15]

Things change considerably if we consider a weak *attractive* interaction. Since in this case there is not a second-order (one-loop) finite fixed point, in Ref. 6 the computation of the beta function was carried to third order (two loops). The authors found a $O(1)$ third-order fixed point. This result, if reliable, would be of extreme physical interest because it would signal a behavior completely different from the Luttinger-liquid paradigm. One should expect the opening of a gap in the dispersion relations, while the Luttinger spectrum is gapless, and an exponential decay of the correlation functions, while in the Luttinger case there is only a power-law decay with increasing distance. The problem is, of course, how seriously one should take the very existence of an attractive perturbative fixed point on the basis of the third-order result. The computation of the fourth-order (three-loop) approximation of the β function was discussed in Refs. 16–18. A smaller but still $O(1)$ fixed point was found. Moreover only the first two terms of the β function are universal. The computation of the third term is useful provided there is some evidence of a perturbatively tractable phase

interacting attractively. In this case a precise determination of the renormalized couplings would be important in computing the response functions.

It is useful to make a comparison with the results obtained with the bosonization method. With *bosonization* it is meant the bosonic representation of fermion field operators.[19,20,4,3] This method is in some sense the inverse of the one used to solve exactly the Luttinger model,[14] where bosonic degrees of freedom are expressed in terms of fermionic operators. Probably the most important result of the bosonization is the exact solution of the model with backscattering[3] (g_1 and g_2 terms; see below) in the particular case where $g_1 = -\frac{6}{5}\pi$. Actually the decoupling between charge and spin degrees of freedom, crucial for an exact solution, is open to questions.[21] Moreover there are problems in the limiting procedure employed,[22] and the ladder operators restoring the correct occupation numbers[12] are not discussed. (A version of the bosonization free from this problems was proposed.[23] It should be noted that this version does not deal with the crucial backscattering interaction term: in Ref. 23 only the Luttinger model is considered.) Taking for granted the Luther-Emery solution,[3] the RG method should fill the missing information for value of g_1 in the neighborhood of the exact solution. From the bosonized representation of the interaction it is not difficult to derive the third-order scaling equations,[3,24] and the response functions calculated in Ref. 16 are in good agreement with the results of Ref. 3.

From these considerations one may be tempted to give a heuristic meaning to the large but finite fixed point. In this paper we want to show that this is not the case. The main point is that even the sign of the third-order fixed point depends on small variations of a parameter γ whose value can be arbitrarily chosen, provided $\gamma > 1$. We will show this both using the Gell–Mann Low (GML) and Wilson RG.

The paper is organized as follows. In Sec. II we briefly review the multiplicative GML approach. We explain why it is useful to check the results of this approach using other methods. Recasting the multiplicative procedure into discrete steps, instead of considering the usual Lie equation, we reach our main conclusion. In Sec. III we formulate a Callan-Symanzik (CS) approach to the problem, and compute the β function in the two-loop approximation. The same computation is proposed in Sec. IV, employing the Wilson RG in the

multiscale formulation.[25,9] Finally in Sec. V we come to the conclusions.

II. GELL–MANN LOW APPROACH

We briefly recall the GML multiplicative RG for one-dimensional interacting fermions. We will follow closely Refs. 5 and 6, with the only difference that we find it convenient to adopt a Euclidean formalism. We consider the g-ological model, defined as follows. The kinetic term is taken linear around the Fermi surface defined by the two points $-k_F$ and k_F,

$$H_0 = \sum_{k,\omega,\sigma} (\omega k - k_F) \psi^+_{k,\omega,\sigma} \psi^-_{k,\omega,\sigma},$$

where $\psi^\pm_{k,\omega,\sigma}$ are creation and annihilation operators for right-moving ($\omega = 1$) and left moving ($\omega = -1$) fermions with momentum k and spin σ ($\sigma = \pm 1/2$). We choose units such that $v_F = 1$ (v_F is the velocity at the Fermi surface). The ultraviolet stability is imposed by bandwidth cutoffs: the momenta are restricted to the intervals ($\omega k_F - k_{uv}, \omega k_F + k_{uv}$) for $\psi^\pm_{k,\omega,\sigma}$. We define $E_0 = 2 k_{uv}$. The interaction Hamiltonian is

$$H_{int} = \frac{1}{2L} \sum_{k,p,\omega,\sigma,\sigma'} (g_{1\parallel} \delta_{\sigma,\sigma'} + g_{1\perp} \delta_{\sigma,-\sigma'}) \psi^+_{k_1,\omega,\sigma}$$
$$\times \psi^+_{k_2,-\omega,\sigma'} \psi^-_{k_2 + 2k_F + p,\omega,\sigma'} \psi^-_{k_1,-2k_F - p,-\omega,\sigma}$$
$$+ \frac{1}{2L} \sum_{k,p,\omega,\sigma,\sigma'} (g_{2\parallel} \delta_{\sigma,\sigma'} + g_{2\perp} \delta_{\sigma,-\sigma'}) \psi^+_{k_1,\omega,\sigma}$$
$$\times \psi^+_{k_2,-\omega,\sigma'} \psi^-_{k_2 + p,-\omega,\sigma'} \psi^-_{k_1 - p,\omega,\sigma}$$
$$+ \frac{1}{2L} \sum_{k,p,\omega,\sigma,\sigma'} (g_{4\parallel} \delta_{\sigma,\sigma'} + g_{4\perp} \delta_{\sigma,-\sigma'})$$
$$\times (\psi^+_{k_1,\omega,\sigma} \psi^+_{k_2,\omega,\sigma'} \psi^-_{k_2 + p,\omega,\sigma'} \psi^-_{k_1 - p,\omega,\sigma}). \quad (1)$$

L is the length of the line. The umklapp interaction term (g_3) is neglected, since it is important only in the half–filled band case, which will be excluded. Since $g_{1\parallel} = -g_{2\parallel}$ it is always possible to take $g_{2\perp} = g_{2\parallel} = g_2$, reducing the independent couplings to $g_{1\parallel}$, $g_{1\perp}$, and g_2. For the sake of simplicity it is possible to neglect, g_4, at least as a first approximation; we know from the Mattis model[26] that g_4 does not change the essence of the problem.

In the Euclidean formalism the free propagator in momentum space is given by

$$G_\omega(k) = \frac{1}{-ik_0 + \omega k_1}, \quad (2)$$

where k_0 is the energy, k_1 the momentum (measured from the Fermi surface), $k = (k_0, k_1)$, and $\omega = 1(-1)$ for right (left) moving fermions. The renormalization procedure is a prescription that defines new couplings for a theory with a lowered UV cutoff E_0. In the limit $E_0 \to 0$ we obtain the renormalized couplings. If G^R_ω is the interacting propagator, the d function is defined by the relation

$$G^R_\omega(k) = d\left(\frac{k_1}{k_{uv}}, \frac{k_0}{E_0}\right) G_\omega(k).$$

The multiplicative constants z and z_i ($i = 1\parallel, 1\perp, 2$) that relate d and the adimensional vertex functions $\tilde{\Gamma}_i$ for different values of the cutoff are defined by

$$d\left(\frac{k_1}{k'_{uv}}, \frac{k_0}{E'_0}, g'\right) = z\left(\frac{E'_0}{E_0}, g\right) d\left(\frac{k_1}{k_{uv}}, \frac{k_0}{E_0}, g\right)$$

$$\tilde{\Gamma}_i\left(\frac{k_j}{k'_{uv}}, \frac{w_j}{E'_0}, g'\right) = z_i^{-1}\left(\frac{E'_0}{E_0}, g\right) \tilde{\Gamma}_i\left(\frac{k_j}{k_{uv}}, \frac{w_j}{E_0}, g\right)$$

$$g'_i = g_i z^{-2}\left(\frac{E'_0}{E_0}, g\right) z_i\left(\frac{E'_0}{E_0}, g\right), \quad (3)$$

where $E'_0 < E_0$ is the lowered cutoff, and g and g' denote respective the old and new couplings, respectively. The invariant couplings g^R_i are defined by

$$g^R_i\left(\frac{E}{E_0}, g\right) = g_i z^{-2}\left(\frac{E}{E_0}, g\right) z_i\left(\frac{E}{E_0}, g\right).$$

g^R_i are invariant in the sense that

$$g^R_i\left(\frac{E}{E'_0}, g'\right) = g^R_i\left(\frac{E}{E_0}, g\right). \quad (4)$$

The couplings g'_i for the theory with UV cutoff E'_0 are defined by

$$g'_i = g^R_i\left(\frac{E'_0}{E_0}, g\right). \quad (5)$$

A differential equation for g^R_i is readily derived, and is the standard Lie equation,

$$\frac{d}{dx} g^R_i(x,g) = \frac{1}{x} \frac{d}{d\xi} g^R_i[\xi, g^R(x,g)]_{\xi=1},$$

where $x = E'_0/E_0$. We are interested in the scaling limit $x \to 0$. The two-loop results are

$$\frac{dg^R_{1\parallel}}{dx} = \frac{1}{x}\left[\frac{1}{\pi} g^{R2}_{1\perp} + \frac{1}{2\pi^2} g^R_{1\parallel} g^{R2}_{1\perp}\right],$$

$$\frac{dg^R_{1\perp}}{dx} = \frac{1}{x}\left[\frac{1}{\pi} g^R_{1\parallel} g^R_{1\perp} + \frac{1}{4\pi^2} (g^{R2}_{1\parallel} g^R_{1\perp} + g^{R3}_{1\perp})\right], \quad (6)$$

$$\frac{dg^R_2}{dx} = \frac{1}{x}\left[\frac{1}{2\pi} g^{R2}_{1\perp} + \frac{1}{4\pi^2} g^R_{1\parallel} g^{R2}_{1\perp}\right].$$

For spin independent interaction ($g_{1\parallel} = g_{1\perp} = g_1$) the nontrivial fixed point is found for $g^\star_1 = -2\pi$.

We now want to recover this result iterating by discrete steps the procedure that defines the new couplings when the cutoff is lowered: we aim to study the dependence on the scaling parameter. Let $\gamma > 1$. In proper units we put $E_0 = \gamma^0$ and $g_{i,0} = g_i(E_0)$ for $i = 1\parallel, 1\perp, 2$. $g_{i,-1}$ is defined as [see Eq. (5)]

$$g_{i,-1} = g_i^R \left(\frac{\gamma^{-1}}{\gamma^0}, g_{j,0} \right) = g_i^R \left(\frac{\gamma^{-1}}{E_0}, g \right),$$

where $g_{(0)} = g(E_0) = g$. The procedure is iterated in the following way: for $n < 0$, we define

$$g_{i,n-1} = g_i^R \left(\frac{\gamma^{n-1}}{\gamma^n}, g_{j,n} \right), \quad i = 1\|, 1\perp, 2.$$

From Eqs. (4) and (5) we have that $g_{i,n}$ for $n = -1$, $-2 \ldots$ are the couplings corresponding to the cutoff sequence $\{\gamma^n\}$:

$$g_i^R \left(\frac{\gamma^{n-1}}{\gamma^n}, g_{j,n} \right) = g_i^R \left(\frac{\gamma^{n-1}}{\gamma^0}, g_{j,0} \right) = g_i(\gamma^{n-1}).$$

In the limit $n \to -\infty$, we obtain the renormalized couplings. We have

$$g_{1\|,n-1} = g_{1\|,n} - \frac{g_{1\perp,n}^2}{\pi} \ln \gamma + \frac{1}{\pi^2} g_{1\|,n} g_{1\perp,n}^2 \left(\ln^2 \gamma - \frac{1}{2} \ln \gamma \right)$$

$$g_{1\perp,n-1} = g_{1\perp,n} - \frac{1}{\pi} g_{1\perp,n} g_{1\|,n} \ln \gamma$$

$$+ \frac{1}{2\pi^2} (g_{1\|,n}^2 g_{1\perp,n} + g_{1\perp,n}^3) \left(\ln^2 \gamma - \frac{1}{2} \ln \gamma \right)$$

$$g_{2,n-1} = g_{2,n} - \frac{1}{2\pi} g_{1\perp,n}^2 \ln \gamma$$

$$+ \frac{1}{2\pi^2} g_{1\|,n} g_{1\perp,n}^2 \left(\ln^2 \gamma - \frac{1}{2} \ln \gamma \right). \quad (7)$$

It is easily checked that in the limit $\gamma \to 1^+$ Eqs. (6) are recovered. In general the fixed point depends on γ (let us remember that it is a third-order fixed point). For $\gamma \neq \sqrt{e}$, if $g_{1\|} = g_{1\perp} = g_1$, we have

$$g_1^\star = \frac{\pi}{(\ln \gamma - 1/2)}. \quad (8)$$

When $\gamma \to 1$ we obtain the previous result $g_1^\star = -2\pi$. Lie equations (6) should not be fundamental, and we find no reason to use the continuous RG instead of its discrete version. The dependence on γ will be discussed in Sec. V.

As a final comment on this method, we note that Eqs. (3) rely on neglecting small contributions that would not allow one to set multiplicative relations where the z factors do not depend on the external momenta. For example, in the case of the one-loop approximation of the four-point vertex function, proportional to

$$- \frac{1}{2\pi} \ln \left(\frac{k_0}{E_0} \right) + \frac{1}{4\pi} \ln \left(1 + \frac{k_0^2}{E_0^2} \right)$$

(k_0 is the external energy; see Fig. 1), the second term is neglected. This is to say that the vertex functions can be divided into scaling terms and ''not scaling'' terms. The first
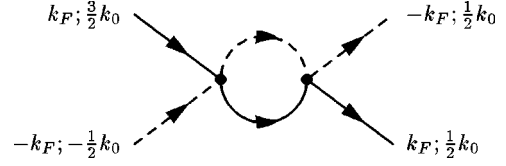

FIG. 1. One-loop diagram contributing to the four-point vertex function. The full (dashed) lines represent right (left) moving fermions. The value of the graph is proportional to $-(1/2\pi)\ln(k_0/E_0) + (1/4\pi)\ln[1 + (k_0^2/E_0^2)]$.

were taken into account while the second were not discussed in Refs. 5 and 6. For this reason we find useful to check Eqs. (6) using other methods.

III. CALLAN-SYMANZIK APPROACH

Within the framework of the multiplicative RG, it is not difficult to formulate a Callan-Symanzik approach for our problem. We follow a common procedure: first we renormalize the theory in the UV with a fixed (renormalized) IR cutoff m; then we compute the beta function and study the IR behavior for $m \to 0$. This approach, devised for a field theory, in our case may be considered unnecessary. Nevertheless we consider it a way to support the GML result.

The IR regularized free propagator is defined by inserting a bare mass m_0 in propagator (2),

$$G_\omega(k, m_0^2) = \frac{ik_0 + \omega k_1}{k^2 + m_0^2},$$

where $k^2 = k_0^2 + k_1^2$ and again $\omega = 1(-1)$ for right (left) moving fermions. We know that the Luttinger model with a local interaction is not renormalizable in the UV[27] (this is also seen from the exact solution[14]). In order to impose the UV stability we choose a nonlocal interaction whose strength decreases with increasing distance. The interaction Hamiltonian of the model is

$$H_{\text{int}} = \sum_{\omega, \sigma = \sigma'} \int d^2x \, d^2y \, \psi_{x,\omega,\sigma}^+ \psi_{y,-\omega,\sigma'}^+ V_{1\|}(x-y)$$

$$\times \psi_{y,\omega,\sigma'}^- \psi_{x,-\omega,\sigma}^- + \sum_{\omega, \sigma \neq \sigma'} \int d^2x \, d^2y \, \psi_{x,\omega,\sigma}^+ \psi_{y,-\omega,\sigma'}^+$$

$$\times V_{1\perp}(x-y) \psi_{y,\omega,\sigma'}^- \psi_{x,-\omega,\sigma}^-$$

$$+ \sum_{\omega, \sigma, \sigma'} \int d^2x \, d^2y \, \psi_{x,\omega,\sigma}^+ \psi_{y,-\omega,\sigma'}^+ V_2(x-y)$$

$$\times \psi_{y,-\omega,\sigma'}^- \psi_{x,\omega,\sigma}^- + \sum_{\omega, \sigma \neq \sigma'} \int d^2x \, d^2y$$

$$\times \psi_{x,\omega,\sigma}^+ \psi_{y,\omega,\sigma'}^+ V_4(x-y) \psi_{y,\omega,\sigma'}^- \psi_{x,\omega,\sigma}^-,$$

where $\psi_{x,\omega,\sigma}^\pm$ are the fermion field operators in coordinate space. The potentials V_i may be chosen, for instance, as

$$V_i(x) = \frac{g_i}{4} p \, e^{-p|x_1|} \delta(x_0), \quad i = 1\|, 1\perp, 2, 4, \quad (9)$$

where $p>0$ is fixed; x_0 and x_1 are the time and space coordinates. In momentum space the model is the same as Eq. (1), with the only difference that the bandwidth cutoffs are replaced by the nonlocal couplings

$$g_i \rightarrow g_i \frac{p^2}{k_1^2+p^2},$$

where k_1 is the exchanged momentum in the given interaction vertex and p is introduced in Eq. (9). In the limit $p \rightarrow \infty$ we recover the local couplings of Eq. (1).

Fermion loops are logarithmically divergent. The theory is regularized introducing a cutoff Λ by the means of the standard Schwinger parametrization:

$$\frac{1}{k^2+m_0^2} = \int_0^\infty d\alpha\, e^{-\alpha(k^2+m_0^2)} \rightarrow \int_{\Lambda^{-2}}^\infty d\alpha\, e^{-\alpha(k^2+m_0^2)}.$$

In order to renormalize the theory we find it convenient to follow the scheme for the local ($p=\infty$) case, even if when p is finite we make more subtractions than strictly necessary. It is a simple exercise of standard power counting to find the superficial degree of divergence D for the n-point vertex functions in the local case:

$$D(\Gamma_n)=2-\frac{n}{2}.$$

We renormalize the couplings ($g_i \rightarrow g_i^R$), the mass ($m_0 \rightarrow m$) and the wave function ($\psi^\pm \rightarrow \psi^{\pm R}$). The multiplicative constant Z is formally introduced by the relation $\psi^\pm = Z^{1/2}\psi^{\pm R}$. Let Γ_n^R be the renormalized proper n-point vertex functions. The relation between bare and renormalized vertex functions is

$$\Gamma_n(\mathbf{q},m_0,g,\Lambda)=Z^{-n/2}\Gamma_n^R(\mathbf{q},m,g^R), \qquad (10)$$

where $\mathbf{q}$ denotes the $n-1$ independent external momenta of Γ_n, and $g=\{g_{1\|},g_{1\perp},g_2,g_4\}$. Of course Γ_n are functions of the spin and ω indices attached to the external fields. For simplicity we have not indicated this explicitly in Eq. (10). For the four-point functions the different possible cases are labeled by a single index $i=1\|,1\perp,2,4$. It should be noted that while $\Gamma_{4,i}$ do not depend on the value of ω, Γ_2 does, so we need an ω label for this vertex function. It proves useful to introduce the reduced two-point vertex function $\hat{\Gamma}_2(k)$:

$$\hat{\Gamma}_2(k)=(ik_0+\omega k_1)\Gamma_{2,\omega}(k). \qquad (11)$$

In the local case $\hat{\Gamma}_2(k)$ does not depend on ω. In the nonlocal case the nonvanishing terms in the infrared limit will be ω independent, so we will neglect the ω dependence of $\hat{\Gamma}_2(k)$. The normalization conditions, which define g_i^R, m, and the finite part (zero-loop term) of Z, are

$$\Gamma_{4,i}^R(\mathbf{0})=g_i^R, \qquad (12a)$$

$$\hat{\Gamma}_2^R(0)=m^2, \qquad (12b)$$

$$\frac{1}{2k_0}\frac{\partial}{\partial k_0}\hat{\Gamma}_2^R(k)\bigg|_{k=(0,0)}=1. \qquad (12c)$$

The CS equations are derived considering insertions of operators related to the derivatives of the vertex functions with respect to the IR cutoff m_0. To this end we introduce the operator O:

$$O(z)=\sum_{\omega,\sigma}\int\frac{d^2x}{2\pi}\frac{\psi_{x,\omega,\sigma}^+\psi_{z,\omega,\sigma}^-}{x_0-z_0-i\omega(x_1-z_1)}.$$

The corresponding source term in the action has the form $\int d^2x\, v(x)O(x)$ with $[v]=2$. In momentum space the operator O is

$$\tilde{O}(q)=\sum_{\omega,\sigma}\int\frac{d^2k}{(2\pi)^2}\frac{\psi_{k+q,\omega,\sigma}^+\psi_{k,\omega,\sigma}^-}{[i(k_0+q_0)+\omega(k_1+q_1)]},$$

where $\psi_{k,\omega,\sigma}$ are the field operators in momentum space, and q is the external momentum of the inserted $\tilde{O}$ operators. When O is inserted into a vertex function Γ_n, the value of D for $\Gamma_{n,O}$ is

$$D(\Gamma_{n,O})=2-\frac{n}{2}+([O]-2)=-\frac{n}{2}.$$

The previous relation means that no new UV divergences appear due to O insertions (we do not consider $\Gamma_{0,O}$. We remember that the vertex functions with insertions are defined as usual by the Legendre transformation on the field source only and not on the source of the inserted operators). Since O does not introduce new divergences, we have $O=ZO^R$. The insertion of s operator O in a vertex function Γ_n will be denoted with $\Gamma_{n,s}$. In analogy with Eq. (11) we define $\hat{\Gamma}_{2,s}(q,\mathbf{k})$ and $\hat{\Gamma}_{2,s}^R(q,\mathbf{k})$, where $\mathbf{k}$ denotes the s external momenta of the s inserted $\tilde{O}$ operators. Equation (10) generalizes into

$$\Gamma_{n,s}(\mathbf{q},\mathbf{k},m_0,g,\Lambda)=Z^{-n/2}Z^s\Gamma_{n,s}^R(\mathbf{q},\mathbf{k},m,g^R). \qquad (13)$$

It is easily deduced that

$$\frac{\partial}{\partial m_0^2}\Gamma_n(\mathbf{q})=\Gamma_{n,1}(\mathbf{q},0), \qquad (14)$$

where the insertion of $\tilde{O}$ on the right-hand side is made at zero momentum, as indicated. From Eqs. (13), (14), and (11), we have

$$\hat{\Gamma}_{2,1}^R(0)=1,$$

$$m^2=Zm_0^2, \qquad (15)$$

where the second relation follows from $\hat{\Gamma}_2^R(0)=Z\hat{\Gamma}_2(0)$. From Eq. (14) we have

$$m\frac{\partial}{\partial m}\Gamma_n(\mathbf{q},m_0,g,\Lambda)\bigg|_{g,\Lambda}=m\frac{\partial m_0^2}{\partial m}\bigg|_{g,\Lambda}\Gamma_{n,1}(\mathbf{q},0,m_0,g,\Lambda). \qquad (16)$$

From Eqs. (16) and (13), with the definitions

$$\gamma_1=\frac{1}{Z}m\frac{\partial Z}{\partial m}\bigg|_{g,\Lambda},$$

$$\beta_i = m \left. \frac{\partial g_i^R}{\partial m} \right|_{g,\Lambda}$$

(γ_1, defined in the previous equations should not be confused with the RG rescaling factor γ), we obtain

$$\left(m \frac{\partial}{\partial m} + \sum_i \beta_i(g^R) \frac{\partial}{\partial g_i^R} - \frac{n}{2} \gamma_1 \right) \Gamma_n^R(\mathbf{q},m,g^R)$$

$$= Z \left(m \left. \frac{\partial m_0^2}{\partial m} \right|_{g_0,\Lambda} \right) \Gamma_{n,1}^R(\mathbf{q},0,m,g^R). \qquad (17)$$

It is easy to eliminate any reference to the bare theory. From Eq. (17), written for $n=2$, we have

$$\left(m \frac{\partial}{\partial m} + \sum_i \beta_i(g^R) \frac{\partial}{\partial g_i^R} - \gamma_1 \right) \hat{\Gamma}_2^R(q,m,g^R)$$

$$= Z \left(m \left. \frac{\partial m_0^2}{\partial m} \right|_{g_0,\Lambda} \right) \hat{\Gamma}_{2,1}^R(q,0,m,g^R).$$

From Eqs. (12b) and the first of Eqs. (15), we conclude

$$Z \left(m \left. \frac{\partial m_0^2}{\partial m} \right|_{g_0,\Lambda} \right) = (2-\gamma_1)m^2,$$

so that Eq. (17) can be written as

$$\left(m \frac{\partial}{\partial m} + \sum_i \beta_i(g^R) \frac{\partial}{\partial g_i^R} - \frac{n}{2} \gamma_1 \right) \Gamma_n^R(\mathbf{q},m,g^R)$$

$$= (2-\gamma_1)m^2 \Gamma_{n,1}^R(\mathbf{q},0,m,g^R).$$

The generalization of Eq. (17) to the case of s insertions is immediate:

$$\left[m \frac{\partial}{\partial m} + \sum_i \beta_i(g^R) \frac{\partial}{\partial g_i^R} + \left(-\frac{n}{2}+s \right) \gamma_1 \right] \Gamma_{n,s}^R(\mathbf{q},\mathbf{k},m,g^R)$$

$$= (2-\gamma_1)m^2 \Gamma_{n,s+1}^R(\mathbf{q},\mathbf{k},0,m,g^R).$$

Having set the general definitions and relations of the CS approach we can proceed. We will limit ourselves to the computation of the β function, which is our problem. The normalization conditions (12) fix the zero-loop terms in the loopwise expansion of g_i, m_0, and Z:

$$g_i^R = g_i^{(0)},$$

$$m^2 = m_0^{2(0)},$$

$$Z^{(0)} = 1.$$

One-loop calculations are easily done. Of course $m_0^{2(1)}=0$ and $Z^{(1)}=0$. This implies that up to one loop $\gamma_1=0$ (that is, $\gamma_1 = O(g^2)$). It is convenient to write the results for the couplings in terms of g_4, $g_{1\perp}$, g_2 and $\tilde{g} \equiv g_2 - g_{1\parallel}$. We find

$$g_{1\perp}^{(1)} = -\frac{1}{\pi} [\ln m - (\ln n 2 + \ln p - 1)] g_{1\perp}^R g_2^R$$

$$+ \frac{1}{\pi} \left[\ln m - \left(\ln 2 + \ln p - \frac{1}{2} \right) \right] g_{1\perp}^R \tilde{g}^R - \frac{1}{2\pi} g_{1\perp}^R g_4^R,$$

$$g_2^{(1)} = -\frac{1}{2\pi} [\ln m - (\ln 2 + \ln p - 1)] g_{1\perp}^{R2}$$

$$\tilde{g}^{(1)} = \frac{1}{2\pi} \left[\ln m - \ln \Lambda + \frac{1}{2}(1 + C + \ln 2) \right] g_{1\perp}^{R2},$$

$$g_4^{(1)} = -\frac{1}{4\pi} g_{1\perp}^{R2}, \qquad (18)$$

where C is the Euler constant. In Eq. (18) we note the presence of p. However simply on the basis of dimensional analysis we can exclude that p will appear in the final result. Two-loop calculations are tedious, and we will omit the details. We calculate only the singular terms in m since we do not plan to go beyond the two-loop approximation. We report the results for Z and $g_{1\perp}$:

$$Z^{(2)} = \frac{1}{2^3 \pi^2} \ln m (g_{1\perp}^{R2} + g_{1\parallel}^{R2} + 2g_2^{R2} - 2g_{1\parallel}^R g_2^R), \qquad (19)$$

$$g_{1\perp}^{(2)} = \frac{1}{2\pi^2} [\ln^2 m - 2\ln m (\ln 2 + \ln p - 1)] g_{1\perp}^R g_2^{R2}$$

$$+ \left\{ \frac{1}{2\pi^2} \ln m - \frac{1}{\pi^2} \left[\ln^2 m - 2\ln m \left(\ln 2 + \ln p - \frac{3}{4} \right) \right] \right\}$$

$$\times g_{1\perp}^R g_2^R \tilde{g}^R + \frac{1}{2\pi^2} \ln m \, g_{1\perp}^R g_2^R g_4^R - \frac{1}{2\pi^2} \ln m \, g_{1\perp}^R \tilde{g}^R g_4^R$$

$$+ \frac{1}{2\pi^2} \left[\ln^2 m - 2\ln m \left(\ln 2 + \ln p - \frac{3}{4} \right) \right] g_{1\perp}^{R3}$$

$$+ \frac{1}{2\pi^2} \left[\ln^2 m - 2\ln m \left(\ln 2 + \ln p - \frac{1}{2} \right) \right] g_{1\perp}^R \tilde{g}^{R2}$$

$$- \frac{1}{2^2 \pi^2} \ln m (g_{1\perp}^{R2} + g_{1\parallel}^{R2} + 2g_2^{R2} - 2g_{1\parallel}^R g_2^R). \qquad (20)$$

From Eqs. (20) and (18) we derive $\beta_{1\perp}^{(2)}(g^R)$. The final result is

$$\beta_{1\perp}(g^R) = \frac{1}{\pi} g_{1\parallel}^R g_{1\perp}^R + \frac{1}{4\pi^2} (g_{1\parallel}^{R2} g_{1\perp}^R + g_{1\perp}^{R3}) + O(g^4). \qquad (21)$$

It can be noted that p does not appear in Eq. (21), as expected. A crucial use in deriving Eq. (21) is made of Eq. (19), which is responsible for the cancellations expected from the exact solutions of the Luttinger and Mattis models,[13,14,26] and for the anomalous behavior of the theory. Of course the anomalous exponent $\eta = \gamma_1(g^\star)$ derived from Eq. (19) when $0 < g_1 \ll 1$ is in agreement with the exact solution of the Luttinger model, where $g_1 = 0$ and $g_4 = 0$.

Equation (21) is the same as Eq. (6) for the $g_{1\perp}$ coupling. For g_2 and $g_{1\parallel}$ the same conclusion holds: the β function of the GML method is recovered. The present CS approach, admittedly too involved, has perhaps the only value in that no use is made of approximate multiplicative relations.

IV. WILSON APPROACH

The multiscale formulation of the Wilson RG (Ref. 25) is particularly well suited to study the running of the coupling constants by discrete steps. The application of this method to interacting one-dimensional fermionic systems started with Refs. 8 and 10, and was thoroughly developed and applied to various problems.[28–31] Here we give a short and simplified account of the method, and refer to the cited papers for the details.

In the coordinate space the free propagator $G_\omega(x)$ for ω particles (again $\omega = \pm 1$ and $v_F = 1$) is

$$G_\omega(x) = \frac{1}{(2\pi)^2} \int dk_0\, dk_1 \frac{e^{-i(k_0 x_0 + k_1 x_1)}}{-ik_0 + \omega k_1}.$$

Actually it is not necessary to start with a kinetic term linearized around the Fermi surface: the RG can deal with realistic quadratic dispersion relations.[8] This simplification is, however, inessential for our purposes. Let p be an arbitrary momentum scale which, for instance, may be chosen equal to the inverse of the range of the potential. The propagator is decomposed in the sum

$$G_\omega(x) = \sum_{h=-\infty}^{1} G_\omega^{(h)}(x),$$

with

$$G_\omega^{(1)}(x) = \frac{1}{(2\pi)^2} \int dk\, e^{-ikx} \frac{1 - e^{-p^{-2}(k_0^2 + k_1^2)}}{-ik_0 + \omega k_1},$$

$$G_\omega^{(h)}(x) = \frac{1}{(2\pi)^2} \int dk \frac{e^{-ikx}}{-ik_0 + \omega k_1}$$
$$\times [e^{-p^{-2}\gamma^{-2h}(k_0^2 + k_1^2)} - e^{-p^{-2}\gamma^{-2h+2}(k_0^2 + k_1^2)}],$$
(22)

where $h \le 0$, $\gamma > 1$ and $kx = k_0 x_0 + k_1 x_1$. This decomposition divides the UV from the IR singularity of the propagator: $G_\omega^{(1)}$ is singular in the UV while $\sum_{h=-\infty}^{0} G_\omega^{(h)} = G_\omega^{IR}$ is singular in the IR. It is important to note that $G_\omega^{(h)}(x)$, the propagator on scale h, for $h \le 0$ has an UV and an IR cutoff: $G_\omega^{(h)}(x)$ is essentially different from 0 only for $x \sim \gamma^{-h}$ ($k \sim \gamma^h$ in momentum space).

One imagines that this decomposition stems from a similar decomposition of the fields,

$$\psi_{x,\omega,\sigma}^{\pm} = \sum_{h=-\infty}^{1} \psi_{x,\omega,\sigma}^{\pm(h)},$$

such that the pairings in the Grassmannian Wick rule are

$$\int P(d\psi_\omega^{(h)}) \psi_{x,\omega,\sigma}^{+(h)} \psi_{y,\omega',\sigma'}^{-(h')} \equiv \langle \psi_{x,\omega,\sigma}^{+(h)} \psi_{y,\omega',\sigma'}^{-(h')} \rangle$$
$$\equiv \delta_{\omega,\omega'} \delta_{\sigma,\sigma'} \delta_{h,h'} G_\omega^h(x-y).$$

We are interested in studying the IR effective potential $V^{(0)}$ arising from the integration of the UV component $\psi_\omega^{(1)}$ from the effective potential $V_{\text{eff}}(\varphi)$ defined by

$$e^{-V_{\text{eff}}(\varphi)} = \frac{1}{\mathcal{N}} \int P(d\psi) e^{-V(\psi + \varphi)},$$

where $\mathcal{N}$ is a normalization constant and V is the interaction potential. The ultraviolet integration was actually performed for the spinless model.[10] In the following we start directly with $V^{(0)}$.

The core of this method consists of a procedure that, integrating out fields from higher to lower *scales* $h (h \to -\infty)$, constructs a well-defined dynamical system of running coupling constants g_h, whose iteration map is the β *functional*. The operators $\mathcal{L}$ and $\mathcal{R} = 1 - \mathcal{L}$ are introduced. $\mathcal{R}$ is the usual renormalization operator of the BPHZ scheme: its action on a given vertex Γ, in momentum space for instance, is given by $\mathcal{R}(\Gamma) = \Gamma - t^\Gamma(\Gamma)$, where t^Γ denotes the Taylor series with respect to the external momenta of Γ up to order $D(\Gamma)$, if $D(\Gamma)$ is the Γ superficial degree of divergence. Of course $\mathcal{L}(\Gamma) = t^\Gamma(\Gamma)$.

The couplings g_h on a given scale h are defined by an inductive scheme. Let us assume we have constructed the effective potential $V^{(h)}(\psi_\omega^{(\le h)}, g_{h+1}, \ldots, g_0)$ on scale h, where $\psi_\omega^{(\le h)} = \sum_{n \le h} \psi_\omega^{(n)}$ and $g_{h+1}, \ldots, g_0$ are the previously defined couplings on scales $h+1, \ldots, 0$. We define

$$\bar{V}^{(h)}(\psi_\omega^{(\le h)}, g_h) \equiv \mathcal{L} V^{(h)}(\psi_\omega^{(\le h)}, g_{h+1}, \ldots, g_0).$$

The previous relation introduces the g_h and relates them to the $g_{h+1}, \ldots, g_0$ through the beta functional B_h: $g_h = g_{h+1} + B_h(g_{h+1}, \ldots, g_0)$. The effective potential $V^{(h-1)}$ on scale $h-1$ is defined by

$$e^{-V^{(h-1)}(\psi^{(\le h-1)})}$$
$$\equiv \frac{1}{\mathcal{N}'} \int P(d\psi^{(h)}) e^{-\mathcal{L} V^{(h)}(\psi^{(\le h)}) - \mathcal{R} V^{(h)}(\psi^{(\le h)})}. \quad (23)$$

Of course $V^{(h-1)} = V^{(h-1)}(\psi_\omega^{(\le h-1)}, g_h, \ldots, g_0)$. The procedure is then iterated. The starting point is given by the couplings g_0 of $\mathcal{L} V^{(0)}$. The final goal is to find a region in the space of parameters g_0 where each initial value generates a trajectory $g_h = g_{h+1} + B_h(g_{h+1}, \ldots, g_0)$ such that the Schwinger functions are analytical in g_h.

Unfortunately this scheme in our problem requires emendation. From the second-order result it becomes clear that α_h and ζ_h grow too quickly independently on the initial conditions. The point is that we know that the interacting propagator has an anomalous behavior: asymptotically for large distances it decays faster than the free propagator. The wave-function renormalization necessary to cure this problem is accomplished by an inductive procedure that redefines step by step the free measure of the functional integral and the couplings by the means of a sequence of parameters Z_h with $h = 0, -1, \ldots$. Let us assume we have introduced

$Z_h, Z_{h+1}, \ldots, Z_0$ and applied our procedure integrating out the scales from 0 to $h+1$ ($h<0$). We obtain an effective potential $\tilde{V}^{(\leq h)}$ [different from $V^{(\leq h)}$, defined by Eq. (23)]. We denote by $P_{Z_h}(\psi_\omega^{(h)})$, $P_{Z_h}(\psi_\omega^{(\leq h-1)})$, and $\tilde{P}_{Z_h}(\psi_\omega^{(h)})$ the free measures with propagators, respectively, $G_\omega^{(h)}/Z_h$, $G_\omega^{(\leq h-1)}/Z_h$, and $\tilde{G}_\omega^{(h)}/Z_h$, where the last one is the modified propagator on scale h and $G_\omega^{(\leq h-1)} = \sum_{i\leq h-1} G_\omega^{(i)}$. $\hat{V}^{(h-1)}$ is defined by

$$\int P_{Z_h}(d\psi_\omega^{(\leq h-1)})e^{-\hat{V}^{(h-1)}(\sqrt{Z_h}\psi_\omega^{(\leq h-1)})}$$

$$= \int P_{Z_h}(d\psi_\omega^{(\leq h-1)})\tilde{P}_{Z_h}(d\psi_\omega^{(h)})e^{-\tilde{V}^{(\leq h)}(\sqrt{Z_h}\psi_\omega^{(\leq h)})}. \quad (24)$$

$\hat{V}^{(h-1)}(Z_h^{1/2}\psi^{(\leq h-1)})$ has the form

$$\hat{V}^{(\leq h-1)}(Z_h^{1/2}\psi_\omega^{(\leq h-1)})$$

$$= (\mathcal{L}+\mathcal{R})\hat{V}^{(\leq h-1)}(Z_h^{1/2}\psi_\omega^{(\leq h-1)})$$

$$= Z_h\left\{ \nu_{h-1}\sum_{\omega,\sigma}\int\frac{d^2k}{(2\pi)^2}\psi_{k,\omega,\sigma}^{(\leq h-1)+}\psi_{k,\omega,\sigma}^{(\leq h-1)-}\right.$$

$$+ \zeta_{h-1}\sum_{\omega,\sigma}\int\frac{d^2k}{(2\pi)^2}\psi_{k,\omega,\sigma}^{(\leq h-1)+}(-ik_0)\psi_{k,\omega,\sigma}^{(\leq h-1)-}$$

$$\left. + \alpha_{h-1}\sum_{\omega,\sigma}\int\frac{d^2k}{(2\pi)^2}\psi_{k,\omega,\sigma}^{(\leq h-1)+}(\omega k_1)\psi_{k,\omega,\sigma}^{(\leq h-1)-}\right\}$$

$$+ \cdots.$$

Now we add and subtract from $\hat{V}^{(h-1)}$ the term $\propto Z_h\zeta_{h-1}\psi_{k,\omega,\sigma}^{+(\leq h-1)}(\omega k_1)\psi_{k,\omega,\sigma}^{-(\leq h-1)}$ and insert the term $\propto Z_h\zeta_{h-1}\psi_{k,\omega,\sigma}^{+(\leq h-1)}(-ik_0+\omega k_1)\psi_{k,\omega,\sigma}^{-(\leq h-1)}$ in the free measure. Let $P'_{Z_h}(\psi_\omega^{(\leq h-1)})$ be the measure changed this way. We define $Z_{h-1}=Z_h(1+\zeta_{h-1})$, and write

$$\int P_{Z_h}(d\psi_\omega^{(\leq h-1)})e^{-\hat{V}^{(h-1)}(\sqrt{Z_h}\psi_\omega^{(\leq h-1)})}$$

$$= \int P'_{Z_h}(d\psi_\omega^{(\leq h-1)})e^{-V'^{(h-1)}(\sqrt{Z_h}\psi_\omega^{(\leq h-1)})} \quad (25)$$

$$= \int P_{Z_{h-1}}(d\psi_\omega^{(\leq h-2)})\tilde{P}_{Z_{h-1}}(d\psi_\omega^{(h-1)})$$

$$\times e^{-V'^{(h-1)}(\sqrt{Z_h}\psi_\omega^{(\leq h-1)})} \quad (26)$$

$$= \int P_{Z_{h-1}}(d\psi_\omega^{(\leq h-2)})\tilde{P}_{Z_{h-1}}(d\psi_\omega^{(h-1)})$$

$$\times e^{-\tilde{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi_\omega^{(\leq h-1)})}. \quad (27)$$

In Eq. (25) $V'^{(h-1)}$ is obtained from $\hat{V}^{(h-1)}$ dropping the ζ_{h-1} term and substituting $\alpha_{h-1}-\zeta_{h-1}$ for α_{h-1}. In Eq. (26) $\tilde{P}_{Z_{h-1}}(d\psi_\omega^{(h-1)})$ is the free measure with propagator $\tilde{G}_\omega^{(h-1)}/Z_{h-1}$ defined such that the remaining part of the free measure is exactly $P_{Z_{h-1}}(d\psi_\omega^{(\leq h-2)})$. Finally Eq. (27) de-
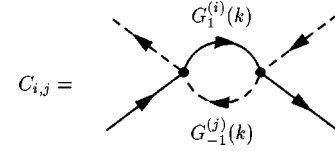

FIG. 2. One-loop contribution defining $C_{i,j}$. k is the internal momentum. The two propagators are on scales i and j.

fines $\tilde{V}^{(h-1)}$, and the right-hand side has the same structure as the right-hand side of Eq. (24), so the procedure may be reiterated. The starting points are $Z_0=1$ and $\tilde{G}_\omega^{(0)}=G_\omega^{(0)}$. The relations between the couplings of $\mathcal{L}\hat{V}^{(\leq h-1)}(\psi_\omega^{(\leq h-1)})$, $g'_{i,h-1}$ ($i=1,2,4$), α_{h-1} and ζ_{h-1}, and the couplings of $\mathcal{L}\tilde{V}^{(\leq h-1)}(\psi_\omega^{(\leq h-1)})$, $g_{i,h-1}$ and δ_{h-1}, are easily found:

$$\delta_{h-1}=\frac{Z_h}{Z_{h-1}}(a_{h-1}-z_{h-1}),$$

$$g_{i,h-1}=\left(\frac{Z_h}{Z_{h-1}}\right)^2 g'_{i,h-1}, \quad i=1,2,4.$$

Of course ζ_{h-1} is no longer present in $\tilde{V}^{(\leq h-1)}(\psi_\omega^{(\leq h-1)})$, but Z_{h-1} is introduced. The replacement $\alpha_h \rightarrow \delta_h$ drastically improves the convergence properties in the limit $h\rightarrow -\infty$.

Now we have the full recipe to proceed. Needless to say, a crucial use of the linked cluster theorem will be made. For brevity only the calculation for the g_1 coupling is sketched (we take $g_{1\perp}=g_{1\parallel}=g_1$).

Let $C_{i,j}$ denote the loop of Fig. 2,

$$C_{i,j}=C_{i-j}=\int\frac{d^2k}{(2\pi)^2}G_\omega^{(i)}(k)G_{-\omega}^{(j)}(k), \quad (28)$$

where $i,j\leq 0$, and $G_\omega^{(i)}$ is the propagator in momentum space on scale i. It is immediate to verify that the right-hand side of Eq. (28) does not depend on ω nor on the scale p [see Eq. (22)], and that it is a function only of the difference $i-j$. In particular $C_{i,i}=C_0$, $i\leq 0$. The second-order calculation for g_1 gives

$$g_{1,h-1}=g_{1,h}+\frac{1}{2}K\left(C_{h,h}g_{1,h}^2+2\sum_{h<j\leq 0}C_{h,j}g_{1,j}^2\right). \quad (29)$$

K is a combinatorial factor: $K=4$. The previous relations give the second-order approximation for the β *functional*. Since we are interested in the β *function*, in Eq. (29) we write $g_{1,j}$ for $j>h$ as functions of $g_{1,h}$. This inversion generates a correction to the higher orders, in particular to the third order. We have [up to $O(g_1^4)$ terms]

$$g_{1,h-1}=g_{1,h}+\frac{1}{2}K\left(C_{h,h}+2\sum_{h<j\leq 0}C_{h,j}\right)g_{1,h}^2-K^2\sum_{h<i\leq 0}C_{h,i}$$

$$\times\left(\sum_{h<j\leq i}C_{j,j}+2\sum_{h<k\leq i}\sum_{k<j\leq 0}C_{k,j}\right)g_{1,h}^3. \quad (30)$$
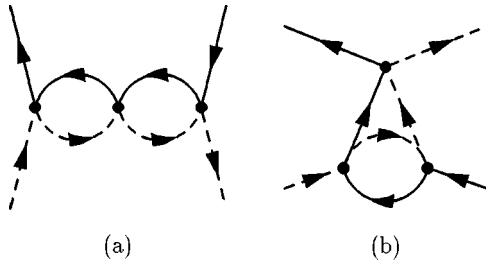
Using Eq. (22) for $G_\omega^{(h)}$, we find

FIG. 3. Two-loop graphs contributing to the β function: D_1 diagrams (a) and D_2 (b) give a simply related contribution: $D_1 = -2D_2$.

$$C_{h,h} + 2\sum_{h<j\leqslant 0} C_{h,j} = \frac{1}{2\pi}[+\ln\gamma - \ln(1+\gamma^{2-2h})$$

$$+\ln(1+\gamma^{-2h})]$$

$$= -\frac{1}{2\pi}\ln\gamma + O(\gamma^{2h}). \quad (31)$$

From Eqs. (30) and (31) we obtain the second-order *discrete* β function, up to γ^{2h} terms (let us remember that $\gamma > 1$, and that we are interested in the $h \to -\infty$ limit):

$$g_{1,h-1} = g_{1,h} - \frac{\ln\gamma}{\pi}g_{1,h}^2 + O(g^3).$$

The second line of Eq. (30) gives the corrections to the third-order result. We find

$$\sum_{h<i\leqslant 0} C_{h,i}\left(\sum_{h<j\leqslant i} C_{j,j} + 2\sum_{h<k\leqslant i}\sum_{k<j\leqslant 0} C_{k,j}\right)$$

$$= \frac{1}{2\pi}\sum_{h<i\leqslant 0} C_{h,i}[-\ln(1+\gamma^{2i})+(i-h)\ln\gamma],$$

where the right-hand side is easily calculated:

$$\left|\sum_{h<i\leqslant 0} C_{h,i}\ln(1+\gamma^{2i})\right| \leqslant A\gamma^h \sum_{h<i\leqslant 0} 1, \quad (32)$$

$$\sum_{h<i\leqslant 0}(i-h)C_{h,i} = \frac{\ln 2}{4\pi}. \quad (33)$$

A is a constant. We neglect the right-hand side of Eq. (32). Equation (33) is also derived neglecting terms exponentially vanishing with h. Putting all together we find a correction **c** to the third order given by

$$\mathbf{c} = -\frac{2\ln 2\ln\gamma}{\pi^2}g_{1,h}^3.$$

The computation of the third order is tedious. We will limit ourselves to note that exists a contribution $\propto \ln 2\ln\gamma$ of two-loop diagrams that cancels exactly **c**. This is important because there is no such term in Eqs. (6), (7), or (21). It comes from the diagrams D_1 and D_2 of Fig. 3, which are related by $D_1 = -2D_2$. We will consider the simpler D_1. The contributions to D_1 given by

$$\sum_{h<i\leqslant 0} C_{h,h}2C_{h,i} + \sum_{h<i\leqslant 0}\sum_{h<j<i} 2C_{h,i}2C_{h,j}$$

$$+ \sum_{h<i\leqslant 0}\sum_{h<j,k<1} C_{h,i}2C_{j,k}$$

$$+ \sum_{h<i\leqslant 0}\sum_{h<j<i}\sum_{h<k\leqslant j} 2C_{h,j}2C_{k,i}$$

amount to

$$\frac{1}{(2\pi)^2}\ln 2\ln\gamma - \frac{1}{2\pi}\sum_{h<i<0} 2C_{i,h}\ln(1+\gamma^{2i}). \quad (34)$$

The second term of Eq. (34) can be neglected as for Eq. (32), and the first term gives the desired cancellation (taking into account the combinatorial factors). The final result is the same as Eqs. (7) or (21), with $g_{1\parallel} = g_{1\perp} = g_1$. Again we find the fixed point $g_1^\star$ of Eq. (8), and we recover Eqs. (6) in the limit $\gamma \to 1$.

V. CONCLUSIONS

We have computed the third-order (two-loop) approximation of the β function for a one-dimensional model of interacting fermions, aiming in particular at studying the case of attractive interaction. An existing result,[6,7] derived making use of tacitly assumed approximations, pointed out a $O(1)$ fixed point. We tried to support this conclusion setting a Callan–Symanzik approach and using the Wilson RG formulated as in Refs. 25, 8, and 10. In each case we recovered the aforementioned result.

An attempt to pursue further the examination of the problem was made in Refs. 16–18, where the fourth-order approximation, which is by no means universal, was computed. We propose a different approach focused on the study of the dependence on γ, the rescaling factor of the RG group. A similar idea was discussed in Ref. 32, where the dependence of the fixed points on the parameters of the RG was analyzed. Our simple idea is that the stability of the result with respect to γ should indicate how reliable one should consider the perturbative result. It was expected that a third-order result would be dependent on γ, but we found too strong a dependence: the fixed point happens to change sign if $\gamma > \sqrt{e}$, which is still ~ 1 (of course taking $\gamma \gg 1$ and, at the same time, truncating at the third order would be questionable). Nothing similar happens to the nontrivial fixed points for repulsive interaction $\{g_1^\star = 0, g_2^\star\}$, which are in some sense insensitive to the value of γ.

Which conclusions can be drawn? It is useful to compare the one-dimensional interacting Fermi gas with the well-known Kondo problem. It can be noted that the scaling equation for g_1 (Fermi gas) and the one for the impurity coupling λ (Kondo model) have the same structure (see, e.g., Ref. 33 for a review in the modern language of conformal field theory). The Kondo effect was thoroughly investigated. The ferromagnetic case corresponds to a Fermi gas with repulsive interaction ($g_1 > 0$): the RG flow is such that $\lambda \to 0$. If the coupling is antiferromagnetic the system flows toward a strong coupling phase ($\lambda \to \infty$).[34] Moreover the infrared divergences induce a scale, the Kondo temperature T_K, char-

acterizing the low-energy physics.

In our case the particular instability of the perturbative result with respect to γ, a dimensionless parameter without a physical meaning, should indicate that the RG flow does not actually stop at a finite value, and suggests a conclusion similar to the previous one. In our case the characteristic scale should be a gap Δ for the spin degrees of freedom, whose expression, according to Ref. 35, should have for small coupling an expression of the type $\Delta \propto \sqrt{g_1} \exp(-1/g_1)$.

ACKNOWLEDGMENTS

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