Quasiparticle localization in superconductors with spin-orbit scattering

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We develop a theory of quasiparticle localization in superconductors in situations without spin rotation invariance. We discuss the existence and properties of superconducting phases with localized/delocalized quasiparticle excitations in such systems in various dimensionalities. Implications for a variety of experimental systems, and to the properties of random Ising models in two dimensions, are briefly discussed.

I. INTRODUCTION

A powerful probe of the properties of a superconductor is obtained by studying the low-temperature dynamics of the quasiparticles. In this context, we proposed[1] that all ground-state phases of disordered superconductors can be characterized, at zero temperature, by their quasiparticle transport properties. The two general possibilities are that the quasiparticle excitations may be delocalized, analogous to a metal, or be localized analogous to an insulator. Previous papers[2–5] have developed a theory of localization of quasiparticles in superconductors in situations with spin rotation invariance. In this paper, we consider the case where the spin is not conserved. This may happen, for instance, in a singlet superconductor in the presence of spin-orbit scattering. Another example is provided by a triplet superconductor where the quasiparticles can exchange spin with the condensate, and hence do not have conserved spin. Indeed, a number of superconducting systems, such as, for instance, the heavy fermion superconductors, are both strongly disordered and have strong spin-orbit scattering (and perhaps even triplet pairing). Thus, in order to understand the possibility of quasiparticle localization in such systems, it is necessary to develop a theory that includes spin-orbit scattering in an essential way. Besides, by analogy with what happens in normal metals, spin-orbit scattering is expected to have profound effects on localization phenomena.

Because the quasiparticle charge density is also not conserved in a superconductor, the only conserved quantity carried by the quasiparticles (at low energies) is the energy density itself. In the presence of impurity scattering, the quasiparticle charge and spin densities in such a superconductor do not diffuse, as they are not conserved. Energy diffusion is possible, though. The corresponding transport quantity is the quasiparticle thermal conductivity.

From a theoretical point of view, quasiparticles in a superconductor with nonconserved spin are more appropriately thought of as real (Majorana) fermions. Thus the problem we consider here is one of localization of Majorana fermions. While localization issues of complex (conventional) fermions have been explored in considerable detail, surprisingly, there has been very little theoretical work on corresponding issues for Majorana fermions. As we argue, the superconductor with nonconserved spin provides a natural experimental realization of such a system. We examine the possible phases (as characterized by quasiparticle transport) and the associated phase transitions. It is of interest to distinguish between situations with and without time-reversal symmetry $\mathcal{T}$, and we will consider each separately. (In the notation of Ref. 6, these correspond to class DIII and class D, respectively).

While the superconductor is our primary motivation, we note also that Majorana fermions arise in other contexts as well. A well-known example is the two-dimensional Ising model. There have been several studies[7] of the properties of the two-dimensional Ising model in the presence of randomness in the bond strengths, though there are still several poorly understood issues. The implications of this work for that problem will be considered briefly towards the end of the paper.

We first show the existence, in two dimensions, of stable ''metallic'' and ''insulating'' phases inside the superconductor with delocalized and localized quasiparticle excitations, respectively. The stability of the ''metallic'' phase in two dimensions has already been alluded to in Ref. 4. We emphasize that both phases are superconducting—they are distinguished by the nature of thermal transport due to the quasiparticles. These two phases are separated by a phase transition which is a ''metal-insulator'' transition inside a superconductor. In Ref. 1, we discussed possible experimental realizations of such phase transitions. The universal critical properties of this transition depend, of course, on whether time reversal is a good symmetry or not. In $\mathcal{T}$-noninvariant systems, the insulating phases may be further characterized in terms of their values of the Hall thermal conductance. The dimensionless ratio $3h\kappa_{xy}/\pi^2 k_B^2 T$ is quantized in units of 1/2. Phases with different values of this quantized Hall thermal conductance are topologically distinct, and are separated by phase transitions. In Fig. 1, we show a schematic phase diagram. Note that, in the case with no $\mathcal{T}$, apart from the ''metal-insulator'' transition, there are also transitions be-

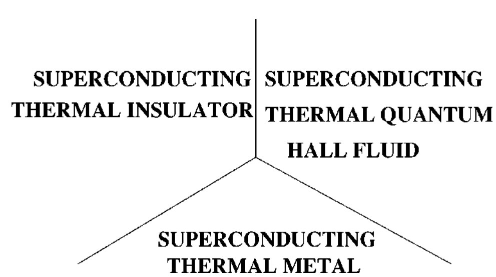

FIG. 1. Schematic zero-temperature phase diagram for the two-dimensional superconductor in the absence of both spin rotation and time-reversal invariances.

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tween the ''metal'' and the ''quantum Hall'' phase, and from the insulator with $\kappa_{xy}/T=0$ to the ''quantum Hall'' phase. Further, there is a multicritical point, where all three phases come together. In three-dimensional systems, ''metallic'' and ''insulating'' phases are again possible with the transition between the two being in a universality class for localization. In the rest of the paper, we will substantiate these claims, and analyze the properties of each phase in further detail.

II. MODELS AND GENERAL FORMALISM

Consider a general lattice Hamiltonian for the quasiparticles in a superconductor with strong spin-orbit scattering:

$$\mathcal{H}=\sum_{ij} t_{ij}^{\alpha\beta}c_{i\alpha}^{\dagger}c_{j\beta}+\Delta_{ij}^{\alpha\beta}c_{i\alpha}^{\dagger}c_{j\beta}^{\dagger}+\text{H.c.} \tag{1}$$

Here i,j refer to the sites of a lattice, and α,β are spin indices. We assume that t and Δ are both short ranged in space.

We will focus on two cases—with and without time-reversal symmetry ($\mathcal{T}$). If present, time reversal is imposed through an antiunitary ''time-reversal'' operator $\mathcal{T}$ which transforms the c operators as

$$\mathcal{T}\begin{bmatrix} c_{i\uparrow} \\ c_{i\downarrow} \end{bmatrix}=\begin{bmatrix} c_{i\downarrow} \\ -c_{i\uparrow} \end{bmatrix}=i\sigma_y c. \tag{2}$$

Note that $\mathcal{T}\boldsymbol{\sigma}^*\mathcal{T}^{-1}=-\boldsymbol{\sigma}$ (with $\boldsymbol{\sigma}$ a vector of Pauli matrices) so that the electron spin is odd under time reversal. With time-reversal invariance present we require that

$$\mathcal{T}\mathcal{H}\mathcal{T}^{-1}=\mathcal{H}. \tag{3}$$

Note that $\mathcal{H}$ has no special symmetries (other than possibly $\mathcal{T}$). In particular, neither the charge nor spin is conserved. It is convenient then to work with Majorana fermions $\eta_{1i\alpha},\eta_{2i\alpha}$ defined through

$$c_{i\alpha}=\frac{1}{\sqrt{2}}(\eta_{i1\alpha}+i\eta_{i2\alpha}). \tag{4}$$

The Hamiltonian when expressed in terms of the η fermions takes the form

$$\mathcal{H}=\eta^T H\eta, \tag{5}$$

with $\eta=\eta_{ia\alpha}$, $(a,b=1,2)$ and $H=H_{ij}^{ab,\alpha,\beta}$ is a matrix in $(ij),(\alpha\beta),ab$ space. By definition, H is Hermitian and moreover $H^T=-H$, so that H is pure imaginary. Thus the problem of quasiparticle localization in a superconductor with spin-orbit scattering is, in essence, one of localization of Majorana fermions. In particular, in the case where even $\mathcal{T}$ is not a good symmetry, the Hamiltonian Eq. (5) is the most general one describing noninteracting Majorana fermions in a disordered system.

Time-reversal symmetry is easily imposed on the Majorana Hamiltonian Eq. (5). Under the action of the antiunitary operator $\mathcal{T}$, it is readily seen that

$$\mathcal{T}\eta_{ia\alpha}=i(\sigma_y)_{\alpha\beta}(\tau_z)_{ab}\eta_{ib\beta}. \tag{6}$$

Here $a,b=1,2$ and $\vec{\tau}$ is a Pauli matrix in ab space. The time-reversal invariance condition Eq. (3) combined with Eq. (6) implies

$$\sigma_y\tau_z H^*\sigma_y\tau_z=H. \tag{7}$$

We note in passing that for a superconductor of spin *polarized* fermions (appropriate, say, in the $A-1$ phase of superfluid $3-He$) in the absence of time-reversal invariance, the general Hamiltonian can still be written in the form $\mathcal{H}=\eta^T H\eta$, with H an antisymmetric and pure imaginary matrix in position (i,j) and ''particle-hole'' (ab) space. But for spinless quasiparticles the condition for time-reversal invariance is different, leading to different symmetries. Specifically, with time-reversal invariance the lattice Hamiltonian depends on a *real* symmetric hopping matrix, t_{ij}, and a *real* antisymmetric gap matrix, Δ_{ij}. In this case, when reexpressed in terms of Majorana fermions, the Hamiltonian becomes $\mathcal{H}=\eta^T H\eta$ with $H=t\tau_y+i\Delta\tau_x$. This Hamiltonian matrix can equivalently be expressed as $H=iA\tau^+ +\text{c.c.}$, with $\tau^+=\tau_x+i\tau_y$ and $A=t+\Delta$ an *arbitrary* real matrix. This ''off-diagonal'' form is very different than the two cases with spinful electrons, and in fact belongs to a different symmetry class—a class studied by Gade and Wegner.[8] Henceforth we focus exclusively on the spinful case.

We are interested in understanding the nature of energy transport (in both spinful cases, with and without $\mathcal{T}$) by the excitations described by the Hamiltonian. For this purpose, it is actually convenient to adopt the following trick. We consider two identical copies of the system. To describe the two copies, we introduce two Majorana fields η and ζ and consider

$$\tilde{\mathcal{H}}=\eta^T H\eta+\zeta^T H\zeta. \tag{8}$$

It is now possible to combine the η and ζ fields into a single complex fermion f:

$$f=\frac{1}{\sqrt{2}}(\eta+i\zeta), \tag{9}$$

$$f^\dagger=\frac{1}{\sqrt{2}}(\eta-i\zeta). \tag{10}$$

Then, we have

$$\tilde{\mathcal{H}}=f^\dagger Hf. \tag{11}$$

Note that the number of f particles is conserved. It is then possible to consider transport of this conserved f-number density. This may be quantified by a conductivity σ_f. As the Hamiltonian $\tilde{\mathcal{H}}$ describes noninteracting f particles, the thermal conductivity κ_f of the f particles is related to σ_f by a Weiedemann-Franz law (as $T\to 0$):

$$\frac{\kappa_f}{T\sigma_f}=\frac{\pi^2 k_B}{3}. \tag{12}$$

Since the Hamiltonian $\tilde{\mathcal{H}}$ represents just two identical copies of the original system described by $\mathcal{H}$, it is clear that the thermal conductivity (κ) of the η particles is exactly half that of the f particles:

$$\kappa = \frac{\kappa_f}{2}. \quad (13)$$

Combining this with Eq. (12), we see that calculation of κ is reduced to calculation of σ_f. Similarly, all the thermodynamic properties of $\mathcal{H}$ may be obtained by halving the corresponding property calculated with $\tilde{\mathcal{H}}$.

It is convenient to define the f-Green's function

$$G_{ij}^{ab,\alpha\beta}(E) = \langle i a \alpha | \frac{1}{E - H + i\epsilon} | j b \beta \rangle, \quad (14)$$

where ϵ is a positive infinitesimal and E is the energy (measured from the Fermi energy). The condition $H^* = -H$ immediately implies that

$$G^*(E) = -G(-E), \quad (15)$$

where we have suppressed all the indices on G. In particular, for states at the Fermi energy, we have

$$G^*(E=0) = -G(E=0), \quad (16)$$

so that $G(E=0)$ is pure imaginary. This Green's function can be conveniently expressed, as usual, as a functional integral over Grasmann variables $f_{ia\alpha}, \overline{f}_{ia\alpha}$ with the action

$$S = i\overline{f}(H - i\epsilon) f. \quad (17)$$

The symmetry Eq. (16) implies that this same generating functional can be used to calculate transport quantities.[3]

III. SYSTEMS WITH TIME-REVERSAL SYMMETRY

The discussion has so far been completely general. We now specialize to the case with $\mathcal{T}$ symmetry.

A. Metallic phase

We first consider the situation in which the disorder is weak. We assume that there is a finite, nonzero mean free path l_e set by the impurity strength in such a manner that the f-particle motion is diffusive on larger scales. (In terms of the original physical system, this corresponds to diffusion of energy.) We are interested in describing the effects of quantum interference on this diffusive motion. We assume also that, in the absence of quantum interference effects, the density of states at the Fermi energy of this diffusive system is finite and nonzero. In that case, it is possible to follow standard techniques to derive a replica nonlinear sigma model field theory to describe the physics at length scales large compared to the mean free path l_e. As the procedure is sufficiently well known, we merely state the results. The field theory is described by the action

$$\mathcal{S} = \int d^d x \frac{1}{2g} \mathrm{Tr}[(\nabla O)^T (\nabla O)] - \epsilon\, \mathrm{Tr}(O + O^T), \quad (18)$$

where $O(x) \in O(2n)$ is a $2n \times 2n$ orthogonal matrix-valued field with n the number of replicas. When $\epsilon = 0$ the action has a global $O(2n) \times O(2n)$ symmetry, $O \rightarrow AOB$ with A and B orthogonal matrices, which is broken down to the diagonal $O(2n)$ $(A^T = B)$ by the ϵ term. The coupling constant g is inversely proportional to the bare f conductivity σ_f^0 (i.e., the conductivity on the scale of the mean free path). The density of quasiparticle states at the Fermi energy is given by

$$\rho = \frac{\rho_0}{4n} \langle \mathrm{Tr}(O + O^T) \rangle. \quad (19)$$

Consider a renormalization-group transformation where short distance fluctuations are integrated out, and the coordinate x is rescaled as $x \rightarrow x' = x e^{-l}$. The leading quantum interference corrections to diffusion can now be obtained from the known[9] perturbative β function of this field theory in the replica limit. The result, in two dimensions, is

$$\frac{dg}{dl} = -\frac{g^2}{4\pi}. \quad (20)$$

Note that g decreases as l is increased. Thus the perturbation theory (in powers of g) gets better at large length scales. For a system of size L, at zero temperature, we may integrate the flow equation up to a scale l^* given by $l_e e^{l^*} \sim L$ to get

$$g(L) = \frac{g_0}{1 + (g_0/4\pi)\ln(L/l_e)}, \quad (21)$$

where g_0 is the bare value of g. For large L, this therefore gives

$$g(L) \approx \frac{4\pi}{\ln(L/l_e)}. \quad (22)$$

Thus $g(L)$ goes to zero logarithmically with the system size. As σ_f is inversely proportional to g, it follows that σ_f *diverges* logarithmically with the system size. At finite temperature, in an infinite system, it is natural to expect that the quantum interference effects will be cut off at a finite dephasing length scale $L_\phi \sim T^{-p}$ due to interaction effects not included in the model. We therefore have

$$\sigma_f \sim \ln\left(\frac{1}{T}\right) \quad (23)$$

at the lowest temperatures. As this also determines the thermal conductivity $\kappa(T)$ of the original system, we have

$$\frac{\kappa}{T} \sim \ln\left(\frac{1}{T}\right). \quad (24)$$

The considerations above establish the existence of a metallic phase with delocalized quasiparticle excitations in two dimensions in the model Hamiltonian Eq. (1) describing the superconductor in the presence of time-reversal invariance, but no spin rotation invariance. In striking contrast to normal metals, quantum interference effects also lead to singular corrections to the density of states in a superconductor.[3] In our previous work,[3] we demonstrated this in the spin rotation invariant cases. We now show that the density of states is *enhanced* in the situation considered in this paper. In particular, we show that in two dimensions it actually *diverges* in the thermodynamic limit.

To see this, consider the action Eq. (18) at finite ϵ. The density of states is obtained from

$$\frac{\rho}{\rho_0} = -\frac{1}{4n}\left[\frac{\partial \mathcal{F}}{\partial \epsilon}\right]_{\epsilon=0}, \tag{25}$$

where $\mathcal{F}$ is the free-energy density defined through

$$\exp(-L^d \mathcal{F}) = \int \mathcal{D}O \exp(-\mathcal{S}). \tag{26}$$

The flow of ϵ under the renormalization group (RG) may be obtained straightforwardly. To leading order in g the result is

$$\frac{d\epsilon}{dl} = \left(2 + \frac{g}{8\pi}\right)\epsilon. \tag{27}$$

This is readily integrated to give

$$\epsilon(l) = \epsilon(0)\exp\left(2l + \frac{1}{8\pi}\int_0^l g(l')dl'\right). \tag{28}$$

The free-energy density scales according to

$$\mathcal{F}(g(0),\epsilon(0)) = e^{-2l}\mathcal{F}(g(l),\epsilon(l)), \tag{29}$$

where we have specialized to two dimensions. For a system of size L, run the RG until a scale l^* such that $l_e e^{l^*} \approx L$. The density of states is then

$$\begin{aligned}\rho(L) &= -\frac{e^{-2l^*}}{4n}\frac{\partial \mathcal{F}(g(l^*),\epsilon(l^*))}{\partial \epsilon(0)} \\ &= -\exp\left(\frac{1}{8\pi}\int_0^{l^*} dl' g(l')\right)\frac{1}{4n}\frac{\partial \mathcal{F}(g(l^*),\epsilon(l^*))}{\partial \epsilon(l^*)}.\end{aligned} \tag{30}$$

After scaling out to l^* the mean free path is comparable to the system size, so that $-(1/4n)[\partial\mathcal{F}(g(l^*),\epsilon(l^*))/\partial\epsilon(l^*)] \to \text{const.}$ Therefore

$$\rho(L) \sim \exp\left(\frac{1}{8\pi}\int_0^{l^*} dl' g(l')\right). \tag{31}$$

For large l, $g(l) \approx 4\pi/l$. Thus we have

$$\rho(L) \sim \sqrt{\ln\frac{L}{l_e}}. \tag{32}$$

Thus the density of states at the Fermi energy diverges. The behavior of the density of states as a function of energy in an infinite system may also be found similarly (or simply guessed from the equation above) to be

$$\rho(E) \sim \sqrt{\ln\frac{1}{E}}. \tag{33}$$

The Einstein relation can now be used to infer that the heat diffusion constant D also diverges on approaching the Fermi energy as

$$D(E) \sim \sqrt{\ln\frac{1}{E}}. \tag{34}$$

Thus the metallic phase has an infinite heat diffusion constant at zero temperature in two dimensions.

The divergence of the density of states has obvious consequences for the low-temperature thermodynamic properties of this phase. For instance, the specific heat behaves as

$$C(T) \sim T\sqrt{\ln\frac{1}{T}}. \tag{35}$$

In three-dimensional systems, the stability of the metallic phase can be established by simple power-counting arguments. Quantum interference effects are then irrelevant at long length scales. The thermal conductivity then goes to zero linearly with the temperature at low T:

$$\frac{\kappa_{3D}}{T} \to \text{const.} \tag{36}$$

The density of states at the Fermi energy is finite, and nonzero. However, quantum interference effects do lead to a singular $\sqrt{|E|}$ cusp in the density of states as a function of the energy (see Ref. 1 for an analogous discussion in the superconductor with conserved spin) so that

$$\rho(E) - \rho(E=0) \sim -\sqrt{|E|}. \tag{37}$$

Note that the density of states increases with decreasing energy as also happens in $d=2$.

B. Insulating phase

For strong disorder, it is possible to have a phase with localized quasiparticle excitations in any finite dimension. In the terminology of Ref. 1, this is a superconducting ‘‘thermal insulator.’’ In this phase, the ratio κ/T goes to zero with the temperature. The density of quasiparticle states also goes to zero at the Fermi energy. To see this, consider the lattice Hamiltonian Eq. (11) in the extreme limit where the f particles are localized to a single site. The Hamiltonian for a single site is constrained to be of the form $H = a\sigma_y\tau_x + b\tau_y$ with a,b real. This has two eigenvalues given by $\pm\sqrt{a^2+b^2}$. Consider now the case where the distribution of a,b has finite, nonzero weight at $a=b=0$. Then, the density of states $\rho(E)$ averaged over the disorder vanishes as $|E|$. If the distribution has vanishing weight at $a=b=0$, then $\rho(E)$ vanishes faster than linearly. Including hopping between the sites should not change this result so long as we are in the localized phase. (As with the superconductors with conserved spin, having a finite density of states requires a diverging weight at $a=b=0$ which is presumably unphysical, and nongeneric.) We thus conclude that the density of quasiparticle states vanishes, at least as fast as $|E|$ in the localized phase.

IV. TIME-REVERSAL BROKEN SYSTEMS

We now move on to systems without time-reversal symmetry. As mentioned earlier, this corresponds to studying the general localization properties of a noninteracting disordered system of Majorana fermions. It is therefore appropriate to call the phase with extended states a ‘‘Majorana metal,’’ and the phase with localized states a ‘‘Majorana insulator.’’

A. Majorana metal

To address issues such as the stability and properties of the Majorana metal, it is useful to think in terms of a replica sigma model field theory which permits a systematic study of quantum interference corrections to diffusive energy transport. This is readily done using standard techniques. The result is

$$\mathcal{S}=\int d^2x\frac{1}{2g}\mathrm{Tr}(\partial Q)^2-\epsilon\,\mathrm{Tr}(\sigma_y Q), \tag{38}$$

where $Q=V^T\sigma_y V$ is a $2n\times 2n$ matrix, V is an $O(2n)$ matrix and n is again the replica index. When $\epsilon=0$, the action is invariant under the group $O(2n)$, $Q\rightarrow W^TQW$, with W an orthogonal matrix. With nonzero ϵ, invariance of the action requires $W=\exp(iG)$ with G pure imaginary and antisymmetric of the form, $W=S\sigma_y+iA$. Here S is an n by n real symmetric matrix and A is n by n real antisymmetric. Since $S+iA$ is n by n Hermitian, the group has evidently been broken down to $U(n)$ by the ϵ term. Thus $\mathcal{S}$ describes a nonlinear sigma model on the manifold $O(2n)/U(n)$. The coupling constant g is again inversely proportional to the (longitudinal) f-particle conductance. The density of states of the quasiparticles may be obtained from

$$\rho=\frac{1}{2n}\langle \mathrm{Tr}(\sigma_y Q)\rangle. \tag{39}$$

Actually, on symmetry grounds, there is a topological term allowed in the action. This follows from the observation that $\Pi_2[O(2n)/U(n)]=Z$ is nontrivial. This will be important to understand the behavior of the thermal Hall conductivity later.

The perturbative RG flows of the couplings g and ϵ in two dimensions in the replica limit are

$$\frac{dg}{dl}=-\frac{g^2}{8\pi}, \tag{40}$$

$$\frac{d\epsilon}{dl}=\left(2+\frac{g}{8\pi}\right)\epsilon. \tag{41}$$

We may use these to extract the physical properties of the Majorana metal phase in exactly the same manner as in the previous section. We therefore simply state the results. First note that the (marginal) irrelevance of the coupling g implies the stability of the Majorana metal in $d=2$. As the temperature goes to zero, the (longitudinal) thermal conductivity κ_{xx} in a finite system of size L behaves as

$$\frac{\kappa_{xx}}{T}\sim\ln\left(\frac{L}{l_e}\right). \tag{42}$$

In an infinite system at finite T, where the quantum interference is cut off by dephasing due to interaction effects, we have

$$\frac{\kappa_{xx}}{T}\sim\ln\left(\frac{1}{T}\right). \tag{43}$$

Similarly, the density of states at the Fermi energy now diverges logarithmically with the system size:

$$\rho(L)\sim\ln\frac{L}{l_e}. \tag{44}$$

As a function of energy in an infinite system, we have

$$\rho(E)\sim\ln\frac{1}{E}. \tag{45}$$

This leads, for instance, to a specific heat which depends on temperature as

$$\frac{C(T)}{T}\sim\ln\frac{1}{T}. \tag{46}$$

Since κ_{xx}/T and the density of states diverge in the same manner with energy, determining the behavior of the thermal diffusion coefficient would require going to second order in the perturbative RG. The diffusion coefficient could then also diverge, but not more rapidly than double logarithmically with energy.

In three dimensions, the behavior of the superconducting metal phase is qualitatively the same irrespective of whether time-reversal symmetry is present or not. Therefore the discussion in the previous section of the three-dimensional case applies here as well.

B. Majorana insulator

At strong disorder, in any dimension, it is possible to find phases where the quasiparticle excitations are localized. The (longitudinal) quasiparticle heat conductivity κ_{xx} goes to zero rapidly with the temperature, in such a phase. We will call this the Majorana insulator, as this corresponds to a localized phase of Majorana fermions. The density of states of the Majorana insulator may be found by an argument analogous to the one in Sec. III B. In brief, we consider a single site Hamiltonian for two Majorana species (corresponding to a single spinless complex fermion) which is constrained to be of the form $H=a\tau_y$ with a real and random. If the distribution of a has a finite, nonzero weight at $a=0$, then there is a finite density of states at zero energy. We again expect this to hold throughout the localized phase, so that the density of states at the Fermi energy can be nonzero. Note that, like in a conventional insulator in nonsuperconducting systems, a density of states that vanishes at the Fermi energy is also possible.

C. Majorana quantum Hall phase

In the absence of time-reversal symmetry, there is potentially another transport property which can be used to distinguish zero-temperature phases: the thermal Hall conductivity κ_{xy}. In particular, in the Majorana insulator in two dimensions, the ratio κ_{xy}/T approaches quantized values as the temperature goes to zero. Phases with different values of the quantized thermal Hall conductance are *topologically* distinct and are separated by phase transitions. In recent work with X. G. Wen,[10] we studied the physics of this quantum Hall system in some detail.

From the point of view of the replica nonlinear sigma model field theory discussed earlier in this section, the existence of insulating phases with quantized thermal Hall con-

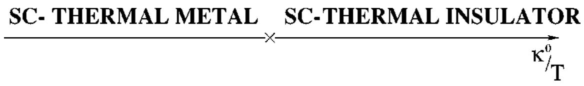

FIG. 2. Schematic zero-temperature phase diagram for the three-dimensional superconductor in the presence of spin-orbit scattering.

ductance can be attributed to the presence of a topological term in the action. This takes the form

$$\mathcal{S}_{top}=\int d^2x\frac{\theta}{16\pi}\mathrm{Tr}(Q\partial_xQ\partial_yQ). \tag{47}$$

The quantity multiplying θ is equal to $i\times integer$ for any given configuration of the Q field. Thus the partition function is periodic under $\theta\to\theta+2\pi m$ for any integer m. Physically, just as in the Pruisken field theory for the conventional integer quantum Hall transition, θ is proportional to the (bare) value of the f-particle Hall conductivity σ_{xy}^f. As shown in Ref. 10, this is in turn proportional to the ratio of the thermal Hall conductivity to the temperature.

V. PHASE DIAGRAM AND TRANSITIONS

We now discuss the (zero-temperature) phase diagram and phase transitions for the dirty superconductor with nonconserved spin. Again, we consider the cases with and without $\mathcal{T}$ separately.

A. Time-reversal invariant systems

As argued in the previous section, metallic and insulating phases are possible in both $d=2$ and $d=3$. A schematic phase diagram as a function of the bare thermal conductivity is as shown below in Fig. 2.

The critical points separating the two phases are in universality classes for Anderson localization. There are some interesting differences between the scaling properties of the two- and three-dimensional systems. In three dimensions, the ratio κ/T, the thermal diffusion coefficient D, and the density of states ρ_0 at the Fermi energy are finite constants in the metal and are zero in the insulator. It is therefore natural to expect that these will go to zero continuously (with some universal critical exponent) as the transition is approached from the metallic side. In two dimensions, however, in the metal, these quantities are all infinite in the limit of zero temperature and infinite system size. (They are zero in the insulator.) Exactly at the transition point, conventional scaling arguments[11] imply a constant value for κ/T. The behavior of the density of states and diffusion coefficient is more unclear. It is perhaps natural to suggest that these will be a constant, though we certainly cannot rule out other possibilities.

B. Time-reversal noninvariant systems

We first consider the three-dimensional case. The phase diagram is similar to that above for the time-reversal invariant system. However, as the density of states may be nonzero in the insulator, we expect that it generically is nonzero at the transition point as well.

The two-dimensional case has a richer phase diagram due to the possibility of quantum Hall phases. For simplicity, we show, in Fig. 1, only the two insulating phases with $\kappa_{xy}=0$ and $\kappa_{xy}/T=\frac{1}{2}$ (in units of $\pi^2k_B^2/3h$). Note that there are three distinct phase transitions, and a multicritical point in this phase diagram. First consider the Majorana metal-insulator transition. Both κ_{xx}/T and ρ_0 are infinite in the Majorana metal. In the Majorana insulator, κ_{xx}/T is zero while ρ_0 may be nonzero. As in the $\mathcal{T}$ invariant case, conventional scaling arguments imply a finite value for κ_{xx}/T at the transition point. For the density of states, it is natural to guess that it is finite at the transition, though again it is hard to rule out other possibilities.

Now consider the transition from the Majorana metal to the quantum Hall phase. This transition is in the same universality class as the transition from the Majorana metal to the insulator with $\kappa_{xy}=0$. The quantum Hall phase is a Majorana insulator with $\kappa_{xy}/T=\frac{1}{2}$. The distinction between these two insulating phases is due to the presence of (heat) current carrying edge states—these are expected to be unimportant in determining the properties of the transition to the metal. Below, we will provide a more formal argument in support of this claim.

Finally, consider the transition between the insulator with $\kappa_{xy}=0$ and the one with $\kappa_{xy}/T=\frac{1}{2}$. A theory for this transition is obtained by considering a particular realization of the two phases. Consider a system of spinless fermions paired into a p_x+ip_y superconducting state in two dimensions. In Ref. 10, it was shown that such a superconductor has a quantized thermal Hall conductivity $\kappa_{xy}/T=\frac{1}{2}$ if the chemical potential is positive (the ''BCS'' limit), and has $\kappa_{xy}=0$ if the chemical potential is negative (''molecular'' limit). Thus, as the chemical potential is varied through zero, there is a transition from the Majorana quantum hall phase to the Majorana insulator. Reference 10 also examined the theory for this transition, and argued that, at least at weak disorder, it is correctly described by a theory of relativistic Majorana fermions with random mass. It is well known that the random mass is irrelevant at the pure free Majorana fixed point.[7] Thus the critical theory is known *exactly* in this case. If we make the important assumption that there is a unique fixed point describing the Majorana insulator-quantum Hall transition, then the arguments above identify it with the free relativistic Majorana fixed point. There is, however, some reason to question this assumption—see Sec. VI.

Some more insight into the phase diagram and the transitions comes from considering the properties of the replica nonlinear sigma model field theory describing the quasiparticles in the two-dimensional superconductor with spin-orbit scattering, and no $\mathcal{T}$. This is described by the action Eq. (38) supplemented with the topological term Eq. (47). The small g regime is described by the perturbative calculations of the previous section. Note that the presence of the topological term does not affect those results. Indeed, the parameter θ plays no role in perturbation theory, and does not renormalize. Thus there is a line of fixed points in the g,θ plane at $g=0$ with θ arbitrary. As the value of θ is proportional to the value of κ_{xy}/T, this ratio varies continuously in the Majorana metal.

Now consider the behavior of the sigma model at large g. It is natural to expect that, in this limit, the physics is captured by a strong-coupling expansion, and that the resulting phase corresponds, physically, to a localized phase. (For conceptual purposes, it may be convenient to think in terms of a

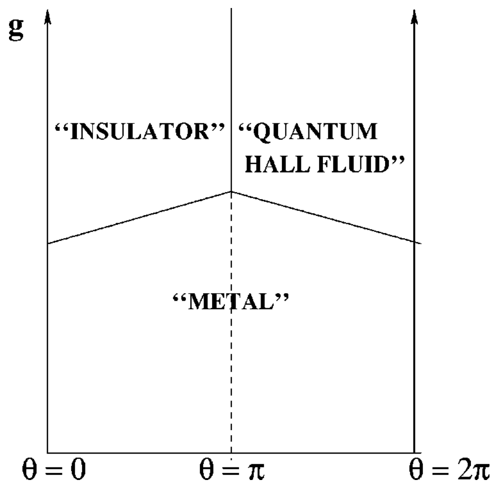

FIG. 3. Phase diagram of the sigma model field theory describing the two-dimensional superconductor with no spin rotation or time-reversal invariances.

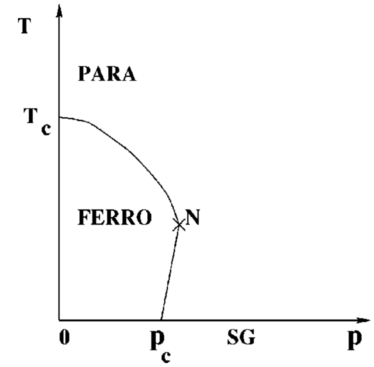

FIG. 4. Phase diagram of the two-dimensional Ising model with random bond strengths. p is the concentration of negative bonds, and T is the temperature. The point N is the Nishimori multicritical point. SG refers to the spin-glass phase which is believed to exist only at zero temperature.

lattice regularized version of the field theory.) First consider $\theta=0$. The corresponding localized phase has $\kappa_{xy}=0$. Similarly, the insulator with $\theta=2\pi$ has $\kappa_{xy}/T=\frac{1}{2}$. Just as in other sigma model field theories with topological terms, the large g phase for $0\leq\theta<\pi$ is expected to be continuously connected to the large g phase at $\theta=0$, i.e., it is an insulator with $\kappa_{xy}=0$. Similarly the large g phase with $\pi<\theta\leq 2\pi$ is expected to be continuously connected to the large g phase with $\theta=2\pi$, i.e., an insulator with $\kappa_{xy}/T=\frac{1}{2}$. The transition between the two localized phases occurs at $\theta=\pi$. The small g metallic phase is separated from these localized phases at large g by a phase boundary. The symmetry $\theta\rightarrow 2\pi-\theta$ implies that this phase boundary be symmetric about the $\theta=\pi$ line (see Fig. 3).

In the field theory, the transitions between the small g metal and the large g insulator at $\theta=0,2\pi$ are in the same universality class as the physics is invariant under $\theta\rightarrow\theta+2\pi$. What about the transition from the metal to the insulator at other values of θ? We suggest that, for $\theta\neq\pi$, these are in the same universality class as the transition at $\theta=0$. This then implies that κ_{xy} is continuous across the metal-insulator transition. Finally, the point where the metal-insulator phase boundary crosses the $\theta=\pi$ line in the phase diagram of the field theory corresponds to the multicritical point where the metal and the two insulating phases come together.

VI. RANDOM BOND ISING MODEL IN $d=2$

In this section, we briefly discuss the possible implications of this paper to the finite temperature properties of classical random bond Ising models in two dimensions. It is well known[7] that this system admits a description in terms of noninteracting disordered Majorana fermions. Specifically, the partition function for the Ising model can be written as a functional integral over Majorana fields that live in two dimensions. This, in turn, may be reinterpreted as a generating function for the wave functions of a quadratic quantum Hamiltonian of Majorana fermions in two spatial dimensions. This Majorana Hamiltonian has no special symmetries—the system is therefore in the universality class of the two-dimensional superconductor with neither spin rotation nor time-reversal invariance.

It is therefore natural to try to identify various phases of the Ising model with corresponding ($T=0$) phases in the two-dimensional superconductor with no spin rotation or time-reversal invariance. First consider the pure Ising model. Both the high- and low-temperature phases correspond to gapped phases in terms of the Majorana fermions. Thus these correspond to insulating phases of the Majorana fermions. Disorder in the Ising model, obtained by making some of the bonds random, would tend to fill up the gap. However, if the disorder is weak, the resulting low-energy states will be strongly localized. Thus, at weak disorder, both the high- and low-temperature phases of the Ising model correspond to localized phases of the Majorana fermions. How then do we distinguish between the two? The distinction is topological, with the quantized value of the thermal Hall conductance differing by 1/2 between the two phases. Strong support for this suggestion is obtained by examining the properties of the system near the transition between the two phases. In the pure Ising model, the critical point is described by a massless relativistic Majorana theory. Moving off criticality introduces a mass for the fermions with the sign of the mass distinguishing the two phases. The result that the dimensionless κ_{xy} jumps by 1/2 at the transition can now be established by direct calculation. The critical theory is identical to the one describing the transition, at weak disorder, in the p_x+ip_y superconductor of spinless fermions between the BCS and molecular limits. We thus identify the Ising transition at weak disorder with the thermal quantum Hall transition.

Now consider introducing strong (bond) randomness into the Ising model. If all the bonds are still positive, the transition is believed to be always controlled by the pure Ising fixed point. The situation is more interesting if some of the bonds are made negative. As the concentration of negative bonds is increased, there is a multicritical point (the Nishimori point, see Fig. 4) (Ref. 12) on the phase boundary, beyond which the transition is no longer controlled by the pure Ising fixed point. The properties of this multicritical point have been the focus of a number of investigations[13] over the last many years. Despite this, there is no detailed understanding of the theory of this point, or of the properties at the phase boundary at lower temperatures. What does the Nishimori point correspond to in the language of the Majorana fermions? There appear to be two distinct possibilities—we will discuss evidence supporting either below.

(i) For the problem of localization of Majorana fermions in two dimensions, the assumption that there is a unique fixed point controlling the thermal quantum Hall transition leads to its identification with the free relativistic Majorana theory. This assumption, which is perhaps natural in the fermion language, then points to the identification of the Nishimori point with the multicritical point separating the Majorana metal, insulator, and the quantum Hall phase. If true, then it is natural to expect that the ferromagnetic phase transition in the Ising model at low temperatures below the Nishimori point is actually the Majorana insulator-metal transition. Some evidence in support of this is provided by the numerical results of Ref. 14 which found no signs of localization of the fermions in the nonferromagnetic phase at low temperature close to the phase boundary. On the other hand, note that the paramagnetic phase at high temperature and weak disorder corresponds to a localized phase. If the scenario outlined above is correct, then we are led to infer the existence of a finite temperature phase transition outside the ferromagnetic phase in the Ising model associated with a delocalization of the Majorana fermions. It is unclear what this transition means in the Ising language, and even whether it happens at all. A natural candidate would have been a transition to a spin-glass phase at low temperature—however, there is strong numerical evidence for the absence of spin-glass order at finite temperature in two-dimensional Ising systems. It seems possible that a delocalized (metallic) phase of Majorana fermions would correspond to a phase in which both the Ising spin and the dual Ising disorder parameter are simultaneously zero.

(ii) A different scenario is obtained by dropping the assumption that there is a unique fixed point controlling the thermal quantum Hall transition. Instead, for the Ising model, we assume that the only finite temperature phase transition is associated with the destruction of the ferromagnetic order. If the concentration of negative bonds is large enough to destroy the ferromagnetism at zero temperature, we assume that the resultant state is a spin glass. These assumptions are perhaps reasonable expectations for the Ising model. Then, the finite temperature ferromagnetic and paramagnetic phases are, in the fermion language both localized phases which are distinguished by their thermal Hall conductance. The existence of the Nishimori point then implies the existence of a multicritical point (at strong disorder) in the phase boundary between these two localized phases. Then, the thermal quantum Hall transition will be controlled by the free relativistic massless Majorana theory at weak disorder, but by a different fixed point at strong disorder. The spin-glass phase in the Ising model presumably corresponds to the Majorana metal. However, it may seem a bit puzzling, in this scenario, why the Majorana metal which is a stable phase of disordered Majorana fermions in two dimensions is only realized in a set of measure zero (a line at $T=0$) in the Ising phase diagram.

Which one of these two scenarios is actually realized and the resultant consequences both for the Ising model and the Majorana localization problem, we leave as an intriguing open question.

VII. DISCUSSION

In this paper, we have discussed the physics of localization of quasiparticles in a superconductor in situations where the quasiparticle spin is not a good quantum number. Our discussion was entirely based on models of noninteracting quasiparticles—including the effects of interactions is an important and interesting issue which we leave for future work. Even within the noninteracting theory, we have found a rather rich phase diagram, and a number of as yet unexplored phase transitions. Several experimental systems to which this work is of relevance can be imagined—here we briefly mention a few.

(i) A particularly attractive prospect for probing the physics discussed here is in a type-II superconductor in a strong magnetic field in the presence of spin-orbit scattering impurities. At low fields, the system will be in the superconducting ''insulator phase.'' With increasing field, under conditions discussed in Ref. 1, there will be a transition (at zero temperature) to a superconducting ''metal'' phase. This transition can be probed, for instance, by measurements of the low-temperature quasiparticle heat conductivity.[1]

(ii) A number of recent experiments[15] have measured low-temperature heat transport by the quasiparticles in the heavy fermion superconductors. We note that as these systems typically have strong spin-orbit scattering, the results of this paper are of potential importance. Some of the heat conductivity measurements have been motivated by the possibility of identifying the correct pairing symmetry in these superconductors. Such identification could be seriously hampered by the localization issues discussed here. In particular, if the disorder is strong enough to localize the quasiparticles, it is hard to infer whether the pure system has a gap to quasiparticle excitations or not from heat transport measurements alone.

(iii) The properties of superfluid $He-3$ in porous media have been the subject of some experimental studies.[16] In this context, it is interesting to ask if the fermionic $He-3$ quasiparticles in the superfluid are localized or delocalized at zero temperature. This may, perhaps, again be probed by heat transport experiments. It is also interesting to consider the properties of superfluid $He-3$ on a disordered two-dimensional substrate. In the A phase, at weak disorder, a thermal quantum Hall effect[10] is predicted. With increasing disorder, it is possible (though not necessary) that there is a zero-temperature phase transition where the quantization of the thermal Hall conductivity is destroyed *before* the superfluidity is destroyed. If this happens, this would be an experimental realization of the thermal quantum Hall transition discussed in the previous section.[10]

Finally, the general problem of localization of Majorana fermions arises in other contexts as well—a specific example being the random bond Ising model in two dimensions as discussed briefly in Sec. VI. Progress in the localization problem may therefore provide a route to further our understanding of the random bond Ising model.

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