Vortex lines in films: Fields and interactions

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General expressions are given for the magnetic field and energy of arbitrary arrangements of straight and curved vortices in an anisotropic superconductor film of finite thickness within anisotropic London theory. As examples we consider the magnetic field and interaction of straight perpendicular vortex lines in films of finite thickness.

I. INTRODUCTION

The magnetic field and energy of arrangements of parallel or curved Abrikosov vortices in type-II superconductors are conveniently calculated by the London theory. This linear theory has the advantage that the magnetic field $\mathbf{b}(\mathbf{r})$ and current density $\mathbf{j}(\mathbf{r})$ are sums of individual vortex contributions, and the energy is a quadratic expression of $\mathbf{b}$ and $\mathbf{j}$ which may be written as a double sum over the vortex positions.[1] In addition, the London theory is easily generalized to anisotropic superconductors by replacing the isotropic penetration depth λ by three penetration depths λ_a, λ_b, and λ_c, where $\lambda_a=\lambda_b=\lambda_{ab}$ in uniaxial symmetry. London theory assumes all vortex cores (of radius $\approx\xi$, the coherence length) to be well separated and the Ginzburg-Landau (GL) parameter $\kappa=\lambda/\xi\gg 1$ to be large.

The general solution of London theory for arbitrary vortex arrangements that have been given so far, apply only to infinitely large superconductors, or to vortices far away from the specimen surface. In the present paper we generalize these expressions to the presence of two parallel surfaces, i.e., to infinite superconductor films or plates of arbitrary thickness d.

The problem of perpendicular vortices in a very thin film ($d\ll\lambda$) was solved first by Pearl[2] for vortices perpendicular to the film. Such vortices interact mainly via their stray field, which extends far into the vacuum and which in the film causes a current density $\mathbf{j}$ that decreases as $1/r$ at large distances r. The force between two such Pearl vortices with one quantum of flux Φ_0 is unscreened and of long range, $j\Phi_0 d \propto 1/r$. Pearl's theory of "pancake vortices" has been generalized to arrangements of parallel thin superconducting layers by Clem,[3] who presented an elegant description of strongly anisotropic layered superconductors, e.g., high-T_c superconductors. The interaction between straight vortices in films of finite thickness was calculated in Refs. 4 and 5.

The stray field outside a film containing vortices should be known for comparison with data obtained by various experimental methods. In Ref. 6 the magnetic field and order parameter of one vortex in a superconducting half space was calculated from GL theory for comparison with scanning-tunneling-microscopy data, but the contribution of the inner stray field (see below) was overlooked in that work. In a recent paper[7] the magnetic field of a vortex inside a highly anisotropic superconductor with a surface parallel to the ac plane and perpendicular to the vortex line, was derived from anisotropic London theory and compared with experiments using a scanning superconducting quantum interference device microscope.

In the present paper we calculate the magnetic field and energy of arbitrary arrangements of curved or straight vortices in a superconductor film of finite thickness d from anisotropic London theory. To do this we shall use the general recipe described in Ref. 8.

First, the boundary condition that no current leaves the surface is satisfied by adding to the magnetic field of the real vortices inside the film, the field generated by appropriate image vortices located outside the film. The sum of these two fields inside the film satisfies the London equation with the correct vortex singularities, its currents at the surface flow parallel to the surface, and its parallel field component is zero at the surface. The perpendicular field component, however, is discontinuous at the surface since the vortex and image fields are valid only inside the superconductor. Therefore, one has to add a third field, which compensates this discontinuity and makes the magnetic field continuous across the surface. This third field satisfies the Laplace equation $\nabla^2\mathbf{b}=0$ outside the superconductor (outer stray field) and the homogeneous London equation $\nabla^2\mathbf{b}=\lambda^{-2}\mathbf{b}$ inside the superconductor (inner stray field). One may say that outside the film this third field is the stray field caused by a layer of magnetic monopoles on the surface of the superconductor. A general expression for this "magnetic surface charge" density is given in Sec. III.

This stray field in general has also a parallel component at the surface, which causes the vortex field to widen like a trumpet when it approaches the surface, see Fig. 1. In addition, there may be a widening also of the vortex core (the tube of suppressed superconducting order parameter), which was calculated from GL theory in Ref. 6. But for large $\kappa \gg 1$ this core widening only changes the cutoff of the logarithmic singularity of the vortex field and should thus be difficult to observe at distances larger than several ξ from the surface.

To fix ideas, we give here the magnetic field of a single straight vortex centered on the z axis (at $x=y=0$) in a su-

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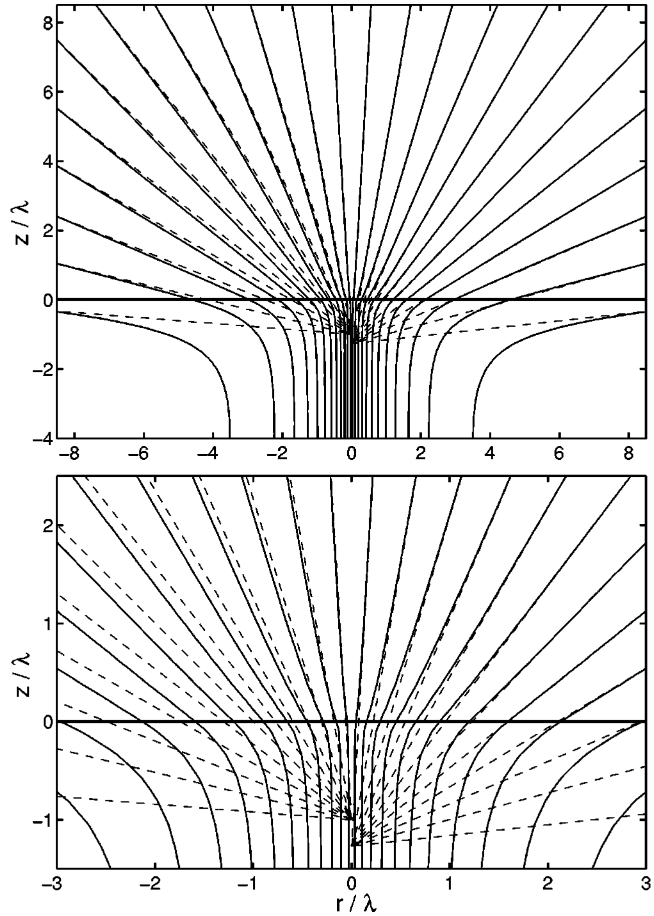

FIG. 1. Magnetic field lines of a straight vortex in a superconducting half space. London theory, $\kappa=20$. The dashed lines give the radial field lines of a magnetic charge of size $2\Phi_0$ positioned on the vortex axis at a depth $-z_0=\lambda$ (left half, the far field) and $-z_0=1.27\lambda$ (right half, a better fit to the near field). The lower plot enlarges the center of the upper plot.

perconducting half space (or very thick film) and in a film of arbitrary thickness d surrounded by vacuum. In this cylindrical geometry, the magnetic field $\mathbf{b}=\nabla\times\mathbf{A}$ conveniently follows from a vector potential, which has only a φ component, $\mathbf{A}=\hat{\boldsymbol{\varphi}}A(r,z)$ [$r=\sqrt{x^2+y^2}$, $\varphi=\arctan(y/x)$] yielding the two field components

$$b_r(r,z)=-\frac{\partial}{\partial z}A,\quad b_z(r,z)=-\frac{1}{r}\frac{\partial}{\partial r}(Ar). \tag{1}$$

The vector potential inside the superconductor satisfies the modified London equation

$$\nabla\times\nabla\times\mathbf{A}=\lambda^{-2}\left(\frac{\Phi_0}{2\pi r}\hat{\boldsymbol{\varphi}}-\mathbf{A}\right), \tag{2}$$

and $\nabla\times\nabla\times\mathbf{A}=0$ in the vacuum. The current density, too, has only a φ component, $\mathbf{j}=\mu_0^{-1}\nabla\times\mathbf{b}=\hat{\boldsymbol{\varphi}}j(r,z)$ with

$$j(r,z)=\mu_0\lambda^{-2}\left[\frac{\Phi_0}{2\pi r}-A(r,z)\right]. \tag{3}$$

For a superconductor filling the half space $z\leqslant 0$ and vacuum at $z>0$, one finds by the method of Ref. 8,

$$A(r,z)=\Phi_0\int\frac{d^2k}{4\pi^2}\frac{-i\mathbf{k}\cdot\hat{\mathbf{r}}}{k^2}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{1+k^2\lambda^2}f(k,z)$$
$$=\frac{\Phi_0}{2\pi\lambda^2}\int_0^\infty dk\frac{J_1(kr)}{k^2+\lambda^{-2}}f(k,z), \tag{4}$$

$$b_r(r,z)=\Phi_0\int\frac{d^2k}{4\pi^2}\frac{-i\mathbf{k}\cdot\hat{\mathbf{r}}}{k^2}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{1+k^2\lambda^2}g(k,z)$$
$$=\frac{\Phi_0}{2\pi\lambda^2}\int_0^\infty dk\frac{J_1(kr)}{k^2+\lambda^{-2}}g(k,z), \tag{5}$$

$$b_z(r,z)=\Phi_0\int\frac{d^2k}{4\pi^2}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{1+k^2\lambda^2}f(k,z)$$
$$=\frac{\Phi_0}{2\pi\lambda^2}\int_0^\infty dk\frac{kJ_0(kr)}{k^2+\lambda^{-2}}f(k,z), \tag{6}$$

with $g(k,z)=-\partial f(k,z)/\partial z$ and

$$f(k,z)=\frac{\tau}{k+\tau}e^{-kz},\quad z>0,$$

$$f(k,z)=1-\frac{k}{k+\tau}e^{\tau z},\quad z\leqslant 0, \tag{7}$$

where $\tau=\sqrt{k^2+\lambda^{-2}}$ and $J_0(x)$ and $J_1(x)$ are Bessel functions. In this section we use the notation $\mathbf{r}=(x,y)$, $\hat{\mathbf{r}}=r/r$, $\mathbf{k}=(k_x,k_y)$, and $k^2=k_x^2+k_y^2$. The unity in Eq. (7) stems from the vortex and its image, which together yield the field of an infinite straight vortex. The other terms stem from the outer and inner stray field. The finite vortex core may be considered by inserting a cutoff factor $\exp(-\xi^2k^2)$ in Eqs. (4)–(6) (Ref. 1) or, more accurately, by replacing the variable r by $\sqrt{r^2+2\xi^2}$. This Clem model[9] for the vortex core was confirmed to be a very good approximation to the exact numerical solution of the GL theory for periodic vortex lattices.[10] Various cutoffs are discussed in detail in Ref. 11.

Figure 1 shows the magnetic field lines of this vortex inside and outside the superconductor, plotted as contour lines of $rA(r,z)$ with quadratic level spacing.[12] The dashed lines in the left half of Fig. 1 show the field of a magnetic monopole of strength $2\Phi_0$ positioned at $x=y=0$, $z=z_0=-\lambda$. This radial field, $\mathbf{b}=-\nabla\Phi(\mathbf{r})$ with $\Phi=(\Phi_0/2\pi)[x^2+y^2+(z-z_0)^2]^{-1/2}$, describes the field far above the superconductor, cf. Sec. III and Ref. 7. Interestingly, the field closer to the vortex tip is much better fitted by choosing $z_0=-1.27\lambda$, see the dashed lines in the right half of Fig. 1.

The same expressions (4)–(6) apply also to the vortex in a thin film of finite thickness d, filling the space $-d\leqslant z\leqslant 0$, but now one has (see Sec. III)

$$f(k,z)=\begin{cases}c_1e^{-kz}, & z>0\\ 1+c_2e^{\tau z}+c_3e^{-\tau z}, & -d\leqslant z\leqslant 0\\ c_1e^{k(z+d)}, & z<-d,\end{cases} \tag{8}$$

with

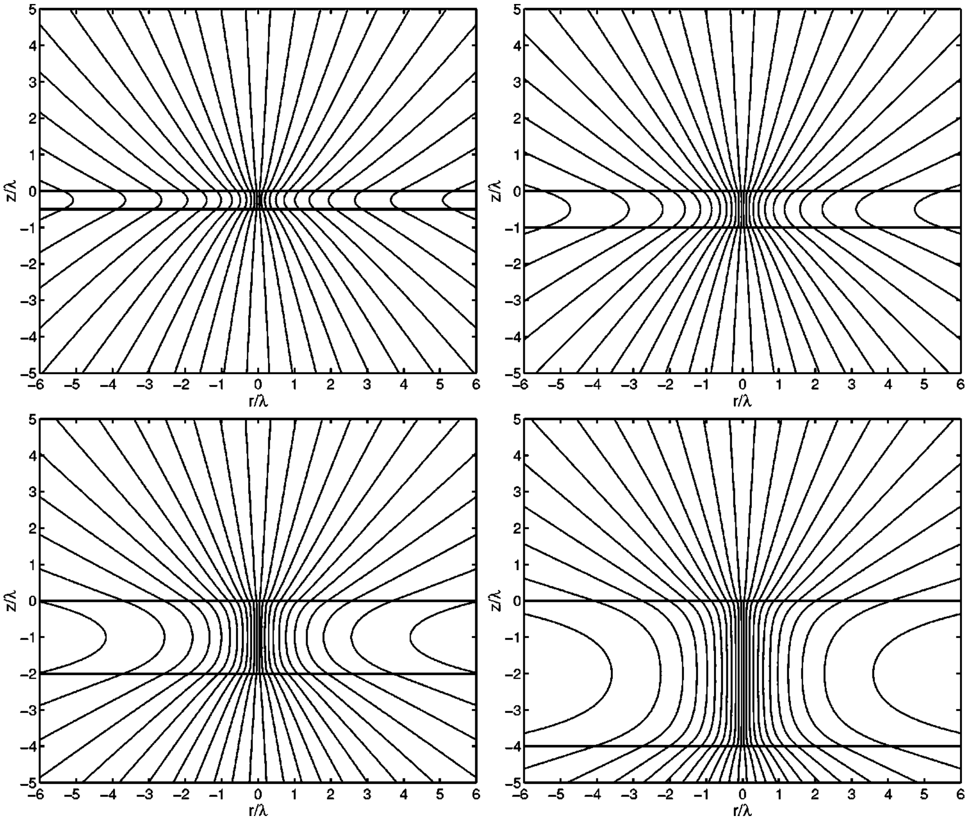

FIG. 2. Magnetic field lines of a straight vortex in superconducting films of various thicknesses $d/\lambda=0.5$, 1, 2, 4. London theory, $\kappa=20$. The film occupies the space $-d\leqslant z\leqslant 0$.

$$c_1(k)=[(k+\tau)e^{\tau d}+(k-\tau)e^{-\tau d}-2k]\frac{\tau}{c_4},$$

$$c_2(k)=[k-\tau-(k+\tau)e^{\tau d}]\frac{k}{c_4},$$

$$c_3(k)=[-k-\tau+(k-\tau)e^{-\tau d}]\frac{k}{c_4},$$

$$c_4(k)=(k+\tau)^2e^{\tau d}-(k-\tau)^2e^{-\tau d}. \tag{9}$$

One easily verifies that $1+c_2+c_3=c_1$, as required by continuity of $\mathbf{b}$ at $z=0$ and $z=-d$, and that $f(k,z)=f(k,d-z)$ as required by symmetry. Our Eqs. (8) and (9) are identical to the bulky Eqs. (4)–(11) of Ref. 5. The magnetic field lines of this vortex are shown for films of various thicknesses in Fig. 2.

II. GENERAL SOLUTION

In this section we derive general expressions for the magnetic field and energy of arbitrary arrangements of straight or curved vortices in a superconducting film of thickness d, with planar surfaces parallel to each other and to the xy plane, occupying the region $-d\leqslant z\leqslant 0$. The superconductor is assumed to be anisotropic, with the c axis along z, and is characterized by the penetration depths λ_{ab} and λ_c for currents parallel and perpendicular to the ab plane, respectively.

Any configuration of vortex lines in the film is characterized by a vectorial vorticity distribution $\boldsymbol{\nu}(\mathbf{r})$ or its Fourier transform $\boldsymbol{\nu}(\mathbf{k})$ (Ref. 13) defined as

$$\boldsymbol{\nu}(\mathbf{k})=\int d^3r e^{-i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\nu}(\mathbf{r}). \tag{10}$$

Since vortex lines form closed loops or lines that begin and terminate at the film surfaces one has $\boldsymbol{\nabla}\cdot\boldsymbol{\nu}(\mathbf{r})=0$ or $\mathbf{k}\cdot\boldsymbol{\nu}(\mathbf{k})=0$. For vortex lines with vanishing core diameter, $\boldsymbol{\nu}(\mathbf{k})$ is given by a sum of line integrals along the vortices,

$$\boldsymbol{\nu}(\mathbf{k})=\sum_j \oint d\boldsymbol{l}_j e^{i\mathbf{k}\cdot\mathbf{r}_j}. \tag{11}$$

Inside the film the magnetic field $\mathbf{b}^{\mathrm{film}}(\mathbf{r})$ satisfies the anisotropic London equation,

$$\nabla\times[\boldsymbol{\Lambda}(\nabla\times\mathbf{b}^{\mathrm{film}})]+\mathbf{b}^{\mathrm{film}}=\phi_0\boldsymbol{\nu}, \tag{12}$$

where the tensor $\boldsymbol{\Lambda}$ is given by $\Lambda_{\alpha\beta}=\Lambda_\alpha\delta_{\alpha\beta}$ $(\alpha,\beta=x,y,z)$ with $\Lambda_x=\Lambda_y=\lambda_{ab}^2$ and $\Lambda_z=\lambda_c^2$. Outside the film, assuming vacuum, the magnetic field can be derived from a scalar potential that satisfies the Laplace equation, that is

$$\mathbf{b}^{\mathrm{vac}}=-\nabla\Phi,\quad \nabla^2\Phi=0. \tag{13}$$

The boundary conditions at the surfaces between the superconductor and the vacuum ($z=0$ and $z=-d$) are that the perpendicular component of the current vanishes and that the magnetic field is continuous.

We solve the London equation, Eq. (12), by the method of images introduced in Ref. 8. First a vortex distribution $\boldsymbol{\nu}^{\mathrm{vi}}(\mathbf{r})$ is defined in all space $(-\infty<z<\infty)$ such that the current generated by it does not cross the film surfaces and that inside the film the vorticity is prescribed. As shown in Ref. 8, $\boldsymbol{\nu}^{\mathrm{vi}}$ consists of the vortex distribution $\boldsymbol{\nu}$ and its specular

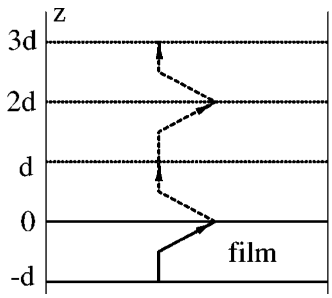

FIG. 3. Periodic arrangement of a distorted vortex line (solid line) and its images (dashed lines) for a film. Two periods are shown for this example.

images. For our film this distribution is periodic in the z direction with period $2d$, see Fig. 3. For the basic interval $-d \leqslant z \leqslant d$ one has

$$\boldsymbol{\nu}^{\rm vi}(\mathbf{r}) = \boldsymbol{\nu}(\mathbf{r}), \quad -d \leqslant z \leqslant 0,$$

$$\boldsymbol{\nu}^{\rm vi}_{\perp}(x,y,z) = -\boldsymbol{\nu}_{\perp}(x,y,-z), \quad 0 \leqslant z \leqslant d,$$

$$\nu^{\rm vi}_z(x,y,z) = \nu_z(x,y,-z), 0 \leqslant z \leqslant d, \tag{14}$$

where $\perp$ stands for the vector component parallel to the xy plane. The magnetic field inside the film is then

$$\mathbf{b}^{\rm film} = \mathbf{b}^{\rm vi} + \mathbf{b}^{\rm stray}. \tag{15}$$

Here $\mathbf{b}^{\rm vi}$ is the field produced by the vortex distribution and its images and is obtained by solving London equation in all space with $\boldsymbol{\nu}^{\rm vi}(\mathbf{r})$ as the field source. The stray field inside the film, $\mathbf{b}^{\rm stray}$, is a solution of the homogeneous London equation. As shown in Ref. 8, $\mathbf{b}^{\rm stray}$ is required to make $\mathbf{b}^{\rm film}$ continuous at the film surfaces.

The field $\mathbf{b}^{\rm vi}$ is more conveniently expressed in terms of Fourier transforms.[1] Since $\boldsymbol{\nu}^{\rm vi}(\mathbf{r})$ is periodic in the z direction with period $2d$, so is the field produced by it, $\mathbf{b}^{\rm vi}$. Thus, its Fourier transform, defined as in Eq. (10), is nonvanishing only for discrete values of $k_z = k_m \equiv m\pi/d$, $m = 0, \pm 1, \pm 2, \ldots$. The Fourier transform of $\boldsymbol{\nu}^{\rm vi}(\mathbf{r})$ is given by

$$\boldsymbol{\nu}^{\rm vi}(\mathbf{k}_{\perp}, k_m) = \int d^2 r_{\perp} e^{-i\mathbf{k}_{\perp}\cdot\mathbf{r}_{\perp}} \int_{-d}^{d} dz e^{-ik_m z} \boldsymbol{\nu}^{\rm vi}(\mathbf{r}_{\perp}, z). \tag{16}$$

Using Eqs. (14), $\boldsymbol{\nu}^{\rm vi}_{\perp}(\mathbf{k}_{\perp}, k_m)$ can be expressed in terms of the vortex distribution inside the film as

$$\boldsymbol{\nu}^{\rm vi}_{\perp}(\mathbf{k}_{\perp}, k_m) = \int d^2 r_{\perp} e^{-i\mathbf{k}_{\perp}\cdot\mathbf{r}_{\perp}} \int_{-d}^{0} dz$$
$$(-2i)\sin(k_m z)\boldsymbol{\nu}_{\perp}(\mathbf{r}_{\perp}, z),$$

$$\nu^{\rm vi}_z(\mathbf{k}_{\perp}, k_m) = \int d^2 r_{\perp} e^{-i\mathbf{k}_{\perp}\cdot\mathbf{r}_{\perp}} \int_{-d}^{0} dz$$
$$2\cos(k_m z)\nu_z(\mathbf{r}_{\perp}, z). \tag{17}$$

The field $\mathbf{b}^{\rm vi}$ is then given by

$$b^{\rm vi}_l(\mathbf{k}_{\perp}, k_m) = g_l(\mathbf{k}_{\perp}, k_m)\nu^{\rm vi}_l(\mathbf{k}_{\perp}, k_m),$$

$$b^{\rm vi}_t(\mathbf{k}_{\perp}, k_m) = g_t(\mathbf{k}_{\perp}, k_m)\nu^{\rm vi}_t(\mathbf{k}_{\perp}, k_m),$$

$$b^{\rm vi}_z(\mathbf{k}_{\perp}, k_m) = g_l(\mathbf{k}_{\perp}, k_m)\nu^{\rm vi}_z(\mathbf{k}_{\perp}, k_m), \tag{18}$$

where we have defined longitudinal and transverse components

$$b^{\rm vi}_l(\mathbf{k}) = k_x b^{\rm vi}_x(\mathbf{k}) + k_y b^{\rm vi}_y(\mathbf{k}) = -k_z b^{\rm vi}_z(\mathbf{k}),$$

$$b^{\rm vi}_t(\mathbf{k}) = k_x b^{\rm vi}_y(\mathbf{k}) - k_y b^{\rm vi}_x(\mathbf{k}), \tag{19}$$

and similarly for $\nu^{\rm vi}_l$ and $\nu^{\rm vi}_t$. The g functions in Eqs. (18) are given by

$$g_l(k) = \frac{\phi_0}{1 + k^2\lambda^2_{ab}}, \tag{20}$$

$$g_t(k_{\perp}, k_z) = \frac{\phi_0}{1 + k^2_{\perp}\lambda^2_c + k^2_z\lambda^2_{ab}}. \tag{21}$$

To obtain the fields $\mathbf{b}^{\rm stray}$ and $\mathbf{b}^{\rm vac}$ (the inner and outer stray fields) in terms of the vortex distribution we solve, respectively, the homogeneous London equation and Laplace equation and require that the magnetic field is continuous at the film surfaces. The results are

$$\mathbf{b}^{\rm stray}_{\perp}(\mathbf{r}) = \int \frac{d^2 k_{\perp}}{4\pi^2} e^{i\mathbf{k}_{\perp}\cdot\mathbf{r}_{\perp}} i\mathbf{k}_{\perp}[\gamma^+ e^{\tau z} + \gamma^- e^{-\tau z}],$$

$$b^{\rm stray}_z(\mathbf{r}) = \int \frac{d^2 k_{\perp}}{4\pi^2} e^{i\mathbf{k}_{\perp}\cdot\mathbf{r}_{\perp}} \frac{k^2_{\perp}}{\tau}[\gamma^+ e^{\tau z} - \gamma^- e^{-\tau z}], \tag{22}$$

where $-d \leqslant z \leqslant 0$ and $\tau = \sqrt{k^2_{\perp} + \lambda^{-2}_{ab}}$, and

$$\mathbf{b}^{\rm vac}(\mathbf{r}) = \int \frac{d^2 k_{\perp}}{4\pi^2} e^{i\mathbf{k}_{\perp}\cdot\mathbf{r}_{\perp}}[i\mathbf{k}_{\perp} - k_{\perp}\hat{\mathbf{z}}]\phi^+ e^{-k_{\perp} z},$$
$$z \geqslant 0,$$

$$\mathbf{b}^{\rm vac}(\mathbf{r}) = \int \frac{d^2 k_{\perp}}{4\pi^2} e^{i\mathbf{k}_{\perp}\cdot\mathbf{r}_{\perp}}[i\mathbf{k}_{\perp} + k_{\perp}\hat{\mathbf{z}}]\phi^- e^{k_{\perp} z},$$
$$z \leqslant -d. \tag{23}$$

The coefficients γ and ϕ are

$$\gamma^-(\mathbf{k}_{\perp}) = \frac{\tau[\mathcal{A}(k_{\perp} - \tau)e^{-\tau d} - \mathcal{B}(k_{\perp} + \tau)]}{k_{\perp}\mathcal{C}},$$

$$\gamma^+(\mathbf{k}_{\perp}) = \frac{\tau[\mathcal{A}(k_{\perp} + \tau)e^{\tau d} - \mathcal{B}(k_{\perp} - \tau)]}{k_{\perp}\mathcal{C}}, \tag{24}$$

$$\phi^+(\mathbf{k}_{\perp}) = (\tau/k_{\perp}\mathcal{C})$$
$$\times\{-2k_{\perp}\mathcal{B} + [(k_{\perp} + \tau)e^{\tau d} + (k_{\perp} - \tau)e^{-\tau d}]\mathcal{A}\},$$

$$\phi^-(\mathbf{k}_{\perp}) = (-\tau e^{k_{\perp} d}/k_{\perp}\mathcal{C})$$
$$\times\{-2k_{\perp}\mathcal{A} + [(k_{\perp} + \tau)e^{\tau d} + (k_{\perp} - \tau)e^{-\tau d}]\mathcal{B}\}, \tag{25}$$

where

$$\mathcal{C}(k_\perp)=e^{-\tau d}(k_\perp-\tau)^2-e^{\tau d}(k_\perp+\tau)^2 \tag{26}$$

[cf. Eq. (9)] and $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A}(\mathbf{k}_\perp)=\frac{1}{2d}\sum_m g_l(\mathbf{k}_\perp,k_m)\,\nu_z^{\rm vi}(\mathbf{k}_\perp,k_m), \tag{27}$$

$$\mathcal{B}(\mathbf{k}_\perp)=\frac{1}{2d}\sum_m e^{-ik_m d}g_l(\mathbf{k}_\perp,k_m)\,\nu_z^{\rm vi}(\mathbf{k}_\perp,k_m). \tag{28}$$

According to Eqs. (18) and (19), $\mathcal{A}$ and $\mathcal{B}$ are, respectively, the Fourier transforms of $b_z^{\rm vi}(\mathbf{r})$ at the surfaces $z=0$ and $z=-d$. These functions describe two surface layers of magnetic monopoles, or magnetic surface charges, which generate the inner and outer stray fields. Note that because of $k_m=m\pi/d$ one has in Eq. (28) $\exp(-ik_m d)=(-1)^m$.

The total energy of the vortex distribution can be written as

$$E=E_{\rm film}+E_{\rm vac}, \tag{29}$$

where $E_{\rm film}$ is the sum of the kinetic energy of the supercurrents and of the magnetic field within the film, and $E_{\rm vac}$ is the energy of the magnetic stray field in the vacuum,

$$E_{\rm film}=\frac{1}{2\mu_0}\int d^2r_\perp\int_{-d}^{0}dz$$

$$[(\nabla\times\mathbf{b}^{\rm film})\Lambda(\nabla\times\mathbf{b}^{\rm film})+(b^{\rm film})^2],$$

$$E_{\rm vac}=\frac{1}{2\mu_0}\int d^2r_\perp\left[\int_0^\infty dz(b^{\rm vac})^2+\int_{-\infty}^{-d}dz(b^{\rm vac})^2\right]. \tag{30}$$

Inserting Eq. (15) in Eq. (30), and using the boundary conditions it follows that, as shown in Ref. 8, the cross terms (containing both $\mathbf{b}^{\rm vi}$ and $\mathbf{b}^{\rm stray}$) vanish and $E_{\rm film}$ can be written as

$$E_{\rm film}=E_{\rm vi}+E_{\rm stray}, \tag{31}$$

where $E_{\rm vi}$ is the energy of vortex-vortex and vortex-image interactions and $E_{\rm stray}$ is the London energy of the stray field within the film and its currents. Using Eqs. (18), (22), and (23) we find that

$$E_{\rm vi}=\frac{\phi_0^2}{4\mu_0}\int\frac{d^2k_\perp}{4\pi^2}\frac{1}{2d}\sum_m\sum_\alpha G_\alpha(\mathbf{k}_\perp,k_m)|\nu_\alpha^{\rm vi}(\mathbf{k}_\perp,k_m)|^2, \tag{32}$$

where $\alpha=x,y,z$ and

$$G_x(\mathbf{k})=G_y(\mathbf{k})=\frac{1}{1+k_\perp^2\lambda_c^2+k_z^2\lambda_{ab}^2},$$

$$G_z(\mathbf{k})=\frac{1+k^2\lambda_c^2}{(1+k^2\lambda_{ab}^2)(1+k_\perp^2\lambda_c^2+k_z^2\lambda_{ab}^2)}, \tag{33}$$

$$E_{\rm stray}=\frac{1}{2\mu_0}\int\frac{d^2k_\perp}{4\pi^2}\frac{k_\perp^2}{\tau}\times[(1-e^{-2\tau d})|\gamma^+|^2+(e^{2\tau d}-1)|\gamma^-|^2], \tag{34}$$

and

$$E_{\rm vac}=\frac{1}{2\mu_0}\int\frac{d^2k_\perp}{4\pi^2}k_\perp(e^{-2k_\perp d}|\phi^-|^2+|\phi^+|^2). \tag{35}$$

III. STRAIGHT VORTEX LINES

As an application of the general results discussed above we consider straight vortex lines perpendicular to the film surfaces. For any distribution of such lines, the vorticity $\boldsymbol{\nu}^{\rm vi}(\mathbf{r})$ has only a z component and is independent of z. Thus, according to Eq. (14),

$$\nu_z^{\rm vi}(\mathbf{k}_\perp,k_m)=2d\,\delta_K(k_m)\int d^2r_\perp e^{-i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\nu_z(\mathbf{r}_\perp), \tag{36}$$

where δ_K denotes the Kronecker delta function. The field of these vortex lines and of their images has only a z component, is independent of z, and is related to $\nu_z^{\rm vi}$ by Eq. (18). According to Eq. (28), the coefficients $\mathcal{A}$ and $\mathcal{B}$ are equal since only the term $k_m=0$ contributes. In this case γ and ϕ from Eqs. (24) and (25) reduce to

$$\gamma^+=-e^{\tau d}\gamma^-,\quad \gamma^-=\frac{\phi^+}{1-e^{\tau d}},\quad \phi^-=-e^{k_\perp d}\phi^+,$$

$$\phi^+=\mathcal{A}\tau\frac{(k_\perp+\tau)e^{\tau d}+(k_\perp-\tau)e^{-\tau d}-2k_\perp}{k_\perp\mathcal{C}}. \tag{37}$$

The expressions for the energies $E_{\rm vi}$, $E_{\rm stray}$, and $E_{\rm vac}$ simplify considerably in this case: $E_{\rm vi}$ is just the total energy (per thickness d) of the vortex-line distribution calculated as if the lines were infinite, and $E_{\rm stray}$ and $E_{\rm vac}$ read

$$E_{\rm stray}=\frac{1}{\mu_0}\int\frac{d^2k_\perp}{4\pi^2}\frac{k_\perp^2\coth(\tau d/2)}{\tau}|\phi^+|^2,$$

$$E_{\rm vac}=\frac{1}{\mu_0}\int\frac{d^2k_\perp}{4\pi^2}k_\perp|\phi^+|^2. \tag{38}$$

Next we calculate the magnetic field of a single vortex line passing through the origin. In this case one has

$$\nu_z^{\rm vi}(\mathbf{k}_\perp,k_m)=2d\,\delta_K(k_m), \tag{39}$$

and

$$b_z^{\rm vi}(\mathbf{k}_\perp,k_m)=2d\,\delta_K(k_m)\frac{\phi_0}{1+k_\perp^2\lambda_{ab}^2},$$

$$\mathcal{A}=\frac{\phi_0}{1+k_\perp^2\lambda_{ab}^2}. \tag{40}$$

The magnetic field $b_z^{\rm vi}$ is just that of an infinite vortex line. The stray fields inside the film and in the vacuum follow from Eqs. (22) and (23), using Eqs. (37) and Eqs. (40). The expressions thus obtained are presented in Eqs. (4)–(6) and (8) and are in agreement with the results of Ref. 5.

Now we consider the multipole expansion of $\mathbf{b}^{\text{vac}}(\mathbf{r})$, which is useful at large distances from the film. We write

$$\Phi(\mathbf{r})=\frac{Q_0}{r}+Q_2\frac{z}{r^3}+Q_4\frac{3z^2-r^2}{r^5}, \tag{41}$$

where $r=(x^2+y^2+z^2)^{1/2}$ is the distance from the origin and Q_0, Q_2 and Q_4 are, respectively, the monopole, dipole and quadrupole moments. To show this and to obtain the moments we proceed as follows. According to Eqs. (23), the vacuum stray field for $z>0$ is given by

$$\mathbf{b}^{\text{vac}}=-\boldsymbol{\nabla}\Phi,$$

$$\Phi(\mathbf{r})=-\int\frac{d^2k_\perp}{4\pi^2}e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\phi^+e^{-k_\perp z},\quad z\geqslant 0. \tag{42}$$

First we expand ϕ^+ in Eq. (42) for small $k_\perp$. Using Eqs. (37), (40), and (26) we find

$$\phi^+(k_\perp)=-\frac{2\pi}{k_\perp}(Q_0+Q_2k_\perp+Q_4k_\perp^2+\cdots), \tag{43}$$

with

$$Q_0=\frac{\phi_0}{2\pi},$$

$$Q_2=-\frac{\phi_0\lambda_{ab}}{2\pi}\coth(d/2\lambda_{ab}),$$

$$Q_4=-\frac{\phi_0\lambda^2}{2\pi}[1-\coth(d/\lambda_{ab})\coth(d/2\lambda_{ab})]. \tag{44}$$

Next we insert in Eq. (42) the identities

$$k_\perp^n e^{-k_\perp z}=(-1)^n\frac{\partial^n e^{-k_\perp z}}{\partial z^n}, \tag{45}$$

$$\int\frac{d^2k_\perp}{(2\pi)^2}\frac{e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp-k_\perp z}}{k_\perp}=2\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2}=\frac{1}{2\pi r}. \tag{46}$$

Carrying out the derivatives with respect to z we obtain Eq. (41).

The multipole expansion converges rapidly for $r\gg\lambda$ in films that are not too thin ($d\leqslant\lambda$) and for $r\gg\lambda^2/d$ in thin films. For bulk samples ($d\gg\lambda$) containing one vortex at the origin, the multipole moments are $Q_0=\phi_0/2\pi$, $Q_2=-\phi_0\lambda/2\pi$ and $Q_4=0$. This result is similar to that obtained in Ref. 7 for the vacuum stray field of a vortex line terminating at the surface parallel to the ac plane. An identical field exists in the vacuum for $z<-d$, with r interpreted as the distance to the vortex core emerging at $z=-d$.

This result means that at large distances from a thick film containing one vortex, the magnetic field looks as if it would originate from a magnetic monopole located inside the superconductor at a depth $|Q_2/Q_0|=\lambda$, see Fig. 1. The strength of this monopole (or magnetic charge) is $2\Phi_0$ (two quanta of flux) as can be seen from Eqs. (41) and (44) yielding a potential $\Phi(\mathbf{r})=(2\Phi_0/4\pi r)(1-z\lambda/r^2)$. The factor 2 in $2\Phi_0$ comes from the fact that the magnetic field lines of a vortex emerging at a planar surface and carrying a flux Φ_0, can spread only into the *half* space above the superconductor, i.e., over a solid angle of 2π. As opposed to this, the field lines of a''free'' magnetic monopole realized at the tip of a long thin coil or a ferromagnetic whisker, can spread over the *full* solid angle of 4π.

The vacuum stray field can also be thought as resulting from a surface density of magnetic charge. It follows from Eqs. (42) and from the identity Eq. (46) that the charge density at the $z=0$ surface is the Fourier transform of $\sigma^+(\mathbf{k}_\perp)=k_\perp\phi^+(k_\perp)/2\pi$. This result is not restricted to a single straight vortex line, but is valid for any distribution of vortices in the film. A similar surface charge density exists on the $z=-d$ surface with ϕ^+ replaced by ϕ^-.

Next we calculate the energy of interaction of a pair of straight vortex lines perpendicular to the film surfaces, one at the origin and the other one displaced from it by a vector $\mathbf{R}$ along the xy plane. In this case the vorticity contains two terms,

$$\nu_z^{\text{vi}}(\mathbf{k}_\perp,k_m)=2\delta_K(k_m)(1+e^{-i\mathbf{k}_\perp\cdot\mathbf{R}}), \tag{47}$$

yielding

$$\mathcal{A}=\frac{\phi_0}{1+k_\perp^2\lambda_{ab}^2}(1+e^{-i\mathbf{k}_\perp\cdot\mathbf{R}}). \tag{48}$$

The contribution to the interaction energy from E_{vi} is just that of two infinite vortex lines, namely,

$$E_{\text{vi}}^{\text{int}}=\frac{\phi_0^2 d}{2\pi\lambda^2\mu_0}K_0(R/\lambda). \tag{49}$$

The interaction energies from the stray field and current in the film and from the vacuum stray field are given, respectively, by the R-dependent part of Eqs. (34) and (35), using Eqs. (48) and $\mathcal{A}=\mathcal{B}$. We find that the large-distance behavior of these quantities can also be expressed in terms of the multipole moments of a single vortex as

$$E_{\text{stray}}^{\text{int}}=-\frac{4\pi 2\lambda Q_0Q_2}{\mu_0\quad R^3},$$

$$E_{\text{vac}}^{\text{int}}=\frac{4\pi}{\mu_0}\left(\frac{Q_0^2}{R}-\frac{\lambda^2Q_0^2+2Q_0Q_4}{R^3}\right). \tag{50}$$

The monopole contribution in $E_{\text{vac}}^{\text{int}}$ is just the classical result obtained by Pearl.[2]

IV. CONCLUSIONS

In conclusion, we derived general expressions for the magnetic field and energy of arbitrary arrangements of vortices in a superconducting film or plate of thickness d satisfying the uniaxially anisotropic London theory with the c axis perpendicular to the film. This general solution is composed of contributions of the vortices and their images and of an inner and outer stray field, which is caused by the surfaces of the film. The energies of the vortex and stray field terms contribute separately, i.e., there are no mixed terms. As an example, we consider straight vortices perpendicular to the film and present figures of the magnetic field lines of one straight vortex inside and outside films of various thicknesses. The magnetic field far from the film and the interaction between two vortices may be expressed in terms of the monopole, dipole, and quadrupole moments of each vortex.

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