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Nonlinear second- and first-sound wave equations in ^{3}He-^{4}He mixtures

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We derive nonlinear Burgers equations for first and second sound in mixtures of ^{3}He-^{4}He, using a reductive perturbation method and obtain expressions for the nonlinear and dissipation coefficients. We further find a diffusion equation for a coupled temperature-concentration mode. The amplitude of first (second) sound generated from second (first) sound in mixtures is also derived. Our derivation includes the dependence of thermodynamical quantities on temperature, pressure, and ^{3}He concentration, and is valid up to a first order in terms of the isobaric expansion coefficient. We show that close to the λ line the nonlinearity of second sound in mixtures is enhanced as compared with pure ^{4}He.

I. INTRODUCTION

Second-sound measurements provide a convenient way to investigate both static and dynamic critical properties of the superfluid transition in ^{4}He as well as in ^{4}He-^{3}He mixtures.[1,2] They can provide information on the superfluid density and second-sound damping deep into the critical region.[3] However, in this region second sound propagation becomes highly nonlinear. Near the λ line, the nonlinear coefficient is negative and its magnitude diverges.[4–6] Hence, even a small amplitude wave quickly develops trailing shock. This effect is especially pronounced near the mixture tricritical point.[7] Therefore, to obtain useful information from a second sound measurement it has to be analyzed within the framework of a suitable nonlinear model. In principle, such a model is provided by the superfluid thermohydrodynamic equations formulated first by Khalatnikov.[8] However, these form a complicated system of coupled partial differential equations (PDE's), not in itself directly suitable for data analysis, and some approximation scheme is necessary. The general approach is to try to decouple the equations, and to obtain simple wave equations for each of the modes of the system, which nevertheless capture the essential nonlinearities. For pure ^{4}He this program is fairly easily implemented,[8] because the propagating modes, first and second sound, are to a good approximation orthogonal. Oscillations in the temperature are accompanied by essentially negligible oscillations in the pressure, and vice versa. This is primarily the result of a small thermal expansion coefficient. In ^{3}He-^{4}He mixtures, however, the situation is more complicated. Second sound in mixtures involves significant oscillations in the ^{3}He concentration X. Thus, while in pure ^{4}He the nonlinear coefficient can readily be obtained setting the expansion coefficient to zero, in mixtures a more careful approach is necessary.

A systematic way to derive appropriate nonlinear wave equations from the complete two fluid thermohydrodynamic equations is provided by the reductive perturbation method (RPM), developed by Taniuti *et al.*[9] This method leads to a set of Burgers equations[10] for the sound modes. It also gives explicit expressions for the nonlinear and damping coefficients, α and μ, in terms of the static and transport properties of the mixture, without the need to make drastic simplifications of the hydrodynamic equations. The objective of this paper is to obtain expressions for the nonlinear and dissipation coefficients for the second- and first-sound modes in the framework of hydrodynamics. Therefore expressions for the nonlinear and damping coefficient for second (first) sound $\alpha_2\,(\alpha_1),\mu_2\,(\mu_1)$ are obtained. (Here and in the following we use the subscript 1 for first sound, 2 for second sound.) We also find that the hydrodynamic equations contain a diffusive mode for mixtures. The nonlinear coefficients are purely functions of the static properties of the fluid, which are reasonably well known. The damping coefficients, however, are functions of the transport coefficients such as the thermal conductivity and the mass diffusion constant. Many (but not all) of the transport coefficients for mixtures have been experimentally investigated during the past decades.[11,12] The diffusive mode depends on the effective thermal conductivity and static properties, on which we have enough information.[11]

The organization of this paper is as follows. In Sec. II we summarize the results of the RPM. We then show in Sec. III how the hydrodynamic equations are brought into a RPM form. In Sec. IV A we present the results for α_2 and μ_2, and their limiting expression near λ line. In Sec. IV B the nonlinear and dissipation coefficients for first sound is derived. In Sec. IV C a diffusion equation for a coupled temperature-concentration mode is derived. In Sec. V we show that first (second) sound can be generated from second (first) sound in mixtures and calculate their local amplitude. In Sec. VI we conclude the paper by pointing out the summary of the main results. Appendix A contains the matrix forms of the hydrodynamic equations. The superfluid hydrodynamic equations are more compactly formulated in terms of the specific entropy and the mass fraction of ^{3}He, rather than in terms of molar quantities. Since the experimental results are usually reported on a per mole basis, we include a conversion table in an Appendix B, together with a table of the various symbols defined in this paper and some numerical values for different parameters.

II. METHOD OF PERTURBATIVE SOLUTION OF NONLINEAR WAVE EQUATION

Consider a set of general one dimensional parabolic partial differential equations written as

$$\frac{\partial}{\partial t}U+A\frac{\partial}{\partial x}U+K_1\frac{\partial}{\partial x}\left(K_2\frac{\partial}{\partial x}U\right)=0 \tag{1}$$

in which U is a vector with the n components u_i and A and K_i are $n\times n$ matrices that depend on those components. The set of PDE's can be decoupled using an expansion of the vector U and the matrix A around U_0, an unperturbed vector

$$U=U^{(0)}+\varepsilon U^{(1)}+\varepsilon^2U^{(2)}+\cdots,$$

$$\begin{aligned}A&=A(U_0)+\varepsilon A_1(U)\big|_{U=U_0}+\cdots\\&=A^{(0)}+\varepsilon U^{(1)}\cdot\nabla_uA^{(0)}+\cdots,\end{aligned}$$

where ε is the parameter of expansion. Using this expansion for U and the coordinate transformation known as Gardner-Morikawa transformations,

$$\xi=\varepsilon(x-\lambda_i^{(0)}t),\tau=\varepsilon^2t \tag{2}$$

the matrix equation (1) decouples into a set of equations for a scalar function $\phi^{(1)}$

$$\frac{\partial}{\partial\tau}\phi_i^{(1)}+\alpha_1\phi_i^{(1)}\frac{\partial}{\partial\xi}\phi_i^{(1)}+\mu_i\frac{\partial^2}{\partial\xi^2}\phi_i^{(1)}=0. \tag{3}$$

Here, $\lambda_i^{(0)}$ are the eigenvalues of the matrix $A^{(0)}$, and R_i are the corresponding eigenvectors. The $\lambda_i^{(0)}$ are the propagation velocities for the modes of the linearized, dissipationless system. The functions $\phi_i^{(1)}$ can be viewed as the expansion coefficients for $U^{(1)}$, $U^{(1)}=\Sigma_i\phi_i^{(1)}R_i$.

The reductive perturbation method provides us with explicit expressions for the nonlinear and damping coefficients in terms of the eigenvectors, and the first derivatives of $\lambda^{(0)}$ with respect to the field r_i. In what follows, we choose the representation $\langle L_i|$ and $|R_i\rangle$ to distinguish between left and right eigenvectors. Then, we have

$$\alpha_i=\langle\nabla_u\lambda_i^{(0)}|R_i\rangle, \tag{4}$$

where $\lambda_i^{(0)}$ is the eigenvalue for the matrix A corresponding to the eigenvector $|R_i\rangle$, $\langle\nabla_u|=(\partial/\partial u_1,\partial/\partial u_2,\ldots,\partial/\partial u_n)$, and the dissipation coefficient μ_i is

$$\mu_i=\frac{\langle L_i|K_0|R_i\rangle}{\langle L_i|R_i\rangle}, \tag{5}$$

where $K_0=(K_1K_2)|_{U=U_0}$.

In the original coordinate frame (x,t) the PDE becomes

$$\frac{\partial}{\partial t}\phi+(\lambda_i^{(0)}+\alpha\phi)\frac{\partial}{\partial x}\phi+\mu\frac{\partial^2}{\partial x^2}\phi=0, \tag{6}$$

where we have dropped the (1) superscript and the subscript i for simplicity. Equation (6) is a Burgers[10,13] equation with a nonlinear coefficient α and a dissipation coefficient μ. $\lambda_i^{(0)}$ is the speed of the wave when the nonlinearity α is suppressed to zero. For an initially symmetric pulse, a shock front is formed when $\alpha>0$, in the opposite case a shock tail develops.

III. SUPERFLUID HYDRODYNAMICS IN MIXTURES

As the starting point for our derivation we take the dissipative superfluid hydrodynamic equations given by[8]

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}=0, \tag{7}$$

$$\frac{\partial j_i}{\partial t}+\frac{\partial\pi_{ik}}{\partial x_k}=\frac{\partial}{\partial x_k}\left[\eta\left(\frac{\partial v_{ni}}{\partial x_k}+\frac{\partial v_{nk}}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_{nk}}{\partial x_k}\right)\right]+\frac{\partial}{\partial x_i}[\zeta_1\nabla\cdot(\mathbf{j}-\rho\mathbf{v}_n)+\zeta_2\nabla\cdot\mathbf{v}_n], \tag{8}$$

$$\frac{\partial}{\partial t}(\rho c)+\nabla\cdot(\rho c\mathbf{v}_n)=\nabla\left[\rho D\left(\nabla_c+\frac{k_T}{T}\nabla T+\frac{k_P}{P}\nabla P\right)\right], \tag{9}$$

$$\frac{\partial}{\partial t}\rho\sigma+\nabla\left[\rho\sigma\mathbf{v}_n+\frac{1}{T}\left(\mathbf{q}-\frac{Z\mathbf{g}}{\rho}\right)\right]=0, \tag{10}$$

$$\frac{\partial}{\partial t}\mathbf{v}_s+\nabla\left(\mu-\frac{Z}{\rho}c+\frac{1}{2}\mathbf{v}_s^2\right)=\nabla[\zeta_3\nabla\cdot(\mathbf{j}-\rho\mathbf{v}_n)+\zeta_4\nabla\cdot\mathbf{v}_n]. \tag{11}$$

Here $\mathbf{j}=\rho\mathbf{v}=\rho_n\mathbf{v}_n+\rho_s\mathbf{v}_s$, $\mathbf{v}_n(\mathbf{v}_s)$ is the velocity of normal (super)fluid component, ρ_n (ρ_s) is the density of the normal (super)fluid component, ρ is the density of the mixture, v is the fluid velocity. For future use we also define the counterflow velocity, $\mathbf{w}=\mathbf{v}_n-\mathbf{v}_s$. The generalized pressure is given by $\pi_{ik}=P\delta_{ik}+\rho_nv_{ni}v_{nk}+\rho_sv_{si}v_{sk}$, σ is the specific entropy, c is the mass concentration of ^{3}He in the mixture, T is the temperature, Z is the chemical potential difference of ^{4}He and ^{3}He, $Z=\rho(\mu_3-\mu_4)$, the ζ_i are second viscosity coefficients, and η is the first viscosity coefficient and we have

$$d\mu=\frac{1}{\rho}dP-\sigma dT+\frac{Z}{\rho}dc-\frac{1}{2}\frac{\rho_n}{\rho}dw^2. \tag{12}$$

The correction to the density and entropy up to the order w^2 can be found from Eq. (12) to be

$$\rho(T,c,P,w)=\rho(T,c,P)+\frac{1}{2}w^2\rho^2\frac{\partial}{\partial P}\frac{\rho_n}{\rho}, \tag{13}$$

$$\sigma(T,c,P,w)=\sigma(T,c,P)+\frac{1}{2}w^2\frac{\partial}{\partial T}\frac{\rho_n}{\rho}, \tag{14}$$

where $w=v_n-v_s$ is the counterflow velocity. The heat flux and impurity current in Eq. (10) are

$$-q=T^2\left[\frac{\partial}{\partial T}\left(\frac{Z}{\rho T}\right)-\frac{k_T}{T}\frac{\partial}{\partial c}\left(\frac{Z}{\rho T}\right)\right]g+\kappa\nabla T \tag{15}$$

and

$$-g=\rho D\left(\nabla c+\frac{k_T}{T}\nabla T+\frac{k_P}{P}\nabla P\right), \tag{16}$$

where D is the diffusion coefficient, κ is the thermal conductivity, k_TD is the thermal diffusion coefficient, and k_pD is the coefficient of barodiffusion. Note that here we have not made any assumptions for the dependence of the thermodynamical quantities so that they can depend on concentration and pressure as well as temperature.

Next, we write the resulting equations in a matrix form

$$B\frac{\partial}{\partial t}U+A'\frac{\partial}{\partial x}U+\frac{\partial}{\partial x}\left(K'\frac{\partial}{\partial x}U\right)=0, \tag{17}$$

where the vector U is

$$U=\begin{pmatrix} P \\ T \\ c \\ v \\ w \end{pmatrix} \tag{18}$$

the 5×5 matrices B, A', and K' are given in Appendix A.

To put Eq. (17) into Eq. (1), we multiply the former by the inverse of B from left. The matrix A of Eq. (1) is given by

$$A=B^{-1}A'. \tag{19}$$

The complete inversion of B and evaluation of Eq. (19) is rather cumbersome. However, since we only need the eigenvectors and eigenvalues of $A^{(0)}$, and the first derivative of the latter with respect to the fields, it suffices to keep terms in A that are at most linear in v and w. We expand matrix A to first order in w and v, as

$$A=A^{(0)}+wA^{(1w)}+vA^{(1v)}. \tag{20}$$

The calculation details of matrices $A^{(0)}$, $A^{(1w)}$, and $A^{(1v)}$ can be found in Appendix A.

The zero order (in v and w) eigenvalues for each of the modes are calculated from

$$A^{(0)}|R\rangle=\lambda^{(0)}|R\rangle, \tag{21}$$

and the first order of perturbation is given by

$$\lambda^{(1w)}=\frac{1}{\langle L|R\rangle}\langle L|A^{(1w)}|R\rangle, \tag{22}$$

$$\lambda^{(1v)}=\frac{1}{\langle L|R\rangle}\langle L|A^{(1v)}|R\rangle. \tag{23}$$

The nonlinear coefficient, α, can then be found by evaluating Eq. (4)

$$\alpha=\frac{\partial\lambda^{(0)}}{\partial P}r_1+\frac{\partial\lambda^{(0)}}{\partial T}r_2+\frac{\partial\lambda^{(0)}}{\partial c}r_3+\lambda^{(1v)}r_4+\lambda^{(1w)}r_5, \tag{24}$$

where r_i are the components of the vector $|R\rangle$. The dissipation coefficient can be calculated using

$$\mu_i=\frac{\langle L_i|B^{-1}K'|R_i\rangle}{\langle L_i|R_i\rangle} \tag{25}$$

(note that in the low-temperature literature the second sound damping μ_i is usually defined as $\mu_i=\frac{-1}{2}D_i$).

IV. RESULTS

A. Second sound in mixtures

The matrix A in Eq. (19) has four nonzero eigenvalues, namely $\pm u_{20}$ and $\pm u_{10}$, the velocities of left and right propagating second and first sound. The fifth eigenvalue is zero, corresponding to a diffusive mode. We will first present the results for second sound. For the second-sound velocity we find

$$u_{20}^2=\frac{\rho_s}{\rho_n}\left(\frac{\bar{\sigma}^2}{\frac{\partial\sigma}{\partial T}}+c^2\frac{\partial}{\partial c}\frac{Z}{\rho}\right)(1+d)^{-1}, \tag{26}$$

where

$$d=\frac{\rho_s}{\rho_n}\left[\left(\frac{c}{\rho}\frac{\partial\rho}{\partial c}\right)^2+2\hat{\bar{\sigma}}\frac{1}{\rho}\frac{\partial\rho}{\partial T}\frac{c}{\rho}\frac{\partial\rho}{\partial c}\right]\simeq\frac{\rho_s}{\rho}\left(\frac{c}{\rho}\frac{\partial\rho}{\partial c}+\hat{\bar{\sigma}}\frac{1}{\rho}\frac{\partial\rho}{\partial T}\right)^2.$$

Here

$$\bar{\sigma}=\sigma-c\frac{\partial\sigma}{\partial c}\ \text{ and }\ \hat{\bar{\sigma}}=\frac{\bar{\sigma}}{\partial\sigma/\partial T}.$$

The limit $c=0$ gives

$$u_{20}^2|_{c=0}=\frac{\rho_s}{\rho_n}\sigma^2\left(\frac{\partial\sigma}{\partial T}\right)^{-1} \tag{27}$$

which is the velocity of second sound in pure ^{4}He. With $\beta=(1/\rho)(\partial\rho/\partial T)=0$, Eq. (26) reduce to the expression derived by Khalatnikov.[8] The importance of considering the thermal expansion coefficient β in ^{4}He is stressed in Ref. 14.

The right eigenvector for the second-sound mode is

$$|R_2\rangle=\begin{bmatrix} \rho_s u_{20}b\Gamma \\ \frac{\rho_s}{\rho}\frac{1}{u_{20}}\hat{\bar{\sigma}}q \\ c\frac{\rho_s}{\rho}\frac{1}{u_{20}} \\ \frac{\rho_s}{\rho}b\Gamma \\ 1 \end{bmatrix}, \tag{28}$$

where

$$q=1-\beta\chi\frac{u_{20}^2}{\bar{\sigma}}\Gamma,$$

$$\Gamma=\left(1-\frac{u_{20}^2}{\partial P/\partial\rho}\right)^{-1}\approx\left[1-\left(\frac{u_{20}}{u_{10}}\right)^2\right]^{-1},$$

$$b=\chi+\hat{\bar{\sigma}}\beta,\ \text{ and }\ \chi=\frac{c}{\rho}\frac{\partial\rho}{\partial c}.$$

This eigenvector is normalized with respect to the counterflow velocity, w, and therefore ϕ in Eq. (6), represents w. This normalization is of course not unique. A transformation to an equation for one of the other fields, P, T, etc., is easily accomplished by scaling with the appropriate component of $|R_2\rangle$. This leaves the overall structure of the Burgers equation unchanged, but leads to a rescaling of the nonlinear coefficient. It is thus necessary to give α a subscript to indicate to which of the otherwise equivalent Burgers equations it belongs. For example, the nonlinear coefficient for a Burgers equation for the temperature amplitude α_T is related to that for the counterflow velocity α_w by

$$\alpha_{2T}=u_{20}\frac{\rho}{\rho_s}\,\frac{\hat{\overline{\sigma}}^{-1}}{q(1+b\Gamma)}\,\alpha_{2v_n}. \tag{29}$$

Using the relation between w and v_n,

$$v_n=\frac{\rho_s}{\rho}\,(1+b\Gamma)w, \tag{30}$$

which can be found from the definitions of w and v in terms of v_n and v_s, and $|R_2\rangle$, we find

$$\alpha_{2w}=\frac{\rho_s}{\rho}\,(1+b\Gamma)\alpha_{2v_n}. \tag{31}$$

Using Eq. (22) the value for the first order corrections to the eigenvalue for the second-sound mode becomes

$$\begin{aligned}\lambda^{(1w)}\langle L_2|R_2\rangle={}&\xi_T\hat{\overline{\sigma}}\left[2\,\frac{\rho}{\rho_n}-\Gamma d-\chi\beta\frac{u_{20}^2}{\bar{\bar{\sigma}}}\,\frac{\rho}{\rho_n}\left(2-\frac{\rho_s}{\rho}\,\Gamma\chi^2\right)\right]\\&+\xi_P\rho u_{20}^2\Gamma b\left(2\,\frac{\rho}{\rho_n}-\Gamma d\right)+c\,\xi_c\left(\frac{\rho}{\rho_n}-\Gamma d\right)\\&+4\,\frac{\rho_s}{\rho_n}\,(1+\Gamma b)+\frac{\rho_s}{\rho}\,\Gamma(1+b\Gamma)d\end{aligned} \tag{32}$$

where

$$\xi_P=\frac{\partial}{\partial P}\,\frac{\rho_s}{\rho},\xi_T=\frac{\partial}{\partial T}\,\frac{\rho_s}{\rho},\ \text{and}\ \ \xi_c=\frac{\partial}{\partial c}\,\frac{\rho_s}{\rho},$$

and

$$\lambda^{(1v)}\langle L_2|R_2\rangle=2+\Gamma b+\Gamma[2\Gamma-(1+\Gamma b)]d, \tag{33}$$

where $\langle L_2|R_2\rangle=2-\Gamma d+2\Gamma^2 d$. In the limit $u_{20}\ll u_{10}$, $\langle L_2|R_2\rangle=2+d$. Substituting the second sound velocity u_{20} for $\lambda^{(0)}$ in Eq. (24), and using the eigenvector (28) for the second sound mode, the expression for the nonlinear coefficient for second sound in mixtures becomes

$$\begin{aligned}\alpha_{2w}={}&u_{20}\frac{\partial u_{20}}{\partial P}\,\rho_s b+\frac{\rho_s}{\rho}\,\frac{1}{u_{20}}\,\frac{\partial u_{20}}{\partial T}\,\hat{\overline{\sigma}}q+\frac{c}{u_{20}}\,\frac{\partial u_{20}}{\partial c}\,\frac{\rho_s}{\rho}\\&+\frac{\rho_s}{\rho}\,b\lambda^{(1v)}+\lambda^{(1w)},\end{aligned} \tag{34}$$

where $\lambda^{(1w)}$ and $\lambda^{(1v)}$ are given in Eqs. (32) and (33), respectively.

For the purpose of studies of critical phenomena (near λ line) we reduce the coefficient α_2 by using the asymptotical behavior $\rho_s\ll\rho_n\approx\rho, d\ll 1, u_{20}^2\ll u_{10}^2\approx\partial P/\partial\rho$. Here we also drop the dependence of ρ on T (Ref. 17) (i.e., $\beta=0$). The expression for the nonlinear coefficient of second sound in this limit in terms of molar concentration X, which is in a more convenient form for experimental purposes, becomes

$$\begin{aligned}\alpha_{2v_n}={}&\frac{M_s}{M_4}\,\frac{\overline{S}T}{C_{XP}}\left(\frac{1}{u_{20}}\,\frac{\partial u_{20}}{\partial T}+\frac{1}{\rho_s/\rho}\,\frac{\partial}{\partial T}\,\frac{\rho_s}{\rho}\right)\\&+\frac{M_s}{M_4}\,X\left(\frac{1}{u_{20}}\,\frac{\partial u_{20}}{\partial X}+\frac{1}{2}\,\frac{1}{\rho_s/\rho}\,\frac{\partial}{\partial X}\,\frac{\rho_s}{\rho}\right)\end{aligned} \tag{35}$$

in which the diverging terms are

$$\frac{\partial}{\partial T}\,\frac{\rho_s}{\rho},\frac{\partial}{\partial X}\,\frac{\rho_s}{\rho},\frac{\partial u_{20}}{\partial T},\ \text{and}\ \ \frac{\partial u_{20}}{\partial X}.$$

Here C_{XP} is the specific heat of mi$\overline{\text{x}}$tures at constant concentration and pressure.[16,1] The ratio $\overline{S}T/C_{XP}$ is larger for mixtures specially close to the tricritical point. Therefore the nonlinear coefficient is negative and large close to the λ line and the concentration terms enhance the nonlinearity.

The asymptotic behavior of superfluid density is given by $\rho_s/\rho=k(X)t_\lambda^{f(X)}$, where $f(X)\approx 2/3$ and $k(X)$ change slowly as a function of concentration X.[2,1] Therefore close to the λ line (but $X<0.55$), the asymptotic form of α_{2v_n} in terms of reduced parameters becomes

$$\alpha_{2v_n}\approx-\frac{M_s}{M_4}\,\frac{T}{T_\lambda}\,\frac{\overline{S}}{C_{XP}}\,t_\lambda^{-1}, \tag{36}$$

where $t_\lambda=1-T/T_\lambda$. However, near the tricritical point $k(X)=k_0x_t^{1/3}$[1] and therefore

$$\alpha_{2v_n}\approx-\frac{M_s}{M_4}\left(\frac{T}{T_\lambda}\,\frac{\overline{S}}{C_{XP}}\,t_\lambda^{-1}+\frac{X}{X_t}\,x_t^{-1}\right), \tag{37}$$

where $x_t=1-X/X_t$ and X_t is concentration of ^{3}He at the tricritical point. Therefore the nonlinearity is more noticeable near the tricritical point.

For the case of pure ^{4}He, i.e., $c=0$, the corrections to the second sound mode eigenvalue [Eq. (32) and (33)] become

$$\lambda^{(1w)}|_{c=0}=\xi_T\hat{\sigma}\Gamma^2\frac{\rho}{\rho_n}+\xi_P\rho u^2\rho b\Gamma^3\frac{\rho}{\rho_n}+2\,\frac{\rho_s}{\rho_n}\,\Gamma^2(1+b\Gamma), \tag{38}$$

where $b|_{c=0}=\beta\sigma(\partial\sigma/\partial T)_P^-$ and

$$\lambda^{(1v)}|_{c=0}=\Gamma^2(1+b\Gamma/2). \tag{39}$$

Therefore the nonlinear coefficient simplifies to

$$\begin{aligned}\alpha_{2w}|_{c=0}={}&u_{20}\frac{\partial u_{20}}{\partial P}\,\rho_s\beta\hat{\overline{\sigma}}+\frac{\rho_s}{\rho}\,\frac{1}{u_{20}}\,\frac{\partial u_{20}}{\partial T}\,\hat{\overline{\sigma}}+\xi_T\hat{\sigma}\Gamma^2\frac{\rho}{\rho_n}\\&+\xi_P\rho u_{20}^2\rho b\Gamma^3\frac{\rho}{\rho_n}+2\,\frac{\rho_s}{\rho_n}\,\Gamma^2(1+b\Gamma)+\frac{\rho_s}{\rho}\,b\Gamma^3\\&\times\left(1+\frac{1}{2}\,b\Gamma\right),\end{aligned} \tag{40}$$

which in the limit $\beta=0$ and $u_{20}\ll u_{10}$ reduces to the Khalatnikov's expression $\alpha'_{2w\,K\,\mathrm{hal}}=\sigma(\partial T/\partial d)\ln[u_{20}^2(\partial\sigma/\partial T)]$.[8,5]

The dissipation coefficient is calculated from the Eq. (25). The general expression for μ_2 is very long and not illuminating, however, it is more useful to get the expression close to the λ line. Therefore the dissipation coefficient can be simplified to

$$-2\mu_2=\frac{\rho_s}{\rho_n}\left\{\left[\left(\frac{4}{3}\,\eta+\zeta_2\right)(1+b)/\rho-(\zeta_1+\zeta_4)\right](1+b)\right.$$
$$\left.-\rho\zeta_3\right\}+\frac{\kappa}{T}\,\frac{\rho_s}{\rho_n}\,\frac{\hat{\bar{\sigma}}^2}{\rho u_{20}^2}+\frac{\rho_s/\rho_n}{u_{20}^2}\,\frac{\partial}{\partial c}\,\frac{Z}{\rho}\,D\left[\left(\frac{k_T}{T}\,\hat{\bar{\sigma}}+c\right)^2\right.$$
$$\left.+\frac{k_P}{P}\,pu_{20}^2 b_\rho\left(\frac{k_T}{T}\,\hat{\bar{\sigma}}+c\right)\right] \tag{41}$$

which is also valid above the λ line for He-L. Therefore above the λ line the equation for second sound becomes a diffusion equation with Eq. (41) as its diffusion parameter.

The dissipation coefficient for the pure He-II limit $(c=0)$ will become

$$-2\mu_2|_{c=0}=\left(\frac{1}{\partial\sigma/\partial T}\right)\left(\frac{k}{T_\rho}\right)+\left[\left(\frac{4}{3}\,\eta+\zeta_2\right)(1+b\Gamma)/\rho-\rho\zeta_3\right.$$
$$\left.-(\zeta_1+\zeta_4)\right]\frac{\rho_s}{\rho_n}\,(1+b\Gamma) \tag{42}$$

with $k_P\ D=0$ and $k_T\ D=0$ for $c=0$, which is consistent with the existing results for pure ^{4}He (Ref. 15) when $\beta=0$.

Therefore we have derived a nonlinear second-sound equation

$$\frac{\partial w}{\partial t}+(u_{20}+\alpha_{2w}w)\,\frac{\partial w}{\partial x}-\mu_2\frac{\partial^2 w}{\partial x^2}=0. \tag{43}$$

In terms of a measurable quantity for example the temperature amplitude T', the Burgers equation is similar to Eq. (43) but with α_{2w} replaced by α_{2T}

$$\frac{\partial T'}{\partial t}+(u_{20}+\alpha_{2T}T')\,\frac{\partial T'}{\partial x}-\mu_2\frac{\partial^2 T'}{\partial x^2}=0, \tag{44}$$

where α_{2T} is given by Eq. (29).

B. First sound

For the first-sound velocity we find

$$u_{10}^2=\frac{\partial P}{\partial\rho}\,(1+d)+u_{20}^2 d. \tag{45}$$

The right eigenvector for the first sound mode of matrix A (19) is

$$|R_1\rangle=\begin{pmatrix} \rho u_{10} \\ -\dfrac{\rho_s}{\rho_n}\,\dfrac{b}{u_{10}}\,\hat{\bar{\sigma}}-\beta\,\dfrac{u_{10}}{\partial\sigma/\partial T}\,\dfrac{1+(\rho_s/\rho)\chi b}{1+d} \\ -c\,\dfrac{\rho_s}{\rho}\,b/u_{10} \\ 1 \\ -\dfrac{\rho}{\rho_n}\,b \end{pmatrix}. \tag{46}$$

For first sound, nonlinear effects only become significant close to the transition. Neglecting the expansion coefficient, β, terms of order $(\rho_s/\rho)^2$, and using $u_{20}\ll u_{10}$, the perturbative corrections to the eigenvalue become

$$\lambda^{(1w)}=\chi^2\frac{\rho\rho_s}{\rho_n^2}\,(\xi_T\hat{\bar{\sigma}}+c\,\xi_c)-\chi u_{10}^2\frac{\rho^2}{\rho_n^2}\,\xi_P-2\chi\,\frac{\rho_s}{\rho_n}\left(1-\frac{\rho_s}{\rho_n}\,\chi\right) \tag{47}$$

and

$$\lambda^{(1v)}=1-\frac{\chi}{1+d}. \tag{48}$$

With these results and $|R_1\rangle$ (46) the nonlinear coefficient can be calculated which gives

$$\alpha_{1v}=-\frac{\rho_s\rho^2}{\rho_n^3}\,\frac{b\chi^2}{1+d}\,(\xi_T\hat{\bar{\sigma}}+c\,\xi_c)+\frac{\rho^2}{\rho_n^3}\,\frac{\chi u_{10}^2}{1+d}\,\xi_P$$
$$+\left(2\,\frac{\rho_s\rho}{\rho_n^2}-1\right)\frac{\chi}{1+d}+1+\rho u_{10}\frac{\partial u_{10}}{\partial P}$$
$$-\frac{\rho_s}{\rho_n}\,\frac{b}{u_{10}}\left(\hat{\bar{\sigma}}\frac{\partial u_{10}}{\partial T}+c\,\frac{\partial u_{10}}{\partial c}\right), \tag{49}$$

where $u_{10}^2\approx(\partial P/\partial\rho)(1+d)$ is the velocity (squared) of first sound and r_2 is the second component of the eigenvector $|R_1\rangle$ Eq. (46). The dissipation becomes

$$-2(1+d)\mu_1=\frac{\rho_s}{\rho_n}\,b\zeta_1+\left(1-2\,\frac{\rho_s}{\rho_n}b\right)\left(\frac{4}{3}\,\eta+\zeta_2\right)\Big/\,\rho+\zeta_3\left(\frac{\rho_s}{\rho_n}\right)^2(b+\beta\hat{\bar{\sigma}})\chi\rho+\zeta_4\left[b\,\frac{\rho_s\rho}{\rho_n^2}+\left(\frac{\rho_s}{\rho_n}\right)^2\beta\hat{\bar{\sigma}}\right]$$
$$+\frac{\kappa}{T}\,\frac{\hat{\bar{\sigma}}^2}{\rho\bar{\sigma}^2}\left(\beta^2u_{10}^2+2\chi\beta\,\frac{\rho_s}{\rho_n}\,\bar{\sigma}\right)+2\left(\frac{k_T}{T}\right)^2\frac{D}{c^2}\,\beta\chi\left(\frac{\hat{\bar{\sigma}}}{\bar{\sigma}}u_{20}^2-\hat{\bar{\sigma}}^3\frac{\rho_s}{\rho_n}\right)+\frac{k_P}{P}\,\frac{k_T}{T}\,\frac{D}{c^2}\,\rho\hat{\bar{\sigma}}u_{10}^2$$
$$\times\left[\beta\hat{\bar{\sigma}}-\frac{\rho_n}{\rho_s}\,u_{20}^2-\left(\frac{u_{20}}{u_{10}}\right)^2(b+u_{10}^2\chi^2\beta/\bar{\sigma})+\frac{\rho_s}{\rho_n}\,b\hat{\bar{\sigma}}\right]+\frac{k_T}{T}\,\frac{D}{c}\,\chi u_{10}^2\frac{\hat{\bar{\sigma}}}{\bar{\sigma}}\left[\beta+2\left(\frac{u_{20}}{u_{10}}\right)^2\beta+\frac{\rho_s}{\rho_n}\,\bar{\sigma}(\chi-\hat{\bar{\sigma}}\beta)/u_{10}^2\right]$$
$$+\frac{k_P}{P}\,\frac{D}{c}\,\rho u_{10}^2\left[\chi-b\left(\frac{u_{20}}{u_{10}}\right)^2+\overline{\hat{\sigma}\sigma}b\,\frac{\rho_s}{\rho_n}/u_{10}^2\right]+\frac{\rho_s}{\rho_n}\,D\chi b. \tag{50}$$

The Burgers equation for first sound will then become

$$\frac{\partial v}{\partial t}+(u_{10}+\alpha_{1v}v)\frac{\partial v}{\partial x}-\mu_1\frac{\partial^2 v}{\partial x^2}=0, \tag{51}$$

$$\frac{\partial P}{\partial t}+\left(u_{10}+\frac{1}{\rho u_{10}}\alpha_{1v}p\right)\frac{\partial P}{\partial x}-\mu_1\frac{\partial P}{\partial x^2}=0. \tag{52}$$

Note that the relation between α_{1v} and α_{1P} is

$$\alpha_{1P}=\frac{1}{\rho u_{10}}\alpha_{1v}. \tag{53}$$

For pure ^{4}He, the nonlinear coefficient becomes

$$\alpha_{1v}|_{c=0}=1+\rho u_{10}\frac{\partial u_{10}}{\partial P} \tag{54}$$

which is equivalent to the Khalatnikov's result that gives α_{1P} for pure ^{4}He,[8,5,6]

The first sound dissipation coefficient for pure ^{4}He becomes

$$-2\mu_1|_{c=0}=\frac{\rho_s}{\rho_n}b\zeta_1+\frac{4}{3}\frac{(\eta+\zeta_2)}{\rho}+\frac{\kappa}{T}\beta^2u_{10}^2\frac{1}{\rho\left(\frac{\partial\sigma}{\partial T}\right)^2}$$

$$+\zeta_4\left[b\frac{\rho_s\rho}{\rho_n^2}+\left(\frac{\rho_s}{\rho_n}\right)^2\beta\hat{\sigma}\right]\frac{\rho_s}{\rho_n}, \tag{55}$$

where here $b=\beta\hat{\sigma}$.

C. The diffusive mode

The fifth eigenvalue of the matrix $A^{(0)}$ is zero, and the corresponding eigenvector is given by

$$|R_5)=\left[0,-\frac{c}{\bar{\sigma}}\frac{\partial}{\partial c}\frac{Z}{\rho},1,0,0\right], \tag{56}$$

wherefrom the nonlinear coefficient results in

$$\alpha_5=0. \tag{57}$$

Therefore this corresponds to a mode which describes temperature-concentration diffusion, while P_5', v_5 and w_5 remain zero. Calculation of dissipation coefficient results in

$$\left(1+\frac{c^2}{\hat{\bar{\sigma}}\bar{\sigma}}\frac{\partial}{\partial c}\frac{Z}{\rho}\right)\mu_5=\frac{c^2}{T\rho\bar{\sigma}^2}\left(\frac{\partial}{\partial c}\frac{Z}{\rho}\right)\kappa_{\text{eff}}+2\frac{\sigma}{\bar{\sigma}^2}cDT\left[c\frac{\partial}{\partial c}\frac{\sigma}{c}\right.$$

$$\left.+\frac{k_T}{T}\frac{\partial}{\partial c}\frac{Z}{\rho}\right], \tag{58}$$

where

$$\kappa_{\text{eff}}=\kappa+\rho D\frac{T}{(\partial/\partial c)(Z/\rho)}\left[c\frac{\partial}{\partial c}\frac{\sigma}{c}+\frac{k_T}{T}\frac{\partial}{\partial c}\frac{Z}{\rho}\right]^2. \tag{59}$$

Therefore for the fifth mode the equation for concentration will become

$$\frac{\partial c_5'}{\partial t}=\mu_5\frac{\partial^2 c_5'}{\partial x^2} \tag{60}$$

and

$$T_5'=-\frac{c}{\sigma}\frac{\partial}{\partial c}\frac{Z}{\rho}c_5'.$$

V. GENERATION OF FIRST SOUND FROM SECOND SOUND AND VICE VERSA

As shown in previous sections, first (second) sound mode carries not only pressure (temperature) fluctuations, but also temperature (pressure) and concentration fluctuations too. Using the eigenvector for each mode we can find the relations between each fields.

Using Eq. (28) we see that for the second sound mode the temperature, concentration and pressure fluctuations are

$$T_2'=\frac{\rho_s}{\rho}\frac{\hat{\bar{\sigma}}}{u_{20}}\left[1-\left(\chi\beta\frac{u_{20}^2}{\sigma}\right)\right]w_2, \tag{61}$$

$$c_2'=\frac{c}{u_{20}}\frac{\rho_s}{\rho}w_2, \tag{62}$$

and

$$P_2'=u_{20}\rho_s\Gamma(\hat{\bar{\sigma}}\beta+\chi)w_2, \tag{63}$$

where w_2 (i.e., counterflow velocity) is the solution to Burgers equation (43). The expression for the pressure fluctuations of the second sound can also be expressed as

$$P_2'=u_{20}^2b\Gamma\rho\frac{1}{\hat{\bar{\sigma}}}\left(1-\chi\beta\frac{u_{20}^2}{\bar{\sigma}}\right)^{-1}T_2'. \tag{64}$$

The fluctuations in temperature and concentration result in fluctuations in the total density, which in tern, cause the pressure to fluctuate. Therefore it is possible to measure the pressure fluctuations (i.e. first sound) of second sound instead of temperature for the detection of second sound in mixtures. However the pressure amplitude is suppressed by a factor of u_{20}^2 as compared with the amplitude of the temperature fluctuation. For example for a temperature amplitude of about 10 μK, one may expect to have a pressure amplitude of about 10 Pa.

The temperature and concentration amplitudes due to first sound mode can also be calculated using the eigenvector Eq. (46),

$$T_1'=-\left(\frac{\rho_s}{\rho_n}\frac{b}{u_{10}}\hat{\bar{\sigma}}+\beta\frac{u_{10}}{\partial\sigma/\partial T}\frac{1+(\rho_s/\rho)\chi b}{1+d}\right)\frac{1}{\rho u_{10}}P_1' \tag{65}$$

and

$$c_1'=c\frac{\rho_s}{\rho^2}\frac{b}{u_{10}^2}P_1', \tag{66}$$

where u_{10} is the velocity of first sound and P_1' is the first-sound pressure amplitude. Therefore one can measure second-sound thermal wave form incoming first-sound pressure fluctuations.

VI. CONCLUSION

In this work we have systematically derived nonlinear wave equations for the propagation of second and first sound in ^{3}He-^{4}He mixtures, with their limit for pure ^{4}He. Our derivation is valid to first order in the expansion coefficient β, and velocities v and w. Using the reductive perturbation method we showed that the second- and first-sound wave equations are Burgers equations and we derived expressions for the nonlinear coefficients α_2 and α_1, and the dissipation coefficient μ_2 and μ_1. We observed that α_2 for mixtures is larger than pure ^{4}He and therefore second sound in mixtures propagates highly nonlinearly close to the transition temperatures. This nonlinear is even more pronounced close to the tricritical point. We also found a diffusive mode for a coupled impurity-temperature in mixtures and derived its diffusion parameter.

As a final remark, the derived value for α_2 can also be used to find the velocity of second sound in the presence of a heat current in mixtures. In this case $w_0 = Q/\rho_s(1+b\Gamma)ST$, where Q is the heat current and w_0 is the resulting counterflow velocity of course this discussion does not consider the change in the critical temperature in the presence of a heat current.

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APPENDIX A

This appendix contains the expressions for the matrices of Sec. III of this paper. At the linear limit, i.e., for $w=0$ and $v=0$, the matrices B and A' are

$$B^{(0)} = \begin{bmatrix} \frac{\partial\rho}{\partial P} & \frac{\partial\rho}{\partial T} & \frac{\partial\rho}{\partial c} & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ \rho\frac{\partial\sigma}{\partial P} & \rho\frac{\partial\sigma}{\partial T} & \rho\frac{\partial\sigma}{\partial c} & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{\rho_n}{\rho} \end{bmatrix}, \qquad \text{(A1)}$$

$$A'^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \rho\frac{\partial P}{\partial\rho} & A'^{(0)}_{15} \\ 0 & 0 & 0 & -\beta\frac{\partial P}{\partial\rho}\frac{\partial T}{\partial\sigma} & A'^{(0)}_{25} \\ 0 & 0 & 0 & 0 & c\frac{\rho_s}{\rho} \\ \frac{1}{\rho} & 0 & 0 & 0 & 0 \\ -\frac{1}{\rho_n}\chi & \frac{\rho}{\rho_n}\overline{\sigma} & c\frac{\rho}{\rho_n}\frac{\partial}{\partial c}\frac{Z}{\rho} & 0 & 0 \end{bmatrix}, \qquad \text{(A2)}$$

where

$$A'^{(0)}_{15} = -\rho_s\frac{[\beta\overline{\sigma}-(\partial\sigma/\partial T)\chi]}{(\partial\rho/\partial P)(\partial\sigma/\partial T)}, \quad A'^{(0)}_{25} = \left(\frac{1}{\rho}\frac{\partial\rho}{\partial P}\overline{\sigma}+\beta\chi\right) \times\frac{\partial P}{\partial\rho}\frac{\partial T}{\partial\sigma}, \quad \beta = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{c,P} \text{ and } \chi = \frac{c}{\rho}\left(\frac{\partial\rho}{\partial c}\right)_{P,T}.$$

The matrices B and A' can be expanded around their background i.e., Eqs. (A1) and (A2) as

$$B = B^{(0)} + wB^{(1w)} + vB^{(1v)},$$

$$A' = A'^{(0)} + wA'^{(1w)} + vA'^{(1v)},$$

where

$$B^{(1w)} = \begin{bmatrix} 0 & 0 & 0 & \rho & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma\rho_s \\ 0 & 0 & 0 & 0 & c\rho_s \\ \frac{1+\chi}{\rho} & -\overline{\sigma} & -c\frac{\partial}{\partial c}\frac{Z}{\rho} & 0 & 0 \end{bmatrix}, \quad B^{(1v)} = 0, \qquad \text{(A3)}$$

$$A'^{(1w)} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\xi_P\rho^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\rho\xi_T \\ 0 & 0 & 0 & 0 & 0 \\ \xi_P & \xi_T & \xi_c & 0 & 0 \end{bmatrix}, \qquad \text{(A4)}$$

$$A'^{(1v)}$$

$$=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\,\frac{\rho_n\rho_s}{\rho} \\ \frac{\partial(\sigma\rho_s)}{\partial P} & \frac{\partial(\sigma\rho_s)}{\partial T} & \frac{\partial(\sigma\rho_s)}{\partial c} & 0 & 0 \\ c\,\frac{\partial\rho_s}{\partial P} & c\,\frac{\partial\rho_s}{\partial T} & \rho_s+c\,\frac{\partial\rho_s}{\partial c} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\rho_n}{\rho} & -\frac{\rho_n\rho_s}{\rho^2}. \end{bmatrix}. \tag{A5}$$

The first order correction terms to the matrix A is calculated from

$$A^{(1v)}=B^{(0)^{-1}}A'^{(1v)}, \tag{A6}$$

$$A^{(1w)}=B^{(0)^{-1}}A'^{(1w)}-B^{(0)^{-1}}B^{(1w)}B^{(0)^{-1}}A'^{(0)}. \tag{A7}$$

The dissipation matrix evaluated at $w=0$ limit becomes

$$K'=-\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{3}\,\eta+\zeta_2 & K'_{25} \\ -\frac{k_P\psi}{P} & -\frac{k_T\psi}{T}+\frac{\kappa}{T} & -\psi & 0 & 0 \\ \frac{k_P}{P}\,\rho D & \frac{k_T}{T}\,\rho D & \rho D & 0 & 0 \\ 0 & 0 & 0 & \zeta_4 & K'_{55} \end{bmatrix}, \tag{A8}$$

where

$$K'_{25}=\frac{\rho_s}{\rho}\left(\frac{4}{3}\,\eta-\rho\zeta_1+\zeta_2\right),$$

$$K'_{55}=\frac{\rho_s}{\rho}\,(\zeta_4-\zeta_3\rho),\quad \psi=\rho D\left(\frac{\partial}{\partial T}\,\frac{Z}{\rho}-\frac{k_T}{T}\,\frac{\partial}{\partial c}\,\frac{Z}{\rho}\right),$$

and D is the diffusion coefficient. The matrix K is found from the matrix operation $K=B^{-1}K'$.

APPENDIX B

This appendix contains definitions of some terms and abbreviations used in the paper and the conversion of some quantities defined in this paper from c into X

$$\beta=\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{P,c}. \tag{B1}$$

The measurements of thermal expansivity β for pure ^{4}He has extensively been reported by Maynard[19] and Niemela and Donnelly.[20] The range of variation of β strongly depends on temperature as well as pressure. From Ref. 19 we see that near the superfluid transition temperature, $\beta\approx-0.023(1/K)$ at SVP and $\beta\approx-0.14(1/K)$ at 25 atm. Therefore one should consider the corrections due to the thermal expansion coefficient specially at higher pressure.[14] The data for the thermal expansion coefficient in mixtures of ^{3}He-^{4}He are reported in Refs. 17 and 18 at saturated vapor pressure. The thermal expansion is larger than the case of pure ^{4}He even at lower temperature so it is very important to consider it.

$$\chi=\frac{c}{\rho}\left(\frac{\partial\rho}{\partial c}\right)_{P,T}=\frac{M_s}{M_4}\,\chi X,\chi X=\frac{X}{\rho}\left(\frac{\partial\rho}{\partial X}\right)_{P,T}$$

and numerically $\chi\approx1$.

$$\xi_P=\frac{\partial}{\partial P}\,\frac{\rho_s}{\rho},\xi_T=\frac{\partial}{\partial T}\,\frac{\rho_s}{\rho},\xi_c=\frac{\partial}{\partial c}\,\frac{\rho_s}{\rho}$$

these parameters diverge near T_λ.

The parameter d defined in this paper is the generalization of the original definition of Khlatnikov,[8] to the case when the thermal expansion is considered

$$d=\frac{\rho_s}{\rho_n}\,b^2,$$

where $b=\chi+\hat{\overline{\sigma}}\beta=M_s/M_4[\chi X+(\overline{S}T/C_{XP})\beta]$ and at the limit of pure ^{4}He it becomes $b|_{c=0}=\beta\sigma(\partial T/\partial\sigma)_P$. The relation between the entropy per mole σ and entropy per gram S is given by

$$S=\sigma M_s.$$

Furthermore

$$\overline{S}(T,X)=S-X\left(\frac{\partial S}{\partial X}\right)_{TP},$$

$$c\,\frac{\partial\sigma}{\partial c}=\frac{X}{M_4}\left(\frac{\partial S}{\partial x}+S\,\frac{M_4-M_3}{M_s}\right),$$

and

$$\overline{\sigma}=\sigma-c\,\frac{\partial\sigma}{\partial c}=\frac{1}{M_4}\,\overline{S}$$

The numerical data for the entropy is reported in Ref. 16.

The potential Z is expressed in terms of chemical potentials of ^{3}He and ^{4}He as[8]

$$Z=\rho(\mu_3-\mu_4).$$

The speeds of second and first sound are

$$u_{20}^2=\frac{M_s}{M_4^2}\,\frac{\rho_s}{\rho_n}\left[\frac{\overline{S}^2T}{C_{XP}}+X^2\left(\frac{\partial\Phi}{\partial X}\right)_{TP}\right](1+d)^{-1},$$

$$u_{10}^2=\frac{\partial P}{\partial\rho}\,(1+d)+u_{20}^2d.$$

The speed of second sound is $u_{20}\approx u_0=20(m/s)$ below T_λ. However, close to T_λ it approaches zero as $u_{20}=u_0[(T_\lambda-T)/T]^{1/3}$. The speed of first sound is $u_{10}\approx240(m/s)$ and therefore the ratio $u_{10}/u_{20}\ll1$ close to T_λ^1. There the factor Γ defined as

$$\Gamma=\left(1-\frac{u_{20}^2}{\partial P/\partial\rho}\right)^{-1}\approx\left[1-\left(\frac{u_{20}}{u_{10}}\right)^2\right]^{-1}$$

becomes numerically $\Gamma\approx 1+0.01$ inside the superfluid region and $\Gamma\approx 1$ near T_λ.

The quantity q is defined as a correction to the temperature component of the second-sound mode eigenvector

$$q=1-\beta\chi\frac{u_{20}^2}{\bar{\sigma}}\Gamma=1-M_s\chi x\beta\frac{u_{20}^2}{\bar{S}}\Gamma.$$

Here M_3, M_4, and $M_s=M_3X+M_4(1-X)$ stand for molar mass of ^{3}He, ^{4}He and a mixture with concentration X, respectively. The molar concentration X or the mass concentration $c=(M_3/M_s)X$ are zero for the case of pure ^{4}He.

The relations for concentration susceptibility and entropy are

$$c^2\partial_c\frac{Z}{\rho}=\frac{M_s}{M_4^2}X^2\frac{\partial\Phi}{\partial X},$$

$$\frac{\overline{\sigma}^2}{\partial\sigma/\partial T}=\frac{M_s}{M_4^2}\frac{\overline{S^2}T}{C_{XP}},$$

$$\hat{\overline{\sigma}}=\frac{\overline{\sigma}}{\partial\sigma/\partial T}=\frac{M_s}{M_4}\frac{\overline{S}T}{C_{XP}},\overline{\sigma}=\frac{\sigma}{\partial\sigma/\partial T}.$$

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[1] G. Ahlers, in *The Physics of Liquid and Solid Helium*, edited by K. H. Benneman and J. B. Ketterson (Wiley, New York, 1976), Pt. 1; G. Ahlers, Phys. Rev. A **10**, 1670 (1974).

[2] T. Kobayashi, F. Uehara, T. Fushimi, A. Tominaga, and Y. Narahara, J. Phys. C **9**, 3573 (1976); F. Uehara, J. Low Temp. Phys. **24**, 769 (1976); G. Terui, *ibid.* **25**, 645 (1976).

[3] L. S. Goldner and G. Ahlers, Phys. Rev. B **45**, 13 129 (1992); **43**, 12 861 (1991); L. S. Goldner, G. Ahlers, and R. Mehrotra, Physica B **165**, 539 (1990); M. J. Adriaans, D. R. Swanson, and J. A. Lipa, *ibid.* **194**, 733 (1994).

[4] M. Lutset, S. Nemirovskii, and A. Tsoi, Zh. Eksp. Teor. Fiz. **81**, 249 (1981) [Sov. Phys. JETP **54**, 127 (1981)].

[5] S. Nemirovskii, Usp. Fiz. Nauk **161**, 51 (1990) [Sov. Phys. Usp. **33**, 420 (1990)].

[6] N. N. Sibel'den, V. B. Stopachinskii, V. A. Tsvetkov, and B. Etien, Pis'ma Zh. Eksp. Teor. Fiz. **32**, 224 (1980) [JETP Lett. **32**, 206 (1980)].

[7] M. Mohazzab, N. Mulders, M. Larson, and A. Nash, J. Low Temp. Phys. **113**, 1031 (1988).

[8] I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965).

[9] T. Taniuti, Suppl. Progr. Theor. Phys. **55**, 1 (1974); T. Taniuti and K. Nishihara, Nonlinear Waves (Pitman, Boston, 1983).

[10] J. M. Burgers, Adv. Appl. Mech. **1**, 171 (1948).

[11] G. Ruppeiner, M. Ryschkewitsch, and H. Meyer, J. Low Temp. Phys. **41**, 179 (1980); R. P. Behringer and H. Meyer, *ibid.* **46**, 407 (1982); **46**, 435 (1982); L. H. Cohen, M. L. Dingus, and H. Meyer, *ibid.* **61**, 79 (1985); C. M. Lai and T. A. Kithens, in *Proceedings of the 13th International Conference on Low Temperature Physics LT-13*, edited by K. D. Timmerhaus, W. J. O'Sullivan, and E. F. Hammel (Plenum, New York, 1974), Vol. 1, p. 576; H. Meyer, J. Low Temp. Phys. **70**, 219 (1988); C. Howald, X. Qin, H. S. Nahm, and H. Meyer, *ibid.* **86**, 375 (1992); D. B. Roe, G. Ruppeiner, and H. Meyer, *ibid.* **27**, 747 (1977).

[12] Reference 1 includes a review of the past results on transport properties of pure ^{4}He and mixtures and a discussion about the dissipation coefficients for sound modes in He-II and He-I.

[13] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1976).

[14] S. Braun, A. Kluwick, and M. S. Cramer, Phys. Fluids **8**, 2513 (1996).

[15] S. Kitabatake and Y. Sawada, J. Phys. Soc. Jpn. **45**, 345 (1978).

[16] S. T. Islander and W. Zimmermann, Jr., Phys. Rev. A **7**, 188 (1973).

[17] H. A. Kierstead, J. Low Temp. Phys. **24**, 497 (1976).

[18] D. Gestrich, R. Walsworth, and H. Meyer, J. Low Temp. Phys. **54**, 37 (1984); S. Wang, C. Howald, and H. Meyer, *ibid.* **79**, 151 (1989); X. Qin, C. Howald, and H. Meyer, *ibid.* **87**, 731 (1992).

[19] J. Maynard, Phys. Rev. B **14**, 3868 (1976); D. L. Elwell and H. Meyer, Phys. Rev. **164**, 245 (1967); H. Kierstead, *ibid.* **162**, 153 (1967).

[20] J. J. Niemela and R. J. Donnelly, J. Low Temp. Phys. **98**, 1 (1995).