Determination of the critical exponents for the λ transition of ^{4}He by high-temperature expansion

Massimo Campostrini*
Dipartimento di Fisica dell'Università di Pisa and INFN, I-56126 Pisa, Italy

Andrea Pelissetto†
Dipartimento di Fisica dell'Università di Roma I and INFN, I-00185 Roma, Italy

Paolo Rossi‡ and Ettore Vicari§
Dipartimento di Fisica dell'Università di Pisa and INFN, I-56126 Pisa, Italy

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We determine the critical exponents for the XY universality class in three dimensions, which is expected to describe the λ transition in ^{4}He. They are obtained from the analysis of high-temperature series computed for a two-component $\lambda\phi^4$ model. The parameter λ is fixed such that the leading corrections to scaling vanish. We obtain $\nu=0.67166(55)$, $\gamma=1.3179(11)$, and $\alpha=-0.0150(17)$. These estimates improve previous theoretical determinations and agree with the more precise experimental results for liquid helium.

The renormalization-group approach to cooperative transitions is one of the most significant successes in theoretical physics. It has explained a wide range of phenomena in many different fields, ranging from statistical physics to elementary particle physics. It should be noted however that the quantitative experimental support for the theory rests primarily on its predictions for critical phase transitions. To test the theory one needs the most accurate possible values for universal quantities. Here the ^{4}He phase transition is of great utility because of the weakness of the singularity of the compressibility of the fluid and of the availability of extremely pure samples. Moreover, the possibility of performing the experiments in space, and therefore in a microgravity environment, reduces the gravity-induced broadening of the transition. Recently a Space Shuttle experiment[1] performed a very precise measurement of the heat capacity of liquid helium to within 2 nK from the λ transition obtaining an extremely accurate estimate of the exponent α:

$$\alpha=-0.01285\pm0.00038. \tag{1}$$

This estimate is extremely precise and represents a challenge for theorists, who, until now, have not been able to compute critical exponents at this level of accuracy.

One of the oldest approaches to the study of critical phenomena is based on high-temperature (HT) expansions. In this approach the main hindrance to a precise determination of universal quantities is the presence of confluent corrections with noninteger exponents. For instance the specific heat is supposed to behave as

$$C_p=At^{-\alpha}(1+Bt^{\Delta}+Ct\ldots)+D+Et+\cdots \tag{2}$$

for $t\equiv|T-T_c|/T_c\to0$, with $\Delta\approx0.5$. The presence of nonanalytic terms introduces a large (and dangerously undetectable) systematic error in the results of the HT series analysis. In order to obtain precise estimates of the critical parameters, the approximants of the HT series should properly allow for the confluent nonanalytic corrections. Integral (also called differential) approximants[2] are, in principle, able to describe a behavior of the type (2) (see, e.g., Ref. 3 for a review). However, the extensive numerical work that has been done for the Ising model shows that in practice, with the series of moderate length that are available today, no analysis is able to predict and take into account non-analytic correction-to-scaling terms.[4,5,7,6,8,9] In order to effectively keep in account these confluent corrections, one should use biased approximants, fixing the value of β_c and of the first nonanalytic exponent Δ (see, e.g., Refs. 10, 7, and 11–14).

To overcome these difficulties, in the early 1980s, Chen, Fisher, and Nickel[6] realized the importance of studying families of models (specified by some auxiliary parameter) which are candidates for belonging to the same universality class. The hope was the possibility of locating a parameter value at which the leading nonanalytic corrections vanish. If the leading nonanalytic terms are no longer present, one expects a faster convergence, and therefore more precise and reliable estimates of the critical quantities. The method was applied to the double-Gaussian and to the Klauder models, both belonging to the Ising universality class and depending continuously on a real parameter; it was shown that a Hamiltonian for which the leading corrections are suppressed — we will name it ''improved'' Hamiltonian — could indeed be found.[6,15,16] The crux of the method is the precise determination of the optimal value of the parameter appearing in the Hamiltonian. In Refs. 6 and 15 the partial differential approximant technique was used; however the errors on the improved Hamiltonian were relatively large and the final results represented only a modest improvement with respect to standard (and much simpler) analyses using biased approximants.

In the past few years it has been understood that improved Hamiltonians can be determined with high accuracy by means of Monte Carlo simulations.[17–20] Using finite-size scaling methods that are very sensitive to confluent corrections, it has been possible to determine various improved Hamiltonians that belong to the Ising universality class, and

TABLE I. Estimates of the critical exponents. See text for the explanation of the symbols in the first column. We indicate with an asterisk (*) the estimates that have been obtained using the hyperscaling relation $2-\alpha-3\nu=0$ or the scaling relation $\gamma-(2-\eta)\nu=0$. bcc and sc refer to the body-centered cubic and to the simple cubic lattices, respectively.

	γ	ν	η	α
IHT [this work]	1.3179(7+4)	0.67166(33+22)	0.0381(2+1)	−0.0150(10+7)*
HT, sc (Ref. 12)	1.325(3)	0.675(2)	0.037(7)*	−0.025(6)*
HT, bcc (Ref. 12)	1.322(3)	0.674(2)	0.039(7)*	−0.022(6)*
HT (Ref. 26)				−0.014(9), −0.022(6)
MC (Ref. 25)	1.319(2)*	0.6723(3)(8)	0.0381(2)(2)	−0.0169(33)*
MC (Ref. 27)	1.316(3)*	0.6721(13)	0.0424(25)	−0.0163(39)*
MC (Ref. 28)	1.315(12)	0.6693(58)	0.035(5)	−0.008(17)*
MC (Ref. 29)	1.307(14)*	0.662(7)	0.026(6)	−0.014(21)*
MC (Ref. 30)	1.323(2)	0.670(2)	0.025(7)*	−0.010(6)*
$d=3$ g-exp (Ref. 31)	1.3169(20)	0.6703(15)	0.0354(25)	−0.011(4)*
ϵ-exp, free (Ref. 31)	1.3110(70)	0.6680(35)	0.0380(50)	−0.004(11)*
ϵ-exp, bc (Ref. 31)	1.317	0.671	0.0370	−0.013*
^{4}He (Ref. 1)		0.67095(13)*		−0.01285(38)
^{4}He (Ref. 32)		0.6705(6)		−0.0115(18)*
^{4}He (Ref. 33)		0.6708(4)		−0.0124(12)*

correspondingly precise estimates of critical quantities have been obtained. It is important to notice that the estimates obtained using different improved Hamiltonians agree within the quoted error bars, confirming the correctness of the error estimates. Similar methods have been applied to the determination of critical exponents for dilute polymers in good solvents.[21]

The possibility of determining precisely the ''improved'' value of the parameter for a family of Hamiltonians has recently revived the program of Ref. 6. In Ref. 22 we performed an extensive analysis for the Ising universality class. Critical exponents and many other universal quantities were determined by analyzing HT series for three different improved Hamiltonians. For each improved model, we obtained very accurate estimates[23] that were in good agreement among each other, confirming the correctness of the quoted errors. This analysis showed that, once a precise estimate of the improved parameter is available, the HT series analysis gives results of a quality comparable with or better than the best Monte Carlo simulations.

As is well known, see, e.g., Ref. 24, the λ transition of liquid helium is expected to be in the XY universality class. In order to check if this is really the case, it is important to have precise theoretical estimates that can be compared with the experimental results. Recently, Hasenbusch and Török[25] performed a high-precision simulation of the $O(2)\,\phi^4$ model, obtaining an accurate estimate of the exponent ν: $\nu=0.6723(3)(8)$. Using the hyperscaling relation $\alpha=2-3\nu$, they obtained

$$\alpha=-0.0169\pm0.0033. \tag{3}$$

This estimate is larger than the experimental result (1), although the difference is barely bigger than the error. It is therefore important to further improve the accuracy of the theoretical estimates in order to understand whether this small discrepancy is significant. For our purposes Ref. 25 is particularly relevant since it provides a quite precise determination of an improved Hamiltonian belonging to the XY universality class. In this paper we study this improved model and obtain very precise estimates of the critical exponents from the HT series analysis. They considerably improve previous HT estimates, showing the effectiveness of the approach. In particular we obtain for α

$$\alpha=-0.0150\pm0.0017, \tag{4}$$

which is in better agreement with the experimental result, although still slightly larger. The small discrepancy between the Monte Carlo and the experimental result is significantly reduced, providing support to the fact that the λ transition belongs to the same universality class of the XY model. The full set of estimates together with other recent results is reported in Table I.

We consider a simple cubic lattice and the Hamiltonian

$$\mathcal{H}=-\beta\sum_{\langle x,y\rangle}\vec{\phi}_x\cdot\vec{\phi}_y+\sum_x\,[\vec{\phi}_x^{\,2}+\lambda(\vec{\phi}_x^{\,2}-1)^2], \tag{5}$$

where $\langle x,y\rangle$ labels a lattice link, and $\vec{\phi}_x$ is a real two-component vector defined on lattice sites. Hasenbusch and Török have performed an extensive simulation of this model and have obtained an estimate of the value of λ, λ*, at which the leading corrections vanish. They quote

$$\lambda^*=2.10\pm0.01\pm0.05, \tag{6}$$

where the two errors are respectively the statistical and the systematic one. We have considered the susceptibility χ and the second-moment correlation length ξ defined by

$$\chi = \sum_x \langle \vec{\phi}_0 \cdot \vec{\phi}_x \rangle, \tag{7}$$

$$\xi = \frac{1}{6\chi} \sum_x |x|^2 \langle \vec{\phi}_0 \cdot \vec{\phi}_x \rangle. \tag{8}$$

Using the linked-cluster expansion technique, we have generated the HT expansion of these two quantities to 20th order for arbitrary values of λ. From the analysis of χ and ξ we can obtain directly estimates of γ and ν. In the analysis we used inhomogeneous integral approximants (IA). Second- and third-order IA's turned out to be the most stable, especially when we biased the approximants[16] requiring the presence of two symmetric singularities at $\beta = \pm\beta_c$. This is a natural requirement, since it can be proved that, on bipartite lattices, $\beta = -\beta_c$ is also a singular point associated to the antiferromagnetic critical behavior.[34] The results we quote were obtained using this type of approximants.

In Table I we give our estimates of the exponents γ and ν that have been obtained analyzing the HT series for the improved Hamiltonian (5) with $\lambda = \lambda^*$. We quote two errors: the first one is related to the spread of the approximants, while the second one gives the variation of the exponent when λ varies between 2.04 and 2.16, cf. Eq. (6). It should be noted that the first error is somewhat larger than the second one, and therefore it is important to further extend the HT series. However, since the second error is far from negligible, a substantial reduction of the uncertainty also requires a more accurate determination of λ^*. Note that this might also reduce the first error, since it would enable us to work closer to the exactly improved Hamiltonian. We do not present details on the generation and analysis of the HT series and we refer the reader to Appendix A of Ref. 22.

From the estimate of ν we can obtain the exponent α assuming the validity of the hyperscaling relation

$$\alpha = 2 - 3\nu. \tag{9}$$

In principle it should be possible to determine α directly from the specific heat, or from the singularity of the susceptibility at the antiferromagnetic point.[34] We tried this second method obtaining only a rough estimate: $\alpha = -0.02(2)$.

From the estimates of γ and ν it is possible to obtain the exponent η, using

$$\gamma = \nu(2-\eta). \tag{10}$$

However, it is not clear how to set the error bar on the result. One can use the independent-error formula taking into account the error on γ and ν, but this may be an overestimate since γ and ν are correlated. To obtain an estimate of η with a smaller, yet reliable, error bar, we used the so-called critical-point renormalization method (see, e.g., Ref. 2 and references therein). The value that is quoted in Table I has been obtained with this method. It is compatible with the estimate obtained using the scaling relation (10), but it has a smaller error bar.

In Table I we report a summary of the most precise estimates that have been obtained in the past few years. When only ν or α was reported, we used the relation (9) to obtain the missing exponent. Analogously if only η or γ was quoted, the second exponent was obtained using the scaling relation (10); in this case the uncertainty was obtained using the independent-error formula. The results we quote have been obtained from the analysis of the HT series of the *XY* model (HT), by Monte Carlo simulations (MC) or by field-theory methods. The field-theory results have been derived by resumming the perturbative expansion in fixed dimension $d=3$ (g-expansion), or the the expansion in $\epsilon = 4-d$. For the ϵ expansion we quote two numbers, corresponding to an unconstrained analysis (''free''), and to a constrained analysis (''bc'') in which the two-dimensional values of the exponents are taken into account. Our final results for γ and ν improve the existing theoretical estimates. They are somewhat lower than previous HT results but they are in full agreement with the field theory and the most recent MC estimates, as well as with the experimental results for the λ transition, showing the expected universality.

*Electronic address: campo@mailbox.difi.unipi.it
†Electronic address: Andrea.Pelissetto@roma1.infn.it
‡Electronic address: rossi@mailbox.difi.unipi.it
§Electronic address: vicari@mailbox.difi.unipi.it

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