

Index distribution of random matrices with an application to disordered systems

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We compute the distribution of the number of negative eigenvalues (the index) for an ensemble of Gaussian random matrices, by means of the replica method. This calculation has important applications in the context of statistical mechanics of disordered systems, where the second derivative of the potential energy (the Hessian) is a random matrix whose negative eigenvalues measure the degree of instability of the energy surface. An analysis of the probability distribution of the Hessian index is therefore relevant for a geometric characterization of the energy landscape in disordered systems. The approach we use here is particularly suitable for this purpose, since it addresses the problem without any *a priori* assumption on the random matrix ensemble and can be naturally extended to more realistic, non-Gaussian distributions.

I. INTRODUCTION

The importance of random matrix theory (RMT) can hardly be overstated. Since its initial development by Wigner and Dyson to deal with the spectrum of many-body quantum systems,^{1,2} it has found applications in areas of physics as diverse as disordered systems, chaos, and quantum gravity, to name just a few.³⁻⁶ Most of the time RMT has been used as a very powerful tool for the study of the energy-level fluctuations of quantum systems. In this case, the matrix that RMT is modeling is of course the quantum Hamiltonian of the system.

However, there is a different context where RMT can be very useful, namely the study of the statistical properties of classical disordered systems. By disordered systems we mean not only those cases where quenched disorder is directly present in the Hamiltonian, as in spin-glasses, random field models or neural networks, but also systems whose physical behavior at low temperatures is heavily influenced by the self-induced disorder of their typical configurations, as for example, supercooled liquids and structural glasses. In all these systems the properties of the energy landscape, or energy surface, are known to be far from trivial. In particular, the presence of many local minima of the potential energy is one of the most distinctive features of this class of systems.^{7,8} An obvious consequence of this fact is that the energy surface displays many extensive regions with unstable negative curvature and therefore has very nontrivial stability properties.^{9,10} In this context, a key object becomes the matrix of the second derivatives of the Hamiltonian, normally called Hessian, which encodes all the stability attributes of the energy landscape.

The study of the statistical properties of the Hessian has been an important issue both in the theory of mean-field spin glasses and in liquid theory. In the former case, it is often possible to analyze the Hessian in the stationary points of the free energy, having therefore important information on the shape and stability of the thermodynamic states.⁷ In liquids, on the other hand, the Hessian of the potential energy is the key object in the context of the instantaneous normal modes approach,^{11,12} where the average spectrum of the Hessian is directly connected to many physical observables of the sys-

tem. In particular, it has been argued that there exists a deep relation between the diffusion properties of a liquid and the negative unstable eigenvalues of the average Hessian.¹²

It is evident that in the above context an application of RMT to the study of the statistical properties of the Hessian can be potentially very useful. An important remark is the following: the Hessian is a matrix that in general depends on the configuration of the system and possibly also on the quenched disorder, when this is present. The basic idea is to derive from the distribution of the configurations and from the distribution of the disorder an effective probability distribution for the Hessian, which can then be studied in the context of RMT (a recent example of this strategy can be found in Ref. 13).

Besides, it is clear from the former discussion that an important issue is the analysis of the negative eigenvalues of the Hessian, since their presence is related to regions of unstable negative curvature of the energy surface and thus possibly to the boundaries of different basins of attractions in the phase space. In particular, the number of negative eigenvalues of the Hessian, called the *index*, is the first and easiest measure of instability. As a consequence, all the tools devised for the investigation of the index in RMT are particularly relevant in the context of statistical mechanics of disordered systems. The average value of the index is trivially related to the average spectrum of the Hessian by a simple integration. On the other hand, a more interesting and less trivial quantity is the *probability distribution* of the index. Indeed, while the average index gives a measure of the overall degree of instability of the energy surface, the knowledge of the fluctuations of the index around its average value allows a more profound and complete geometric description of the energy landscape.

In this paper, we compute the probability distribution of the index for an ensemble of Gaussian random matrices with a diagonal shift. This ensemble provides the simplest possible model for the Hessian of a disordered system at a given energy and represents the ideal context where to develop the technical aspects of this kind of computation. Moreover, in the Gaussian context we are able to give nontrivial physical interpretations of our results.

In order to compute the index distribution, we use a fer-

mionic replica method. In the past, the replica method has been applied to recover standard results in RMT, with variable success. Recently the interest of the community has focused again on this method^{14–16} and some indications of the mathematical consistency of the method have been provided, even if some strong criticisms still persist.¹⁷ The present computation offers an interesting example where the replica method can be applied to obtain exact results, which are not easily available in the standard RMT literature.

There is also another important reason for using the replica method in the computation of the index, which is related to the physical relevance of the Hessian discussed above. As we have seen, RMT can be used once an effective probability distribution for the Hessian has been worked out from the distribution of the configurations and from the distribution of the quenched disorder. This effective distribution will not be Gaussian in general (unless we consider some very particular models) and typically it will not belong to the standard ensembles considered by ordinary RMT. By means of the replica method we have in principle no need to assume any specific form of the distribution.

The paper is organized as follows. In Sec. II, we compute the average determinant for matrices of the Gaussian Orthogonal Ensemble as a warm-up exercise to fix notation and ideas. We then proceed in Sec. III to the main part of the paper, where we calculate the average index distribution by means of the replica method, in the limit of large matrices. In Sec. IV, we apply the previous analysis to the specific case of a mean-field spin-glass model, where the Hessian is exactly a Gaussian random matrix. Finally, in Sec. V, we discuss the general relevance of our results and state our conclusions. The contribution of replica symmetry broken saddle-point solutions is calculated in the Appendix.

II. A PRELIMINARY CALCULATION

Consider the matrix

$$M_{ij} = J_{ij} - E \delta_{ij}, \quad (1)$$

where J_{ij} is an N -dimensional real and symmetric random matrix with the Gaussian distribution function

$$\mathcal{P}[J] = 2^{-N/2} \left(\frac{N}{\pi} \right)^{N^2/2} \exp \left(-\frac{N}{4} \text{Tr} J^2 \right). \quad (2)$$

We have introduced a diagonal shift E in order to mimic what in general happens in disordered systems, where M represents the Hessian of the Hamiltonian. In this context, we expect to find very few negative eigenvalues of M at low energies, because of the dominance of minima at very low energies. This is the effect of the shift E in (1) and we therefore shall refer in the following to E as the *energy*.

The average density of eigenvalues, or spectrum, of M is defined by

$$\begin{aligned} \rho(\lambda; E) &= -\frac{1}{\pi N} \text{Im} \overline{\text{Tr}(\lambda - M + i\epsilon)^{-1}} \\ &= -\frac{1}{\pi N} \text{Im} \frac{\partial}{\partial \lambda} \overline{\log \det(\lambda - M + i\epsilon)}, \end{aligned} \quad (3)$$

where the bar indicates the average over distribution (2). It is well known that for the Gaussian ensembles the spectrum ρ in the limit $N \rightarrow \infty$ is given by a semicircle centered around $\lambda = -E$, that is

$$\rho(\lambda; E) = \frac{1}{2\pi} \sqrt{4 - (\lambda + E)^2}, \quad (4)$$

while ρ is zero outside the semicircle support.³

In order to fix our notation and to acquire some familiarity with the method we will use, we compute in this section the average determinant of M . In general this is *not* a self-averaging quantity, in the sense that fluctuations around the mean value do not decrease in the limit $N \rightarrow \infty$. The correct object to average is in principle the logarithm of the determinant, as it appears in the definition of ρ , since this is an extensive quantity. However, it is a particular property of the Gaussian case that the determinant *is* self-averaging at the leading order, so that the calculation of $\overline{\det M}$ is an interesting and simple warm-up exercise for what we want to show later.

Using a standard procedure we can write the determinant by means of a Gaussian integral over N -dimensional fermionic vectors $(\bar{\psi}, \psi)$

$$\det M = \int d\bar{\psi} d\psi \exp \left[-\sum_{i,j=1}^N \bar{\psi}_i \psi_j (J_{ij} - E \delta_{ij}) \right], \quad (5)$$

If we now average over the J , perform a Hubbard-Stratonovich transformation and integrate out the fermions,^{18,19} we obtain

$$\overline{\det M} = \int dq e^{NS(q,E)}, \quad (6)$$

with

$$S(q,E) = -\frac{1}{2} q^2 + \log(-E - iq). \quad (7)$$

This integral can be solved exactly in the limit $N \rightarrow \infty$ by means of the steepest descent method.²⁰ The action S has two saddle points in the complex plane, given by

$$q_{\pm} = \frac{i}{2} E \pm \frac{1}{2} \sqrt{4 - E^2}. \quad (8)$$

According to the value E of the energy, only one or both the saddle points may contribute to the integral so that we can distinguish three different regimes:

$E < -2$: For these values of the energy the only stationary point contributing to the integral is q_+ and we have

$$\overline{\det M} = e^{NS(q_+, E)} = 2^{-N} (|E| - \sqrt{E^2 - 4})^N e^{N(|E| + \sqrt{E^2 - 4})^2/8}. \quad (9)$$

In this energy regime, the spectrum ρ has support completely contained in the positive semiaxis and we thus expect the average determinant to be positive, as it is.

$E > 2$: The support of the spectrum is now entirely contained in the negative semiaxis, so we expect all eigenvalues

of the matrix to be negative. In this case, the relevant saddle point turns out to be q_- , and we thus find for the determinant

$$\begin{aligned} \overline{\det M} &= e^{NS(q_-, E)} \\ &= (-1)^N 2^{-N} (|E| - \sqrt{E^2 - 4})^N e^{N(|E| + \sqrt{E^2 - 4})^2/8}, \end{aligned} \quad (10)$$

with the correct prefactor $(-1)^N$ indicating that all eigenvalues are negative.

$-2 < E < +2$: In this regime the situation is qualitatively different. Indeed in this case the integration path can be deformed in a path passing through the two saddle points q_+ and q_- and the real part of the action S is the same in each one of them. Therefore, the two saddle points both contribute to the integral and we have

$$\overline{\det M} = e^{NS(q_+, E)} + e^{NS(q_-, E)} = (-1)^{N\alpha(E)} e^{N(E^2 - 2)/4 + \log 2}, \quad (11)$$

where

$$\alpha(E) = \frac{1}{\pi} \arctg\left(\frac{-\sqrt{4 - E^2}}{E}\right) + \frac{1}{4\pi} E \sqrt{4 - E^2}. \quad (12)$$

At these values of the energy the spectrum of M is partly contained in the negative semiaxis, so that a nontrivial fraction of the eigenvalues is negative. The interesting point is that the interplay between the two saddle points gives rise to the correct sign of the determinant. Indeed, it is easy to check that $\alpha(E)$ is exactly the mean fraction of negative eigenvalues of M , that is

$$\alpha(E) = \int_{-\infty}^0 d\lambda \rho(\lambda; E). \quad (13)$$

Note that the mechanism we have described above, given by the interplay between the two saddle points, is crucial in order to obtain the correct result for the determinant of M .

III. THE INDEX DISTRIBUTION

The index \mathcal{I}_M of a matrix M , defined as the number of its negative eigenvalues, can be computed from the following formula²¹

$$\mathcal{I}_M = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} [\log \det(M - i\epsilon) - \log \det(M + i\epsilon)]. \quad (14)$$

The meaning of this relation is quite clear: the function $f(z) = \log \det(M - z)$ has a cut on the real axis at each eigenvalue of M , such that by means of the limit in Eq. (14) we are crossing as many cuts as negative eigenvalues are present. Besides, this formula can be simply obtained by integrating the nonaveraged spectrum (3) from minus infinity up to zero. In the case we are considering, the index is a function of the energy E and its average value is given by $N\alpha(E)$ [Eq. (12)].

We are interested in calculating the average probability distribution of the index, at a given energy E , that is the probability $P(K; E)$ to have a matrix M with index \mathcal{I}_M equal to K , at energy E

$$P(K; E) = \overline{\delta[K - \mathcal{I}_M(E)]}. \quad (15)$$

In the following, it will be important to distinguish between the *extensive* index K , which is a positive integer between 0 and N , and the *intensive* one $k = K/N$, which takes values in the continuous interval $[0, 1]$, and whose probability distribution is

$$p(k; E) = \overline{\delta[k - \mathcal{I}_M(E)/N]} = NP(Nk; E). \quad (16)$$

Note that the limit $N \rightarrow \infty$ is well defined only for $p(k; E)$.

From Eqs. (14) and (15) we get

$$P(K; E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu e^{-i\mu K} \overline{G(\mu, E)}, \quad (17)$$

where

$$G(\mu, E) = \det^{\mu/2\pi}(M - i\epsilon) \det^{-\mu/2\pi}(M + i\epsilon). \quad (18)$$

We now make use of the replica method to represent the powers of the determinants in $G(\mu, E)$ as analytic continuations of integer powers

$$\det^{\pm\mu/2\pi}(M \mp i\epsilon) = \lim_{n_{\pm} \rightarrow \pm\mu/2\pi} \det^{n_{\pm}}(M \mp i\epsilon). \quad (19)$$

By introducing two different sets of N -dimensional fermionic vectors $(\bar{\chi}_{\pm}^r, \chi_{\pm}^r)$ with $r = 1, \dots, n_{\pm}$, we can rewrite the determinants as

$$\begin{aligned} \det^{\pm\mu/2\pi}(M \mp i\epsilon) &= \lim_{n_{\pm} \rightarrow \pm\mu/2\pi} \int D\bar{\chi}_{\pm}^r D\chi_{\pm}^r \\ &\quad \times \exp\left[-\sum_{r=1}^{n_{\pm}} \bar{\chi}_{\pm}^r (M \mp i\epsilon) \chi_{\pm}^r\right], \end{aligned} \quad (20)$$

where the sums over the matrix indices i, j are hereafter always understood. We can write everything in a more compact fashion by introducing the Grassmann vectors $(\bar{\psi}_a, \psi_a)$, with $a = 1, \dots, (n_+ + n_-)$, defined as (see also Ref. 22)

$$(\psi_1, \dots, \psi_{(n_+ + n_-)}) \equiv (\chi_+^1, \dots, \chi_+^{n_+}, \chi_-^1, \dots, \chi_-^{n_-}), \quad (21)$$

together with the matrix

$$\epsilon_{ab} \equiv \text{diag}(\underbrace{\epsilon, \dots, \epsilon}_{n_+}, \underbrace{-\epsilon, \dots, -\epsilon}_{n_-}). \quad (22)$$

Note that both ψ_a and ϵ_{ab} have replica dimension $n \equiv (n_+ + n_-) \rightarrow 0$. In this way, we have for G

$$G(\mu, E) = \lim_{n_{\pm} \rightarrow \pm \mu/2\pi} \int D\bar{\psi} D\psi \times \exp \left[- \sum_{ab=1}^n \bar{\psi}_a (M \delta_{ab} - i \epsilon_{ab}) \psi_b \right]. \quad (23)$$

The average of G over the distribution of J can be computed in the standard way^{18,19,15} by a generalization of the procedure of the previous section, the main difference being the fact that we have an extra replica dimension, so that the variable q must be replaced by a matrix Q_{ab} . We, thus, obtain the σ model^{18,19} in the form

$$\overline{G(\mu, E)} = \int DQ e^{NS(Q, E)}, \quad (24)$$

with

$$S(Q, E) = -\frac{1}{2} \text{Tr} Q^2 + \log \det(-\hat{E} - iQ), \quad (25)$$

and $\hat{E}_{ab} = E \delta_{ab} + i \epsilon_{ab}$. Note the similarity with Eqs. (6) and (7). The matrix Q is an $n \times n$ self-dual real-quaternion matrix.^{19,15} It has $2n^2 - n$ degrees of freedom, and is diagonalized by transformations of the symplectic group $Sp(n)$. Note that the symmetry properties of the matrix Q are a direct consequence of having used a purely fermionic representation in Eqs. (20)–(23).²³ In Eq. (25), we see for the first time the role of ϵ as a symmetry breaking field. The matrix \hat{E} has an upper block of size n_+ , which contains $+i\epsilon$ and a lower one of size n_- with $-i\epsilon$, so that the action is only invariant under $Sp(n)/Sp(n_+) \times Sp(n_-)$, and the full invariance under $Sp(n)$ is only recovered in the limit $\epsilon \rightarrow 0$. However, how *exactly* the symmetry breaking affects the calculation will become clearer below.

We can evaluate the integral (24) by means of the steepest descent, or saddle point, method, which becomes exact for large N . The saddle point equation for the matrix Q reads

$$Q = i(\hat{E} + iQ)^{-1}.$$

This equation can be solved assuming for Q a diagonal form, $Q_{ab} = z_a \delta_{ab}$. We have two different sets of equations, one set for the elements belonging to the upper block, $z_a^{(u)}$, and a second set for the elements of the lower block $z_a^{(l)}$. The only difference between the two sets is, of course, the sign of ϵ ,

$$z_a^{(u)} = i(E + i\epsilon + iz_a^{(u)})^{-1} \text{upper block} \quad (26)$$

$$z_a^{(l)} = i(E - i\epsilon + iz_a^{(l)})^{-1} \text{lower block}.$$

Each one of these two sets of equations has two solutions, $z_{\pm}^{(u)}$ for the upper block, $z_{\pm}^{(l)}$ for the lower one, namely

$$z_{\pm}^{(u)} = \frac{i}{2}(E + i\epsilon) \pm \frac{1}{2} \sqrt{4 - (E + i\epsilon)^2}, \quad (27)$$

$$z_{\pm}^{(l)} = \frac{i}{2}(E - i\epsilon) \pm \frac{1}{2} \sqrt{4 - (E - i\epsilon)^2}.$$

For all values of the energy such that $|4 - E^2| \gg \epsilon$ these solutions can be expanded in powers of ϵ and read

$$z_{\pm}^{(u)} = q_{\pm} - \epsilon \left(\frac{1}{2} \pm i \frac{E}{2\sqrt{4 - E^2}} \right) + O(\epsilon^2), \quad (28)$$

$$z_{\pm}^{(l)} = q_{\pm} + \epsilon \left(\frac{1}{2} \pm i \frac{E}{2\sqrt{4 - E^2}} \right) + O(\epsilon^2),$$

where q_{\pm} are given in Eq. (8).

There are some important things to note here, related to the fact that the presence of ϵ crucially modifies the mutual relevance of the different saddle points. We have seen in the previous section that in the regime $-2 < E < 2$ both the saddle points q_+ and q_- contribute to the integral. This is true also in the present case, when a value $\epsilon \neq 0$ is considered: for each z_a the integration path can be deformed in a path passing through z_+ and z_- and in principle both the saddle points must be taken into account. However, when we look at the real part of the action S , we now discover that the contribution of one saddle point is exponentially dominant over the other by a factor $\exp(-N\epsilon)$. This is in contrast with the case of the previous section, where the real part of S was the same in the two saddle points.

The crucial point is that, due to opposite sign of ϵ in the upper and lower blocks, the real part of the action is tilted in opposite ways in the two blocks and, as a consequence, the dominant saddle point becomes z_+ for the upper block and z_- for the lower one. We now start to understand the way in which ϵ works as a symmetry breaking field: without ϵ the two saddle points have the same weight in the integral and we have to consider both of them. With ϵ , the weights are modified in opposite ways for the upper and lower blocks. In order to apply the steepest descent method we must perform the limit $N \rightarrow \infty$ *before* the limit $\epsilon \rightarrow 0$, and this selects *just one* different saddle point for each of the two different blocks, dumping completely the nondominant contribution. As a result, when at the end $\epsilon \rightarrow 0$ we have selected q_+ for the upper block and q_- for the lower one. This is very reminiscent of what happens in statistical physics, where, in order to break a symmetry by means of an external field, the thermodynamic limit must be performed before sending the field to zero.

On the other hand, for energies $|E| > 2$, the effect of ϵ is harmless, there is no qualitative change from the situation described in the previous section and the same kind of saddle point for the upper and lower block contributes to the integral.

We can now proceed in our computation. We will focus first on the region $-2 < E < 2$, where the typical spectrum is not positive defined and where we thus expect a more interesting index distribution. According to the above discussion on the dominant saddle points, we must consider the following form for the saddle point solution Q_{SP} :

$$Q_{\text{SP}} = \text{diag}(\underbrace{z_+^{(u)}, \dots, z_+^{(u)}}_{n_+}, \underbrace{z_-^{(l)}, \dots, z_-^{(l)}}_{n_-}). \quad (29)$$

This form is invariant under the unbroken group $Sp(n_+) \times Sp(n_-)$ of replica symmetry transformations, and

in this sense we shall refer to it as a replica symmetric (RS) saddle point.^{10,14} We note that Eq. (18) is invariant under the simultaneous action of complex conjugation and inversion of μ , which after replicating becomes $n_{\pm} \rightarrow n_{\mp}$, and that our saddle point satisfies this invariance. If we plug expression (29) into Eq. (24), we obtain after taking the limit $\epsilon \rightarrow 0$

$$\begin{aligned} \overline{G(\mu, E)} &= (-E - iq_+)^{N\mu/2\pi} (-E - iq_-)^{-N\mu/2\pi} \\ &\quad \times e^{-N\mu(q_+^2 - q_-^2)/4\pi} \\ &= \exp[i\mu N\alpha(E)], \end{aligned} \quad (30)$$

where $\alpha(E)$ is the average fraction of negative eigenvalues given by Eq. (12). From Eq. (17) we finally get the probability $p(k, E)$ in the limit $N \rightarrow \infty$,

$$p(k, E) = \delta[k - \alpha(E)]. \quad (31)$$

This result is very reasonable, but also rather trivial: the probability distribution of the intensive index is a δ function peaked on its average value in the limit $N \rightarrow \infty$. In order to observe a nontrivial behavior we need to consider the scaling with N , that is, the distribution of the index for large but finite N . This is particularly important if we are interested in the distribution of the extensive index, as for example in the case of disordered systems, where we want to know the change in the probability of different stationary points when variations of the index of order one, not of order N , are considered.

To go beyond result (31), we must consider fluctuations around the saddle point (29). The general procedure is discussed in the Appendix. As expected there are three kinds of fluctuations: within the upper block, within the lower block, and those which mix the two blocks. Their corresponding eigenvalues and degeneracies are,

$$\begin{aligned} \omega_u &= 1 + z_+^{(u)} z_+^{(u)} \\ &= (1 + q_+^2) - \frac{\epsilon q_+^2}{\sqrt{1 - E^2/4}} + O(\epsilon^2) \quad d_u = 2n_+^2 - n_+, \\ \omega_l &= 1 + z_-^{(l)} z_-^{(l)} = (1 + q_-^2) - \frac{\epsilon q_-^2}{\sqrt{1 - E^2/4}} + O(\epsilon^2) \\ &\quad d_l = 2n_-^2 - n_-, \end{aligned} \quad (32)$$

$$\omega_m = 1 + z_+^{(u)} z_-^{(l)} = \frac{\epsilon}{\sqrt{1 - E^2/4}} + O(\epsilon^2) \quad d_m = 4n_+ n_-.$$

The first two sets of eigenmodes are massive modes, in the sense that their eigenvalues are $O(1)$. The third set are soft modes: for vanishing ϵ they would correspond to zero modes associated to the restoration of the $Sp(n_+ + n_-)$ symmetry; for small nonzero ϵ they become soft vibrations. In a formulation of the problem in terms of the nonlinear σ model^{18,19,24} the massive modes would correspond to longitudinal variations of Q (i.e., variations such that $Q \cdot \delta Q + \delta Q \cdot Q \neq 0$), while soft/zero modes would correspond to transverse variations.²⁷ Integrating over the fluctuations, we obtain

$$\begin{aligned} \overline{G(\mu, E)} &= \omega_u^{-(n_+^2 - n_+/2)} \omega_l^{-(n_-^2 - n_-/2)} \omega_m^{-2n_+ n_-} \\ &\quad \times \exp[i\mu N\alpha(E)]. \end{aligned} \quad (33)$$

In the replica limit $n_{\pm} \rightarrow \pm \mu/2\pi$, this quantity becomes

$$\begin{aligned} \overline{G(\mu, E)} &= \exp\left[i\mu N\alpha(E) - \frac{\mu^2}{2\pi^2} \log\left(\frac{\sqrt{\omega_u \omega_l}}{\omega_m}\right) \right. \\ &\quad \left. + \frac{\mu}{4\pi} \log\left(\frac{\omega_u}{\omega_l}\right) \right]. \end{aligned} \quad (34)$$

From Eq. (17) we obtain the distribution for the extensive and intensive index for finite but large N :

$$P(K, E) = \sqrt{\frac{1}{2\pi\Delta(E)}} \exp\left[-\frac{[K - N\alpha(E) + \beta(E)]^2}{2\Delta(E)} \right], \quad (35)$$

$$p(k, E) = \sqrt{\frac{N^2}{2\pi\Delta(E)}} \exp\left[-\frac{N^2[k - \alpha(E) + \beta(E)/N]^2}{2\Delta(E)} \right]. \quad (36)$$

These are Gaussian distributions peaked on the average value $\alpha(E)$. Indeed the shift,

$$\beta(E) = \frac{1}{2\pi} \arctg\left(\frac{E}{\sqrt{4 - E^2}}\right), \quad (37)$$

is of order one and is not relevant at large enough values of N . The variance $\Delta(E)$ is given by

$$\Delta(E) = \frac{1}{\pi^2} \log\left(\frac{\sqrt{\omega_u \omega_l}}{\omega_m}\right), \quad (38)$$

that is

$$\Delta(E) = \frac{1}{\pi^2} \log[2\pi^2 \epsilon^{-1} \rho_0(E)^2], \quad (39)$$

where we have defined $\rho_0(E) \equiv \rho(\lambda=0; E)$ [see Eq. (4)]. This result for the variance can also be obtained by the method of orthogonal polynomials where ϵ plays the role of a high frequency cutoff.^{25,26,4}

The fact that expression (39) still depends on ϵ can seem rather unphysical, especially when we consider the fact that the limit $\epsilon \rightarrow 0$ has to be performed. However, we have to remember that we are looking at finite N corrections, and this very fact makes the parameters ϵ and N no longer independent. In this way, the presence of ϵ translates in a more physical N dependence and this allows us to compute the scaling of the index distribution with the matrix size N . Before discussing the result we have obtained for the index distribution, we have therefore to address the problem of the relation between ϵ and N .

There are mainly two different reasons why ϵ and N are related. First, as we have previously noted, there is a precise interplay between the two limits, $N \rightarrow \infty$ and $\epsilon \rightarrow 0$, when the saddle point approximation is used in order to solve integral (24): the symmetry breaking due to ϵ works only if $\epsilon \rightarrow 0$ after $N \rightarrow \infty$, as in any thermodynamic calculation. If N is kept finite, we need a value of ϵ big enough to guarantee the

dominance of one saddle point over the other. We have seen that the role of ϵ is to modify the real part of the action in such a way that along the integration path γ one saddle point is weighted more than the other. However, if ϵ is too small, also the nondominant saddle point may give a non-negligible contribution to the integral. To avoid this fact we need the secondary contribution to be suppressed also at finite N and to vanish when the limit $N \rightarrow \infty$ is considered. The suppression factor is given, at order ϵ , by

$$e^{-N[S(z_+^{(u)}) - S(z_-^{(u)})]} = e^{-2\pi N \epsilon \rho_0(E)}, \quad (40)$$

for the upper block (for the lower block an analogous expression is valid). In order for the suppression factor to vanish it must hold

$$\epsilon N \rightarrow \infty, \quad N \rightarrow \infty. \quad (41)$$

This imposes a lower bound for ϵ when N is finite. A natural general choice is therefore to assume

$$\epsilon = \frac{1}{N^{1-\delta(N)}}, \quad (42)$$

where the exponent $\delta(N)$ has to satisfy the relation $\delta(N) \log N \rightarrow \infty$. The simplest possibility is, of course, a constant value of δ . However, as we shall argue immediately below, this would not be consistent with the second condition we have to impose on ϵ .

The second bound for ϵ comes from the following observation. When we perform our calculation with a finite value of N and of ϵ , there are of course two different kinds of corrections to the asymptotic exact result: the first kind is related to the saddle-point approximation and brings corrections that scale as inverse powers of N . The second is related to the nonzero value of ϵ and brings corrections that scales as powers of ϵ . Consistency requires that in the final result the error introduced by considering a finite value of ϵ must be of the same order as the terms we discard in the expansion in $1/N$. It can be easily shown that the corrections to the index distribution (35) for finite ϵ are of order ϵ^2 , that is

$$G(\mu) = \exp[N\alpha(E) + \beta(E) + O(\epsilon^2)]. \quad (43)$$

On the other hand, by considering the Gaussian fluctuations around the saddle point, we are discarding terms of order $1/N^2$ in the exponent of Eq. (43). Thus, we must impose the condition

$$\epsilon^2 \sim \frac{1}{N^2}. \quad (44)$$

Equation (44) is consistent with Eqs. (42) and (41) only if,

$$\delta(N) \rightarrow 0, \quad \delta(N) \log N \rightarrow \infty, \quad N \rightarrow \infty. \quad (45)$$

In this way, we finally get for the variance the result,

$$\begin{aligned} \Delta(E, N) &= \frac{1}{\pi^2} \log[4 \pi^2 N^{(1-\delta(N))} \rho_0(E)^2] \\ &\sim \frac{1}{\pi^2} \log N + \frac{2}{\pi^2} \log(2 \pi \rho_0), \end{aligned} \quad (46)$$

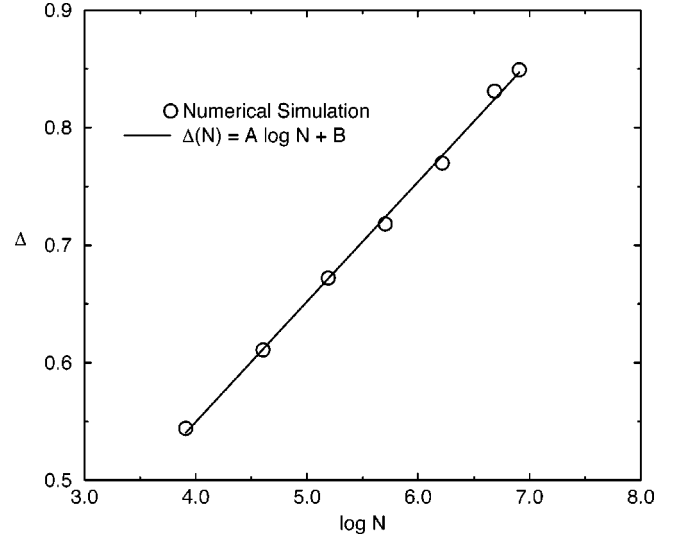


FIG. 1. The variance Δ as a function of $\log N$ for $E=0$, obtained by means of exact numerical diagonalization on the Gaussian orthogonal ensemble. The full line is the linear fit.

where we have taken $\epsilon = 1/2N^{1-\delta}$, the factor $1/2$ being consistent with Eq. (40) at $E=0$. Note that we have disregarded $\delta(N)$ in the right-hand side of Eq. (46) since we are interested in the asymptotic behavior of the variance with respect to N at a given value of the energy E . Indeed $\delta(N)$ represents a vanishing correction to the slope of Δ versus $\log(N)$, while the last term in Eq. (46) gives a finite value for the intercept and has therefore to be kept.

This result agrees very well with numerical simulations: in Fig. 1, we plot the variance Δ as a function of $\log N$, obtained by exact numerical diagonalization. A linear fit gives

$$\begin{aligned} \Delta &= \frac{a}{\pi^2} \log N + \frac{b}{\pi^2} \log(2 \pi \rho_0), \quad a = 1.005 \pm 0.006, \\ & \quad b = 1.993 \pm 0.003. \end{aligned} \quad (47)$$

This same scaling for the variance has been found also in Ref. 25, where a completely different method based on the invariance properties of the Gaussian orthogonal ensemble and the dominance of intrinsic binary correlations was used. In the Appendix, we show in details that the contributions of the other possible saddle point solutions of the whole integral (24) to the index distribution are smaller by inverse powers of $\log N$ in this energy region, therefore the scaling with N is correctly reproduced by Eqs. (35), (36) and (46).

We can finally analyze the significance of our result, Eqs. (35), (36), and (46), in the energy regime $-2 < E < 2$. What we see is that the variance of the probability distribution of the *intensive* index goes to zero in the limit $N \rightarrow \infty$ and this was quite expected, given our former result (31). On the other hand, the variance of the distribution of the *extensive* index diverges logarithmically for $N \rightarrow \infty$. The meaning of this result is the following: on the one hand the probability of finding a matrix with an index density different from the average one, that is with an extensive index $\mathcal{I} \sim N\alpha + O(N)$, is zero in the limit $N \rightarrow \infty$. But, on the other hand, the probability of having a matrix whose index differs from

the average one for a number of negative eigenvalues of order one, i.e., $\mathcal{I} \sim N\alpha + O(1)$, is exactly the same as the probability of having a matrix with the average index, in the limit $N \rightarrow \infty$. As we shall see in the next section, this fact has some very interesting physical consequences in the context of disordered systems.

Let us now look at the other energy regions. First of all, we note that the derivation of Eqs. (28), (32), and (40) holds as long as the energy is such that $\rho_0(E)$ is of $O(1)$. But this condition breaks down when the energy gets close to ± 2 and $\rho_0(E) \ll 1$. In this region the procedure previously adopted to compute the index distribution has to be modified. Indeed, when $\rho_0(E)$ becomes too small the suppression mechanism (40) starts being inefficient, and the saddle point (29) is no longer the only one contributing to the integral. At some point the excitations that were treated as soft modes in Eq. (32) must be considered as zero modes connecting equivalent saddle points: there exists a manifold $Sp(n_+ + n_-)/Sp(n_+) \times Sp(n_-)$ of saddle points and the original replica symmetry under $Sp(n_+ + n_-)$ is restored. At this stage ϵ plays no longer any role and it can be taken to zero. The massive modes are the same as in Eq. (32), and after integrating over them and exactly over the degrees of freedom associated with the zero modes we obtain (up to trivial factors in the replica limit)

$$\overline{G(\mu, E)} = N^{2n_+ + n_-} \mathcal{V}_{n_+ + n_-}^{n_+} \omega_u^{-(n_+^2 - n_+/2)} \omega_1^{-(n_-^2 - n_-/2)} \times \exp[i\mu N\alpha(E)], \quad (48)$$

where $\mathcal{V}_{n_+ + n_-}^{n_+}$ corresponds to the volume of the manifold of saddle-points solutions (see the Appendix). At this point, one has to analytically continue the previous expression for $n_+ \rightarrow \mu/2\pi$, $n_- \rightarrow 0$. The volume $\mathcal{V}_{n_+ + n_-}^{n_+}$ is finite for $n_+ = 0$ and it is zero for positive integers.¹⁴ Its analytic continuation is an oscillatory function of n_+ , with exponentially increasing amplitude¹⁷, so that the presence of this factor in the former equation makes the index distribution non-Gaussian. However, as long as this analytic continuation is finite for non-integer $n_+ \sim 1$, the distribution can be approximated by a Gaussian with variance

$$\Delta(E \sim \pm 2) = \frac{1}{\pi^2} \log[8\pi^4 N \rho_0(E)^3]. \quad (49)$$

We can see from Eq. (49) that the variance still scales as $\log N$. However, when $|E - 2| \sim 1/N^{2/3}$, we have $\rho_0(E) \sim 1/N^{1/3}$ and a further crossover takes place: the variance $\Delta(E)$ becomes of order one meaning that the index distribution is dramatically more peaked around its typical value as we approach $E = \pm 2$. Note also that when $E \sim -2 + 1/N^{2/3}$ the typical index $\alpha(E)$ becomes of order 1, meaning that in this region matrices with $O(1)$ negative eigenvalues are dominant. Summarizing, in the energy regime where the average number of negative eigenvalues is of order one, the fluctuations around the mean value become of order one too.

When the energy is exactly at the threshold values $E = \pm 2$ we have a special case since the saddle point equations for the eigenvalues have a single degenerate solution,

and the harmonic terms in the expansion around the saddle point vanish. It is not difficult to show that the distributions here become

$$P(K, E \rightarrow -2^+) = N^{-1} \delta(K),$$

$$P(K, E \rightarrow 2^-) = N^{-1} \delta(K - N). \quad (50)$$

The calculation in the regions $|E| > 2$ is completely straightforward since ϵ plays no role from the beginning. As mentioned before, the same kind of saddle point has to go in both blocks, so that we have

$$Q_{\text{SP}} = \text{diag}(\underbrace{z_{\pm}^{(u)}, \dots, z_{\pm}^{(u)}}_{n_+}, \underbrace{z_{\pm}^{(l)}, \dots, z_{\pm}^{(l)}}_{n_-}), \quad (51)$$

where the plus (minus) sign corresponds to negative (positive) energies. There is only one kind of massive fluctuation with degeneracy $2n^2 - n$, which goes to zero in the replica limit, and thus the integration over fluctuations gives a trivial prefactor. The final result for the distribution of K is

$$P(K, E) = \begin{cases} N^{-1} \delta(K) & E < -2 \\ N^{-1} \delta(K - N) & E > 2, \end{cases} \quad (52)$$

which coincides with the limiting behavior (50) of the distribution in the region $-2 < E < 2$. Thus, while in the energy region $-2 < E < 2$ values of the index with an $O(1)$ difference from the typical one have a finite probability, here the index distribution is so much peaked on the typical value that even small changes in the index have zero probability.

IV. AN APPLICATION TO DISORDERED SYSTEMS

In this section, we consider a mean-field spin-glass model, that has been extensively studied in the last years and whose thermodynamical as well as dynamical features are very well known, namely the p -spin spherical model.²⁸⁻³² Our aim is to use the results of the calculation we have carried out in the previous section, in order to have a better understanding of the statistical and geometrical properties of the energy landscape for this model.

This problem is by itself relevant, because both the static properties and the peculiar off-equilibrium dynamical behavior of mean field spin glasses, and in particular of this model, are known to be deeply related to the distribution of the minima and of the saddles of the Hamiltonian.^{7,9,10,22} Moreover, it is now commonly accepted that the p -spin spherical model shares many common features with structural glasses, which are presently one of the major challenges for statistical mechanics. Indeed, notwithstanding the completely different form of the Hamiltonians, some structural glasses (in particular fragile glasses) and the p -spin spherical model have a very similar structure of the energy landscape.³³ Therefore, a thorough investigation of the energy landscape for the p -spin spherical model is important also for a better understanding of structural glasses.

As already stated in the Introduction, knowing the index distribution of the Hessian at various energies is equivalent to knowing the fluctuations in the stability of the energy surface. In other words, the index distribution tells us what are the dominant stationary points of the Hamiltonian (or

saddles) at a given energy, and, more importantly, what is the probability distribution around the typical saddles, thus providing an insight on the mutual entropic accessibility of different stationary points. This is what we are going to describe in this last section.

The reason why the p -spin spherical model is particularly appropriate for an application of the above calculation and concepts is the following: when we look at the stationary points of the Hamiltonian of this system, we find that the Hessian matrix M in such stationary points, behaves as a Gaussian random matrix of the same kind as the ones considered in the calculations above. More specifically, if we classify the stationary points of the Hamiltonian in terms of their energy density E , we find that the Hessian $M(E)$ in these stationary points is a random matrix of the form (see for instance Ref. 31)

$$M_{ij}(E) = J_{ij} - E \delta_{ij}, \quad (53)$$

where J_{ij} is an N -dimensional real and symmetric random matrix with the same Gaussian distribution as (2), and where N is the size of the system. The spectrum of the Hessian in the stationary points is therefore,

$$\rho(\lambda; E) = \frac{1}{2\pi} \sqrt{E_{th}^2 - (\lambda + E)^2}, \quad (54)$$

where E_{th} is the so-called *threshold energy*, which depends on the parameters of the model (in the previous sections it was $|E_{th}|=2$). Given the particular shape of the Hessian, we can completely disregard the details of the p -spin spherical model and assume the results obtained in our calculation as the starting point, interpreting these results in terms of probability distributions of the stationary points of the Hamiltonian.

Let us begin our geometric analysis of the energy landscape from very low energies. When $E < -|E_{th}|$ the semicircle is entirely contained in the positive semiaxis and the average determinant of the Hessian is positive: this is the region dominated by minima, as the index distribution (52) shows. Moreover, as we have already noted in the previous section, the probability of finding a stationary point with an index different from the typical one (i.e., 0) is zero. Minima are strongly dominant in this energy regime. A more careful analysis¹⁰ shows that even in this regime there are saddles with nonzero index, but the probability of these objects is exponentially small in N , that is

$$P(K, E) \sim e^{-KN\Omega(E)} \quad K = 1, 2, 3, \dots \quad (55)$$

This result is obtained by considering nonsymmetric contribution to the saddle-point equations (see Ref. 10 and the Appendix) and, consistently with Eq. (52), it gives a contribution too small to be caught by simply analyzing fluctuations around the dominant saddle point. The above result shows that at low energies minima are exponentially dominant over saddles of order one, and even more dominant over saddles with extensive index. In this sense, we shall call this region the *decoupling regime*, since at any energy below the threshold only one kind of stationary points, namely minima, dominates. When we raise the energy, we finally arrive at $E = -|E_{th}|$: here the semicircle touches the zero and the decoupling between different stationary points is no longer

true. Indeed, it can be proved¹⁰ that $\Omega(E_{th})=0$, meaning that at the threshold energy minima and saddles of order one have the same probability.

Thanks to the calculation of the previous section, we are now in the position to answer the following question: What happens *above* the threshold energy? From a simple inspection of the semicircle law it is clear that above the threshold saddles become important, since many negative eigenvalues appear and the average index $N\alpha(E)$ is nonzero. Yet, in order to have information on the degree of decoupling of the stationary points, the simple typical index $N\alpha(E)$ is not enough. The reason is the following: the knowledge of the typical index does not tell us whether at that same energy other stationary points, different from the typical ones, do or do not have nonzero probability. In this sense the mutual entropic accessibility of different stationary points is encoded in the index distribution $P(K, E)$, which reveals to what extent the typical saddles are dominant over the non-typical ones.

From Eq. (35) we see that in this regime not only the dominant stationary points are saddles of order N , but, also, that the probability of finding a minimum is of order e^{-N^2} . The decoupling between minima and dominant saddles is therefore much more dramatic than the one we found below the threshold. On the other hand, because of the divergence of the variance Δ with N [Eq. (46)], we see that there is a mixing among saddles with the same *intensive* index: the probability of having a saddle whose index differs from the average by a number of order one, is the same as the probability of the typical saddles.³⁴ In other words, the main result is that there is no decoupling among saddles with the same intensive index, so that a mixing of different stationary points occurs, while still a decoupling exists between dominant saddles and minima.

Summarizing, we can therefore distinguish two energy regimes where the probability distribution of the stationary points, and therefore the geometric structure of the energy landscape, is very different: a decoupled regime for $E < E_{th}$ and a mixed regime for $E > E_{th}$. Interestingly enough, the threshold energy E_{th} is exactly the asymptotic energy where a purely dynamical transition occurs: below a critical temperature T_d , the ergodicity is broken and the system is no longer able to visit the entire phase space in its time evolution, remaining confined to an energy level higher than the equilibrium one. This ‘‘dynamical energy’’ is equal to E_{th} .^{30–32}

This suggests us to relate the information we have on the distribution of the stationary points, following from the index distribution, to the dynamical physical behavior of the system. Above T_d the equilibrium energy E of the system is higher than threshold value E_{th} and therefore belongs to the mixed regime: the equilibrium landscape explored by the system is dominated by saddles of order N which, as we have shown, are all equally relevant up to variations of the index of order one. This means that all these unstable stationary points are equally accessible to the system in its time evolution. As T_d is approached the equilibrium energy E gets closer and closer to E_{th} , and the properties of the equilibrium landscape change accordingly to the behavior of $P(K, E)$ we have discussed in the previous section: when $E \sim -|E_{th}| + 1/N^{2/3}$ saddles with index of order one become

the most relevant and the variance of the index distribution is now finite. This means that minima start having a finite probability in this energy regime. The range of temperatures where this behavior takes places is of order $1/N^{2/3}$ and shrinks to zero in the thermodynamic limit. Below T_d , the equilibrium energy belongs to the decoupled regime, that is $E < E_{th}$: minima are now dominant and saddles of any order have exponentially vanishing probability. We can therefore interpret T_d as the temperature where a geometric transition occurs from a regime of strong mixing of the stationary points to a regime of equally strong decoupling.

V. CONCLUSIONS

In this paper, we computed the average index distribution for an ensemble of Gaussian random matrices. We find a result which is in optimum agreement with exact numerical diagonalization. This computation is, in our opinion, an interesting example where the fermionic replica method, together with a careful asymptotic expansion of the integrals, gives correct results. We hope that the present work can, therefore, contribute to clarify the role of the replica method in the context of RMT.

Besides, and this was our main purpose, the index distribution provides a really useful tool for investigating the geometric structure of the energy landscape in disordered systems. In the previous section, we applied this tool to the simple case of the p -spin spherical model and discussed the physical consequences of our results. In general, the task of computing the distribution of the index of the Hessian is not as simple as in the p -spin model. The main reason is that the Hessian usually does not behave as a Gaussian random matrix, because, as noted in the Introduction, its distribution is determined both by the distribution of the quenched disorder and by the distribution of the configurations. However, the same procedure we adopted in this paper can also be applied to these more complicated cases, with the appropriate modifications: to compute the index distribution at a given energy E , one has to average over the distribution of the disorder and integrate over the relevant configurations belonging to the manifold of energy E .¹⁰ This is the reason why the method presented in this paper is particularly suitable for this

task, since it addresses the problem without assuming any particular form for the distribution of the Hessian.

Finally, there have been recently some attempts to find a connection between the occurrence of a thermodynamical phase transition and the change in the topology of the configuration space visited by the system at equilibrium.³⁵ For various non-disordered models which present a second-order phase transition it has been shown via numerical simulations that the fluctuations of the curvature of the configuration space exhibit a singular behavior at the transition point. This is similar to the behavior described in the previous section for the p -spin spherical model, where the average fluctuations of the index (39) at the equilibrium energy encounter a dramatic change as the dynamical transition is approached.¹⁰ This suggests first of all that also in disordered systems a connection between thermodynamical behavior and topology of the configuration space exists. Besides, the case of the p spin is also peculiar in this sense: it presents a static phase transition, which is thermodynamically of second order, but it is discontinuous in the order parameter²⁸ and exhibits a purely dynamical transition at a higher temperature.³⁰ As we have shown, in this case a dramatic change of geometrical properties occurs at the dynamical transition, indicating that a more complex situation probably holds for disordered systems which present this sort of behavior.

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APPENDIX

In this Appendix, we calculate the contributions to the index distribution of saddle-point (SP) solutions different from Eq. (29) for energies in the region $-2 < E < 2$.

A general SP solution reads,

$$Q_{\text{SP}} = \text{diag} \left(\underbrace{z_+^{(u)}, \dots, z_+^{(u)}}_{p_+}, \underbrace{z_-^{(u)}, \dots, z_-^{(u)}}_{n_+ - p_+}, \underbrace{z_-^{(l)}, \dots, z_-^{(l)}}_{p_-}, \underbrace{z_+^{(l)}, \dots, z_+^{(l)}}_{n_- - p_-} \right). \quad (\text{A1})$$

For $p_{\pm} \neq 0, n_{\pm}$ these SP are not invariant under the unbroken replica symmetry $Sp(n_+) \times Sp(n_-)$ of Eq. (24), and have therefore been called replica symmetry broken (RSB) solutions,¹⁴ even though the symmetry is subsequently restored by zero modes. The action (25) at the saddle-point solution (A1) reads, after taking the replica limit,

$$iN\alpha(E)[\mu - 2\pi(p_+ - p_-)].$$

Let us consider now the fluctuations.¹⁵ There are sixteen different normal modes. They are labeled by the pair of indices (α, σ) , where the index $\sigma = \pm 1$ indicates the upper and lower block, and the index $\alpha = \pm 1$ indicates the sub-block with solution z_{α} . The eigenvalues are thus denoted by $\omega_{(\alpha, \sigma)(\alpha', \sigma')}$. There are three kinds of fluctuations:

(i) Massive modes: these correspond to $\alpha = \alpha'$ for any σ and σ' , with eigenvalues $\omega_{(\alpha\sigma)(\alpha\sigma')} = (1 + q_{\alpha}^2)$.

(ii) Zero modes: for $\alpha = -\alpha'$ and $\sigma = \sigma'$. They are present for any RSB solution.

(iii) Soft modes: for $\alpha = -\alpha'$ and $\sigma = -\sigma'$, with eigenvalue $\omega_{(\alpha\sigma)(-\alpha-\sigma)} = \alpha\sigma\epsilon/\pi\rho_0(E)$.

Let us consider SP's with $p_+ = p_- = p$, which respect the symmetry of the problem under simultaneous complex conjugation and inversion of μ . For these, after integrating out the fluctuations, we obtain

$$\overline{G(\mu, E)}_{p_{\pm}=p} = \lim_{n_{\pm} \rightarrow \pm \mu/2\pi} \mathcal{V}_{n_+}^p \mathcal{V}_{n_-}^p [\pi\rho_0^2(E)]^{4p^2} \overline{G(\mu, E)}_0, \quad (\text{A2})$$

where $\overline{G(\mu, E)}_0$ is the result from the RS solution (34) [with $\epsilon \sim 1/N^{1-\delta}$, see Eq. (45)] and the volume of the zero mode manifolds $\mathcal{V}_{n_{\pm}}^p$ is given by

$$\mathcal{V}_n^p = [4\pi^3 \rho_0^2(E)]^{2p(n-p)} F_n^p, \\ F_n^p = \frac{\Gamma(1+n)}{\Gamma(1+p)\Gamma(1+n-p)} \prod_{j=1}^p \frac{\Gamma(1+2j)}{\Gamma[1+2(n-j+1)]}. \quad (\text{A3})$$

We need now to determine the zero-modes volume in the replica limit. Using the property of the Gamma function $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, and noting that $n_{\pm} \rightarrow \pm \mu/2\pi \sim N/\Delta(E)$, so we want the large $|n_{\pm}|$ limit, we find that

$$F_{n_+}^p \rightarrow \left[\frac{(-1)^{p^2} \pi^{-4p^2}}{\Gamma^2(1+p)} \prod_{j=1}^p \Gamma^2(1+2j) \right] \left(\frac{\mu}{2\pi} \right)^{2p^2-p} \sin^p \mu. \quad (\text{A4})$$

We can now use Eq. (17) to obtain the contribution to the index distribution of the RSB solutions. For the simplest one $p=1$ we get,

$$P_1(K, E) = [4\pi^2 \rho_0^2(E)]^{-4} \sqrt{\frac{2}{\pi \Delta^3(E)}} \\ \times \left([K - N\alpha(E) + 1] \exp\left\{ -\frac{[K - N\alpha(E) + 1]^2}{2\Delta(E)} \right\} \right. \\ \left. - [K - N\alpha(E) - 1] \exp\left\{ -\frac{[K - N\alpha(E) - 1]^2}{2\Delta(E)} \right\} \right). \quad (\text{A5})$$

For $|E| < 2$ this is an $O(1/\log N)$ contribution to the distribution (36) obtained from the RS solution. In contrast to the case of correlation functions,¹⁴ contributions from higher RSB saddle points do not vanish, but give contributions decreasing by powers of $\log N$.

Let us turn now to the external regions $|E| > 2$. The RSB solutions here are those with $p_+ = (n_- - p_-) = p > 0$. Evaluating Eq. (25) in these SP's we find that their contributions are suppressed by

$$\exp\left\{ -pN \left[|E| \sqrt{E^2 - 4} + \log\left(\frac{E^2}{2} + \frac{|E|}{2} \sqrt{E^2 - 4} + 1 \right) \right] \right\},$$

as was already found in Ref. 10.

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