Exact diagonalization of the generalized supersymmetric *t*-*J* model with boundaries

Heng Fan* and Miki Wadati
Department of Physics, Graduate School of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan

Xiao-man Wang
Technology Department of Library, Northwest University, Xi'an 710069, China

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We study the generalized supersymmetric *t*-*J* model with boundaries in three different gradings: fermionic, fermionic, bosonic (FFB), bosonic, fermionic, fermionic (BFF), and fermionic, bosonic, fermionic (FBF). Starting from the trigonometric *R* matrix, and in the framework of the graded quantum inverse scattering method, we solve the eigenvalue problems for the supersymmetric *t*-*J* model. Detailed calculations are presented to obtain the eigenvalues and Bethe ansatz equations of the supersymmetric *t*-*J* model with boundaries in three different backgrounds.

I. INTRODUCTION

One-dimensional strongly correlated electron models, such as the *t*-*J* model, have been attracting a great deal of interest in the context of high-T_c superconductivity. The Hamiltonian of the *t*-*J* model includes the near-neighbor hopping (t) and antiferromagnetic exchange (J):[1,2]

$$H=\sum_{j=1}^{L}\left\{-t\mathcal{P}\sum_{\sigma=\pm1}(c_{j,\sigma}^{\dagger}c_{j+1,\sigma}+\text{H.c.})\mathcal{P}+J\left(\mathbf{S}_j\mathbf{S}_{j+1}-\frac{1}{4}n_n n_{j+1}\right)\right\}. \quad (1)$$

It is known that this model is supersymmetric and integrable for $J=\pm 2t$.[3,4] The supersymmetric *t*-*J* model was also studied in Refs. 5–9; for a review, see Ref. 10 and references therein. Essler and Korepin showed that the one-dimensional Hamiltonian can be obtained from the transfer matrix of the two-dimensional supersymmetric exactly solvable lattice model.[7,9] They used the graded quantum inverse scattering method (QISM) (Refs. 11 and 12) and obtained the eigenvalues and eigenvectors for the supersymmetric *t*-*J* model with periodic boundary conditions in three different backgrounds; for related works, see, for example, Ref. 13. In this paper, we shall start from the trigonometric *R* matrix which is a generalization of the *R* matrix used in Ref. 9. The Hamiltonian is also a generalization of the supersymmetric *t*-*J* model. We shall consider the reflecting boundary condition cases. By using the graded QISM, we obtain the eigenvalues of the transfer matrix with boundaries in three different backgrounds.

The exactly solvable models are generally solved by imposing periodic boundary conditions. Recently, solvable models with reflecting (open) boundary conditions have been extensively studied.[14–39] Besides the original Yang-Baxter equation,[40,41] the reflection equations also play a key role in proving the commutativity of the transfer matrices under reflecting boundary conditions.[14,15] The Hamiltonian includes nontrivial boundary terms which are determined by the boundary *K* matrices.

In our previous paper,[42] we used the algebraic Bethe ansatz method to solve the eigenvalue and eigenvector problems of the supersymmetric *t*-*J* model with reflecting boundary conditions in the framework of the graded QISM [fermionic, fermionic, and bosonic (FFB) grading]. Here we shall extend the results in Ref. 42. We start from the trigonometric *R* matrix proposed by Perk and Schultz[43] and change the formulas to the graded case. Three kinds of grading are imposed, so there are three *R* matrices for different grading. Solving the graded reflection equation, we give general diagonal solutions. There are altogether four kinds of different boundary conditions for each choice of grading. Using the graded algebraic Bethe ansatz method in three possible gradings FFB, bosonic, fermionic, fermionic (BFF), and fermionic, bosonic, fermionic (FBF), we obtain the eigenvalues of the transfer matrix with general diagonal boundary matrices.

The graded method was proposed in Ref. 44, and it was applied for the reflection equation in Ref. 19, and later was applied to fermionic models.[20,21] In this paper, we shall use the graded reflection equation to study the supersymmetric *t*-*J* model. For the supersymmetric *t*-*J* model, the spin of the electrons and the charge ''hole'' degrees of freedom play a very similar role, forming a graded superalgebra with two fermions and one boson. The holes obey boson commutation relations, while the spinons are fermions; see Ref. 10 and references therein. The graded approach has the advantage of making a clear distinction between bosonic and fermionic degrees of freedom. So it is interesting to study the supersymmetric *t*-*J* model with reflecting boundary conditions by the graded algebraic Bethe ansatz method. In this paper, we give a detailed analysis for the Bethe ansatz in three different backgrounds. We should mention that the trigonometric *R* matrix related to the supersymmetric *t*-*J* model with reflecting boundary conditions was studied in Refs. 22 and 23 by using the usual reflection equation; the results have also been extended to more general cases.[24,25] And the thermodynamic limit of the Bethe ansatz was calculated in Ref. 26. The finite-size corrections in the supersymmetric *t*-*J* model with boundary fields are presented in Ref. 37. The integrable bulk Hamiltonian was derived previously by Karowski and Foer-

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ster and by Gonzales-Ruiz.[22,23] Bariev also showed that it is integrable and studied physical properties for the Hermitian case.[8]

As mentioned in Ref. 9, the formulas and the results for three different gradings are significantly different, so we shall write out in detail the graded algebraic Bethe ansatz for the generalized supersymmetric *t*-*J* model with four kinds of boundaries.

The paper is organized as follows: In Sec. II, we review the supersymmetric *t*-*J* model and its generalization. We start from the Perk-Shultz[43] model and change it to the graded case. In Sec. III, the general solutions of the reflection equation are presented. In Sec. IV, in the FFB grading, we use the algebraic Bethe ansatz method to obtain the eigenvalues and eigenvectors of the transfer matrix with boundaries. In Secs. V and VI, we study the case of BFF grading and FBF grading. Section VII includes a brief summary and some discussions.

II. SUPERSYMMETRIC *t*-*J* MODEL AND ITS GENERALIZATION

We first review the supersymmetric *t*-*J* model. For convenience, we adopt the notations in Ref. 9. The Hamiltonian of the supersymmetric *t*-*J* model is given as

$$H=-t\sum_{j=1}^{N}\sum_{\sigma=\pm}[c_{j,\sigma}^{\dagger}(1-n_{j,-\sigma})c_{j+1,\sigma}(1-n_{j+1,-\sigma})+c_{j+1,\sigma}^{\dagger}(1-n_{j+1,-\sigma})c_{j+1,\sigma}(1-n_{j,-\sigma})]+J\sum_{j=1}^{N}\left[S_j^zS_{j+1}^z+\frac{1}{2}(S_j^{\dagger}S_{j+1}+S_jS_{j+1}^{\dagger})-\frac{1}{4}n_jn_{j+1}\right]. \tag{2}$$

This form is an equivalent expression of the Hamiltonian (1). The operators $c_{j,\sigma}$ and $c_{j,\sigma}^{\dagger}$ mean the annihilation and creation operators of electrons with spin σ on a lattice site *j*, and we assume that the total number of lattice sites is *N*, $\sigma=\pm$ representing spin down and up, respectively. These operators are canonical Fermi operators satisfying anticommutation relations

$$\{c_{j,\sigma}^{\dagger},c_{j,\tau}\}=\delta_{ij}\delta_{\sigma\tau}. \tag{3}$$

We denote by $n_{j,\sigma}=c_{j,\sigma}^{\dagger}c_{j,\sigma}$ the number operator for the electron on a site *j* with spin σ, and by $n_j=\Sigma_{\sigma=\pm}n_{j,\sigma}$ the number operator for the electron on a site *j*. The Fock vacuum state $|0\rangle$ satisfies $c_{j,\sigma}|0\rangle=0$. There are altogether three possible electronic states at a given lattice site *j* due to excluding double occupancy:

$$|0\rangle,\quad |\uparrow\rangle_j=c_{j,1}^{\dagger}|0\rangle,\quad |\downarrow\rangle_j=c_{j,-1}^{\dagger}|0\rangle. \tag{4}$$

$S_j^z,S_j,S_j^{\dagger}$ are spin operators satisfying $su(2)$ algebra and can be expressed as

$$S_j=c_{j,1}^{\dagger}c_{j,-1},\quad S_j^{\dagger}=c_{j,-1}^{\dagger}c_{j,},\quad S_j^z=\frac{1}{2}(n_{j,1}-n_{j,-1}). \tag{5}$$

It has been proved that for a special value $J=2t=2$, the Hamiltonian of the supersymmetric *t*-*J* model can be written as the a graded permutation operator[6,7,9]

$$H=-\sum_{j=1}^{N}P_{j,j+1}-2\hat{N}. \tag{6}$$

Here we have omitted a constant term. The total number operator $\hat{N}=\Sigma_{j=1}^{N}n_j$ commutes with the Hamiltonian and is dedicated to the chemical potential. We shall also omit the second term in the following. The graded permutation operator can be represented as

$$P_{ac}^{bd}=\delta_{ad}\delta_{bc}(-1)^{\epsilon_a\epsilon_c}. \tag{7}$$

Here, differently from the nongraded case, we have the Grassmann parities $\epsilon_a=1,0$ representing fermions and bosons, respectively. The Hamiltonian can also be represented by the generators of $u(1|2)$, $su(1|2)$ is a subalgebra of $u(1|2)$,

$$H=-\sum_{j=1}^{N}\left\{\sum_{\sigma}(Q_{j+1,\sigma}^{\dagger}Q_{j,\sigma}+Q_{j,\sigma}^{\dagger}Q_{j+1,\sigma})-2S_j^zS_{j+1}^z-S_jS_{j+1}^{\dagger}-S_{j+1}S_j^{\dagger}+2T_jT_{j+1}\right\}. \tag{8}$$

The generators of the algebra $u(1|2)$ are given by relation (5) and the following:

$$Q_{j,\pm}=(1-n_{j,\mp})c_{j,\pm},$$

$$Q_{j,\pm}^{\dagger}=(1-n_{j,\mp})c_{j,\pm}^{\dagger},\quad T_j=1-\frac{1}{2}n_j. \tag{9}$$

The fundamental representations of these operators take the following form:

$$S_j^z=\begin{bmatrix}-\frac{1}{2}&0&0\\0&\frac{1}{2}&0\\0&0&0\end{bmatrix},\quad T_j=\begin{bmatrix}\frac{1}{2}&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix},$$

$$S_k=e_{21}^k,\quad S_k^{\dagger}=e_{12}^k,\quad Q_{k,1}=e_{32}^k,$$

$$Q_{k,1}^{\dagger}=e_{23}^k,\quad Q_{k,-1}=e_{31}^k,\quad Q_{k,-1}^{\dagger}=e_{13}^k, \tag{10}$$

where e_{ij}^k is a 3×3 matrix acting on the *k*th space with elements $(e_{ij}^k)_{\alpha\beta}=\delta_{i\alpha}\delta_{j\beta}$.

The above Hamiltonian can be obtained from the logarithmic derivative at zero spectral parameter of the transfer matrix constructed by the rational *R* matrix. In this paper, we shall study the trigonometric R matrix. Let us start from the R matrix of the Perk-Schultz model.[43] The nonzero entries of the R matrix are given by

$$\tilde{R}(\lambda)_{aa}^{aa}=\sin(\eta+\epsilon_a\lambda),$$

$$\tilde{R}(\lambda)_{ab}^{ab}=q_{ab}\sin(\lambda),\quad a\neq b,$$

$$\tilde{R}(\lambda)_{ba}^{ab}=e^{i\,\mathrm{sgn}(a-b)\lambda}\sin(\eta),\quad a\neq b, \tag{11}$$

where

$$\mathrm{sgn}(a-b)=\begin{cases}1, & \text{if } a>b,\\ -1, & \text{if } a<b.\end{cases} \tag{12}$$

As mentioned above, ϵ_a is the Grassman parity, $\epsilon_a=0$ for bosons and $\epsilon_a=1$ for fermions. We demand $q_{ab}q_{ba}=1$, in the following, and let $q_{ab}=(-)^{\epsilon_a\epsilon_b}$. This R matrix of the Perk-Schultz model satisfies the usual Yang-Baxter equation

$$\tilde{R}_{12}(\lambda-\mu)\tilde{R}_{13}(\lambda)\tilde{R}_{23}(\mu)=\tilde{R}_{23}(\mu)\tilde{R}_{13}(\lambda)\tilde{R}_{12}(\lambda-\mu). \tag{13}$$

Introducing a diagonal matrix $I_{ac}^{bd}=(-)^{\epsilon_a\epsilon_c}\delta_{ab}\delta_{cd}$, we change the original R matrix to the following form:

$$R(\lambda)=I\tilde{R}(\lambda). \tag{14}$$

Considering the nonzero elements of the R matrix R_{ab}^{cd}, we have $\epsilon_a+\epsilon_b+\epsilon_c+\epsilon_d=0$. One can show that the R matrix satisfies the graded Yang-Baxter equation

$$R(\lambda-\mu)_{a_1a_2}^{b_1b_2}R(\lambda)_{b_1a_3}^{c_1b_3}R(\mu)_{b_2b_3}^{c_2c_3}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{b_2}}$$
$$=R(\mu)_{a_2a_3}^{b_2b_3}R(\lambda)_{a_1b_3}^{b_1c_3}R(\lambda-\mu)_{b_1b_2}^{c_1c_2}(-)^{(\epsilon_{a_1}+\epsilon_{b_1})\epsilon_{b_2}}. \tag{15}$$

In the framework of the QISM, we can construct the L operator from the R matrix as

$$L_{aq}(\lambda)\equiv R_{aq}(\lambda), \tag{16}$$

where a represents the auxiliary space and q represents the quantum space. Thus we have the (graded) Yang-Baxter relation

$$R_{12}(\lambda-\mu)L_1(\lambda)L_2(\mu)=L_2(\mu)L_1(\lambda)R_{12}(\lambda-\mu). \tag{17}$$

Here the tensor product is in the sense of the supertensor product defined as

$$(F\otimes G)_{ac}^{bd}=F_a^bG_c^d(-)^{(\epsilon_a+\epsilon_b)\epsilon_c}. \tag{18}$$

In the rest of this paper, all tensor products are in the supersense. However, there are two kinds of supertensor product; we shall point it out later.

The row-to-row monodromy matrix $T_N(\lambda)$ is defined as the matrix product over the N operators on all sites of the lattice,

$$T_a(\lambda)=L_{aN}(\lambda)L_{aN-1}(\lambda)\cdots L_{a1}(\lambda), \tag{19}$$

where a still represents the auxiliary space, and the tensor product is in the graded sense. Explicitly we write

$$\{[T(\lambda)]^{ab}\}_{\alpha_1\cdots\alpha_N}^{\beta_1\cdots\beta_N}$$
$$=L_N(\lambda)_{a\alpha_N}^{c_N\beta_N}L_{N-1}(\lambda)_{c_N\alpha_{N-1}}^{c_{N-1}\beta_{N-1}}\cdots L_1(\lambda)_{c_2\alpha_1}^{b\beta_1}$$
$$\times(-1)^{\sum_{j=2}^{N}(\epsilon_{\alpha_j}+\epsilon_{\beta_j})\sum_{i=1}^{j-1}\epsilon_{\alpha_i}}. \tag{20}$$

By repeatedly using the Yang-Baxter relation (17), one can prove easily that the monodromy matrix also satisfies the Yang-Baxter relation

$$R(\lambda-\mu)T_1(\lambda)T_2(\mu)=T_2(\mu)T_1(\lambda)R(\lambda-\mu). \tag{21}$$

For periodic boundary conditions, the transfer matrix $\tau_{peri}(\lambda)$ of this model is defined as the supertrace of the monodromy matrix in the auxiliary space. In general case, the supertrace is defined as

$$\tau_{peri}(\lambda)=\mathrm{str}\,T(\lambda)=\sum(-1)^{\epsilon_a}T(\lambda)_{aa}. \tag{22}$$

As a consequence of the Yang-Baxter relation (21) and the unitarity property of the R matrix, we can prove that the transfer matrices commute with each other for different spectral parameters:

$$[\tau_{peri}(\lambda),\tau_{peri}(\mu)]=0. \tag{23}$$

Generally in this sense we mean that the model is integrable. Expanding the transfer matrix in the powers of λ, we can find conserved quantities; the first nontrivial conserved equantity is the Hamiltonian.

For the rational R matrix, it has been proved that the Hamiltonian obtained by taking the first logarithmic derivative at the zero spectral parameter, $H=-i\{d\ln[\tau(\lambda)]/d\lambda\}|_{\lambda=0}=-\sum_{k=1}^{N}P_{k,k+1}$, is equivalent to the Hamiltonian of the supersymmtric t-J model.[9]

Here we shall study the trigonometric case. Noting $R_{ij}(0)=-\sin(\eta)P_{ij}$, the Hamiltonian can be defined as

$$H=\sin(\eta)\frac{d\ln[\tau(\lambda)]}{d\lambda}\bigg|_{\lambda=0}=\sum_{j=1}^{N}H_{j,j+1}, \tag{24}$$

with $H_{j,j+1}\equiv P_{j,j+1}L'_{j,j+1}(0)$.

As an example, we choose fermionic, fermionic, and bosonic (FFB) grading which means $\epsilon_1=\epsilon_2=1$, $\epsilon_3=0$. Explicitly, we can write the R matrix as

$$R(\lambda)=\begin{pmatrix} a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & b(\lambda) & 0 & -c_-(\lambda) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & b(\lambda) & 0 & 0 & 0 & c_-(\lambda) & 0 & 0\\ 0 & -c_+(\lambda) & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & a(\lambda) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & c_-(\lambda) & 0\\ 0 & 0 & c_+(\lambda) & 0 & 0 & 0 & b(\lambda) & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & c_+(\lambda) & 0 & b(\lambda) & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w(\lambda) \end{pmatrix}, \tag{25}$$

where

$$a(\lambda)=\sin(\lambda-\eta),\quad w(\lambda)=\sin(\lambda+\eta),\quad b(\lambda)=\sin(\lambda),\quad c_\pm(\lambda)=e^{\pm i\lambda}\sin(\eta). \tag{26}$$

The rational limit of this R matrix is completely the same as the one used by Essler and Korepin in Ref. 9. In the framework of the QISM, we define the L operator as

$$L_n(\lambda)=\begin{pmatrix} b(\lambda)-[b(\lambda)-a(\lambda)]e_{11}^n & -c_-(\lambda)e_{21}^n & c_-(\lambda)e_{31}^n\\ -c_+(\lambda)e_{12}^n & b(\lambda)-[b(\lambda)-a(\lambda)]e_{22}^n & c_-(\lambda)e_{32}^n\\ c_+(\lambda)e_{13}^n & c_+(\lambda)e_{23}^n & b(\lambda)-[b(\lambda)-w(\lambda)]e_{33}^n \end{pmatrix}. \tag{27}$$

Here e_{ab}^n acts on the nth quantum space.

We denote explicitly the row-to-row monodromy matrix as

$$T(\lambda)=\begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda)\\ A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda)\\ C_1(\lambda) & C_2(\lambda) & D(\lambda) \end{pmatrix}. \tag{28}$$

If we choose the FFB grading, the transfer matrix is then given as

$$\tau(\lambda)_{peri}=-A_{11}(\lambda)-A_{22}(\lambda)+D(\lambda). \tag{29}$$

Thus we can write

$$L'(0)=\begin{pmatrix} 1-[1-\cos(\eta)]e_{11} & i\sin(\eta)e_{21} & -i\sin(\eta)e_{31}\\ -i\sin(\eta)e_{12} & 1-[1-\cos(\eta)]e_{22} & -i\sin(\eta)e_{32}\\ i\sin(\eta)e_{13} & i\sin(\eta)e_{23} & 1-[1-\cos(\eta)]e_{33} \end{pmatrix}. \tag{30}$$

With the help of the fundamental representation of algebra $u(1|2)$, we have

$$\begin{aligned} H_{j,j+1}=&\sum_{\sigma=\pm}[Q_{j,\sigma}Q_{j+1,\sigma}^\dagger+Q_{j,\sigma}^\dagger Q_{j+1,\sigma}]-S_jS_{j+1}^\dagger-S_{j+1}S_j^\dagger+\cos(\eta)[-2S_j^zS_{j+1}^z+2T_jT_{j+1}^\dagger]\\ &+2i\sin(\eta)[-S_j^zT_{j+1}+T_jS_{j+1}^z+S_j^z-S_{j+1}^z+T_j-T_{j+1}]. \end{aligned} \tag{31}$$

As mentioned in the Introduction, this Hamiltonian was previously obtained by Karowski and Foerster and by Gonzales-Ruiz.[22,23] Explicitly, using the fermionic representations (5) and (9), we can write the Hamiltonian of the generalized supersymmetric t-J model as follows:[22,23]

$$\begin{aligned} H=&\sum_{j=1}^N\sum_{\sigma=\pm}[c_{j,\sigma}^\dagger(1-n_{j,-\sigma})c_{j+1,\sigma}(1-n_{j+1,-\sigma})+c_{j+1,\sigma}^\dagger(1-n_{j+1,-\sigma})c_{j+1,\sigma}(1-n_{j,-\sigma})]\\ &-2\sum_{j=1}^N\left[\frac{1}{2}(S_j^\dagger S_{j+1}+S_jS_{j+1}^\dagger)+\cos(\eta)S_j^zS_{j+1}^z-\frac{\cos(\eta)}{4}n_jn_{j+1}\right]+i\sin(\eta)\sum_{j=1}^N[S_j^zn_{j+1}-S_{j+1}^zn_j]. \end{aligned} \tag{32}$$

Here a periodic boundary condition is assumed. We remark that this Hamiltonian is in general not Hermitian.

In this paper, we shall study the reflecting boundary conditions, which may cause nontrivial boundary terms in the Hamiltonian.

III. INTEGRABLE REFLECTING BOUNDARY CONDITIONS AND THE SOLUTIONS OF REFLECTION EQUATION

In this paper, we consider the reflecting boundary condition case. At the end of the 1980s, Sklyanin proposed a systematic approach to handle exactly solvable models with reflecting (open) boundary conditions,[14] which includes a so-called reflection equation proposed by Cherednik:[15]

$$R_{12}(\lambda-\mu)K_1(\lambda)R_{21}(\lambda+\mu)K_2(\mu)=K_2(\mu)R_{12}(\lambda+\mu)K_1(\lambda)R_{21}(\lambda-\mu). \quad (33)$$

For the graded case, the above form of the reflection equation remains the same. We only need to change the usual tensor product to the graded tensor product.[19] We write it explicitly as

$$R(\lambda-\mu)_{a_1a_2}^{b_1b_2}K(\lambda)_{b_1}^{c_1}R(\lambda+\mu)_{b_2c_1}^{c_2d_1}K(\mu)_{c_2}^{d_2}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{b_2}}=K(\mu)_{a_2}^{b_2}R(\lambda+\mu)_{a_1b_2}^{b_1c_2}K(\lambda)_{b_1}^{c_1}R(\lambda-\mu)_{c_2c_1}^{d_2d_1}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{c_2}}. \quad (34)$$

We limit the discussion to diagonal solutions of the reflection equation. Suppose $K(\lambda)_a^b=\delta_{ab}k_a(\lambda)$. Inserting this relation into the reflection equation, we find that there is only one nontrivial relation to be solved:

$$\begin{aligned}&R(\lambda-\mu)_{a_1a_2}^{a_1a_2}R(\lambda+\mu)_{a_2a_1}^{a_1a_2}k(\lambda)_{a_1}k(\mu)_{a_1}\\&+R(\lambda-\mu)_{a_1a_2}^{a_2a_1}R(\lambda+\mu)_{a_1a_2}^{a_1a_2}k(\lambda)_{a_2}k(\mu)_{a_1}\\&=R(\lambda+\mu)_{a_1a_2}^{a_1a_2}R(\lambda-\mu)_{a_2a_1}^{a_1a_2}k(\mu)_{a_2}k(\lambda)_{a_1}\\&+R(\lambda+\mu)_{a_1a_2}^{a_2a_1}R(\lambda-\mu)_{a_1a_2}^{a_1a_2}k(\mu)_{a_2}k(\lambda)_{a_2}.\end{aligned} \quad (35)$$

Suppose $a_2>a_1$, and substitute the exact form of the elements of the R matrix into the above relation. We find a general diagonal solution

$$\frac{k(\lambda)_{a_1}}{k(\lambda)_{a_2}}=\frac{\sin(\xi+\lambda)}{\sin(\xi-\lambda)}e^{-2i\lambda}, \quad (36)$$

where ξ is an arbitrary parameter. In a special limit we can see that the identity is also a solution of the reflection equation. For the cases (FFB, BFF, and FBF grading) we study in this paper, there are two types of solutions to the reflection equation:

$$K_I(\lambda)=\begin{pmatrix}\sin(\xi+\lambda)e^{-2i\lambda}&&\\&\sin(\xi+\lambda)e^{-2i\lambda}&\\&&\sin(\xi-\lambda)\end{pmatrix},$$

$$K_{II}(\lambda)=\begin{pmatrix}\sin(\xi+\lambda)e^{-2i\lambda}&&\\&\sin(\xi-\lambda)&\\&&\sin(\xi-\lambda)\end{pmatrix}. \quad (37)$$

Instead of the monodromy matrix $T(\lambda)$ for periodic boundary conditions, we consider the double-row monodromy matrix

$$\mathcal{T}(\lambda)=T(\lambda)K(\lambda)T^{-1}(-\lambda) \quad (38)$$

for the reflecting boundary conditions. Using the Yang-Baxter relation, and considering the boundary K matrix which satisfies the reflection equation, one can prove that the double-row monodromy matrix $\mathcal{T}(\lambda)$ also satisfies the reflection equation

$$R(\lambda-\mu)_{a_1a_2}^{b_1b_2}\mathcal{T}(\lambda)_{b_1}^{c_1}R(\lambda+\mu)_{b_2c_1}^{c_2d_1}\mathcal{T}(\mu)_{c_2}^{d_2}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{b_2}}=\mathcal{T}(\mu)_{a_2}^{b_2}R(\lambda+\mu)_{a_1b_2}^{b_1c_2}\mathcal{T}(\lambda)_{b_1}^{c_1}R(\lambda-\mu)_{c_2c_1}^{d_2d_1}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{c_2}}. \quad (39)$$

Next, we shall study the properties of the R matrix. We define the supertransposition st as

$$(A^{st})_{ij}=A_{ji}(-1)^{(\epsilon_i+1)\epsilon_j}. \quad (40)$$

As an example, we take the FFB grading, which means $\epsilon_1=\epsilon_2=1,\ \epsilon_3=0$. We can rewrite the above relation explicitly as

$$\begin{pmatrix}A_{11}&A_{12}&B_1\\A_{21}&A_{22}&B_2\\C_1&C_2&D\end{pmatrix}^{st}=\begin{pmatrix}A_{11}&A_{21}&C_1\\A_{12}&A_{22}&C_2\\-B_1&-B_2&D\end{pmatrix}. \quad (41)$$

We also define the inverse of the supertransposition $\overline{st}$ as $\{A^{st}\}^{\overline{st}}=A$.

For the R matrix with all three different gradings, FFB, BFF, and FBF, we can prove directly that the R matrix satisfies the following unitarity and cross-unitarity relations:

$$R_{12}(\lambda)R_{21}(-\lambda)=\rho(\lambda)\times\text{i.d.},$$

$$\rho(\lambda)=\sin(\eta+\lambda)\sin(\eta-\lambda), \quad (42)$$

$$R_{12}^{st_1}(\eta-\lambda)M_1R_{21}^{st_1}(\lambda)M_1^{-1}=\tilde{\rho}(\lambda)\times\text{id.},$$

$$\tilde{\rho}(\lambda)=\sin(\lambda)\sin(\eta-\lambda). \quad (43)$$

Here the matrix M is diagonal and is determined by the R matrix. For three different gradings, the forms of M are different. We have $M=\text{diag}(e^{2i\eta},1,1)$ for FFB grading, $M=\text{diag}(1,1,e^{-2i\eta})$ for BFF grading, and $M=1$ for FBF grading.

In order to construct the commuting transfer matrix with boundaries, besides the reflection equation, we need the dual reflection equation. Generally, the dual reflection equation which depends on the unitarity and cross-unitrarity relations of the R matrix takes different forms for different models. For the models considered in this paper, the cross-unitarity relation remains the same for three different backgrounds. We can write the dual reflection equation in the following form:

$$R_{12}(\mu-\lambda)K_1^+(\lambda)M_1^{-1}R_{21}(\eta-\lambda-\mu)K_2^+(\mu)M_2^{-1}$$
$$=K_2^+(\mu)M_2^{-1}R_{12}(\eta-\lambda-\mu)K_1^+(\lambda)M_1^{-1}R_{21}(\mu-\lambda). \tag{44}$$

One finds that there is an isomorphism between the reflection equation (33) and the dual reflection equation (44):

$$K(\lambda):\to K^+(\lambda)=MK(-\lambda+\eta/2). \tag{45}$$

Here we mean that, given a solution of the reflection equation (33), we can find a solution of the dual reflection equation (44). Note, however, that in the sense of the commuting transfer matrix, the reflection equation and the dual reflection equation are independent of each other.

The transfer matrix with boundaries is defined as

$$t(\lambda)=\mathrm{str}\,K^+(\lambda)\mathcal{T}(\lambda). \tag{46}$$

The commutativity of $t(\lambda)$ can be proved by using unitarity and cross-unitarity relations, the reflection equation, and the dual reflection equation. The detailed proof of the commuting transfer matrix with boundaries for the super (graded) case can be found, for instance, in Refs. 27,28,42,45, etc.

We also define the Hamiltonian by a relation

$$H\equiv\frac{1}{2}\sin(\eta)\frac{d\ln t(\lambda)}{d\lambda}\bigg|_{\lambda=0}$$
$$=\sum_{j=1}^{N-1}P_{j,j+1}L'_{j,j+1}(0)+\frac{1}{2}\frac{\sin(\eta)}{\sin(\xi)}K'_1(0)$$
$$+\frac{\mathrm{str}_a\,K_a^+(0)P_{Na}L'_{Na}(0)}{\mathrm{str}_a\,K_a^+(0)}. \tag{47}$$

We still take the FFB grading as an example, and thus $M=\mathrm{diag}(e^{2i\eta},1,1)$. We have two types of the solutions to the dual reflection equation:

$$K_I^+(\lambda)=\begin{pmatrix}\sin(\xi^+-\lambda)e^{i(2\lambda+\eta)} & & \\ & \sin(\xi^+-\lambda)e^{i(2\lambda-\eta)} & \\ & & \sin(\xi^++\lambda-\eta)\end{pmatrix},$$
$$K_{II}^+(\lambda)=\begin{pmatrix}\sin(\xi^+-\lambda)e^{i(2\lambda+\eta)} & & \\ & \sin(\xi^++\lambda-\eta) & \\ & & \sin(\xi^++\lambda-\eta)\end{pmatrix}, \tag{48}$$

where ξ^+ is also an arbitrary boundary parameter. Since the reflection equation and the dual reflection equation are independent of each other, there are altogether four different types of boundaries determined by boundary K and K^+ matrices: $\{K_I,K_I^+\}$, $\{K_I,K_{II}^+\}$, $\{K_{II},K_I^+\}$, and $\{K_{II},K_{II}^+\}$.

The Hamiltonian of the generalized supersymmetric t-J model with boundaries is written as

$$H=\sum_{j=1}^{N-1}\sum_{\sigma=\pm}[c_{j,\sigma}^\dagger(1-n_{j,-\sigma})c_{j+1,\sigma}(1-n_{j+1,-\sigma})$$
$$+c_{j+1,\sigma}^\dagger(1-n_{j+1,-\sigma})c_{j+1,\sigma}(1-n_{j,-\sigma})]$$
$$-2\sum_{j=1}^{N-1}\left[\frac{1}{2}(S_j^\dagger S_{j+1}+S_jS_{j+1}^\dagger)+\cos(\eta)S_j^zS_{j+1}^z\right.$$
$$\left.-\frac{\cos(\eta)}{4}n_jn_{j+1}\right]+i\sin(\eta)\sum_{j=1}^{N-1}[S_j^zn_{j+1}-S_{j+1}^zn_j]$$
$$-2\cos(\eta)\sum_{j=1}^{N-1}n_j+e^{-i\eta}n_1-e^{-i\eta}n_N+H_1+H_N, \tag{49}$$

where H_1 and H_N are determined by the reflecting matrices. Explicitly, they are

$$H_1^I=\frac{\sin(\eta)}{\sin(\xi)}e^{i\xi}n_1,\quad H_1^{II}=\frac{\sin(\eta)}{2\sin(\xi)}e^{i\xi}n_1-\frac{\sin(\eta)}{\sin(\xi)}e^{i\xi}S_1^z,$$

$$H_N^I=-\frac{\sin(\eta)}{2\sin(\xi^++\eta)}e^{-i(\xi^++\eta)}n_N,$$

$$H_N^{II}=-\frac{\sin(\eta)}{2\sin(\xi^+)}e^{-i\xi^+}n_N+\frac{\sin(\eta)}{\sin(\xi^+)}e^{-i\xi^+}S_N^z. \tag{50}$$

We remark that there are four types of boundary terms in the Hamiltonian.

The solution of the graded reflection equation is identical to that of the nongraded reflection equation, because we focus our attention on the diagonal solutions of the reflection equation, and the two cases for graded and nongraded are completely the same. The solution of the dual reflection equation for the FFB case is similar to the nongraded case in Ref. 23 except for a minus in the last diagonal elements. And the boundary terms appearing in the Hamiltonians (49) and (50) are similar to the previous results[23] (the anisotropic parameter should be redefined as $\eta\equiv-\gamma$).

IV. ALGEBRAIC BETHE ANSATZ METHOD FOR FFB GRADING

In this section, FFB grading is assumed. We shall use the nested algebraic Bethe ansatz method to obtain the eigenvalues of the transfer matrix with boundaries defined above.

A. Commutation relations necessary for the algebraic Bethe ansatz method

We write solution of the dual reflection equation K^+ and the double-row monodromy matrix $\mathcal{T}$, respectively, in the following form:

$$K^+(\lambda)=\mathrm{diag}(k_1^+(\lambda),k_2^+(\lambda),k_3^+(\lambda)), \tag{51}$$

$$\mathcal{T}(\lambda)=\begin{pmatrix} \mathcal{A}_{11}(\lambda) & \mathcal{A}_{12}(\lambda) & \mathcal{B}_1(\lambda) \\ \mathcal{A}_{21}(\lambda) & \mathcal{A}_{22}(\lambda) & \mathcal{B}_2(\lambda) \\ \mathcal{C}_1(\lambda) & \mathcal{C}_2(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}. \tag{52}$$

Instead of $\mathcal{A}_{ab}$, we shall use $\tilde{\mathcal{A}}_{ab}$ in the algebraic Bethe ansatz method so that there will exist only one type wanted term in the commutation relation. The transformation takes the form

$$\mathcal{A}(\lambda)_{ab}=\tilde{\mathcal{A}}(\lambda)_{ab}+\delta_{ab}\frac{e^{-2i\lambda}\sin(\eta)}{\sin(2\lambda+\eta)}\mathcal{D}(\lambda). \tag{53}$$

So the transfer matrix with boundaries can be rewrite as

$$\begin{aligned} t(\lambda)&=-k_1^+(\lambda)\mathcal{A}_{11}(\lambda)-k_2^+(\lambda)\mathcal{A}_{22}(\lambda)+k_3^+(\lambda)\mathcal{D}(\lambda) \\ &=-k_1^+(\lambda)\tilde{\mathcal{A}}_{11}(\lambda)-k_2^+(\lambda)\tilde{\mathcal{A}}_{22}(\lambda)+U_3^+(\lambda)\mathcal{D}(\lambda), \end{aligned} \tag{54}$$

where

$$U_3^+(\lambda)\equiv k_3^+(\lambda)-\frac{e^{-2i\lambda}\sin(\eta)}{\sin(2\lambda+\eta)}[k_1^+(\lambda)+k_2^+(\lambda)]. \tag{55}$$

For type I and II solutions of the dual reflection equation K^+, we have

$$U_3^+(\lambda)=\frac{\sin(2\lambda-\eta)\sin(\xi^++\lambda+\eta)}{\sin(2\lambda+\eta)}, \quad \text{for} \quad K_I^+, \tag{56}$$

$$U_3^+(\lambda)=\frac{\sin(2\lambda-\eta)\sin(\xi^++\lambda)e^{i\eta}}{\sin(2\lambda+\eta)}, \quad \text{for} \quad K_{II}^+. \tag{57}$$

As mentioned above, the double-row monodromy matrix also satisfies the graded reflection equation (39). Setting the indices in that relation to be special values, we can find the commutation relations which are necessary for the algebraic Bethe ansatz method. The detailed calculation is tedious and complicated, so we do not present it here. The result is

$$\mathcal{C}_{d_1}(\lambda)\mathcal{C}_{d_2}(\mu)=-\frac{r_{12}(\lambda-\mu)_{c_2c_1}^{d_2d_1}}{\sin(\lambda-\mu+\eta)}\mathcal{C}_{c_2}(\mu)\mathcal{C}_{c_1}(\lambda), \tag{58}$$

$$\begin{aligned} \mathcal{D}(\lambda)\mathcal{C}_d(\mu)&=\frac{\sin(\lambda+\mu)\sin(\lambda-\mu-\eta)}{\sin(\lambda+\mu+\eta)\sin(\lambda-\mu)}\mathcal{C}_d(\mu)\mathcal{D}(\lambda) \\ &\quad+\frac{\sin(2\mu)\sin(\eta)e^{i(\lambda-\mu)}}{\sin(\lambda-\mu)\sin(2\mu+\eta)}\mathcal{C}_d(\lambda)\mathcal{D}(\mu) \\ &\quad-\frac{\sin(\eta)e^{i(\lambda+\mu)}}{\sin(\lambda+\mu+\eta)}\mathcal{C}_b(\lambda)\tilde{\mathcal{A}}_{bd}(\mu), \end{aligned} \tag{59}$$

$$\begin{aligned} \tilde{\mathcal{A}}_{a_1d_1}(\lambda)\mathcal{C}_{d_2}(\mu)&=\frac{r_{12}(\lambda+\mu+\eta)_{a_1c_2}^{c_1b_2}r_{21}(\lambda-\mu)_{b_1b_2}^{d_1d_2}}{\sin(\lambda+\mu+\eta)\sin(\lambda-\mu)}\mathcal{C}_{c_2}(\mu)\tilde{\mathcal{A}}_{c_1b_1}(\lambda)+\frac{\sin(\eta)e^{-i(\lambda-\mu)}}{\sin(\lambda-\mu)\sin(2\lambda+\eta)}r_{12}(2\lambda \\ &\quad+\eta)_{a_1b_1}^{b_2d_1}\mathcal{C}_{b_1}(\lambda)\tilde{\mathcal{A}}_{b_2d_2}(\mu)-\frac{\sin(2\mu)\sin(\eta)e^{-i(\lambda+\mu)}}{\sin(\lambda+\mu+\eta)\sin(2\lambda+\eta)\sin(2\mu+\eta)}r_{12}(2\lambda+\eta)_{a_1b_2}^{d_2d_1}\mathcal{C}_{b_2}(\lambda)\mathcal{D}(\mu). \end{aligned} \tag{60}$$

Here the indices take values 1 and 2, and the r matrix is defined as

$$r_{12}(\lambda)=\begin{pmatrix} \sin(\lambda-\eta) & 0 & 0 & 0 \\ 0 & \sin(\lambda) & -\sin(\eta)e^{-i\lambda} & 0 \\ 0 & -\sin(\eta)e^{i\lambda} & \sin(\lambda) & 0 \\ 0 & 0 & 0 & \sin(\lambda-\eta) \end{pmatrix}. \tag{61}$$

In fact, the elements of the r matrix are equal to those of the original R matrix when its indices just take the values 1 and 2.

B. Vacuum state

According to the definition of the double-row monodromy matrix, we write it explicitly as

$$\mathcal{T}(\lambda)=\begin{pmatrix}\mathcal{A}_{11}(\lambda) & \mathcal{A}_{12}(\lambda) & \mathcal{B}_1(\lambda)\\ \mathcal{A}_{21}(\lambda) & \mathcal{A}_{22}(\lambda) & \mathcal{B}_2(\lambda)\\ \mathcal{C}_1(\lambda) & \mathcal{C}_2(\lambda) & \mathcal{D}(\lambda)\end{pmatrix}=T(\lambda)K(\lambda)T^{-1}(-\lambda)$$

$$=\begin{pmatrix}A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda)\\ A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda)\\ C_1(\lambda) & C_2(\lambda) & D(\lambda)\end{pmatrix}\begin{pmatrix}k_1(\lambda) & 0 & 0\\ 0 & k_2(\lambda) & 0\\ 0 & 0 & k_3(\lambda)\end{pmatrix}\begin{pmatrix}\bar{A}_{11}(-\lambda) & \bar{A}_{12}(-\lambda) & \bar{B}_1(-\lambda)\\ \bar{A}_{21}(-\lambda) & \bar{A}_{22}(-\lambda) & \bar{B}_2(-\lambda)\\ \bar{C}_1(-\lambda) & \bar{C}_2(-\lambda) & \bar{D}(-\lambda)\end{pmatrix}. \tag{62}$$

For convenience, we can write the inverse of the row-to-row monodromy matrix as

$$T_a^{-1}(-\lambda)=L_{1a}(\lambda)L_{2a}(\lambda)\cdots L_{Na}(\lambda), \tag{63}$$

where we have used the unitarity relation of the R matrix and a whole factor is omitted.

Define the reference state in the nth quantum space and the vacuum $|0\rangle$ as

$$|0\rangle_n=\begin{pmatrix}0\\0\\1\end{pmatrix},\quad |0\rangle=\otimes_{k=1}^{N}|0\rangle_k. \tag{64}$$

By use of the definition of the row-to-row monodromy matrix (19) and its inverse (63), we have

$$A_{ab}(\lambda)|0\rangle=\delta_{ab}\sin^N(\lambda)|0\rangle,$$
$$D(\lambda)|0\rangle=\sin^N(\lambda+\eta)|0\rangle,$$
$$B_a(\lambda)|0\rangle=0,\quad C_a(\lambda)|0\rangle\neq 0, \tag{65}$$

$$\bar{A}_{ab}(-\lambda)|0\rangle=\delta_{ab}\sin^N(\lambda)|0\rangle,$$
$$\bar{D}(-\lambda)|0\rangle=\sin^N(\lambda+\eta)|0\rangle,$$
$$\bar{B}_a(-\lambda)|0\rangle=0,\quad \bar{C}_a(\lambda)|0\rangle\neq 0. \tag{66}$$

So, with the help of $\mathcal{T}$'s definition relation (62), we can show that

$$\mathcal{D}(\lambda)|0\rangle=k_3(\lambda)\sin^{2N}(\lambda+\eta)|0\rangle,$$
$$\tilde{\mathcal{A}}_{ab}(\lambda)|0\rangle=0,\quad a\neq b,$$
$$\mathcal{B}_a(\lambda)|0\rangle=0,$$
$$\mathcal{C}_a(\lambda)|0\rangle\neq 0. \tag{67}$$

To obtain the actions of operator $\tilde{\mathcal{A}}_{aa}$ on the vacuum state, we use the following relation obtained from the Yang-Baxter relation:

$$[T^{-1}(-\lambda)]_{a_2}^{b_2}R(2\lambda)_{a_1b_2}^{b_1c_2}T(\lambda)_{b_1}^{c_1}(-1)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{c_2}}$$
$$=T(\lambda)_{a_1}^{b_1}R(2\lambda)_{b_1a_2}^{c_1b_2}[T^{-1}(-\lambda)]_{b_2}^{c_2}(-1)^{(\epsilon_{a_1}+\epsilon_{b_1})\epsilon_{a_2}}. \tag{68}$$

Actually, we have already used it to obtain the results $\tilde{\mathcal{A}}_{ab}(\lambda)|0\rangle=0,\ a\neq b$.

Then, we have

$$\tilde{\mathcal{A}}_{11}(\lambda)|0\rangle=\left[k_1(\lambda)-k_3(\lambda)\frac{\sin(\eta)e^{-2i\lambda}}{\sin(2\lambda+\eta)}\right]\sin^{2N}(\lambda)|0\rangle$$
$$\equiv W_1(\lambda)\sin^{2N}(\lambda)|0\rangle. \tag{69}$$

For the case I and II reflecting K matrix, we have the same W_1 which takes the form

$$\text{for } K_I \text{ and } K_{II}:\quad W_1(\lambda)=e^{-2i\lambda}\frac{\sin(2\lambda)\sin(\xi+\lambda+\eta)}{\sin(2\lambda+\eta)}. \tag{70}$$

Similarly, we have

$$\tilde{\mathcal{A}}_{22}(\lambda)|0\rangle=\left[k_2(\lambda)-k_3(\lambda)\frac{\sin(\eta)e^{-2i\lambda}}{\sin(2\lambda+\eta)}\right]\sin^{2N}(\lambda)|0\rangle$$
$$\equiv W_2(\lambda)\sin^{2N}(\lambda)|0\rangle. \tag{71}$$

For the case I and II reflecting K matrix, W_2 takes the following forms, respectively:

$$\text{for } K_I:\quad W_2(\lambda)=e^{-2i\lambda}\frac{\sin(2\lambda)\sin(\xi+\lambda+\eta)}{\sin(2\lambda+\eta)}, \tag{72}$$

$$\text{for } K_{II}:\quad W_2(\lambda)=e^{i\eta}\frac{\sin(2\lambda)\sin(\xi-\lambda)}{\sin(2\lambda+\eta)}. \tag{73}$$

C. Bethe ansatz

We construct a set of the eigenvectors of the transfer matrix with reflecting boundary conditions as

$$\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}. \tag{74}$$

Here $F^{d_1\cdots d_n}$ is a function of the spectral parameters μ_j. Applying the transfer matrix (54) on this eigenvector, we find the eigenvalues $\Lambda(\lambda)$ of the transfer matrix $t(\lambda)$ and a set of Bethe ansatz equations. This technique is standard for the algebraic Bethe ansatz method. Apply first $\mathcal{D}$ on the eigenvector defined above, use next the commutation relation (59), consider the value of $\mathcal{D}$ acting on the vacuum state (67). Then we have

$$\mathcal{D}(\lambda)\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}$$

$$=k_3(\lambda)\sin^{2L}(\lambda+\eta)\prod_{i=1}^{n}\frac{\sin(\lambda+\mu_i)\sin(\lambda-\mu_i-\eta)}{\sin(\lambda+\mu_i+\eta)\sin(\lambda-\mu_i)}\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}+\text{u.t.}, \tag{75}$$

where ''u.t.'' means the unwanted terms.

We act $\tilde{\mathcal{A}}_{aa}(\lambda)$ on the assumed eigenvector (74). Using repeatedly the commutation relations (60), we have

$$\tilde{\mathcal{A}}_{aa}(\lambda)\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}$$

$$=\prod_{i=1}^{n}\frac{1}{\sin(\lambda-\mu_i)\sin(\lambda+\mu_i+\eta)}r_{12}(\lambda+\mu_1+\eta)_{ac_1}^{a_1e_1}r_{21}(\lambda-\mu_1)_{b_1e_1}^{ad_1}r_{12}(\lambda+\mu_2+\eta)_{a_1c_2}^{a_2e_2}r_{21}$$

$$\times(\lambda-\mu_2)_{b_2e_2}^{b_1d_2}\cdots r_{12}(\lambda+\mu_n+\eta)_{a_{n-1}c_n}^{a_ne_n}r_{21}(\lambda-\mu_n)_{b_ne_n}^{b_{n-1}d_n}\mathcal{C}_{c_1}(\mu_1)\cdots\mathcal{C}_{c_n}(\mu_n)\tilde{\mathcal{A}}_{a_nb_n}(\lambda)|0\rangle F^{d_1\cdots d_n}+\text{u.t.} \tag{76}$$

Summarizing relations (67), (69), and (71), we obtain

$$\mathcal{A}_{a_nb_n}(\lambda)|0\rangle=\delta_{a_nb_n}W_{a_n}(\lambda)\sin^{2L}(\lambda)|0\rangle. \tag{77}$$

We can rewrite the transfer matrix as

$$t(\lambda)=-k_1^+(\lambda)\tilde{\mathcal{A}}_{11}(\lambda)-k_2^+(\lambda)\tilde{\mathcal{A}}_{22}(\lambda)+U_3^+(\lambda)\mathcal{D}(\lambda)$$

$$=-k_a^+(\lambda)\tilde{\mathcal{A}}_{aa}(\lambda)+U_3^+(\lambda)\mathcal{D}(\lambda). \tag{78}$$

Thus the eigenvalue of the transfer matrix with reflecting boundary condition is written as

$$t(\lambda)\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}$$

$$=U_3^+(\lambda)k_3(\lambda)\sin^{2N}(\lambda+\eta)\prod_{i=1}^{n}\frac{\sin(\lambda+\mu_i)\sin(\lambda-\mu_i-\eta)}{\sin(\lambda+\mu_i+\eta)\sin(\lambda-\mu_i)}\mathcal{C}_{d_1}(\mu_1)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}$$

$$+\sin^{2N}(\lambda)\prod_{i=1}^{n}\frac{1}{\sin(\lambda-\mu_i)\sin(\lambda+\mu_i+\eta)}\mathcal{C}_{c_1}(\mu_1)\cdots\mathcal{C}_{c_n}(\mu_n)|0\rangle t^{(1)}(\lambda)_{d_1\cdots d_n}^{c_1\cdots c_n}F^{d_1\cdots d_n}+\text{u.t.}, \tag{79}$$

where $t^{(1)}(\lambda)$ is the so-called nested transfer matrix, and with the help of the relation (76), it can be defined as

$$t^{(1)}(\lambda)_{d_1\cdots d_n}^{c_1\cdots c_n}=-k_a^+(\lambda)\{r(\lambda+\mu_1+\eta)_{ac_1}^{a_1e_1}r(\lambda+\mu_2+\eta)_{a_1c_2}^{a_2e_2}\cdots r(\lambda+\mu_1+\eta)_{a_{n-1}c_n}^{a_ne_n}\}$$

$$\times\delta_{a_nb_n}W_{a_n}(\lambda)\{r_{21}(\lambda-\mu_n)_{b_ne_n}^{b_{n-1}d_n}\cdots r_{21}(\lambda-\mu_2)_{b_2e_2}^{b_1d_2}r_{21}(\lambda-\mu_1)_{b_1e_1}^{ad_1}\}. \tag{80}$$

We find that this nested transfer matrix can be defined as a transfer matrix with reflecting boundary conditions corresponding to the anisotropic case

$$t^{(1)}(\lambda)=\text{str}\,K^{(1)+}(\tilde{\lambda})T^{(1)}(\tilde{\lambda},\{\tilde{\mu}_i\})$$

$$\times K^{(1)}(\tilde{\lambda})T^{(1)-1}(-\tilde{\lambda},\{\tilde{\mu}_i\}), \tag{81}$$

with the grading $\epsilon_1=\epsilon_2=1$. Here, we denote $\tilde{\lambda}=\lambda+\eta/2$, $\tilde{\xi}=\xi+\eta/2$, $\tilde{\xi}^+=\xi^+-\eta/2$, and the same notation will be used, for instance, $\tilde{\mu}=\mu+\eta/2$. Explicitly we have

$$K^{(1)+}_I(\tilde{\lambda})=\sin(\tilde{\xi}^+-\tilde{\lambda}+\eta)e^{i(2\tilde{\lambda}-\eta)}\begin{pmatrix}e^{i\eta}&\\&e^{-i\eta}\end{pmatrix} \tag{82}$$

and

$$K^{(1)+}_{II}(\tilde{\lambda})=\begin{pmatrix}\sin(\tilde{\xi}^+-\tilde{\lambda}-\eta)e^{2i\tilde{\lambda}}&\\&\sin(\tilde{\xi}^++\tilde{\lambda}-\eta)\end{pmatrix}, \tag{83}$$

corresponding to K_I^+ and K_{II}^+, respectively. We also have

$$K_I^{(1)}(\tilde{\lambda})=e^{-i(2\tilde{\lambda}-\eta)}\frac{\sin(2\tilde{\lambda}-\eta)\sin(\tilde{\xi}+\tilde{\lambda})}{\sin(2\tilde{\lambda})}\times\text{id.}, \tag{84}$$

$$K_{II}^{(1)}(\tilde{\lambda})=\frac{\sin(2\tilde{\lambda}-\eta)e^{i\eta}}{\sin(2\tilde{\lambda})}\begin{pmatrix}\sin(\tilde{\xi}+\tilde{\lambda})e^{-2i\tilde{\lambda}}&\\&\sin(\tilde{\xi}-\tilde{\lambda})\end{pmatrix}, \tag{85}$$

corresponding to K_I and K_{II}. The row-to-row monodromy matrix $T^{(1)}(\tilde{\lambda},\{\tilde{\mu}_i\})$ (corresponding to the periodic boundary condition) is defined as

$$T^{(1)}_{aa_n}(\tilde{\lambda},\{\tilde{\mu}_i\})^{e_1\cdots e_n}_{c_1\cdots c_n} = r(\tilde{\lambda}+\tilde{\mu}_1)^{a_1e_1}_{ac_1} r(\tilde{\lambda}+\tilde{\mu}_2)^{a_2e_2}_{a_1c_2}\cdots r(\tilde{\lambda}+\tilde{\mu}_1)^{a_ne_n}_{a_{n-1}c_n} = L^{(1)}_1(\tilde{\lambda}+\tilde{\mu}_1)L^{(1)}_2(\tilde{\lambda}+\tilde{\mu}_2)\cdots L^{(1)}_n(\tilde{\lambda}+\tilde{\mu}_1). \tag{86}$$

The L operator takes the form

$$L^{(1)}_k(\tilde{\lambda}) = \begin{pmatrix} b(\lambda)-[b(\lambda)-a(\lambda)]e^{11}_k & -c_-(\lambda)e^{21}_n \\ -c_+(\lambda)e^{12}_n & b(\lambda)-[b(\lambda)-a(\lambda)]e^{22}_n \end{pmatrix}. \tag{87}$$

And we also have

$$T^{(1)-1}(-\tilde{\lambda},\{\tilde{\mu}_i\}) = r_{21}(\tilde{\lambda}-\tilde{\mu}_n)^{b_{n-1}d_n}_{b_ne_n}\cdots r_{21} \times(\tilde{\lambda}-\tilde{\mu}_2)^{b_1d_2}_{b_2e_2} r_{21}(\tilde{\lambda}-\tilde{\mu}_1)^{ad_1}_{b_1e_1} = L^{(1)-1}_n(-\tilde{\lambda}+\tilde{\mu}_n)\cdots L^{(1)-1}_2 \times(-\tilde{\lambda}+\tilde{\mu}_2)L^{(1)-1}_1(-\tilde{\lambda}+\tilde{\mu}_1), \tag{88}$$

where we have used the unitarity relation of the r matrix, $r_{12}(\lambda)r_{21}(-\lambda)=\sin(\eta-\lambda)\sin(\eta+\lambda)\times\mathrm{id}$.

In this section, we show that the problem of finding the eigenvalue of the original transfer matrix $t(\lambda)$ reduces to the problem of finding the eigenvalue of the nested transfer matrix $t^{(1)}(\lambda)$. In relation (79), one can see that besides the wanted term which gives the eigenvalue, we also have the unwanted terms which must be canceled so that the assumed eigenvector is indeed the eigenvector of the transfer matrix. With the help of the symmetry property (58) of the assumed eigenvector (74), we find that, if $\mu_1,\ldots,\mu_n$ satisfy the following Bethe ansatz equations, the unwanted terms will vanish:

$$U^+_3(\mu_j)k_3(\mu_j)\sin^{2N}(\mu_j+\eta) \times\prod_{i=1,\neq j}^{n}\sin(\mu_j+\mu_i)\sin(\mu_j-\mu_i-\eta) = -\sin^{2N}(\mu_j)\Lambda^{(1)}(\mu_j), j=1,2,\ldots,n. \tag{89}$$

Here we have used the notation $\Lambda^{(1)}(\lambda)$ to denote the eigenvalue of the nested transfer matrix $t^{(1)}(\lambda)$.

Thus what we should do next is find the eigenvalue of the nested transfer matrix $t^{(1)}$.

D. Nested algebraic Bethe ansatz method

We expect that the eigenvalue of the nested transfer matrix can be solved similarly as that of the original transfer matrix. So we should first prove that the above-defined nested transfer matrix indeed constitutes a commuting family. Note that the grading is $\epsilon_1=\epsilon_2=1$. Actually, because all grading is fermionic, the graded method is simply the same as the usual one.

We note that the r matrix satisfies the unitarity and cross-unitarity relations

$$r_{12}(\lambda)r_{21}(-\lambda)=\sin(\eta+\lambda)\sin(\eta-\lambda)\times\mathrm{id.}, \tag{90}$$

$$r^{st_1}_{12}(2\eta-\lambda)M^{(1)}_1 r^{st_1}_{21}(\lambda)M^{(1)-1}_1=\sin(\lambda)\sin(2\lambda-\eta)\times\mathrm{id.} \tag{91}$$

The matrix $M^{(1)}$ is a diagonal matrix, $M^{(1)}=\mathrm{diag}(e^{2i\eta},1)$.

In order to prove the commutativity of the nested transfer matrices, we need the reflection equation and the dual reflection equation, which take the following forms:

$$r_{12}(\lambda-\mu)K^{(1)}_1(\lambda)r_{21}(\lambda+\mu)K^{(1)}_2(\mu) = K^{(1)}_2(\mu)r_{12}(\lambda+\mu)K^{(1)}_1(\lambda)r_{21}(\lambda-\mu), \tag{92}$$

$$r_{12}(\mu-\lambda)K^{(1)+}_1(\lambda)M^{-1}_1 r_{21}(2\eta-\lambda-\mu)K^{(1)+}_2(\mu)M^{-1}_2 = K^{(1)+}_2(\mu)M^{-1}_2 r_{12}(2\eta-\lambda-\mu)K^{(1)+}_1(\lambda)M^{-1}_1 r_{21}(\mu-\lambda). \tag{93}$$

By a direct calculation, we can prove that the above-defined reflecting matrices $K^{(1)}_I$ and $K^{(1)}_{II}$ satisfy the reflection equation, and also $K^{(1)+}_I$ and $K^{(1)+}_{II}$ satisfy the dual reflection equation.

We know that the following graded Yang-Baxter relation is satisfied:

$$r(\lambda-\mu)L^{(1)}_1(\lambda)L^{(1)}_2(\mu)=L^{(1)}_2(\mu)L^{(1)}_1(\lambda)r(\lambda-\mu). \tag{94}$$

Therefore, we also have the Yang-Baxter relation for the row-to-row monodromy matrix:

$$r(\lambda-\mu)T^{(1)}_1(\lambda,\{\mu_i\})T^{(1)}_2(\mu,\{\mu_i\}) = T^{(1)}_2(\mu,\{\mu_i\})T^{(1)}_1(\lambda,\{\mu_i\})r(\lambda-\mu). \tag{95}$$

Since we alreay know $K^{(1)}$ satisfies the reflection equation (92), we can show that the nested double-row monodromy matrix

$$\mathcal{T}^{(1)}(\lambda,\{\mu_i\})\equiv T^{(1)}(\lambda,\{\mu_i\})K^{(1)}(\lambda)T^{(1)-1}(-\lambda,\{\mu_i\}) \tag{96}$$

also satisfies the the reflection equation

$$r_{12}(\lambda-\mu)\mathcal{T}^{(1)}_1(\lambda,\{\mu_i\})r_{21}(\lambda+\mu)\mathcal{T}^{(1)}_2(\mu,\{\mu_i\}) = \mathcal{T}^{(1)}_2(\nu,\{\mu_i\})r_{12}(\lambda+\mu)\mathcal{T}^{(1)}_1(\lambda,\{\mu_i\})r_{21}(\lambda-\mu). \tag{97}$$

Parallel to the procedures presented above, with the help of unitarity, cross-unitarity relations and the reflection and dual reflection equations, one can prove that the defined nested transfer matrix indeed constitutes a commuting family.

Now, let us use again the algebraic Bethe ansatz method to obtain the eigenvalue $\Lambda^{(1)}(\lambda)$ of the nested transfer matrix $t^{(1)}(\lambda)$. We write the nested double-row monodromy matrix as

$$\mathcal{T}^{(1)}(\lambda,\{\mu_i\})=\begin{pmatrix}\mathcal{A}^{(1)}(\lambda) & \mathcal{B}^{(1)}(\lambda)\\ \mathcal{C}^{(1)}(\lambda) & \mathcal{D}^{(1)}(\lambda)\end{pmatrix}$$
$$=T^{(1)}(\lambda,\{\mu_i\})K^{(1)}(\lambda)T^{(1)-1}(-\lambda,\{\mu_i\})$$
$$=\begin{pmatrix}A^{(1)}(\lambda) & B^{(1)}(\lambda)\\ C^{(1)}(\lambda) & D^{(1)}(\lambda)\end{pmatrix}\begin{pmatrix}k_1^{(1)}(\lambda) & \\ & k_2^{(1)}(\lambda)\end{pmatrix}\times\begin{pmatrix}\overline{A}^{(1)}(-\lambda) & \overline{B}^{(1)}(-\lambda)\\ \overline{C}^{(1)}(-\lambda) & \overline{D}^{(1)}(-\lambda)\end{pmatrix}. \tag{98}$$

For convenience, we introduce again a transformation

$$\mathcal{A}^{(1)}(\lambda)=\tilde{\mathcal{A}}^{(1)}(\lambda)-\frac{\sin(\eta)e^{-2i\lambda}}{\sin(2\lambda-\eta)}\mathcal{D}^{(1)}(\lambda). \tag{99}$$

Because the nested double-row monodromy matrix satisfies the reflection equation (97), we can find the following commutation relations:

$$\mathcal{D}^{(1)}(\lambda)\mathcal{C}^{(1)}(\mu)=\frac{\sin(\lambda-\mu+\eta)\sin(\lambda+\mu)}{\sin(\lambda-\mu)\sin(\lambda+\mu-\eta)}\mathcal{C}^{(1)}(\mu)\mathcal{D}^{(1)}(\lambda)-\frac{\sin(2\mu)\sin(\eta)e^{i(\lambda-\mu)}}{\sin(\lambda-\mu)\sin(2\mu-\eta)}\mathcal{C}^{(1)}(\lambda)\mathcal{D}^{(1)}(\mu)+\frac{\sin(\eta)e^{i(\lambda+\mu)}}{\sin(\lambda+\mu-\eta)}\mathcal{C}^{(1)}(\lambda)\tilde{\mathcal{A}}^{(1)}(\mu), \tag{100}$$

$$\tilde{\mathcal{A}}^{(1)}(\lambda)\mathcal{C}^{(1)}(\mu)=\frac{\sin(\lambda-\mu-\eta)\sin(\lambda+\mu-2\eta)}{\sin(\lambda-\mu)\sin(\lambda+\mu-\eta)}\times\mathcal{C}^{(1)}(\mu)\tilde{\mathcal{A}}^{(1)}(\lambda)+\frac{\sin(\eta)\sin(2\lambda-2\eta)e^{-i(\lambda-\mu)}}{\sin(\lambda-\mu)\sin(2\lambda-\eta)}\times\mathcal{C}^{(1)}(\lambda)\tilde{\mathcal{A}}^{(1)}(\mu)-\frac{\sin(2\mu)\sin(2\lambda-2\eta)\sin(\eta)e^{-i(\lambda+\mu)}}{\sin(\lambda+\mu-\eta)\sin(2\lambda-\eta)\sin(2\mu-\eta)}\times\mathcal{C}^{(1)}(\lambda)\mathcal{D}^{(1)}(\mu), \tag{101}$$

$$\mathcal{C}^{(1)}(\lambda)\mathcal{C}^{(1)}(\mu)=\mathcal{C}^{(1)}(\mu)\mathcal{C}^{(1)}(\lambda). \tag{102}$$

As the reference states for the nesting, we choose

$$|0\rangle_k^{(1)}=\begin{pmatrix}0\\1\end{pmatrix},\quad |0\rangle^{(1)}=\otimes_{k=1}^{n}|0\rangle_k^{(1)}. \tag{103}$$

With the help of the definition (86) and (88), we know the actions of the nested monodromy matrix and the inverse of the monodromy matrix on the reference state:

$$T^{(1)}(\lambda,\{\mu_i\})|0\rangle^{(1)}=\begin{pmatrix}A^{(1)}(\tilde{\lambda}) & B^{(1)}(\tilde{\lambda})\\ C^{(1)}(\tilde{\lambda}) & D^{(1)}(\tilde{\lambda})\end{pmatrix}|0\rangle^{(1)}=\begin{pmatrix}\prod_{i=1}^{n}\sin(\tilde{\lambda}+\tilde{\mu}_i) & 0\\ C^{(1)}(\tilde{\lambda}) & \prod_{i=1}^{n}\sin(\tilde{\lambda}+\tilde{\mu}_i-\eta)\end{pmatrix}|0\rangle^{(1)}, \tag{104}$$

$$T^{(1)-1}(-\lambda,\{\mu_i\})|0\rangle^{(1)}=\begin{pmatrix}\overline{A}^{(1)}(\tilde{\lambda}) & \overline{B}^{(1)}(\tilde{\lambda})\\ \overline{C}^{(1)}(\tilde{\lambda}) & \overline{D}^{(1)}(\tilde{\lambda})\end{pmatrix}|0\rangle^{(1)}=\begin{pmatrix}\prod_{i=1}^{n}\sin(\tilde{\lambda}-\tilde{\mu}_i) & 0\\ C^{(1)}(\tilde{\lambda}) & \prod_{i=1}^{n}\sin(\tilde{\lambda}-\tilde{\mu}_i-\eta)\end{pmatrix}|0\rangle^{(1)}. \tag{105}$$

Repeating almost the same calculation in the former sections, we obtain the results of the nested double-row monodromy matrix acting on the nested vacuum state $|0\rangle^{(1)}$:

$$\mathcal{B}^{(1)}(\tilde{\lambda})|0\rangle^{(1)}=0,\quad \mathcal{C}^{(1)}(\tilde{\lambda})|0\rangle^{(1)}\neq 0, \tag{106}$$

$$\mathcal{D}^{(1)}(\tilde{\lambda})|0\rangle^{(1)}=U_2(\tilde{\lambda})\prod_{i=1}^{n}\times[\sin(\tilde{\lambda}+\tilde{\mu}_i-\eta)\sin(\tilde{\lambda}-\tilde{\mu}_i-\eta)]|0\rangle^{(1)}. \tag{107}$$

Here we use the notation $U_2=k_2^{(1)}$,

$$U_2(\tilde{\lambda})=e^{-i(2\tilde{\lambda}-\eta)}\frac{\sin(2\tilde{\lambda}-\eta)\sin(\tilde{\lambda}+\tilde{\xi})}{\sin(2\tilde{\lambda})} \tag{108}$$

for the K_I case and

$$U_2(\tilde{\lambda})=\frac{e^{i\eta}\sin(2\tilde{\lambda}-\eta)\sin(\tilde{\xi}-\tilde{\lambda})}{\sin(2\tilde{\lambda})} \tag{109}$$

for the K_{II} case.

Using the Yang-Baxter relation, we also have

$$\mathcal{A}^{(1)}(\tilde{\lambda})|0\rangle^{(1)}=k_1^{(1)}(\tilde{\lambda})A^{(1)}(\tilde{\lambda})\overline{A}^{(1)}(-\tilde{\lambda})|0\rangle^{(1)}+k_2^{(1)}(\tilde{\lambda})\times\frac{b(2\tilde{\lambda})}{a(2\tilde{\lambda})-b(2\tilde{\lambda})}[A^{(1)}(\tilde{\lambda})\overline{A}^{(1)}(-\tilde{\lambda})-\overline{D}^{(1)}(-\tilde{\lambda})D^{(1)}(\tilde{\lambda})]|0\rangle^{(1)}$$

$$=\left[k_1^{(1)}(\tilde\lambda)+k_2^{(1)}(\tilde\lambda)\frac{\sin(\eta)e^{-2i\lambda}}{\sin(2\lambda-\eta)}\right]$$
$$\times\prod_{i=1}^{n}[\sin(\tilde\lambda+\tilde\mu_i)\sin(\tilde\lambda-\tilde\mu_i)]|0\rangle^{(1)}$$
$$-\frac{\sin(\eta)e^{-2i\lambda}}{\sin(2\tilde\lambda-\eta)}\mathcal{D}^{(1)}(\tilde\lambda)|0\rangle^{(1)}. \tag{110}$$

With the help of the transformation (99), we find

$$\tilde{\mathcal{A}}^{(1)}(\tilde\lambda)|0\rangle^{(1)}=U_1(\tilde\lambda)\prod_{i=1}^{n}[\sin(\tilde\lambda+\tilde\mu_i)\sin(\tilde\lambda-\tilde\mu_i)]|0\rangle^{(1)}, \tag{111}$$

where we denote

$$U_1(\tilde\lambda)=k_1^{(1)}(\tilde\lambda)+k_2^{(1)}(\tilde\lambda)\frac{\sin(2\eta e^{-2i\tilde\lambda})}{\sin(2\tilde\lambda-\eta)}. \tag{112}$$

Here U_1 takes the following form explicitly: For $K_I(\lambda)$, $U_1(\tilde\lambda)=e^{-2i\tilde\lambda}\sin(\tilde\lambda+\tilde\xi)$. For $K_{II}(\lambda)$, $U_1(\tilde\lambda)=e^{-i(2\tilde\lambda-\eta)}\sin(\tilde\lambda+\tilde\xi-\eta)$.

The nested transfer matrix takes the form

$$t^{(1)}(\tilde\lambda)=\mathrm{str}\,K^{(1)}(\tilde\lambda)\mathcal{T}^{(1)}(\tilde\lambda)$$
$$=-k_1^{(1)+}(\tilde\lambda)\mathcal{A}^{(1)}(\tilde\lambda)-k_2^{(1)+}(\tilde\lambda)\mathcal{D}^{(1)}(\tilde\lambda)$$
$$=-U_1^{+}(\tilde\lambda)\tilde{\mathcal{A}}^{(1)}(\tilde\lambda)-U_2^{+}(\tilde\lambda)\mathcal{D}^{(1)}(\tilde\lambda), \tag{113}$$

where we denote $U_1^{+}=k_1^{(1)+}$,

$$U_2^{+}(\lambda)=k_2^{(1)+}(\lambda)-\frac{\sin(\eta)e^{-2i\tilde\lambda}}{\sin(2\lambda-\eta)}k_1^{(1)+}(\lambda), \tag{114}$$

which means the following.

For the K_I^{+} case,

$$U_1^{+}(\tilde\lambda)=\sin(\tilde\xi^{+}-\tilde\lambda+\eta), \tag{115}$$

$$U_2^{+}(\tilde\lambda)=\frac{\sin(2\tilde\lambda-2\eta)\sin(\tilde\xi^{+}-\tilde\lambda+\eta)}{\sin(2\tilde\lambda-\eta)}e^{i(2\tilde\lambda-\eta)}. \tag{116}$$

For the K_{II}^{+} case,

$$U_1^{+}(\tilde\lambda)=\sin(\tilde\xi^{+}-\tilde\lambda+\eta)e^{2i\tilde\lambda},$$

$$U_2^{+}(\tilde\lambda)=\frac{\sin(\tilde\lambda+\tilde\xi^{+})\sin(2\tilde\lambda-2\eta)}{\sin(2\tilde\lambda-\eta)}. \tag{117}$$

Following the standard algebraic Bethe ansatz method, we assume that the eigenvector of the nested transfer matrix is constructed as $\mathcal{C}(\tilde\mu_1^{(1)})\mathcal{C}(\tilde\mu_2^{(1)})\cdots\mathcal{C}(\tilde\mu_m^{(1)})|0\rangle^{(1)}$. Applying the nested transfer matrix (113) on this eigenvector, using repeatedly the commutation relations (100) and (101), we have the eigenvalue

$$\Lambda^{(1)}(\tilde\lambda)=-U_1^{+}(\tilde\lambda)U_1(\tilde\lambda)$$
$$\times\prod_{i=1}^{n}[\sin(\tilde\lambda+\tilde\mu_i)\sin(\tilde\lambda-\tilde\mu_i)]$$
$$\times\prod_{l=1}^{m}\left\{\frac{\sin(\tilde\lambda-\tilde\mu_l^{(1)}-\eta)\sin(\tilde\lambda+\tilde\mu_l^{(1)}-2\eta)}{\sin(\tilde\lambda-\tilde\mu_l^{(1)})\sin(\tilde\lambda+\tilde\mu_l^{(1)}-\eta)}\right\}$$
$$-U_2^{+}(\tilde\lambda)U_2(\tilde\lambda)$$
$$\times\prod_{i=1}^{n}[\sin(\tilde\lambda+\tilde\mu_i-\eta)\sin(\tilde\lambda-\tilde\mu_i-\eta)]$$
$$\times\prod_{l=1}^{m}\left\{\frac{\sin(\tilde\lambda-\tilde\mu_l^{(1)}+\eta)\sin(\tilde\lambda+\tilde\mu_l^{(1)})}{\sin(\tilde\lambda-\tilde\mu_l^{(1)})\sin(\tilde\lambda+\tilde\mu_l^{(1)}-\eta)}\right\}, \tag{118}$$

where $\tilde\mu_1^{(1)},\ldots,\tilde\mu_m^{(1)}$ should satisfy the following Bethe ansatz equation:

$$\frac{U_1^{+}(\tilde\mu_j^{(1)})U_1(\tilde\mu_j^{(1)})}{U_2^{+}(\tilde\mu_j^{(1)})U_2(\tilde\mu_j^{(1)})}\frac{\sin(2\tilde\mu_j^{(1)}-2\eta)}{\sin(2\tilde\mu_j^{(1)})}$$
$$\times\prod_{i=1}^{n}\frac{\sin(\tilde\mu_j^{(1)}+\tilde\mu_i)\sin(\tilde\mu_j^{(1)}-\tilde\mu_i)}{\sin(\tilde\mu_j^{(1)}+\tilde\mu_i-\eta)\sin(\tilde\mu_j^{(1)}-\tilde\mu_i-\eta)}$$
$$=\prod_{l=1,\neq j}^{m}\frac{\sin(\tilde\mu_j^{(1)}-\tilde\mu_l^{(1)}+\eta)\sin(\tilde\mu_j^{(1)}+\tilde\mu_l^{(1)})}{\sin(\tilde\mu_j^{(1)}-\tilde\mu_l^{(1)}-\eta)\sin(\tilde\mu_j^{(1)}+\tilde\mu_l^{(1)}-2\eta)},$$
$$j=1,\ldots,m. \tag{119}$$

We already know the exact form of $\Lambda^{(1)}$, so we can change the former Bethe ansatz equation presented in relation (89) as follows:

$$1=\frac{U_2^{+}(\tilde\mu_j)U_2(\tilde\mu_j)}{U_3^{+}(\tilde\mu_j)U_3(\tilde\mu_j)}\frac{\sin^{2N}(\mu_j)}{\sin^{2N}(\mu_j+\eta)}$$
$$\times\prod_{l=1}^{m}\frac{\sin(\tilde\mu_j-\tilde\mu_l^{(1)}+\eta)\sin(\tilde\mu_j+\tilde\mu_l^{(1)})}{\sin(\tilde\mu_j-\tilde\mu_l^{(1)})\sin(\tilde\mu_j+\tilde\mu_l^{(1)}-\eta)},$$
$$j=1,\ldots,n. \tag{120}$$

The eigenvalue of the transfer matrix $t(\lambda)$ with reflecting boundary condition (46) is obtained as

$$\Lambda(\lambda)=U_3^{+}(\lambda)U_3(\lambda)\sin^{2N}(\lambda+\eta)$$
$$\times\prod_{i=1}^{n}\frac{\sin(\lambda+\mu_i)\sin(\lambda-\mu_i-\eta)}{\sin(\lambda+\mu_i+\eta)\sin(\lambda-\mu_i)}+\sin^{2N}(\lambda)$$
$$\times\prod_{i=1}^{n}\frac{1}{\sin(\lambda-\mu_i)\sin(\lambda+\mu_i+\eta)}\Lambda^{(1)}(\tilde\lambda). \tag{121}$$

Here, for convenience, we give a summary of the values U and U^{+}.

Case I:

$$U_1^{+}(\tilde\lambda)=\sin(\tilde\xi^{+}-\tilde\lambda+\eta)e^{2i\tilde\lambda},$$

$$U_2^+(\tilde{\lambda})=\frac{\sin(2\tilde{\lambda}-2\eta)\sin(\tilde{\xi}^+-\tilde{\lambda}+\eta)}{\sin(2\tilde{\lambda}-\eta)}e^{i(2\tilde{\lambda}-\eta)},$$

$$U_3^+(\lambda)=\frac{\sin(2\lambda-\eta)\sin(\xi^++\lambda+\eta)}{\sin(2\lambda+\eta)}. \quad (122)$$

Case II:

$$U_1^+(\tilde{\lambda})=\sin(\tilde{\xi}^+-\tilde{\lambda}+\eta)e^{2i\tilde{\lambda}},$$

$$U_2^+(\tilde{\lambda})=\frac{\sin(\tilde{\lambda}+\tilde{\xi}^+)\sin(2\tilde{\lambda}-2\eta)}{\sin(2\tilde{\lambda}-\eta)},$$

$$U_3^+(\lambda)=\frac{\sin(2\lambda-\eta)\sin(\xi^++\lambda)}{\sin(2\lambda+\eta}e^{i\eta}. \quad (123)$$

Case I:

$$U_1(\tilde{\lambda})=\sin(\tilde{\lambda}+\tilde{\xi})e^{-2i\tilde{\lambda}},$$

$$U_2(\tilde{\lambda})=e^{-i(2\tilde{\lambda}-\eta)}\frac{\sin(2\tilde{\lambda}-\eta)\sin(\tilde{\lambda}+\tilde{\xi})}{\sin(2\tilde{\lambda})},$$

$$U_3(\lambda)=\sin(\xi-\lambda). \quad (124)$$

Case II:

$$U_1(\tilde{\lambda})=\sin(\tilde{\lambda}+\tilde{\xi}-\eta)e^{-i(2\tilde{\lambda}-\eta)},$$

$$U_2(\tilde{\lambda})=e^{2\eta}\frac{\sin(2\tilde{\lambda}-\eta)\sin(\tilde{\xi}-\tilde{\lambda})}{\sin(2\tilde{\lambda})},$$

$$U_3(\lambda)=\sin(\xi-\lambda). \quad (125)$$

Since U and U^+ are independent of each other, there are four combinations for $\{U,U^+\}$ such as $\{I,I\}$, $\{I,II\}$, $\{II,I\}$, and $\{II,II\}$.

In the special limit $\xi\to -i\infty$, the solution of the reflection equation becomes identity; our result should be reduced to the results obtained by Foerster and Karowski.[22] And in the rational limit, the results are equivalent to the previous results.[42,37]

V. ALGEBRAIC BETHE ANSATZ FOR BFF GRADING

A. First-level Bethe ansatz

For the case of BFF grading, the calculations proceed parallel to the case of FFB. However, for the nested algebraic Bethe ansatz method, the low-level r matrix is BF grading which is significantly different from the FF grading r matrix. Actually, as we observed in the last section, the graded method is the same as the usual method for the FF grading r matrix. We shall study the supersymmetric t-J model in the BFF grading.

The R matrix is now

$$R(\lambda)=\begin{pmatrix} w(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & b(\lambda) & 0 & c_-(\lambda) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & b(\lambda) & 0 & 0 & 0 & c_-(\lambda) & 0 & 0\\ 0 & c_+(\lambda) & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & a(\lambda) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & -c_-(\lambda) & 0\\ 0 & 0 & c_+(\lambda) & 0 & 0 & 0 & b(\lambda) & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -c_+(\lambda) & 0 & b(\lambda) & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\lambda) \end{pmatrix}. \quad (126)$$

The diagonal solutions of the dual reflection equation are

$$K_I^+(\lambda)=\begin{pmatrix} \sin(\xi^+-\lambda)e^{i(2\lambda-\eta)} & & \\ & \sin(\xi^+-\lambda)e^{i(2\lambda-\eta)} & \\ & & \sin(\xi^++\lambda-\eta)e^{-2i\eta} \end{pmatrix},$$

$$K_{II}^+(\lambda)=\begin{pmatrix} \sin(\xi^+-\lambda)e^{i(2\lambda-\eta)} & & \\ & \sin(\xi^++\lambda-\eta) & \\ & & \sin(\xi^++\lambda-\eta)e^{-2i\eta} \end{pmatrix}, \quad (127)$$

where ξ^+ is an arbitrary boundary parameter.

We still denote solution of the dual reflection equation K^+ and the double-row monodromy matrix $\mathcal{T}$, respectively, in the following forms:

$$K^+(\lambda)=\mathrm{diag}(k_1^+(\lambda),k_2^+(\lambda),k_3^+(\lambda)), \quad (128)$$

$$\mathcal{T}(\lambda)=\begin{pmatrix}\mathcal{A}_{11}(\lambda) & \mathcal{A}_{12}(\lambda) & \mathcal{B}_{1}(\lambda)\\ \mathcal{A}_{21}(\lambda) & \mathcal{A}_{22}(\lambda) & \mathcal{B}_{2}(\lambda)\\ \mathcal{C}_{1}(\lambda) & \mathcal{C}_{2}(\lambda) & \mathcal{D}(\lambda)\end{pmatrix}. \tag{129}$$

In order to obtain the commutation relations, we need the following transformation:

$$\mathcal{A}(\lambda)_{ab}=\tilde{\mathcal{A}}(\lambda)_{ab}-\delta_{ab}\frac{e^{-2i\lambda}\sin(\eta)}{\sin(2\lambda-\eta)}\mathcal{D}(\lambda). \tag{130}$$

Because the double-row monodromy matrix satisfies the reflection equation, we obtain the following commutation relations after some tedious calculations:

$$\mathcal{C}_{d_1}(\lambda)\mathcal{C}_{d_2}(\mu)=\frac{r_{12}(\lambda-\mu)_{c_2c_1}^{d_2d_1}}{\sin(\lambda-\mu-\eta)}(-1)^{1+\epsilon_{d_1}+\epsilon_{c_2}+\epsilon_{c_1}\epsilon_{c_2}}\mathcal{C}_{c_2}(\mu)\mathcal{C}_{c_1}(\lambda), \tag{131}$$

$$\mathcal{D}(\lambda)\mathcal{C}_{d}(\mu)=\frac{\sin(\lambda+\mu)\sin(\lambda-\mu+\eta)}{\sin(\lambda+\mu-\eta)\sin(\lambda-\mu)}\mathcal{C}_{d}(\mu)\mathcal{D}(\lambda)-\frac{\sin(2\mu)\sin(\eta)e^{i(\lambda-\mu}}{\sin(\lambda-\mu)\sin(2\mu-\eta)}\mathcal{C}_{d}(\lambda)\mathcal{D}(\mu)+\frac{\sin(\eta)e^{i(\lambda+\mu)}}{\sin(\lambda+\mu-\eta)}\mathcal{C}_{b}(\lambda)\tilde{\mathcal{A}}_{bd}(\mu), \tag{132}$$

$$\begin{aligned}\tilde{\mathcal{A}}_{a_1d_1}(\lambda)\mathcal{C}_{d_2}(\mu)=&(-1)^{\epsilon_{a_1}+\epsilon_{d_1}+\epsilon_{c_1}\epsilon_{b_2}+\epsilon_{d_1}\epsilon_{d_2}}\frac{r_{12}(\lambda+\mu-\eta)_{a_1c_2}^{c_1b_2}r_{21}(\lambda-\mu)_{b_1b_2}^{d_1d_2}}{\sin(\lambda+\mu-\eta)\sin(\lambda-\mu)}\mathcal{C}_{c_2}(\mu)\tilde{\mathcal{A}}_{c_1b_1}(\lambda)\\ &-(-1)^{\epsilon_{a_1}(1+\epsilon_{b_1})+\epsilon_{d_1}}\frac{\sin(\eta)e^{-i(\lambda-\mu)}}{\sin(\lambda-\mu)\sin(2\lambda-\eta)}r_{12}(2\lambda-\eta)_{a_1b_1}^{b_2d_1}\mathcal{C}_{b_1}(\lambda)\tilde{\mathcal{A}}_{b_2d_2}(\mu)\\ &+(-1)^{\epsilon_{d_1}+\epsilon_{a_1}(\epsilon_{d_1}+\epsilon_{d_2})}\frac{\sin(2\mu)\sin(\eta)e^{-i(\lambda+\mu)}}{\sin(\lambda+\mu-\eta)\sin(2\lambda-\eta)\sin(2\mu-\eta)}r_{12}(2\lambda+\eta)_{a_1b_2}^{d_2d_1}\mathcal{C}_{b_2}(\lambda)\mathcal{D}(\mu).\end{aligned} \tag{133}$$

Here the indices take values 1 and 2, and the Grassmann parities are BF, $\epsilon_1=0,\epsilon_2=1$. The r matrix is defined as

$$r_{12}(\lambda)=\begin{pmatrix}\sin(\lambda+\eta) & 0 & 0 & 0\\ 0 & \sin(\lambda) & \sin(\eta)e^{-i\lambda} & 0\\ 0 & \sin(\eta)e^{i\lambda} & \sin(\lambda) & 0\\ 0 & 0 & 0 & \sin(\lambda-\eta)\end{pmatrix}. \tag{134}$$

The elements of the r matrix are equal to those of the original R matrix when its indices just take values 1 and 2, and the Grassmann parities also remain the same as before if we just take values 1 and 2. This r matrix has $su(1|1)$ symmetry.

Let the elements of the double-row monodromy matrix act on the vacuum state $|0\rangle$:

$$\begin{aligned}&\mathcal{D}(\lambda)|0\rangle=U_3(\lambda)\sin^{2N}(\lambda-\eta)|0\rangle,\\ &\mathcal{A}_{aa}(\lambda)|0\rangle=W_a(\lambda)\sin^{2N}(\lambda)|0\rangle,\\ &\tilde{\mathcal{A}}_{ab}(\lambda)|0\rangle=0,\quad a\neq b\\ &\mathcal{B}_a(\lambda)|0\rangle=0,\\ &\mathcal{C}_a(\lambda)|0\rangle\neq 0.\end{aligned} \tag{135}$$

Here we have defined

$$U_3(\lambda)=k_3(\lambda),\quad W_a(\lambda)=k_a(\lambda)+\frac{\sin(\eta)e^{-2i\eta}}{\sin(2\lambda-\eta)}k_3(\lambda). \tag{136}$$

Substituting the exact forms of the reflecting type I and type II K matrices into the above relation, we have

$$\begin{aligned}&W_1^I(\lambda)=W_2^I(\lambda)=\frac{\sin(\lambda)\sin(\xi+\lambda-\eta)e^{-i2\lambda}}{\sin(2\lambda-\eta)},\\ &W_1^{II}(\lambda)=\frac{\sin(2\lambda)\sin(\xi+\lambda-\eta)e^{-i2\lambda}}{\sin(2\lambda-\eta)},\\ &W_2^{II}(\lambda)=\frac{\sin(2\lambda)\sin(\xi-\lambda)e^{-i\eta}}{\sin(2\lambda-\eta)}.\end{aligned} \tag{137}$$

The transfer matrix with boundaries for BFF grading is written as

$$\begin{aligned}t(\lambda)&=k_1^+(\lambda)\mathcal{A}_{11}(\lambda)-k_2^+(\lambda)\mathcal{A}_{22}(\lambda)-k_3^+(\lambda)\mathcal{D}(\lambda)\\ &=(-1)^{\epsilon_a}k_a^+(\lambda)\tilde{\mathcal{A}}_{aa}(\lambda)+U_3^+(\lambda)\mathcal{D}(\lambda),\end{aligned} \tag{138}$$

where U_3^+ is defined by

$$U_3^+(\lambda)\equiv k_3^+(\lambda)+\frac{e^{-2i\lambda}\sin(\eta)}{\sin(2\lambda-\eta)}[k_1^+(\lambda)-k_2^+(\lambda)]. \tag{139}$$

For type I and II solutions of the dual reflection equations K^+, we have

$$U_3^+(\lambda)=k_3(\lambda)=\sin(\xi^+ +\lambda-\eta)e^{-2i\eta},\quad \text{for}\quad K_I^+, \tag{140}$$

$$U_3^+(\lambda)=\sin(\xi^+ +\lambda-2\eta)e^{-i\eta},\quad \text{for}\quad K_{II}^+. \tag{141}$$

Using the standard algebraic Bethe ansatz method, applying the above-defined transfer matrix (138) on the ansatz of eigenvector $\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}$, we have

$$\begin{aligned}
t(\lambda)&\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}\\
&=U_3^+(\lambda)U_3(\lambda)\sin^{2N}(\lambda-\eta)\\
&\quad\times\prod_{i=1}^{n}\frac{\sin(\lambda+\mu_i)(\sin\lambda-\mu_i+\eta)}{\sin(\lambda+\mu_i-\eta)\sin(\lambda-\mu_i)}\\
&\quad\times\mathcal{C}_{d_1}(\mu_1)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}+\sin^{2N}(\lambda)\\
&\quad\times\prod_{i=1}^{n}\frac{1}{\sin(\lambda-\mu_i)\sin(\lambda+\mu_i-\eta)}\\
&\quad\times\mathcal{C}_{c_1}(\mu_1)\cdots\mathcal{C}_{c_n}(\mu_n)|0\rangle t^{(1)}(\lambda)_{d_1\cdots d_n}^{c_1\cdots c_n}F^{d_1\cdots d_n}+\text{u.t.},
\end{aligned} \tag{142}$$

where the nested transfer matrix $t^{(1)}(\lambda)$ is defined as

$$\begin{aligned}
t^{(1)}(\lambda)_{d_1\cdots d_n}^{c_1\cdots c_n}&=(-)^{\epsilon_a}k_a^+(\lambda)\\
&\quad\times\{r(\lambda+\mu_1-\eta)_{ac_1}^{a_1e_1}r(\lambda+\mu_2\\
&\quad-\eta)_{a_1c_2}^{a_2e_2}\cdots r(\lambda+\mu_1-\eta)_{a_{n-1}c_n}^{a_ne_n}\}\\
&\quad\delta_{a_nb_n}W_{a_n}(\lambda)\{r_{21}(\lambda-\mu_n)_{b_ne_n}^{b_{n-1}d_n}\cdots r_{21}(\lambda\\
&\quad-\mu_2)_{b_2e_2}^{b_1d_2}r_{21}(\lambda-\mu_1)_{b_1e_1}^{ad_1}\}\\
&\quad\times(-1)\sum_{i=1}^{n}(\epsilon_{a_i}+\epsilon_{b_i})(1+\epsilon_{e_i}).
\end{aligned} \tag{143}$$

Here we have used $\epsilon_a\epsilon_b=\epsilon_c\epsilon_d$ for a nonzero elements of the r matrix r_{ab}^{cd}. We also know that for nonzero r_{ab}^{cd}, we have $\epsilon_a+\epsilon_c=\epsilon_b+\epsilon_d$. Considering $\epsilon_a+\epsilon_a=0$, we can write

$$\begin{aligned}
\epsilon_{a_i}+\epsilon_{b_i}&=\epsilon_{a_i}+2\epsilon_{a_{i+1}}+\cdots+2\epsilon_{a_{n-1}}\\
&\quad+2\epsilon_{b_{n-1}}+\cdots+2\epsilon_{b_{i+1}}+\epsilon_{b_i},\\
&=\sum_{j=1}^{n-i}(\epsilon_{c_{i+j}}+\epsilon_{d_{i+j}}),\quad i=1,\ldots,n-1,
\end{aligned} \tag{144}$$

in which

$$\begin{aligned}
\sum_{i=1}^{n}(\epsilon_{a_i}+\epsilon_{b_i})(1+\epsilon_{e_i})&=\sum_{j=2}^{n}(\epsilon_{c_j}+\epsilon_{e_j})\sum_{i=1}^{j-1}(1+\epsilon_{e_i})\\
&\quad+\sum_{j=2}^{n}(\epsilon_{d_j}+\epsilon_{e_j})\sum_{i=1}^{j-1}(1+\epsilon_{e_i}).
\end{aligned} \tag{145}$$

Thus this nested transfer matrix can still be interpreted as a transfer matrix with reflecting boundary conditions corresponding to the anisotropic case

$$\begin{aligned}
t^{(1)}(\lambda)&=\mathrm{str}\,K^{(1)+}(\tilde{\lambda})T^{(1)}(\tilde{\lambda},\{\tilde{\mu}_i\})\\
&\quad\times K^{(1)}(\tilde{\lambda})T^{(1)-1}(-\tilde{\lambda},\{\tilde{\mu}_i\}),
\end{aligned} \tag{146}$$

with the grading BF $\epsilon_1=0,\epsilon_2=1$, where we denote $\tilde{x}=x-\eta/2$, $x=\lambda,\mu,\xi,\xi^+$. According to the definition, we have nested reflecting matrices

$$K^{(1)}(\tilde{\lambda})\equiv\begin{pmatrix}W_1\left(\tilde{\lambda}+\frac{1}{2}\eta\right) & \\ & W_2\left(\tilde{\lambda}+\frac{1}{2}\eta\right)\end{pmatrix}=\begin{cases}\dfrac{\sin(2\tilde{\lambda}+\eta)\sin(\tilde{\xi}+\tilde{\lambda})}{\sin(2\tilde{\lambda})}e^{-i(2\tilde{\lambda}+\eta)}\times\text{id.}, & \text{for case I},\\ \dfrac{\sin(2\tilde{\lambda}+\eta)e^{-i(2\tilde{\lambda}+\eta)}}{\sin(2\tilde{\lambda})}\mathrm{diag}(\sin(\tilde{\xi}+\tilde{\lambda}),\sin(\tilde{\xi}-\tilde{\lambda})e^{i2\tilde{\lambda}}), & \text{for case II},\end{cases} \tag{147}$$

and

$$K^{(1)+}(\tilde{\lambda})\equiv\begin{pmatrix}k_1^+\left(\tilde{\lambda}+\frac{1}{2}\eta\right) & \\ & k_2^+\left(\tilde{\lambda}+\frac{1}{2}\eta\right)\end{pmatrix}=\begin{cases}\sin(\tilde{\xi}^+-\tilde{\lambda})e^{i2\tilde{\lambda}}\times\text{id.}, & \text{for case I},\\ \mathrm{diag}(\sin(\tilde{\xi}-\tilde{\lambda})e^{i2\tilde{\lambda}},\sin(\tilde{\xi}+\tilde{\lambda})), & \text{for case II}.\end{cases} \tag{148}$$

The row-to-row monodromy matrices $T^{(1)}(\tilde{\lambda},\{\tilde{\mu}_i\})$ and $T^{(1)-1}(-\tilde{\lambda},\{\tilde{\mu}_i\})$ are defined, respectively, as

$$T^{(1)}_{aa_n}(\tilde{\lambda},\{\tilde{\mu}_i\})^{e_1\cdots e_n}_{c_1\cdots c_n}=r(\tilde{\lambda}+\tilde{\mu}_1)^{a_1e_1}_{ac_1}r(\tilde{\lambda}+\tilde{\mu}_2)^{a_2e_2}_{a_1c_2}\cdots r(\tilde{\lambda}+\tilde{\mu}_1)^{a_ne_n}_{a_{n-1}c_n}(-1)^{\sum_{j=2}^{n}(\epsilon_{c_i}+\epsilon_{e_i})\sum_{i=1}^{j-1}(1+\epsilon_{e_i})}$$
$$=L^{(1)}(\tilde{\lambda}+\tilde{\mu}_1)^{a_1e_1}_{ac_1}L^{(1)}(\tilde{\lambda}+\tilde{\mu}_2)^{a_2e_2}_{a_1c_2}\cdots L^{(1)}(\tilde{\lambda}+\tilde{\mu}_1)^{a_ne_n}_{a_{n-1}c_n}(-1)^{\sum_{j=2}^{n}(\epsilon_{c_i}+\epsilon_{e_i})\sum_{i=1}^{j-1}(1+\epsilon_{e_i})}, \tag{149}$$

$$T^{(1)-1}(-\tilde{\lambda},\{\tilde{\mu}_i\})=r_{21}(\tilde{\lambda}-\tilde{\mu}_n)^{b_{n-1}d_n}_{b_ne_n}\cdots r_{21}(\tilde{\lambda}-\tilde{\mu}_2)^{b_1d_2}_{b_2e_2}r_{21}(\tilde{\lambda}-\tilde{\mu}_1)^{ad_1}_{b_1e_1}(-1)^{\sum_{j=2}^{n}(\epsilon_{d_i}+\epsilon_{e_i})\sum_{i=1}^{j-1}(1+\epsilon_{e_i})}$$
$$=L^{(1)-1}_n(-\tilde{\lambda}+\tilde{\mu}_n)^{b_{n-1}d_n}_{b_ne_n}\cdots L^{(1)-1}_2(-\tilde{\lambda}+\tilde{\mu}_2)^{b_1d_2}_{b_2e_2}L^{(1)-1}_1(-\tilde{\lambda}+\tilde{\mu}_1)^{ad_1}_{b_1e_1}(-1)^{\sum_{j=2}^{n}(\epsilon_{d_i}+\epsilon_{e_i})\sum_{i=1}^{j-1}(1+\epsilon_{e_i})}, \tag{150}$$

where we have used the unitarity relation of the r matrix $r_{12}(\lambda)r_{21}(-\lambda)=\sin(\eta-\lambda)\sin(\eta+\lambda)\times\mathrm{id}$. The L operator is obtained from the r matrix and takes the form

$$L^{(1)}_k(\tilde{\lambda})=\begin{pmatrix} b(\lambda)-[b(\lambda)-w(\lambda)]e^{11}_k & c_-(\lambda)e^{21}_n \\ c_+(\lambda)e^{12}_n & b(\lambda)-[b(\lambda)-a(\lambda)]e^{22}_n\end{pmatrix}. \tag{151}$$

We find that the supertensor product in the above-defined monodromy matrix is different from the original definition. Nevertheless, as in the periodic boundary condition case, we can define another graded tensor product as follows:[9]

$$F\overline{\otimes}G^{bd}_{ac}=F^b_aG^d_c(-1)^{(\epsilon_a+\epsilon_b)(1+\epsilon_c)}. \tag{152}$$

Effectively the graded tensor product switches even and odd Grassmann parities. The graded tensor product in the above monodromy matrices follows the newly defined rule.

The L operator satisfies the following Yang-Baxter relation:

$$r(\lambda-\mu)^{b_1b_2}_{a_1a_2}L^{(1)}(\lambda)^{c_1}_{b_1}L^{(1)}(\mu)^{c_2}_{b_2}(-1)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{b_2}}=L^{(1)}(\mu)^{b_2}_{a_2}L^{(1)}(\lambda)^{b_1}_{a_1}r(\lambda-\mu)^{c_1c_2}_{b_1b_2}(-1)^{(\epsilon_{a_1}+\epsilon_{b_1})\epsilon_{b_2}}. \tag{153}$$

Multiplying both sides of this Yang-Baxter relation by $(-1)^{(\epsilon_{a_1}+\epsilon_{c_1})}$, we obtain

$$\hat{r}(\lambda-\mu)^{b_1b_2}_{a_1a_2}L^{(1)}(\lambda)^{c_1}_{b_1}L^{(1)}(\mu)^{c_2}_{b_2}(-1)^{(\epsilon_{b_1}+\epsilon_{c_1})(1+\epsilon_{b_2})}=L^{(1)}(\mu)^{b_2}_{a_2}L^{(1)}(\lambda)^{b_1}_{a_1}\hat{r}(\lambda-\mu)^{c_1c_2}_{b_1b_2}(-1)^{(\epsilon_{a_1}+\epsilon_{b_1})(1+\epsilon_{b_2})}. \tag{154}$$

This is just the graded Yang-Baxter relation in the newly defined graded tensor product. And we have another r matrix

$$\hat{r}(\lambda)^{bd}_{ac}=(-1)^{\epsilon_a+\epsilon_b}r(\lambda)^{bd}_{ac}. \tag{155}$$

For the row-to-row monodromy matrix, we also have

$$\hat{r}(\lambda_1-\lambda_2)^{b_1b_2}_{a_1a_2}T^{(1)}(\lambda_1,\{\mu_i\})^{c_1}_{b_1}T^{(1)}(\lambda_2,\{\mu_i\})^{c_2}_{b_2}(-1)^{(\epsilon_{b_1}+\epsilon_{c_1})(1+\epsilon_{b_2})}$$
$$=T^{(1)}(\lambda_2,\{\mu_i\})^{b_2}_{a_2}T^{(1)}(\lambda,\{\mu_i\})^{b_1}_{a_1}\hat{r}(\lambda_1-\lambda_2)^{c_1c_2}_{b_1b_2}(-1)^{(\epsilon_{a_1}+\epsilon_{b_1})(1+\epsilon_{b_2})}. \tag{156}$$

In order to prove that the nested monodromy matrix is indeed the transfer matrix with reflecting boundary conditions, we need to prove that it constitutes a commuting family. As discussed in the last sections, we should prove that $K^{(1)}$ and $K^{(1)+}$ satisfy something like reflection equations. One can prove that $K^{(1)}$ and $K^{(1)+}$ satisfy the following graded reflection equations in the newly defined graded sense:

$$\hat{r}(\lambda-\mu)^{b_1b_2}_{a_1a_2}K^{(1)}(\lambda)^{c_1}_{b_1}\hat{r}(\lambda+\mu)^{c_2d_1}_{b_2c_1}K^{(1)}(\mu)^{d_2}_{c_2}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})(1+\epsilon_{b_2})}$$
$$=K^{(1)}(\mu)^{b_2}_{a_2}\hat{r}(\lambda+\mu)^{b_1c_2}_{a_1b_2}K^{(1)}(\lambda)^{c_1}_{b_1}\hat{r}(\lambda-\mu)^{d_2d_1}_{c_2c_1}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})(1+\epsilon_{c_2})}, \tag{157}$$

$$\hat{r}(-\lambda+\mu)^{b_1b_2}_{a_1a_2}K^{(1)+}(\lambda)^{c_1}_{b_1}\hat{r}(-\lambda-\mu)^{c_2d_1}_{b_2c_1}K^{(1)+}(\mu)^{d_2}_{c_2}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})(1+\epsilon_{b_2})}$$
$$=K^{(1)+}(\mu)^{b_2}_{a_2}\hat{r}(-\lambda-\mu)^{b_1c_2}_{a_1b_2}K^{(1)+}(\lambda)^{c_1}_{b_1}\hat{r}(-\lambda+\mu)^{d_2d_1}_{c_2c_1}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})(1+\epsilon_{c_2})}. \tag{158}$$

We see that the second relation is consistent with the cross-unitarity relation $\hat{r}^{st_1}_{12}(-\lambda)\hat{r}^{st_1}_{21}(\lambda)=-\sin^2(\lambda)\times\mathrm{id}$. Thus the nested transfer matrix is proved to constitute a commuting family. We can still use the graded algebraic Bethe ansatz method to find its eigenvalue and eigenvector.

B. Algebraic Bethe ansatz method for the BF six-vertex model with boundaries and the final results for the BFF case

Denote the double-row monodromy matrix as

$$\mathcal{T}^{(1)}(\lambda,\{\mu_i\})=\begin{pmatrix}\mathcal{A}^{(1)}(\lambda) & \mathcal{B}^{(1)}(\lambda)\\ \mathcal{C}^{(1)}(\lambda) & \mathcal{D}^{(1)}(\lambda)\end{pmatrix}. \tag{159}$$

For convenience, we need the following transformation:

$$\mathcal{A}^{(1)}(\lambda)=\tilde{\mathcal{A}}^{(1)}(\lambda)-\frac{\sin(\eta)e^{-2i\lambda}}{\sin(2\lambda-\eta)}\mathcal{D}^{(1)}(\lambda). \tag{160}$$

Because the nested double-row monodromy matrix satisfies the reflection equation

$$\hat{r}(\lambda-\mu)_{a_1a_2}^{b_1b_2}\mathcal{T}^{(1)}(\lambda)_{b_1}^{c_1}\hat{r}(\lambda+\mu)_{b_2c_1}^{c_2d_1}\mathcal{T}^{(1)}(\mu)_{c_2}^{d_2}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})(1+\epsilon_{b_2})}$$
$$=\mathcal{T}^{(1)}(\mu)_{a_2}^{b_2}\hat{r}(\lambda+\mu)_{a_1b_2}^{b_1c_2}\mathcal{T}^{(1)}(\lambda)_{b_1}^{c_1}\hat{r}(\lambda-\mu)_{c_2c_1}^{d_2d_1}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})(1+\epsilon_{c_2})}, \tag{161}$$

we have the following commutation relations:

$$\mathcal{D}^{(1)}(\lambda)\mathcal{C}^{(1)}(\mu)=\frac{\sin(\lambda-\mu+\eta)\sin(\lambda+\mu)}{\sin(\lambda-\mu)\sin(\lambda+\mu-\eta)}\mathcal{C}^{(1)}(\mu)\mathcal{D}^{(1)}(\lambda)-\frac{\sin(2\mu)\sin(\eta)e^{i(\lambda-\mu)}}{\sin(\lambda-\mu)\sin(2\mu-\eta)}\mathcal{C}^{(1)}(\lambda)\mathcal{D}^{(1)}(\mu)+\frac{\sin(\eta)e^{i(\lambda+\mu)}}{\sin(\lambda+\mu-\eta)}\mathcal{C}^{(1)}(\lambda)\tilde{\mathcal{A}}^{(1)}(\mu), \tag{162}$$

$$\tilde{\mathcal{A}}^{(1)}(\lambda)\mathcal{C}^{(1)}(\mu)=\frac{\sin(\lambda-\mu+\eta)\sin(\lambda+\mu)}{\sin(\lambda-\mu)\sin(\lambda+\mu-\eta)}\mathcal{C}^{(1)}(\mu)\tilde{\mathcal{A}}^{(1)}(\lambda)-\frac{\sin(\eta)\sin(2\lambda)e^{-i(\lambda-\mu)}}{\sin(\lambda-\mu)\sin(2\lambda-\eta)}\mathcal{C}^{(1)}(\lambda)\tilde{\mathcal{A}}^{(1)}(\mu)+\frac{\sin(2\mu)\sin(2\lambda)\sin(\eta)e^{-i(\lambda+\mu)}}{\sin(\lambda+\mu-\eta)\sin(2\lambda-\eta)\sin(2\mu-\eta)}\times\mathcal{C}^{(1)}(\lambda)\mathcal{D}^{(1)}(\mu), \tag{163}$$

$$\mathcal{C}^{(1)}(\lambda)\mathcal{C}^{(1)}(\mu)=-\frac{\sin(\lambda-\mu+\eta)}{\sin(\lambda-\mu-\eta)}\mathcal{C}^{(1)}(\mu)\mathcal{C}^{(1)}(\lambda). \tag{164}$$

For the local vacuum state $|0\rangle^{(1)}=\overline{\otimes}_{k=1}^{n}|0\rangle_k^{(1)}$, we have

$$\mathcal{B}^{(1)}(\tilde{\lambda})|0\rangle^{(1)}=0,$$

$$\mathcal{C}^{(1)}(\tilde{\lambda})|0\rangle^{(1)}\neq 0,$$

$$\tilde{\mathcal{A}}^{(1)}(\tilde{\lambda})|0\rangle^{(1)}=U_1(\tilde{\lambda})\prod_{i=1}^{n}[\sin(\tilde{\lambda}+\tilde{\mu}_i)\sin(\tilde{\lambda}-\tilde{\mu}_i)]|0\rangle^{(1)},$$

$$\mathcal{D}^{(1)}(\tilde{\lambda})|0\rangle^{(1)}=U_2(\tilde{\lambda})\prod_{i=1}^{n}[\sin(\tilde{\lambda}+\tilde{\mu}_i-\eta)\times\sin(\tilde{\lambda}-\tilde{\mu}_i-\eta)]|0\rangle^{(1)}. \tag{165}$$

Applying the transfer matrix $t^{(1)}(\tilde{\lambda})=U_1^+(\tilde{\lambda})\tilde{\mathcal{A}}^{(1)}(\tilde{\lambda})-U_2^+(\tilde{\lambda})\mathcal{D}^{(1)}(\tilde{\lambda})$ on the ansatz of the eigenvector $\mathcal{C}(\tilde{\mu}_1^{(1)})\mathcal{C}(\tilde{\mu}_2^{(1)})\cdots\mathcal{C}(\tilde{\mu}_m^{(1)})|0\rangle^{(1)}$, we find the eigenvalue of the nested transfer matrix as follows:

$$\Lambda^{(1)}(\tilde{\lambda})=U_1^+(\tilde{\lambda})U_1(\tilde{\lambda})\prod_{i=1}^{n}[\sin(\tilde{\lambda}+\tilde{\mu}_i)\sin(\tilde{\lambda}-\tilde{\mu}_i)]\times\prod_{l=1}^{m}\left\{\frac{\sin(\tilde{\lambda}-\tilde{\mu}_l^{(1)}+\eta)\sin(\tilde{\lambda}+\tilde{\mu}_l^{(1)})}{\sin(\tilde{\lambda}-\tilde{\mu}_l^{(1)})\sin(\tilde{\lambda}+\tilde{\mu}_l^{(1)}-\eta)}\right\}-U_2^+(\tilde{\lambda})U_2(\tilde{\lambda})\prod_{i=1}^{n}[\sin(\tilde{\lambda}+\tilde{\mu}_i-\eta)\times\sin(\tilde{\lambda}-\tilde{\mu}_i-\eta)]\times\prod_{l=1}^{m}\left\{\frac{\sin(\tilde{\lambda}-\tilde{\mu}_l^{(1)}+\eta)\sin(\tilde{\lambda}+\tilde{\mu}_l^{(1)})}{\sin(\tilde{\lambda}-\tilde{\mu}_l^{(1)})\sin(\tilde{\lambda}+\tilde{\mu}_l^{(1)}-\eta)}\right\}, \tag{166}$$

where $\tilde{\mu}_1^{(1)},\ldots,\tilde{\mu}_m^{(1)}$ should satisfy the Bethe ansatz equations

$$\frac{U_1^+(\tilde{\mu}_j^{(1)})U_1(\tilde{\mu}_j^{(1)})}{U_2^+(\tilde{\mu}_j^{(1)})U_2(\tilde{\mu}_j^{(1)})}\times\prod_{i=1}^{n}\frac{\sin(\tilde{\mu}_j^{(1)}+\tilde{\mu}_i)\sin(\tilde{\mu}_j^{(1)}-\tilde{\mu}_i)}{\sin(\tilde{\mu}_j^{(1)}+\tilde{\mu}_i-\eta)\sin(\tilde{\mu}_j^{(1)}-\tilde{\mu}_i-\eta)}=1,\quad j=1,\ldots,m. \tag{167}$$

The eigenvalue of the transfer matrix $t(\lambda)$ with reflecting boundary condition is finally obtained as

$$\Lambda(\lambda)=-U_3^+(\lambda)U_3(\lambda)\sin^{2N}(\lambda-\eta)\times\prod_{i=1}^{n}\frac{\sin(\lambda+\mu_i)\sin(\lambda-\mu_i+\eta)}{\sin(\lambda+\mu_i-\eta)\sin(\lambda-\mu_i)}+\sin^{2N}(\lambda)\prod_{i=1}^{n}\frac{1}{\sin(\lambda-\mu_i)\sin(\lambda+\mu_i-\eta)}\Lambda^{(1)}(\tilde{\lambda}), \tag{168}$$

and $\mu_1, \ldots, \mu_m$ should satisfy the Bethe ansatz equations

$$\frac{\sin(2\tilde{\mu}_j+\eta)}{\sin(2\tilde{\mu}_j-\eta)}\prod_{i=1,\neq j}^{n}\left\{\frac{\sin(\mu_j+\mu_i)\sin(\mu_j-\mu_i+\eta)}{\sin(\tilde{\mu}_j+\tilde{\mu}_i-\eta)\sin(\tilde{\mu}_j-\tilde{\mu}_i-\eta)}\right\} = \frac{\sin^{2N}(\mu_j)}{\sin^{2N}(\mu_j-\eta)}\frac{U_2^+(\tilde{\mu}_j)U_2(\tilde{\mu}_j)}{U_3^+(\mu_j)U_3(\mu_j)} \times\prod_{l=1}^{m}\left\{\frac{\sin(\tilde{\mu}_j-\tilde{\mu}_l^{(1)}+\eta)\sin(\tilde{\mu}_j+\tilde{\mu}_l^{(1)})}{\sin(\tilde{\mu}_j-\tilde{\mu}_l^{(1)})\sin(\tilde{\mu}_j+\tilde{\mu}_l^{(1)}-\eta)}\right\}, \quad j=1,\ldots,n, \tag{169}$$

where $\tilde{\mu}=\mu-\frac{1}{2}\eta$.

Finally, we give a summary of U and U^+ for BFF grading.

Case I:

$$U_1^+(\tilde{\lambda})=\sin(\tilde{\xi}^+-\tilde{\lambda})e^{2i\tilde{\lambda}},$$

$$U_2^+(\tilde{\lambda})=\frac{\sin(2\tilde{\lambda})\sin(\tilde{\xi}^+-\tilde{\lambda})}{\sin(2\tilde{\lambda}-\eta)}e^{i(2\tilde{\lambda}-\eta)},$$

$$U_3^+(\lambda)=\sin(\xi^++\lambda-\eta)e^{-2i\eta}. \tag{170}$$

Case II:

$$U_1^+(\tilde{\lambda})=\sin(\tilde{\xi}^+-\tilde{\lambda})e^{2i\tilde{\lambda}},$$

$$U_2^+(\tilde{\lambda})=\frac{\sin(\tilde{\lambda}+\tilde{\xi}^+-\eta)\sin(2\tilde{\lambda})}{\sin(2\tilde{\lambda}-\eta)},$$

$$U_3^+(\lambda)=\sin(\xi^++\lambda-2\eta)e^{-i\eta}. \tag{171}$$

Case I:

$$U_1(\tilde{\lambda})=\frac{\sin(2\tilde{\lambda}+\eta)\sin(\tilde{\xi}+\tilde{\lambda})}{\sin(2\tilde{\lambda}-\eta)}e^{-i2(\tilde{\lambda}+\eta)},$$

$$U_2(\tilde{\lambda})=\frac{\sin(2\tilde{\lambda}+\eta)\sin(\tilde{\xi}+\tilde{\lambda})}{\sin(2\tilde{\lambda})}e^{-i(2\tilde{\lambda}+\eta)},$$

$$U_3(\lambda)=\sin(\xi-\lambda). \tag{172}$$

Case II:

$$U_1(\tilde{\lambda})=\frac{\sin(2\tilde{\lambda}+\eta)\sin(\tilde{\xi}+\tilde{\lambda}-\eta)}{\sin(2\tilde{\lambda}-\eta)}e^{-i(2\tilde{\lambda}+\eta)},$$

$$U_2(\tilde{\lambda})=\frac{\sin(2\tilde{\lambda}+\eta)\sin(\tilde{\xi}-\tilde{\lambda})}{\sin(2\tilde{\lambda})}e^{-i\eta},$$

$$U_3(\lambda)=\sin(\xi-\lambda). \tag{173}$$

As before U and U^+ are independent of each other, so there are four combinations for $\{U,U^+\}$ such as $\{I,I\}$, $\{I,II\}$, $\{II,I\}$, and $\{II,II\}$.

VI. RESULTS FOR FBF GRADING

The last possible grading is FBF, $\epsilon_1=\epsilon_3=1$, $\epsilon_2=0$. We can analyze it in the same way as the BFF grading. Here we just present the eigenvalue, the corresponding Bethe ansatz equation, and the boundary factors. The eigenvalue of the transfer matrix with reflecting boundary condition is

$$\Lambda(\lambda)=-U_3^+(\lambda)U_3(\lambda)\sin^{2N}(\lambda-\eta)\times\prod_{i=1}^{n}\frac{\sin(\lambda+\mu_i)\sin(\lambda-\mu_i+\eta)}{\sin(\lambda+\mu_i-\eta)\sin(\lambda-\mu_i)}+\sin^{2N}(\lambda)\times\prod_{i=1}^{n}\frac{1}{\sin(\lambda-\mu_i)\sin(\lambda+\mu_i-\eta)}\Lambda^{(1)}(\tilde{\lambda}),$$

$$\Lambda^{(1)}(\tilde{\lambda})=-U_1^+(\tilde{\lambda})U_1(\tilde{\lambda})\prod_{i=1}^{n}[\sin(\tilde{\lambda}+\tilde{\mu}_i)\sin(\tilde{\lambda}-\tilde{\mu}_i)]\times\prod_{l=1}^{m}\left\{\frac{\sin(\tilde{\lambda}-\tilde{\mu}_l^{(1)}-\eta)\sin(\tilde{\lambda}+\tilde{\mu}_l^{(1)})}{\sin(\tilde{\lambda}-\tilde{\mu}_l^{(1)})\sin(\tilde{\lambda}+\tilde{\mu}_l^{(1)}+\eta)}\right\}+U_2^+(\tilde{\lambda})U_2(\tilde{\lambda})\times\prod_{i=1}^{n}[\sin(\tilde{\lambda}+\tilde{\mu}_i+\eta)\sin(\tilde{\lambda}-\tilde{\mu}_i+\eta)]\times\prod_{l=1}^{m}\left\{\frac{\sin(\tilde{\lambda}-\tilde{\mu}_l^{(1)}-\eta)\sin(\tilde{\lambda}+\tilde{\mu}_l^{(1)})}{\sin(\tilde{\lambda}-\tilde{\mu}_l^{(1)})\sin(\tilde{\lambda}+\tilde{\mu}_l^{(1)}+\eta)}\right\}, \tag{174}$$

where $\tilde{\mu}_1^{(1)}, \ldots, \tilde{\mu}_m^{(1)}$ should satisfy the Bethe ansatz equations

$$\frac{U_1^+(\tilde{\mu}_j^{(1)})U_1(\tilde{\mu}_j^{(1)})}{U_2^+(\tilde{\mu}_j^{(1)})U_2(\tilde{\mu}_j^{(1)})}\times\prod_{i=1}^{n}\frac{\sin(\tilde{\mu}_j^{(1)}+\tilde{\mu}_i)\sin(\tilde{\mu}_j^{(1)}-\tilde{\mu}_i)}{\sin(\tilde{\mu}_j^{(1)}+\tilde{\mu}_i+\eta)\sin(\tilde{\mu}_j^{(1)}-\tilde{\mu}_i+\eta)}=1, \quad j=1,\ldots,m, \tag{175}$$

and $\tilde{\mu}_1, \ldots, \tilde{\mu}_n$ should satisfy

$$1=\frac{\sin^{2N}(\mu_j)}{\sin^{2N}(\mu_j-\eta)}\frac{U_2^+(\tilde{\mu}_j)U_2(\tilde{\mu}_j)}{U_3^+(\mu_j)U_3(\mu_j)}\times\prod_{l=1}^{m}\left\{\frac{\sin(\tilde{\mu}_j-\tilde{\mu}_l^{(1)}-\eta)\sin(\tilde{\mu}_j+\tilde{\mu}_l^{(1)})}{\sin(\tilde{\mu}_j-\tilde{\mu}_l^{(1)})\sin(\tilde{\mu}_j+\tilde{\mu}_l^{(1)}+\eta)}\right\}, \quad j=1,\ldots,n. \tag{176}$$

The boundary factors are as follows.

Case I:

$$U_1^+(\tilde{\lambda}) = \sin(\tilde{\xi}^+ - \tilde{\lambda}) e^{2i\tilde{\lambda}},$$

$$U_2^+(\tilde{\lambda}) = \frac{\sin(2\tilde{\lambda})\sin(\tilde{\xi}^+ - \tilde{\lambda})}{\sin(2\tilde{\lambda}+\eta)} e^{i(2\tilde{\lambda}+\eta)},$$

$$U_3^+(\lambda) = \sin(\xi^+ + \lambda - \eta). \tag{177}$$

Case II:

$$U_1^+(\tilde{\lambda}) = \sin(\tilde{\xi}^+ - \tilde{\lambda}) e^{2i\tilde{\lambda}},$$

$$U_2^+(\tilde{\lambda}) = \frac{\sin(\tilde{\lambda}+\tilde{\xi}^+ + \eta)\sin(2\tilde{\lambda})}{\sin(2\tilde{\lambda}+\eta)},$$

$$U_3^+(\lambda) = \sin(\xi^+ + \lambda) e^{-i\eta}. \tag{178}$$

Case I:

$$U_1(\tilde{\lambda}) = \sin(\tilde{\xi} + \tilde{\lambda}) e^{-i2(\tilde{\lambda}+\eta)},$$

$$U_2(\tilde{\lambda}) = \frac{\sin(2\tilde{\lambda}+\eta)\sin(\tilde{\xi}+\tilde{\lambda})}{\sin(2\tilde{\lambda})} e^{-i(2\tilde{\lambda}+\eta)},$$

$$U_3(\lambda) = \sin(\xi - \lambda). \tag{179}$$

Case II:

$$U_1(\tilde{\lambda}) = \sin(\tilde{\xi} + \tilde{\lambda} + \eta) e^{-i(2\tilde{\lambda}+\eta)},$$

$$U_2(\tilde{\lambda}) = \frac{\sin(2\tilde{\lambda}+\eta)\sin(\tilde{\xi}-\tilde{\lambda})}{\sin(2\tilde{\lambda})} e^{-i\eta},$$

$$U_3(\lambda) = \sin(\xi - \lambda). \tag{180}$$

VII. SUMMARY AND DISCUSSION

We have studied the generalized supersymmetric t-J model with boundaries in the framework of the graded quantum inverse scattering method. The trigonometric R matrix of the Perk-Shultz model is changed to the graded one. Solving the reflection equation and the dual reflection equation, we obtain two types of solutions each for three different backgrounds FFB, BFF, and FBF. The transfer matrix is constructed from the R matrix and the reflecting K matrix. The Hamiltonian is the the supersymmetric t-J model with boundary terms. Using the graded algebraic Bethe ansatz method, we obtain the eigenvalues of the transfer matrix for three possible gradings. The corresponding Bethe ansatz equations are obtained.

Comparing our results with the previous results in Ref. 23, we find that the form of Bethe ansatz equations for BFF case in Sec. V is similar to the results obtained in Ref. 23.

It is important to investigate the thermodynamic limit of the results obtained in this paper. There, we may find some physical quantities such as free energy, surface free energy, interfacial tension etc. It is also important to extend the supersymmetric t-J model to a more general supersymmetric case.

Recently, boundary impurity problems have attracted considerable interests.[46–51] Studying the boundary impurities by using three different grading is interesting and will be left for a future study.

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*Permanent address: Institute of Modern Physics, Northwest University, Xi'an 710069, P. R. China.

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