Conductance-phase determination in double-slit transmission across a quantum dot using a Hilbert transform method

R. Englman*
Department of Physics and Applied Mathematics, Soreq NRC, Yavne 81800, Israel and Research Institute, College of Judea and Samaria, Ariel 44284, Israel

A. Yahalom
Faculty of Engineering, Tel Aviv University, Ramat Aviv, Israel

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Recent mesoscopic two-arm experiments involving quantum dots, electron interferometry, and Aharonov-Bohm effects have enabled measurements of both electron transmission probabilities and phases. Unexpected features in the phases as function of the gap voltage U have stimulated several theoretical works. It is shown in this paper that the phases (f) and conductances ($|C|$), appearing both in the experimental and in theoretical studies, are interrelated through integral expressions, causing f and $\ln(|C|)$ to be Hilbert transforms. The empirically found interrelations imply certain analytical properties of the U dependence of wave functions in mesoscopic systems.

I. INTRODUCTION

A few years ago the conductivity and phase of transmitted electrons were jointly determined by employment of a double-slit arrangement in conjunction with a quantum dot.[1–3] The role of the latter, when inserted in one arm of the interferometer, was to induce a measurable phase shift in the component of the electron wave function representing passage along that arm. In the experiments the gate voltage U was varied (keeping other essential parameters fixed) and the transmission amplitude modulus or some related quantity (named ''conductance'' here for short, following Ref. 1, and denoted by $|C|$), and phase f were determined as the system passed through several resonances. Surprise was occasioned by more recent experiments in which the phase increased by π across the resonances (manifested by maxima in the transmission probability) but dropped sharply by the same amount at antiresonances.[4] Several theoretical treatments have been published to account for these results,[5–9] and further efforts of interpretation are in progress.[10,11] These works employ a variety of physical models and compare them to the conductance and phase curves.

In contrast the these works, the present paper points out a consistency relationship between the observed phases and transmission amplitudes, which, whenever it holds, is model independent and is grounded in the analyticity properties of the wave transmission. The relation makes $\ln|C|$ and f Hilbert transform,[12] explicitly shown in Eqs. (3) and (4) below as functions of the externally applied gate voltage parameter. Hints as to the existence of such relationship can indeed be found in several papers (and indeed, with the extension indicated below, the Breit-Wigner formula in Refs. 2, 14, and 15 is a particular instance of the applicability of the Hilbert transform), but neither the precise form (e.g., the relevance of the LOGARITHM of the conductance) nor the conditions for the validity of the theory have been stated. We shall state these conditions in a later section with the aim of channeling future theoretical attempts to building physical models such that the conditions are explicitly taken care of.

The present work is an outgrowth of previous publications in which reciprocal relations between the phase and moduli of a time-dependent wave function and of optical wave fields were obtained and applied.[13,16–19] These relations operated in the time domain. Reciprocal relations in the frequency domain, taking the form of Kramers-Kronig relations, have of course been widely known.[20] These are based on the causality principle, and (although mathematically similar) are logically unrelated to those in Refs. 13, 16, and 19. The present work extends the formalism to a consideration of the analytical properties of the wave function in a mesoscopic system a as function of an external parameter, the gate voltage.

II. COMPLEX CONDUCTANCE

We consider solutions of a Schrödinger equation or the corresponding propagator or Green function. The Schrödinger equation contains the gate voltage U, and therefore the solutions will be functions of U. The same will be true for a generic quantity $C(U)$ that derives from the solutions, like the conductance, the transmission amplitude, or ''the interference term.'' The various ''$|C|$-equivalent'' quantities are listed in Table I, with sources given. We now make the supposition (presently to be confirmed by the observed data) that the conductance $|C|$ is the modulus of a complex quantity

$$C = C(U), \tag{1}$$

which depends on the real variable U. To complete the definition of C we introduce the phase f, as

$$C(U) = |(U)| \exp[i f(U)]. \tag{2}$$

Under certain conditions, the following relations hold between the phase and the conductance:

$$-(1/\pi) P \int_{-\infty}^{\infty} dU' [\ln|C(U')|] \Big/ (U'-U) = \pm f(U) \tag{3}$$

and

TABLE I. Physical quantities represented in this work by $|C|$.

''Modulus'' quantity	Reference
Conductance	1, 7
Transmission amplitude	4, 8
Transmission coefficient	3, 9
Amplitude of Aharonov-Bohm oscillations	4, 6
Interference term amplitude	8

$$(1/\pi)P\int_{-\infty}^{\infty} dU'[f(U')] \Big/ (U'-U) = \pm \ln|C(U)|. \tag{4}$$

Here P denotes the principal value of the integral. $\ln|C(U)|$, and $\pm f(U)$ are ''Hilbert transforms.''[12] We now turn to the conditions for the validity of Eqs. (3) and (4).

Let us assume that C can be analytically continued to be a function of the complex ''gate voltage''

$$W = U + iV, \tag{5}$$

in the sense that if W is inserted into the Schrödinger equation, then the solutions reduce to the physical solutions when $V \to 0$. We thus define C as a complex function of the complex variable W,

$$C = C(W), \tag{6}$$

and suppose that (a) $\ln C(W)$ is analytic in the lower (or upper) half W plane, (b) $\ln C(W)$ tends to zero on a large semicircle in that half plane, and (c) $C(W)$ can have zeros or poles on the real line $V=0$ (Refs. 12, 16, and 19). The lower (or upper) signs are appropriate for functions analytic in the lower (or upper) halves.

Some extension of the formalism is possible. Thus, when the complex conductance behaves for large W as some power W^{-k}, where k may positive or negative, a standard procedure in complex variable theory enables the use of relations (3) and (4) for a modified $C(U)$. This is obtained by multiplying the complex conductance by $(U-i)^k$ [or by $(U+i)^k$], computing the integrands with the modified functions and compensating in the result by subtracting the algebraically evaluated quantities, e.g., $\arg(U-i)^k = -k\arctan(1/U)$. (This procedure was used in Ref. 19, and is analogous to that in Ref. 12, Sec. 11.17.) Of course, when the functional form of $C(W)$ is fully known, the procedure of Eqs. (3) and (4) is unnecessary, but we are mainly concerned with those other cases in which $|C(U)|$ is available only numerically, as data values, or as a complicated function for real U [while its analytical behavior is presumed to be as in (a)–(c)].

The location of ZEROS of $C(W)$ is an important feature in this work, and we discuss it now. In previous works[16–19], in which time was the independent variable, it was established that in several physically significant cases the analyticity conditions are met. Thus a proof was given for the proper analyticity of the ground states of an adiabatically evolving system, including the location of the zeros of the wave function.[19] (The same form of analyticity is also present in coherent and squeezed wave packet states.) Regarding the underlying causes for the location of the zeros, discussions of the zeros of time-dependent wave functions have involved suggestions to link these to the number of modes (p. 209 in Ref. 21) or to the environment-induced coupling between modes,[22] while complex-valued zeros of grand partition functions for finite systems herald phase transitions in the Yang-Lee theory of condensation.[23] In a recent papers, transmission zeros in a one-dimensional channel were shown to regulate phase jumps.[24] The zeros (which lie on the real energy axis) arise from phase interference between waves. Just as in the present study, the analyticity properties of the LOGARITHM of the transmission are linked to the location of the zeros in Ref. 24.

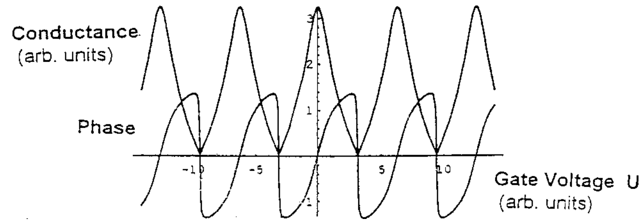

FIG. 1. Symmetric, periodic resonances. Conductances (or transmission probability amplitudes, etc) and phases are plotted against the gap voltage U (all in arbitrary units). The source of the plots in this and the following four figures are shown in Table II.

We further note that in Eq. (4) the phase has the freedom of choice of an additive constant, and in Eq. (3) the conductance that of a multiplicative constant, since the Hilbert transform of a constant is zero. This permits us to treat other quantities related to the conductance on the same footing, as long as they differ from it only through a multiplicative constant. Moreover, when the physical quantities are some powers of each other, then the corresponding derived phases are simply multiples of each other, so that if the unit of the phase is not specified; then the relation in Eq. (3) can be used for all of them. With this understanding, our results hold equally for conductance, Aharonov-Bohm oscillation amplitudes, transmission probabilities, and other quantities. As already stated, we refer to them generically as ''conductance'' (Table I).

It is clear from Eqs. (3) and (4) that [provided the stated conditions (a)–(c) hold] the phase is uniquely given from the conductance, and vice versa. Any physical model or theory needs to account of one type of quantity alone. In the following figures we present graphically several types of phases and conductance amplitudes (*not* the logarithms) as functions of the real gate voltage U, and relate them to published experimental and theoretical results. The quantities plotted by us satisfy Eqs. (3) and (4) and are Hilbert transforms in the U (or W) domain. The actual expressions on which the plots are based are listed in Table II. We can now state the following: *Any (observed) conductance that, as function of (real) U, is numerically similar to any of the* $|C(U)|$*'s in the list and has the same analytical behavior for (complex) W, will also yield a corresponding phase* $f(U)$ *that is numerically similar. Any conductance that is numerically similar, but is analytically dissimilar, can yield a phase that is completely dissimilar*. These properties are reflections of the fact that the conductance derives from equations that are defined for complex values of the gate voltage.

III. GRAPHICAL REPRESENTATIONS

The curves in Fig. 1 have the shapes of the experimental values of Ref. 4 shown in their Figs. 3(b) and 3(c) (or Fig. 2

TABLE II. Sources of the plots in Figs. 1–5. In each case $|C(U)|$ and arg $C(U)$ were plotted. $\ln|C(U)|$, and arg $C(U)(\equiv f)$ are Hilbert transforms.

Figure	$C(U)$
1	$(1+0.95e^{iU})/(1-0.4e^{iU})$
2	$1.01(1+0.95e^{iU})/[1-0.4(1+0.75\sin U)e^{iU}]$
3	$(1.7e^{iU})/(1-0.4e^{iU})$
4	$[(U-3)+i\sqrt{2}]^{-1}$
5	$[U-6+0.35i]^{-1}-[U-14+0.35i]^{-1}$

in Ref. 8) except that the experimental values are somewhat skewed and not quite periodic. The latter property (if not an instrumental effect) can have its origin in the differences between levels of the quantum dots. Since the effects appear to be small, we ignore them in this work.

The strong antiresonances near integral multiple values of π, where $C(U)\approx 0$, and the sudden ''phase lapses'' (Ref. 8) between resonances are evident. The curves in Fig. 2 differ from the previous set only by allowing skewness in the conductances, present in the observational curves of Ref. 4. The phase does not significantly differ from that in Fig. 1.

The conductance curves of Fig. 3 are still oscillatory, but they do not get close to the horizontal axis, i.e., $|C(U)| \gg 0$. Yet, the phases oscillate, contrary to what might have been anticipated. (Note, e.g., the caption to Fig. 2 in Ref. 9). The downward slopes of the phase are now moderate and, in fact, they scale with $[C(\pi)]^{-1}$ [with the zeros of $C(W)$ being simple, as in Table II].

The elementary curves in Fig. 4 resemble some experimental and theoretical curves [e.g., Figs. 4(a) and 4(c) in Ref. 1, Fig. 2(c) in Ref. 4 and Fig. 1 in Ref. 6], indicating that mathematical consistency relations hold between the observational quantities. The curves in Fig. 5 resemble the theoretical curves of Ref. 8 (Fig. 3). The source function shown in Table II has poles in one half-plane only, and has no zeros in the finite portion of the complex plane.

IV. HILBERT TRANSFORM FOR RAW DATA VALUES

In this section we derive the phase directly from the raw observational data of the magnitudes (''conductance'') by the Hilbert transform method (that is, without using an interpolating function). Specifically, we start with DISCRETE data values shown by dots in Fig. 3b of Ref. 4, and employ Eq. (3) on these. A slight problem arises, though, in that the range of integration in Eq. (3) is infinite, while the data points cover only a finite range of the gate voltage. A natural (but perhaps oversimplified) solution of this problem is to assume that the experimental data possess a periodicity (implying, as we have done before, that deviations from strict periodicity are negligible). We have taken as basis, the ''conductance'' (more accurately: ''the magnitude of the Aharonov-Bohm oscillations'') data values in Fig. 3b of Ref. 4 contained in the resonance peak just following the vertical dotted line (since these seem to be the least affected by experimental errors) and used these experimental points adjusted to periodicity. They are shown in Fig. 6(a) below by dots. We have then replaced the infinite integral in Eq. (3) (with the positive sign) by a sum over the experimental values inside the elementary resonance peak and by a further discrete sum over all equivalent, identical peaks. The values for the phases that are thus obtained from the integral in (3) are shown in Fig. 6(b) by stars.

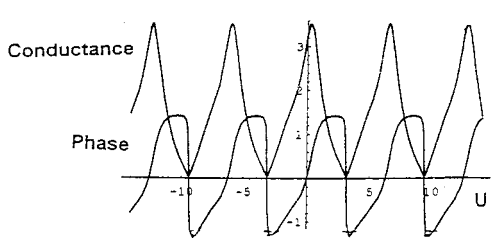

FIG. 2. Skewness effects. (Quantities and units as in Fig. 1.)

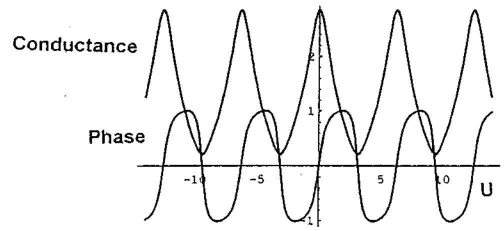

FIG. 3. Conductance not having nodes. (Quantities and units as in Fig. 1.)

These are in reasonably close agreement with the experimental values of the phase, also given by Schuster *et al.*[4] in their Fig. 3c, and shown by us in Fig. 6(b) by dots (again imposing a periodic recurrence of the peaks). The only adjustment that was made in the calculated phase (plotted in units of π) is a constant vertical shift. (We recall our previous remark, in Sec. II, about an arbitrariness of a constant shift in the phase.) The range of the computed phase exceeds the observed one by about 15%: this excess is presumably due to the inaccuracy involved in replacing the principal integral by a sum over the data points (comprising only 16 values). This is probably also the source of the discrepancy near odd-integral values of U. However, the calculated ''phase lapses'' are similar to the observed ones, and the horizontal displacements in the maxima between the conductance and the phase are $\frac{1}{4}$ of the fundamental period, as given by experiment.

Considering that the Hilbert transform method is based on continuous functions, it is gratifying to note its applicability to discrete, raw, numerical data.

V. DISCUSSION

In this work relations have been formulated between observed phases and transmission amplitudes, so that the phase

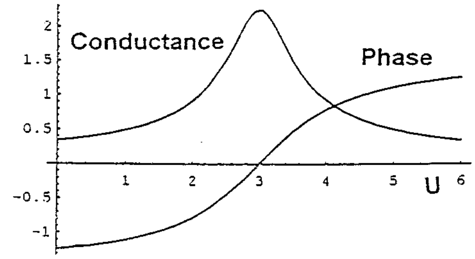

FIG. 4. Lorentzian conductance. (Quantities and units as in Fig. 1.)

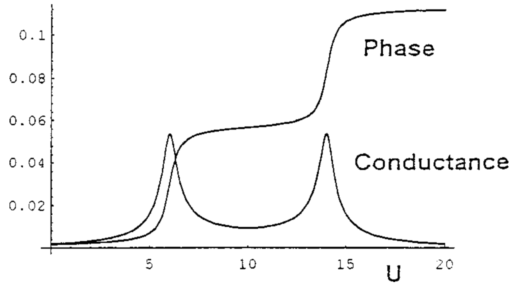

FIG. 5. Stepwise phases. (Quantities and units as in Fig. 1.)

and the log of the ''conductance'' are Hilbert transforms as function of the gap voltage. The relations are contingent upon certain analyticity conditions, rather than absolute. We have found that, to a good accuracy, available (low temperature) experimental and several theoretical curves obey the relationships. In particular, for the remarkable, sharp decreases in the experimental phases between resonances, plotted out in Fig. 6(b), we have found that the slope of the decrease depends on the value of the conductance minimum, positioned midway between the resonances. (Compare Fig. 1 and 3 and Table II.)

The relations do not replace physical models, but provide a check on them. Regarding models in previous works, the transmission amplitude modulus, shown in Ref. 3 as Eq. (1) and based on simplified one-dimensional models of Refs. 14 and 15, is derivable from a complex conductance

$$C(\varphi)\propto[1-r_1r_2\exp(i\varphi)]^{-1}. \tag{7}$$

Here r_1 and r_2 are reflectivities, and φ is the sum of a magnetic flux term and the phase acquired by the partial wave traveling along the ring's arm in the absence of a magnetic field.[15] The latter part is an essentially linear function of the gate voltage U [as shown in Fig. 2(c) in Ref. 3]. Since $|r_1r_2|<1$, this approximate expression has the postulated analytical behavior in U. Similarly, the transmission amplitude of Ref. 9 appearing in their Eq. (1), is essentially the difference of two terms of the form in Eq. (7), in which r_1r_2 take opposite signs: this again has the analyticity properties (a)–(c).

In spite of the agreements found in this paper for every studied case, it may well happen that in some other instances (coming from either future experiments or some new theoretical model) there be discrepancies from the integral relations (3) or (4). According to the theory in this work, these can be assigned to deviations from the analyticity conditions, for whose validity we have no *a priori* reason. [As an example, we might choose to extend $C(U)$ beyond the two Breit-Wigner terms in Table II for Fig. 5, by adding to it several further resonances. Then the new resonances will look like those in Fig. 5(a). But the dispersion relations (3) and (4) can only be applied if the proper analyticity is ensured, that is, if no new zeros have been introduced in the wrong half-plane. This illustrates the role of analyticity, as set out in the italicized sentences of Sec. II.] Let it be remarked, however, that a correction term is available for those cases that have extra zeros in the wrong half-plane.[13,25] The contribution from this term is of a fixed sign (and if the zeros are simple and are either very far from, or very close to the real U axis, their effect is limited).[13]

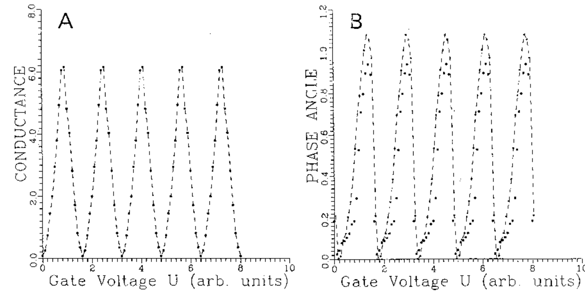

FIG. 6. Discrete application of Hilbert transform. (a) Observed oscillation magnitudes. The values shown by dots are from Ref. 4, here designated as ''Conductance,'' in arbitrary units. Broken lines connect dots, which are the input in the computation. (b) Phase angle (in units of π). The stars show the values for $f(U)$ that are the output from expression (3), after adding a uniform upward shift of approximately $\pi/2$. They are connected by broken lines. The dots show observed values taken from Ref. 4.

It is suggested that future experimental or theoretical work on electron transmission in mesoscopic systems should check for the existence of Hilbert transform relationships between phase and transmission probability. Condition (a) imposes extremely strict constraints on the possible forms of the complex conductance, precluding as it does the existence of branch cuts for $\ln C(W)$ in a half-plane. (These constraints are clearly more stringent than that whose satisfaction is guaranteed by Poincaré's theorem, quoted in Ref. 26.) Our present findings [which are consistent with condition (a)], raises the questions of whether there is any underlying physics in these mesoscopic devices that induces the validity of this analyticity condition, and whether future ones will still preserve it. In principle, the validity would be immediately verifiable from numerical integration of the data according to Eqs. (3) and (4). However, from a practical angle, since the conductance is available only for a finite range of the gate voltage, the infinite integration may not be feasible, and, indeed, the asymptotic behavior of the conductance, necessary to carry out the prescription in Sec. II may be hard to come by.

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*Electronic address: englman@vms.huji.ac.il

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