N-component Ginzburg-Landau Hamiltonian with cubic anisotropy: A six-loop study

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We consider the Ginzburg-Landau Hamiltonian with a cubic-symmetric quartic interaction and compute the renormalization-group functions to six-loop order in $d=3$. We analyze the stability of the fixed points using a Borel transformation and a conformal mapping that takes into account the singularities of the Borel transform. We find that the cubic fixed point is stable for $N>N_c$, $N_c=2.89(4)$. Therefore, the critical properties of cubic ferromagnets are not described by the Heisenberg isotropic Hamiltonian, but instead by the cubic model at the cubic fixed point. For $N=3$, the critical exponents at the cubic and symmetric fixed points differ very little (less than the precision of our results, which is $\lesssim 1\%$ in the case of γ and ν). Moreover, the irrelevant interaction bringing from the symmetric to the cubic fixed point gives rise to slowly decaying scaling corrections with exponent $\omega_2=0.010(4)$. For $N=2$, the isotropic fixed point is stable and the cubic interaction induces scaling corrections with exponent $\omega_2=0.103(8)$. These conclusions are confirmed by a similar analysis of the five-loop ϵ expansion. A constrained analysis, which takes into account that $N_c=2$ in two dimensions, gives $N_c=2.87(5)$.

I. INTRODUCTION

According to the universality hypothesis, critical phenomena can be described in terms of quantities that do not depend on the microscopic details of the system, but only on global properties such as the dimensionality and the symmetry of the order parameter, and the range of the interactions. There exist several physical systems that are characterized by short-range interactions and an N-component order parameter. Because of universality, their critical properties can be studied by using the Ginzburg-Landau ϕ^4 Hamiltonian and by employing standard field-theoretic renormalization-group techniques. When the order parameter has only one component, one obtains the Ising universality class that describes, for instance, the liquid-vapor transition in simple fluids and the transitions of multicomponent fluid systems; in this case the density plays the role of the order parameter. The two-component model (XY model) describes the helium superfluid transition, the Meissner transition in type-II superconductors and some transitions in liquid crystals, while the limit $N\rightarrow 0$ gives the infinite-length properties of dilute polymers in a good solvent.

The critical properties of many magnetic materials are also computed using the N-component Ginzburg-Landau Hamiltonian. Uniaxial (anti-)ferromagnets should be described by the Ising universality class ($N=1$), while magnets with easy-plane anisotropy should belong to the XY universality class. Ferromagnets with cubic symmetry are often described in terms of the $N=3$ Hamiltonian. However, this is correct if the nonrotationally invariant interactions that have only the reduced symmetry of the lattice are irrelevant in the renormalization-group sense. Standard considerations based on the canonical dimensions of the operators indicate that there are two terms that one may add to the Hamiltonian and that are cubic invariant: a cubic hopping term $\Sigma_{\mu=1,3}(\partial_\mu \phi_\mu)^2$ and a cubic interaction term $\Sigma_{\mu=1,3}\phi_\mu^4$. The first interaction was studied in Refs. 1–5. A two-loop $O(\epsilon^2)$ calculation indicates that it is irrelevant at the symmetric point, although it induces slowly decaying crossover effects. We will not consider it here — although it would be worthwhile to perform a more systematic study — since the second term already introduces significant changes in the critical behavior of the system. We will therefore consider a three-dimensional ϕ^4 theory with two quartic couplings:[6,5]

$$\mathcal{H}=\int d^d x\left\{\frac{1}{2}\sum_{i=1}^{N}\left[(\partial_\mu \phi_i)^2+r\phi_i^2\right]+\frac{1}{4!}\sum_{i,j=1}^{N}(u_0+v_0\delta_{ij})\phi_i^2\phi_j^2\right\}. \tag{1.1}$$

The added cubic term breaks explicitly the O(N) invariance of the model, leaving a residual discrete cubic symmetry given by the reflections and permutations of the field components.

The model described by the Hamiltonian (1.1) has been extensively studied. It has four fixed points:[6,5] the trivial Gaussian one, the Ising one in which the N components of the field decouple, the O(N)-symmetric and the cubic fixed points.

The Gaussian fixed point is always unstable, and so is the Ising fixed point.[7] Indeed, in the latter case, it is natural to interpret Eq. (1.1) as the Hamiltonian of N Ising-like systems coupled by the $O(N)$-symmetric term. But this interaction is the sum of the products of the energy operators of the different Ising systems. Therefore, at the Ising fixed point, the crossover exponent associated to the O(N)-symmetric quartic term should be given by the specific-heat critical expo-

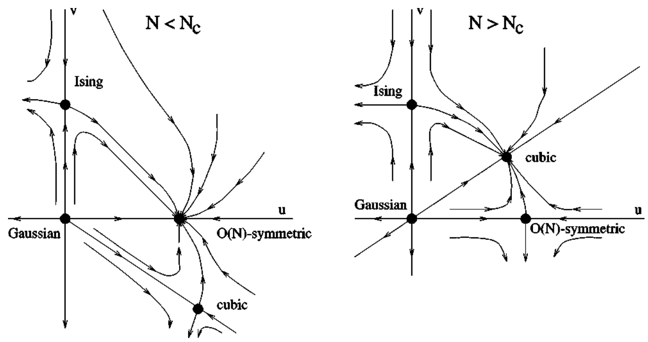

FIG. 1. Renormalization-group flow in the coupling plane (u,v) for $N<N_c$ and $N>N_c$.

nent α_I of the Ising model, independently of N. Since α_I is positive, indeed $\alpha_I=0.1099(7)$ (see, e.g., Ref. 8 and references therein), the Ising fixed point is unstable.

While the Gaussian and the Ising fixed points are unstable for any number of components N, the stability properties of the O(N)-symmetric and of the cubic fixed points depend on N. For sufficiently small values of N, $N<N_c$, the O(N)-symmetric fixed point is stable and the cubic one is unstable. For $N>N_c$, the opposite is true: the renormalization-group flow is driven towards the cubic fixed point, which now describes the generic critical behavior of the system. The O(N)-symmetric point corresponds to a tricritical transition. Figure 1 sketches the flow diagram in the two cases $N<N_c$ and $N>N_c$. At $N=N_c$, the two fixed points should coincide, and logarithmic corrections to the O(N)-symmetric critical exponents are expected. Outside the attraction domain of the fixed points, the flow goes away towards more negative values of u and/or v and finally reaches the region where the quartic interaction no longer satisfies the stability condition. These trajectories should be related to first-order phase transitions.[9,10]

If $N>N_c$, the cubic anisotropy is relevant and therefore the critical behavior of the system is not described by the Heisenberg isotropic Hamiltonian. If the cubic interaction favors the alignment of the spins along the diagonals of the cube, i.e., for a positive coupling v_0, the critical behavior is controlled by the cubic fixed point and the cubic symmetry is retained even at the critical point. On the other hand, if the system tends to magnetize along the cubic axes — this corresponds to a negative coupling v_0 — then the system undergoes a first-order phase transition.[11,5,12,13] Moreover, since the symmetry is discrete, there are no Goldstone excitations in the low-temperature phase. The longitudinal and the transverse susceptibilities are finite for $T<T_c$ and $H\to 0$, and diverge as $|t|^{-\gamma}$ for $t\propto T-T_c\to 0$.[14]

In the limit $N\to\infty$, keeping Nu and v fixed, one can derive exact expressions for the exponents at the cubic fixed point. Indeed, in this limit the model can be reinterpreted as a constrained Ising model,[15] leading to a Fisher renormalization of the Ising critical exponents.[16] One has[17,15,5]

$$\eta=\eta_I+O\left(\frac{1}{N}\right),\quad \nu=\frac{\nu_I}{1-\alpha_I}+O\left(\frac{1}{N}\right), \tag{1.2}$$

where η_I, ν_I, and α_I are the critical exponents of the Ising model (see, e.g., Ref. 8 and references therein for recent estimates of the Ising critical exponents).

If $N<N_c$, the cubic term in the Hamiltonian is irrelevant, and therefore, it generates only scaling corrections $|t|^{\Delta_c}$ with $\Delta_c>0$. However, their presence leads to important physical consequences. For instance, the transverse susceptibility at the coexistence curve (i.e., for $T<T_c$ and $H\to 0$), which is divergent in the O(N)-symmetric case, is now finite and diverges only at T_c as $|t|^{-\gamma-\Delta_c}$.[11,18,19,5,20] In other words, below T_c, the cubic term is a ''dangerous'' irrelevant operator. Note that for N sufficiently close to N_c, irrespective of which fixed point is the stable one, the irrelevant interaction bringing from the unstable to the stable fixed point gives rise to very slowly decaying corrections to the leading scaling behavior.

In three dimensions, a simple argument based on the symmetry of the two-component cubic model[21] shows that the cubic fixed point is unstable for $N=2$, so that $N_c>2$. Indeed, for $N=2$, a $\pi/4$ internal rotation, i.e.,

$$(\phi_1,\phi_2)\to\frac{1}{\sqrt{2}}(\phi_1+\phi_2,\phi_1-\phi_2), \tag{1.3}$$

maps the cubic Hamiltonian (1.1) into a new one of the same form but with new couplings (u_0',v_0') given by

$$u_0'=u_0+\tfrac{3}{2}v_0,\quad v_0'=-v_0. \tag{1.4}$$

This symmetry maps the Ising fixed point onto the cubic one. Therefore, the two fixed points describe the same theory and have the same stability. Since the Ising point is unstable, the cubic point is unstable too, so that the stable point is the isotropic one. In two dimensions, this is no longer true. Indeed, one expects the cubic interaction to be truly marginal for $N=2$ (Refs. 22,23) and relevant for $N>2$,[24] so that $N_c=2$ in two dimensions.

During the years, the model (1.1) has been the object of several studies.[6,18,25–27,3,4,28–30,23,31–40] In the 1970s several computations were done using the ϵ expansion;[6,18,26,27] they predicted $3<N_c<4$, indicating that cubic ferromagnets are described by the $O(N)$-invariant Heisenberg model. However, recent studies have questioned these conclusions. Field-theoretic studies, based on the analysis of the three-loop[31,32] and four-loop series[33,40] in fixed dimension, and of the five-loop expansion in powers of $\epsilon=4-d$ (Refs. 34–37,40) suggest that $N_c\lesssim 3$, although they do not seem to be conclusive in excluding the value $N_c=3$. On the other hand, the results of Ref. 38, obtained from Monte Carlo simulations using finite-size scaling techniques, are perfectly consistent with the value $N_c\approx 3$. The authors of Ref. 38 even suggest that $N_c=3$ exactly.

A further study of this issue is therefore of particular relevance for the ferromagnetic materials characterized by an order parameter with $N=3$. For this purpose we extended the perturbative expansions of the β functions and of the exponents to six loops in the framework of the fixed-dimension field-theoretic approach.[41] These perturbative expansions are only asymptotic. Nonetheless, accurate results can be obtained by employing resummation techniques that use their Borel summability[42] and the knowledge of the large-order behavior.[43,44] For this reason, we have also computed the singularity of the Borel transform that is closest to the origin, extending the calculations of Refs. 43,44.

TABLE I. Summary of the results in the literature. The values of the exponents refer to $N=3$. The subscripts "s" and "c" indicate that the exponent is related to the symmetric and to the cubic fixed point, respectively. H.T. expansion and approximate RG mean, respectively, high-temperature expansion and approximate renormalization-group equations.

	Method	Results
Ref. 26, 1974	ϵ expansion: $O(\epsilon^3)$	$N_c \simeq 3.128$
Ref. 28, 1977	approximate RG	$\nu_s\,\omega_{2,s}=-0.11$, $N_c \simeq 2.3$
Ref. 30, 1981	H.T. expansion: $O(\beta^{10})$	$\nu_s\,\omega_{2,s}=-0.63(10)$, $N_c<3$
Ref. 23, 1982	scaling-field	$N_c \simeq 3.38$
Ref. 33, 1989	$d=3$ expansion: $O(g^4)$	$\omega_{2,c}\simeq 0.008$, $N_c \simeq 2.91$
Ref. 34, 1995	ϵ expansion: $O(\epsilon^5)$	$N_c \simeq 2.958$
Ref. 36, 1997	ϵ expansion: $O(\epsilon^5)$	$\omega_{2,s}=-0.00214$, $\omega_{2,c}=0.00213$, $N_c<3$
Ref. 37, 1997	ϵ expansion: $O(\epsilon^5)$	$N_c \simeq 2.86$
Ref. 38, 1998	Monte Carlo	$\omega_{2,s}=0.0007(29)$, $N_c \approx 3$
Ref. 40, 1999	$d=3$ expansion: $O(g^4)$	$\omega_{2,s}=-0.0081$, $\omega_{2,c}=0.0077$, $N_c=2.89(2)$
This work	ϵ expansion: $O(\epsilon^5)$	$\omega_{2,s}=-0.003(4)$, $\omega_{2,c}=0.006(4)$, $N_c=2.87(5)$
This work	$d=3$ expansion: $O(g^6)$	$\omega_{2,s}=-0.013(6)$, $\omega_{2,c}=0.010(4)$, $N_c=2.89(4)$

The analysis of the perturbative series has been done following closely Ref. 45. We have estimated errors using an algorithmic procedure, trying to be as conservative as possible. This can be immediately realized by comparing our uncertainties with those previously quoted: even though our series are longer, the errors we report are sometimes larger than those of previous studies. Our results confirm previous field-theoretic studies: the $N=3$ isotropic fixed point is indeed unstable and we estimate $N_c=2.89(4)$ from the six-loop fixed-dimension expansion and $N_c=2.87(5)$ from the reanalysis of the five-loop ϵ expansion. For comparison, in Table I we report our estimates together with previous determinations of N_c and of the eigenvalues for $N=3$. It should be noted that the estimates of the critical exponents do not essentially depend on which fixed point is the stable one. Moreover, the tiny difference (smaller than the precision of our results, which is $\lesssim 1\%$ in the case of γ and ν) between the values at the two fixed points would be very difficult to observe, because of crossover effects decaying as t^{Δ} with $\Delta=\omega_{2,c}\nu_c=0.007(3)$. Large corrections to scaling appear also for $N=2$. Indeed, at the XY fixed point (the stable one), we find $\omega_2=0.103(8)$. Thus, even though the cubic interaction is irrelevant, it induces strong scaling corrections behaving as t^{Δ}, $\Delta=\omega_2\nu\approx 0.06$. Therefore, crossover effects are expected in this case, depending on the strength of the cubic term. Finally, we have checked the theoretical predictions for the model in the large-N limit finding good agreement.

We want to mention that, in the limit $N\to 0$, the cubic model (1.1) describes the Ising model with site-diluted disorder.[46–48] However, in this case, the perturbative expansion is not Borel summable.[49–51] Therefore, it is not completely clear how to obtain meaningful results from the perturbative series. An investigation of these problems will be presented elsewhere.

The paper is organized as follows. In Sec. II we present our calculation of the perturbative expansions to six loops in $d=3$. We give the basic definitions, the six-loop series, and the singularity of the Borel transform. In Sec. III we present the analysis of these expansions: we determine the stability of the fixed points and compute the exponents for several values of N. In Sec. IV we present a reanalysis of the ϵ-expansion five-loop series. The new analysis differs from the previous ones in the fact that it uses the large-order behavior of the series at the cubic fixed point. Finally, in the Appendix we report a three-loop ϵ-expansion computation of the zero-momentum four-point couplings at the cubic fixed point in three dimensions.

II. THE FIXED-DIMENSION PERTURBATIVE EXPANSION IN THREE DIMENSIONS

A. Renormalization of the ϕ^4 theory with cubic anisotropy

The fixed-dimension ϕ^4 field-theoretic approach[41] provides an accurate description of the critical properties of $O(N)$-symmetric models in the high-temperature phase (see, e.g., Ref. 52). The method can also be applied to the two-parameter cubic model.[31] The idea is to perform an expansion in powers of appropriately defined zero-momentum quartic couplings. In order to obtain estimates of the universal critical quantities, the perturbative series are resummed exploiting their Borel summability, and then evaluated at the fixed-point values of the couplings.

The theory is renormalized by introducing a set of zero-momentum conditions for the (one-particle irreducible) two-point and four-point correlation functions:

$$\Gamma^{(2)}_{ab}(p)=\delta_{ab}Z_\phi^{-1}[m^2+p^2+O(p^4)], \tag{2.1}$$

$$\Gamma^{(4)}_{abcd}(0)=Z_\phi^{-2}m\left[\frac{u}{3}(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}) + v\;\delta_{ab}\delta_{ac}\delta_{ad}\right]. \tag{2.2}$$

They relate the second-moment mass m, and the zero-momentum quartic couplings u and v to the corresponding Hamiltonian parameters r, u_0, and v_0:

$$u_0=muZ_uZ_\phi^{-2}, \quad v_0=mvZ_vZ_\phi^{-2}. \tag{2.3}$$

In addition, one introduces the function Z_t that is defined by the relation

$$\Gamma^{(1,2)}_{ab}(0)=\delta_{ab}Z_t^{-1}, \tag{2.4}$$

where $\Gamma^{(1,2)}$ is the (one-particle irreducible) two-point function with an insertion of $\frac{1}{2}\phi^2$.

From the pertubative expansion of the correlation functions $\Gamma^{(2)}$, $\Gamma^{(4)}$, and $\Gamma^{(1,2)}$ and the above relations, one derives the functions $Z_\phi(u,v)$, $Z_u(u,v)$, $Z_v(u,v)$, $Z_t(u,v)$ as a double expansion in u and v.

The fixed points of the theory are given by the common zeros of the β-functions

$$\beta_u(u,v)=m\left.\frac{\partial u}{\partial m}\right|_{u_0,v_0},$$

$$\beta_v(u,v)=m\left.\frac{\partial v}{\partial m}\right|_{u_0,v_0}. \tag{2.5}$$

The stability properties of the fixed points are controlled by the eigenvalues ω_i of the matrix

$$\Omega=\begin{pmatrix}\frac{\partial\beta_u(u,v)}{\partial u} & \frac{\partial\beta_u(u,v)}{\partial v}\\ \frac{\partial\beta_v(u,v)}{\partial u} & \frac{\partial\beta_v(u,v)}{\partial v}\end{pmatrix}, \tag{2.6}$$

computed at the given fixed point: a fixed point is stable if both eigenvalues are positive. The eigenvalues ω_i are related to the leading scaling corrections, which vanish as $\xi^{-\omega_i}\sim|t|^{\Delta_i}$ where $\Delta_i=\nu\omega_i$.

One also introduces the functions

$$\eta_\phi(u,v)=\left.\frac{\partial\ln Z_\phi}{\partial\ln m}\right|_{u_0,v_0}=\beta_u\frac{\partial\ln Z_\phi}{\partial u}+\beta_v\frac{\partial\ln Z_\phi}{\partial v}, \tag{2.7}$$

$$\eta_t(u,v)=\left.\frac{\partial\ln Z_t}{\partial\ln m}\right|_{u_0,v_0}=\beta_u\frac{\partial\ln Z_t}{\partial u}+\beta_v\frac{\partial\ln Z_t}{\partial v}. \tag{2.8}$$

Finally, the critical exponents are obtained from

$$\eta=\eta_\phi(u^*,v^*), \tag{2.9}$$

$$\nu=[2-\eta_\phi(u^*,v^*)+\eta_t(u^*,v^*)]^{-1}, \tag{2.10}$$

$$\gamma=\nu(2-\eta). \tag{2.11}$$

B. The six-loop perturbative series

We have computed the perturbative expansion of the correlation functions (2.1), (2.2), and (2.4) to six loops. The diagrams contributing to the two-point and four-point functions to six-loop order are reported in Ref. 53: they are approximately 1000, and it is therefore necessary to handle them with a symbolic manipulation program. For this purpose, we wrote a package in MATHEMATICA.[54] It generates the diagrams using the algorithm described in Ref. 55, and computes the symmetry and group factors of each of them. We did not calculate the integrals associated to each diagram, but we used the numerical results compiled in Ref. 53. Summing all contributions we determined the renormalization constants and all renormalization-group functions.

We report our results in terms of the rescaled couplings[56]

$$u\equiv\frac{16\pi}{3}R_N\bar{u},\quad v\equiv\frac{16\pi}{3}\bar{v}, \tag{2.12}$$

where $R_N=9/(8+N)$, so that the β-functions associated to $\bar{u}$ and $\bar{v}$ have the form $\beta_{\bar{u}}(\bar{u},0)=-\bar{u}+\bar{u}^2+O(\bar{u}^3)$ and $\beta_{\bar{v}}(0,\bar{v})=-\bar{v}+\bar{v}^2+O(\bar{v}^3)$.

The resulting series are

$$\beta_{\bar{u}}=-\bar{u}+\bar{u}^2+\frac{2}{3}\bar{u}\bar{v}-\frac{4(190+41N)}{27(8+N)^2}\bar{u}^3-\frac{400}{81(8+N)}\bar{u}^2\bar{v}-\frac{92}{729}\bar{u}\bar{v}^2+\bar{u}\sum_{i+j\geq3}b^{(u)}_{ij}\bar{u}^i\bar{v}^j, \tag{2.13}$$

$$\beta_{\bar{v}}=-\bar{v}+\bar{v}^2+\frac{12}{8+N}\bar{u}\bar{v}-\frac{308}{729}\bar{v}^3-\frac{832}{81(8+N)}\bar{u}\bar{v}^2-\frac{4(370+23N)}{27(8+N)^2}\bar{u}^2\bar{v}+\bar{v}\sum_{i+j\geq3}b^{(v)}_{ij}\bar{u}^i\bar{v}^j, \tag{2.14}$$

$$\eta_\phi=\frac{8(2+N)}{27(8+N)^2}\bar{u}^2+\frac{16}{81(8+N)}\bar{u}\bar{v}+\frac{8}{729}\bar{v}^2+\sum_{i+j\geq3}e^{(\phi)}_{ij}\bar{u}^i\bar{v}^j, \tag{2.15}$$

$$\eta_t=-\frac{(2+N)}{(8+N)}\bar{u}-\frac{1}{3}\bar{v}+\frac{2(2+N)}{(8+N)^2}\bar{u}^2+\frac{4}{3(8+N)}\bar{u}\bar{v}+\frac{2}{27}\bar{v}^2+\sum_{i+j\geq3}e^{(t)}_{ij}\bar{u}^i\bar{v}^j. \tag{2.16}$$

For $3\leq i+j\leq6$, the coefficients $b^{(u)}_{ij}$, $b^{(v)}_{ij}$, $e^{(\phi)}_{ij}$, and $e^{(t)}_{ij}$ are reported in Tables II, III, IV, and V, respectively.

We have performed several checks of our calculations

(i) $\beta_{\bar{u}}(\bar{u},0)$, $\eta_\phi(\bar{u},0)$, and $\eta_t(\bar{u},0)$ reproduce the corresponding functions of the O(N)-symmetric model.[56,57]

(ii) $\beta_{\bar{v}}(0,\bar{v})$, $\eta_\phi(0,\bar{v})$, and $\eta_t(0,\bar{v})$ reproduce the corresponding functions of the Ising-like ($N=1$) ϕ^4 theory.

(iii) The following relations hold for $N=1$:

$$\begin{aligned}\beta_{\bar{u}}(u,x-u)+\beta_{\bar{v}}(u,x-u)&=\beta_{\bar{v}}(0,x),\\ \eta_\phi(u,x-u)&=\eta_\phi(0,x),\\ \eta_t(u,x-u)&=\eta_t(0,x).\end{aligned} \tag{2.17}$$

(iv) For $N=2$, using the symmetry (1.3) and (1.4), and taking into account the rescalings (2.12), one can easily obtain the identities

$$\begin{aligned}\beta_{\bar{u}}(\bar{u}+\tfrac{5}{3}\bar{v},-\bar{v})+\frac{5}{3}\beta_{\bar{v}}(\bar{u}+\tfrac{5}{3}\bar{v},-\bar{v})&=\beta_{\bar{u}}(\bar{u},\bar{v}),\\ \beta_{\bar{v}}(\bar{u}+\tfrac{5}{3}\bar{v},-\bar{v})&=-\beta_{\bar{v}}(\bar{u},\bar{v}),\\ \eta_\phi(\bar{u}+\tfrac{5}{3}\bar{v},-\bar{v})&=\eta_\phi(\bar{u},\bar{v}),\\ \eta_t(\bar{u}+\tfrac{5}{3}\bar{v},-\bar{v})&=\eta_t(\bar{u},\bar{v}).\end{aligned} \tag{2.18}$$

TABLE II. The coefficients $b_{ij}^{(u)}$, cf. Eq. (2.13).

i,j	$R_N^{-i} b_{ij}^{(u)}$
3,0	$0.27385517+0.075364029\,N+0.0018504016\,N^2$
2,1	$0.67742325+0.027353409\,N$
1,2	$0.4154565+0.0025592148\,N$
0,3	0.090448951
4,0	$-0.27925724-0.091833749\,N-0.0054595646\,N^2$ $+0.000023722893\,N^3$
3,1	$-0.94383662-0.083252807\,N+0.00061860174\,N^2$
2,2	$-0.96497888-0.012460145\,N$
1,3	$-0.42331874-0.0017709429\,N$
0,4	-0.075446692
5,0	$0.35174477+0.13242502\,N+0.011322026\,N^2$ $+0.000054833719\,N^3+8.6768933\times10^{-7}\,N^4$
4,1	$1.5209008+0.19450536\,N+0.0011078614\,N^2$ $+0.000031779782\,N^3$
3,2	$2.2073347+0.065336326\,N+0.0003564925\,N^2$
2,3	$1.5315693+0.010676901\,N$
1,4	$0.56035196+0.0013469481\,N$
0,5	0.087493302
6,0	$-0.51049889-0.21485252\,N-0.023839375\,N^2$ $-0.00050021682\,N^3+2.0167763\times10^{-6}\,N^4$ $+4.4076733\times10^{-8}\,N^5$
5,1	$-2.6984083-0.45068252\,N-0.010821468\,N^2$ $+0.00005796668\,N^3+2.0515456\times10^{-6}\,N^4$
4,2	$-5.1135549-0.26769177\,N-0.0006311751\,N^2$ $+0.000019413374\,N^3$
3,3	$-4.9317312-0.067574712\,N+0.000028278087\,N^2$
2,4	$-2.754683-0.0095836704\,N$
1,5	$-0.86229463-0.001856332\,N$
0,6	-0.1179508

TABLE III. The coefficients $b_{ij}^{(v)}$, cf. Eq. (2.14).

i,j	$R_N^{-i} b_{ij}^{(v)}$
3,0	$0.64380517+0.05741276\,N-0.0017161966\,N^2$
2,1	$1.6853305+0.0030714114\,N$
1,2	1.3138294
0,3	0.3510696
4,0	$-0.76706177-0.089054667\,N$ $+0.000040711369\,N^2-0.000087586118\,N^3$
3,1	$-2.7385841-0.049218875\,N-0.00002623469\,N^2$
2,2	$-3.3477204+0.0075418394\,N$
1,3	-1.8071874
0,4	-0.37652683
5,0	$1.0965348+0.15791293\,N+0.0023584631\,N^2$ $-0.000061471346\,N^3-5.3871247\times10^{-6}\,N^4$
4,1	$4.9865485+0.17572792\,N-0.0020718369\,N^2$ $-0.000019382912\,N^3$
3,2	$8.3645284+0.0039620562\,N+0.00021363122\,N^2$
2,3	$6.8946012-0.0230874\,N$
1,4	2.8857918
0,5	0.49554751
6,0	$-1.7745533-0.30404316\,N-0.0094338079\,N^2$ $+0.000066993864\,N^3-6.5724895\times10^{-6}\,N^4$ $-3.753114\times10^{-7}\,N^5$
5,1	$-9.8298296-0.53384955\,N+0.0022033252\,N^2$ $-0.00013066822\,N^3-2.5959429\times10^{-6}\,N^4$
4,2	$-21.073538-0.16628697\,N-0.000014827682\,N^2$ $+4.4988524\times10^{-6}\,N^3$
3,3	$-23.569724+0.095716867\,N-0.00083903999\,N^2$
2,4	$-14.927998+0.0486813\,N$
1,5	-5.1298717
0,6	-0.74968893

These relations are exactly satisfied by our six-loop series. Note that, since the Ising fixed point is $(0,g_I^*)$ with $g_I^*=1.402(2)$ (Ref. 8), the above symmetry gives us the location of the cubic fixed point $(\frac{5}{3}g_I^*,-g_I^*)$.

(v) In the large-N limit the critical exponents of the cubic fixed point are related to those of the Ising model: $\eta=\eta_I$ and $\nu=\nu_I/(1-\alpha_I)$. One can easily see that for $N\to\infty$, $\eta_\phi(u,v)=\eta_I(v)$, where $\eta_I(v)$ is the perturbative series that determines the exponent η of the Ising model. Therefore, the first relation is trivially true. On the other hand, the second relation $\nu=\nu_I/(1-\alpha_I)$ is not identically satisfied by the series, and is verified only at the critical point.[58]

(vi) We finally note that our series are in agreement with the five-loop results that appeared recently in Ref. 59.

C. Singularity of the Borel transform

Since field-theoretic perturbative expansions are asymptotic, the resummation of the series is essential to obtain accurate estimates of the physical quantities. In three-dimensional ϕ^4 theories one exploits their Borel summability[42] and the knowledge of the large-order behavior of the expansion (see, e.g., Ref. 52).

In the case of the $O(N)$-symmetric ϕ^4 theory, the expansion is performed in powers of the zero-momentum four-point coupling g. The large-order behavior of the series $S(g)=\Sigma s_k g^k$ of any quantity is related to the singularity g_b of the Borel transform that is closest to the origin. Indeed, for large k,

$$s_k \sim k!(-a)^k k^b[1+O(k^{-1})] \quad \text{with} \quad a=-1/g_b. \tag{2.19}$$

The value of g_b depends only on the Hamiltonian, while the exponent b depends on which Green's function is considered. If the perturbative expansion is Borel summable, then g_b is negative. The value of g_b can be obtained from a steepest-descent calculation in which the relevant saddle point is a finite-energy solution (instanton) of the classical field equations with negative coupling.[43,44] If the Borel transform is singular for $g=g_b$, its expansion in powers of g converges only for $|g|<|g_b|$. An analytic extension can be obtained by a conformal mapping,[45] such as

$$y(g)=\frac{\sqrt{1-g/g_b}-1}{\sqrt{1-g/g_b}+1}. \tag{2.20}$$

TABLE IV. The coefficients $e_{ij}^{(\phi)}$, cf. Eq. (2.15).

i,j	$R_N^{-i} e_{ij}^{(\phi)}$
3,0	$0.00054176134 + 0.00033860084\,N + 0.000033860084\,N^2$
2,1	$0.002437926 + 0.00030474076\,N$
1,2	0.0027426668
0,3	0.00091422227
4,0	$0.00099254838 + 0.00070251807\,N + 0.0001018116\,N^2 - 6.5516886\times 10^{-7}\,N^3$
3,1	$0.0059552903 + 0.0012374633\,N - 7.8620264\times 10^{-6}\,N^2$
2,2	$0.01046567 + 0.00031166693\,N$
1,3	0.0071848915
0,4	0.0017962229
5,0	$-0.00036659735 - 0.0002572117\,N - 0.000032026611\,N^2 + 2.2430702\times 10^{-6}\,N^3 - 1.1094045\times 10^{-7}\,N^4$
4,1	$-0.0027494801 - 0.00055434769\,N + 0.000036974267\,N^2 - 1.6641067\times 10^{-6}\,N^3$
3,2	$-0.0064696069 - 0.000070210701\,N + 2.7823882\times 10^{-6}\,N^2$
2,3	$-0.0066046286 + 0.00006759333\,N$
1,4	-0.0032685176
0,5	-0.00065370353
6,0	$0.00069568037 + 0.00056585941\,N + 0.00012057302\,N^2 + 5.7466979\times 10^{-6}\,N^3 - 3.8385183\times 10^{-8}\,N^4 - 1.0441273\times 10^{-8}\,N^5$
5,1	$0.0062611234 + 0.001962173\,N + 0.00010407066\,N^2 - 3.1504745\times 10^{-7}\,N^3 - 1.8794292\times 10^{-7}\,N^4$
4,2	$0.018957129 + 0.0018483045\,N + 0.000012482494\,N^2 - 7.5537784\times 10^{-7}\,N^3$
3,3	$0.027158805 + 0.00059570458\,N + 1.7043408\times 10^{-6}\,N^2$
2,4	$0.020775681 + 0.000041479576\,N$
1,5	0.0083268641
0,6	0.0013878107

TABLE V. The coefficients $e_{ij}^{(t)}$, cf. Eq. (2.16).

i,j	$R_N^{-i} e_{ij}^{(t)}$
3,0	$-0.025120499 - 0.016979919\,N - 0.0022098349\,N^2$
2,1	$-0.11304225 - 0.019888514\,N$
1,2	$-0.13037154 - 0.0025592148\,N$
0,3	-0.044310253
4,0	$0.021460047 + 0.015690833\,N + 0.0024059273\,N^2 - 0.000037238563\,N^3$
3,1	$0.12876028 + 0.029764853\,N - 0.00044686275\,N^2$
2,2	$0.22779178 + 0.0093256378\,N$
1,3	$0.15630733 + 0.0017709429\,N$
0,4	0.039519569
5,0	$-0.022694287 - 0.017985168\,N - 0.0035835384\,N^2 - 0.00013566164\,N^3 - 1.699309\times 10^{-6}\,N^4$
4,1	$-0.17020715 - 0.049785186\,N - 0.0019839454\,N^2 - 0.000025489635\,N^3$
3,2	$-0.40917573 - 0.034538379\,N - 0.00028943185\,N^2$
2,3	$-0.43464785 - 0.0093557007\,N$
1,4	$-0.22065482 - 0.0013469481\,N$
0,5	-0.044400355
6,0	$0.029450619 + 0.024874579\,N + 0.005728397\,N^2 + 0.00031557863\,N^3 - 5.858689\times 10^{-6}\,N^4 - 1.0373506\times 10^{-7}\,N^5$
5,1	$0.26505557 + 0.091343427\,N + 0.0058838593\,N^2 - 0.00010172194\,N^3 - 1.8672312\times 10^{-6}\,N^4$
4,2	$0.80694662 + 0.09798244\,N + 0.00053192615\,N^2 - 0.000012819748\,N^3$
3,3	$1.1638111 + 0.043471034\,N - 0.000017867101\,N^2$
2,4	$0.89481393 + 0.010634231\,N$
1,5	$0.36032293 + 0.001856332\,N$
0,6	0.060363211

In this way the Borel transform becomes a series in powers of $y(g)$ that converges for all positive values of g provided that all singularities of the Borel transform are on the real negative axis.[45] For the $O(N)$-symmetric theory accurate estimates (see, e.g., Ref. 60) have been obtained resumming the available series: the β function[56] is known up to six loops, while the functions η_ϕ and η_t are known to seven loops.[61] A subtle point in this method is the estimate of the uncertainty of the results. Indeed the nonanalyticity of the Callan-Symanzik β function at the fixed-point value g^* (Refs. 41,62–64) may cause a slow convergence of the estimates to the correct fixed-point value. This may lead to an underestimate of the uncertainty that is usually derived from stability criteria. The reason is that this resummation method approximates the β function in the interval $[0,g^*]$ with a sum of analytic functions. Since, for $g=g^*$, the β function is not analytic, the convergence at the endpoint of the interval is slow. However, the comparison of these results with those obtained in other approaches shows that the above nonanalyticity causes only very small effects, which are negligible in most cases. See Refs. 64,8 for a discussion of this issue.

In order to apply a resummation technique similar to that used in Ref. 45 to our six-loop series, i.e., in order to use a Borel transformation and a conformal mapping to get a convergent sequence of approximations, we extended the large-order analysis to the cubic model. In particular we considered the double expansion in $\bar{u}$ and $\bar{v}$ at fixed $z\equiv\bar{v}/\bar{u}$. Then, we studied the large-order behavior of the resulting expansion in powers of $\bar{u}$. This was done following the standard approach described, for example, in Refs. 44,52, i.e., by studying the saddle-point solutions of the cubic model. We will report the calculation elsewhere;[65] here we give only the results.

For $z\equiv\bar{v}/\bar{u}$ fixed, the singularity of the Borel transform closest to the origin, $\bar{u}_b$, is given by

$$\frac{1}{\bar{u}_b} = -a(R_N+z) \qquad \text{for } z>0,$$
$$\frac{1}{\bar{u}_b} = -a\left(R_N+\frac{1}{N}z\right) \qquad \text{for } \frac{-2NR_N}{N+1}<z<0, \tag{2.21}$$

where

$$a = 0.147\,774\,22\ldots, \quad R_N = \frac{9}{N+8}. \tag{2.22}$$

Note that the series in powers of $\overline{u}$ keeping z fixed is not Borel summable for $\overline{u}>0$ and $z<-R_N$. This fact will not be a real limitation for us, since we will only consider values of z such that $\overline{u}_b<0$.

It should be noted that these results do not apply to the case $N=0$. Indeed, in this case, there are additional singularities in the Borel transform.[49–51]

The exponent b in Eq. (2.19) is related to the number of symmetries broken by the classical solution.[44] It depends on the quantity considered. In the cubic model, for $v\neq 0$, we have $b=5/2$ for the function η_ϕ, and $b=7/2$ for the β functions and η_t. For $v=0$, we recover the results of the $O(N)$-symmetric model, that is $b=2+N/2$ for η_ϕ, and $b=3+N/2$ for the β function and η_t.[45]

III. ANALYSIS OF THE FIXED-DIMENSION SIX-LOOP SERIES

In this section we present the analysis of the six-loop series in fixed dimension. The resummation of the series has been done using the method of Ref. 45 and the expression for the Borel singularity we have given in the previous section. Explicitly, given a series

$$R(\overline{u},\overline{v}) = \sum_{k=0} \sum_{h=0} R_{hk}\overline{u}^h\overline{v}^k, \tag{3.1}$$

we have generated a new quantity $E(R)_p(\alpha,b;\overline{u},\overline{v})$ according to

$$E(R)_p(\alpha,b;\overline{u},\overline{v}) = \sum_{k=0}^{p} B_k(\alpha,b;\overline{v}/\overline{u}) \times \int_0^\infty dt\, t^b e^{-t} \frac{y(\overline{u}t;\overline{v}/\overline{u})^k}{[1-y(\overline{u}t;\overline{v}/\overline{u})]^\alpha}, \tag{3.2}$$

where

$$y(x;z) = \frac{\sqrt{1-x/\overline{u}_b(z)}-1}{\sqrt{1-x/\overline{u}_b(z)}+1}, \tag{3.3}$$

and $\overline{u}_b(z)$ is defined in Eq. (2.21). The coefficients $B_k(\alpha,b;\overline{v}/\overline{u})$ are determined by the requirement that the expansion of $E(R)_p(\alpha,b;\overline{u},\overline{v})$ in powers of $\overline{u}$ and $\overline{v}$ gives $R(\overline{u},\overline{v})$ to order p. For each value of α, b, and p, $E(R)_p(\alpha,b;\overline{u},\overline{v})$ provides an estimate of $R(\overline{u},\overline{v})$.

First of all, we have analyzed the stability properties of the $O(N)$-symmetric fixed point. Since $\partial\beta_{\overline{v}}/\partial\overline{u}(\overline{u},0)=0$, the eigenvalues are simply

$$\omega_1 = \frac{\partial\beta_{\overline{u}}}{\partial\overline{u}}(\overline{u}^*,0), \quad \omega_2 = \frac{\partial\beta_{\overline{v}}}{\partial\overline{v}}(\overline{u}^*,0), \tag{3.4}$$

where $\overline{u}^*$ is the fixed-point value of $\overline{u}$. The exponent ω_1 is the usual exponent that is considered in the $O(N)$-symmetric theory, while ω_2 is the eigenvalue that determines the stability of the fixed point.

In order to compute ω_2, for many choices of the four parameters α_1, b_1, α_2, and b_2, we have determined an estimate of $\overline{u}^*$ and ω_2 from the equations

$$E(\beta_{\overline{u}}/\overline{u})_p(\alpha_1,b_1;\overline{u}_p^*(\alpha_1,b_1),0)=0, \tag{3.5}$$

$$\hat{\omega}_2(\alpha_1,b_1,\alpha_2,b_2;p) = E(\partial\beta_{\overline{v}}/\partial\overline{v})_p(\alpha_2,b_2;\overline{u}_p^*(\alpha_1,b_1),0). \tag{3.6}$$

Note that $\overline{u}_p^*(\alpha_1,b_1)$ is determined implicitly by Eq. (3.5), which has been solved numerically for each value of α_1 and b_1.

Then, we have considered sets of approximants such that $\alpha_{1,2}\in[\overline{\alpha}-\Delta\alpha,\overline{\alpha}+\Delta\alpha]$ and $b_{1,2}\in[\overline{b}-\Delta b,\overline{b}+\Delta b]$. The final estimate was obtained averaging over all integer values of α_1, α_2, b_1, and b_2 belonging to these intervals. The results for $N=3$ and several choices of the parameters are reported in Table VI. In order to obtain a final estimate, we should devise a procedure to determine ''optimal'' values for the parameters $\overline{\alpha}$, $\overline{b}$, $\Delta\alpha$, and Δb. Reasonable values of $\overline{\alpha}$ and $\overline{b}$ can be obtained by requiring that the estimates are approximately independent of the number of terms one is considering. It is less clear how to determine the width of the intervals $\Delta\alpha$ and Δb. Indeed, the results are stable, within the quoted errors, for many different choices of these two parameters, while the standard deviation of the estimate, which we use as an indication of the error, strongly depends on the choice one is making. In order to have reasonable error estimates, we have compared our results for $\overline{u}^*$ with the estimates obtained from the analysis of the same series by different authors. For $N=3$, Guida and Zinn-Justin quote 1.390(4).[60] We have therefore chosen our parameters so that we reproduce their errors. More precisely, we choose $\Delta\alpha=2$ and $\Delta b=3$, and quote the error in the final results as two standard deviations. With this choice, we obtain $\overline{u}^*=1.393(4)$, which agrees with the previous estimate and has the same error. Results for other values of N are reported in Table VII. Again, one can verify the good agreement of our estimates of $\overline{u}^*$ with the results of Ref. 60.

In Table VII we also report our results for ω_2. The $O(2)$-symmetric point is stable, since $\omega_2=0.103(8)>0$. On the other hand, the symmetric point is clearly unstable for $N\geqslant 4$. For $N=3$, the analysis gives $\omega_2=-0.013(6)<0$, so that the fixed point is unstable and $N_c<3$. To better understand the reliability of the results, in Fig. 2 we show the distribution of the estimates of ω_2 when we vary α_1, b_1, α_2, and b_2 in the chosen interval. It is evident that the quoted error on ω_2 is quite conservative and that the results become increasingly stable as the number of loops increases.

We have then determined N_c, defined as the value of N for which $\omega_2=0$ at the $O(N)$-symmetric fixed point.[66] The computation was done as before. For each value of the four parameters α_1, b_1, α_2, and b_2, we computed an estimate of N_c, by requiring

TABLE VI. Estimates of the fixed-point value $\overline{u}^*$ and of the critical exponent ω_2 at the $O(N)$-symmetric fixed point for $N=3$. The number p indicates the number of loops included in the analysis, the columns labeled by α and b indicate the intervals of α and b used, ''Final'' reports our final estimate for the given analysis.

	α	b	$p=4$	$p=5$	$p=6$	Final
$\overline{u}^*$	[0.5,2.5]	[5,11]	1.400(6)	1.392(4)	1.392(1)	1.392(2)
	[0.5,2.5]	[5,13]	1.400(6)	1.392(4)	1.392(1)	1.392(2)
	[0.5,2.5]	[3,13]	1.406(17)	1.389(7)	1.394(4)	1.394(8)
	[−0.5,3.5]	[5,11]	1.404(13)	1.392(6)	1.393(2)	1.393(4)
ω_2	[0.5,2.5]	[5,11]	−0.014(10)	−0.009(5)	−0.013(2)	−0.013(4)
	[0.5,2.5]	[5,13]	−0.012(9)	−0.009(4)	−0.012(2)	−0.012(4)
	[0.5,2.5]	[3,13]	−0.017(22)	−0.007(12)	−0.014(7)	−0.014(14)
	[−0.5,3.5]	[5,11]	−0.015(16)	−0.008(7)	−0.013(3)	−0.013(6)

$$\hat{\omega}_2(\alpha_1,b_1,\alpha_2,b_2;p)=0. \tag{3.7}$$

The distribution of the results is shown in Fig. 3. It is evident that $N_c<3$, as of course it should be expected from the analysis of ω_2 at $N=3$. The result is increasingly stable as the number of orders that are included increases. We estimate

$$N_c=2.91(9)\ (4\text{ loops}),\quad 2.91(3)\ (5\text{ loops}),\quad 2.89(2)\ (6\text{ loops}). \tag{3.8}$$

We can thus safely conclude that $N_c=2.89(4)$.

We have then considered the cubic fixed point and studied its stability. Again we used four parameters α_1, b_1, α_2, and b_2. For each choice we first computed an estimate of the critical point $\overline{u}_p^*(\alpha_1,b_1,\alpha_2,b_2)$, $\overline{v}_p^*(\alpha_1,b_1,\alpha_2,b_2)$ solving the equations

$$E(\beta_{\overline{u}}/\overline{u})_p(\alpha_1,b_1;\overline{u}_p^*(\alpha_1,b_1,\alpha_2,b_2),\overline{v}_p^*(\alpha_1,b_1,\alpha_2,b_2))=0, \tag{3.9}$$

$$E(\beta_{\overline{v}}/\overline{v})_p(\alpha_2,b_2;\overline{u}_p^*(\alpha_1,b_1,\alpha_2,b_2),\overline{v}_p^*(\alpha_1,b_1,\alpha_2,b_2))=0. \tag{3.10}$$

Then we determined the elements of the stability matrix Ω from

$$\Omega_{11}=E(\partial\beta_{\overline{u}}/\partial\overline{u})_p(\alpha_1,b_1;\overline{u}_p^*(\alpha_1,b_1,\alpha_2,b_2),\overline{v}_p^*(\alpha_1,b_1,\alpha_2,b_2)), \tag{3.11}$$

$$\Omega_{22}=E(\partial\beta_{\overline{v}}/\partial\overline{v})_p(\alpha_2,b_2;\overline{u}_p^*(\alpha_1,b_1,\alpha_2,b_2),\overline{v}_p^*(\alpha_1,b_1,\alpha_2,b_2)), \tag{3.12}$$

$$\Omega_{12}=\overline{u}_p^*(\alpha_1,b_1,\alpha_2,b_2)E(1/\overline{u}\cdot\partial\beta_{\overline{u}}/\partial\overline{v})_{p-1}\times(\alpha_1,b_1;\overline{u}_p^*(\alpha_1,b_1,\alpha_2,b_2),\overline{v}_p^*(\alpha_1,b_1,\alpha_2,b_2)), \tag{3.13}$$

TABLE VII. Estimates of $\overline{u}^*$ and of the eigenvalue ω_2 at the $O(N)$-fixed point for several values of N and for different orders p of the perturbative series. The last column reports the final estimate.

N	α	b	$p=4$	$p=5$	$p=6$	Final
			$\overline{u}^*$			
2	[−0.5,3.5]	[5,11]	1.422(15)	1.409(7)	1.408(2)	1.408(4)
3	[−0.5,3.5]	[5,11]	1.404(13)	1.392(6)	1.393(2)	1.393(4)
4	[0.5,4.5]	[9,15]	1.372(12)	1.375(2)	1.375(1)	1.375(2)
8	[0.5,4.5]	[9,15]	1.303(6)	1.304(1)	1.305(1)	1.305(2)
∞						1
			ω_2			
2	[−0.5,3.5]	[5,11]	0.099(20)	0.107(11)	0.103(4)	0.103(8)
3	[−0.5,3.5]	[5,11]	−0.015(16)	−0.008(7)	−0.013(3)	−0.013(6)
4	[0.5,4.5]	[9,15]	−0.105(10)	−0.109(3)	−0.111(2)	−0.111(4)
8	[0.5,4.5]	[9,15]	−0.371(8)	−0.379(4)	−0.385(4)	−0.385(8)
∞						−1

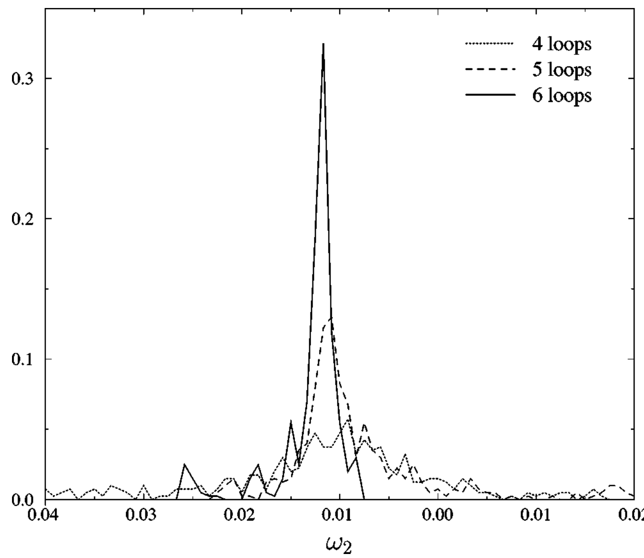

FIG. 2. Distribution of the estimates of ω_2 at the $O(3)$-symmetric fixed point.

$$\Omega_{21}=\overline{v}^*_p(\alpha_1,b_1,\alpha_2,b_2)E(1/\overline{v}\cdot\partial\beta_{\overline{v}}/\partial\overline{u})_{p-1} \times(\alpha_2,b_2;\overline{u}^*_p(\alpha_1,b_1,\alpha_2,b_2),\overline{v}^*_p(\alpha_1,b_1,\alpha_2,b_2)). \tag{3.14}$$

We computed the eigenvalues of Ω, obtaining estimates of the exponents ω_1 and ω_2. As before, we determined $\overline{\alpha}$ and $\overline{b}$ by requiring the stability of the estimates of $\overline{u}^*$, $\overline{v}^*$, ω_1, and ω_2, when varying the order of the series. Errors were computed as before. The results are reported in Table VIII. They show that for $N\geqslant 3$ the cubic fixed point is stable. Note that for $N=3$ we find $\overline{v}^*>0$, which agrees with what is expected in the case of a stable cubic fixed point, see Fig. 1. The distribution of the estimates of $\overline{v}^*$, obtained varying the parameters $\alpha_1,b_1,\alpha_2,b_2$, and reported in Fig. 4, shows that the result is quite stable. In accordance with this scenario, we find $\omega_2=0.010(4)>0$. The quoted error is quite conservative, as it can be seen from Fig. 5 in which we report the distribution of the estimates of ω_2. Note that the results become increasingly stable as the number of loops increases. One may compare our estimate of ω_2 at the cubic fixed point with the four-loop result of Ref. 40, $\omega_2=0.0081$, which is fully consistent with ours.[67]

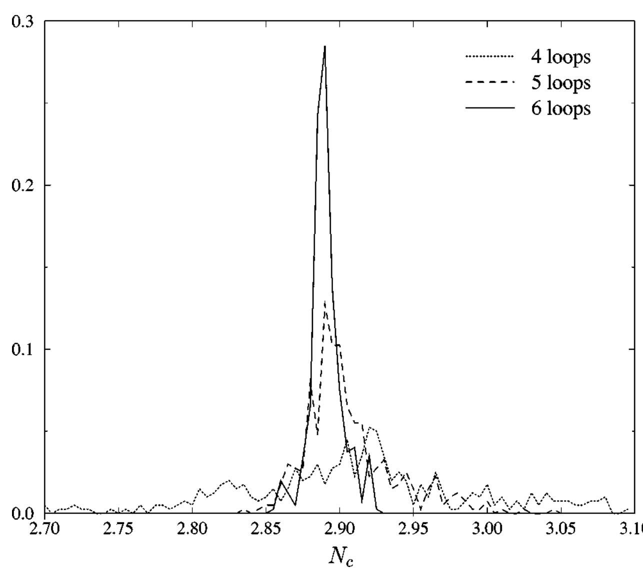

FIG. 3. Distribution of the estimates of N_c. It is determined by requiring $\omega_2=0$ at the $O(N)$-symmetric fixed point.

TABLE VIII. Estimates of the cubic fixed point and of the eigenvalues of the stability matrix for various values of N and orders p of the perturbative series. Our final results correspond to $p=$f.

N	p	$\overline{u}^*$	$\overline{v}^*$	ω_1	ω_2
3	4	1.36(11)	0.04(13)	0.780(11)	0.011(31)
	5	1.328(26)	0.089(28)	0.777(5)	0.009(6)
	6	1.321(9)	0.096(10)	0.781(2)	0.010(2)
	f	1.321(18)	0.096(20)	0.781(4)	0.010(4)
4	4	0.907(72)	0.606(82)	0.711(77)	0.144(78)
	5	0.883(17)	0.639(17)	0.804(48)	0.049(40)
	6	0.881(7)	0.639(7)	0.781(22)	0.076(20)
	f	0.881(14)	0.639(14)	0.781(44)	0.076(40)
8	4	0.448(49)	1.138(65)	0.695(85)	0.211(93)
	5	0.440(14)	1.140(16)	0.831(77)	0.098(47)
	6	0.440(6)	1.136(5)	0.775(44)	0.149(33)
	f	0.440(12)	1.136(10)	0.775(88)	0.149(66)
∞	4	0.182(15)	1.422(24)	0.744(95)	0.185(15)
	5	0.173(6)	1.424(7)	0.783(31)	0.177(6)
	6	0.174(3)	1.417(3)	0.790(9)	0.178(3)
	f	0.174(6)	1.417(6)	0.790(18)	0.178(6)

We computed the exponents at the cubic fixed point, using Eqs. (2.9), (2.10), and (2.11). The results are reported in Table IX. Note that for $N=3$, the exponents at the cubic fixed point do not differ appreciably from those of the isotropic model. A recent reanalysis[60] of the fixed-dimension expansion of the three-dimensional $O(N)$-symmetric models

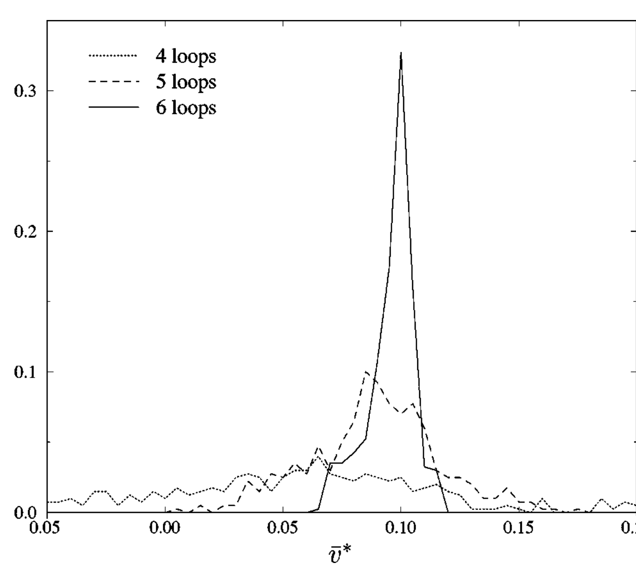

FIG. 4. Distribution of $\overline{v}^*$ of the cubic fixed point for $N=3$.

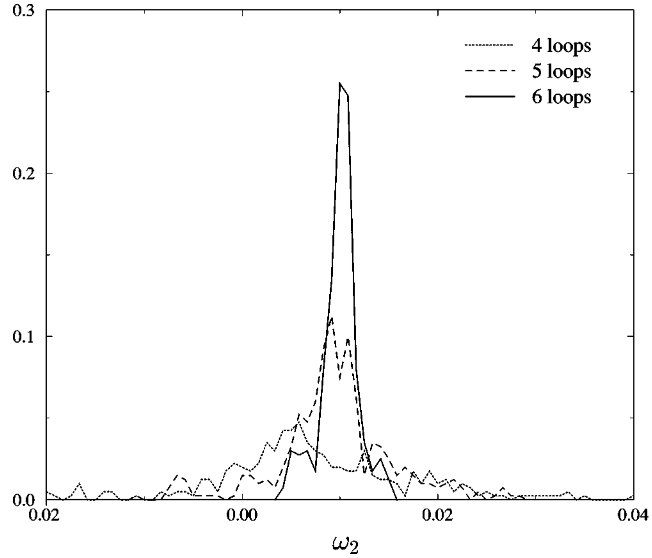

FIG. 5. Distribution of the estimates of ω_2 at the cubic fixed point for $N=3$.

obtained the following estimates for $N=3$: $\gamma=1.3895(50)$, $\nu=0.7073(35)$, and $\eta=0.0355(25)$. These results should be compared with the critical exponents at the cubic fixed point: $\gamma=1.390(12)$, $\nu=0.706(6)$, and $\eta=0.0333(26)$. The difference is smaller than the precision of our results, which is about 1% in the case of γ and ν.

We also checked the exact predictions for the exponents in the large-N limit. For $N\to\infty$ with $\bar{v}$ and $\bar{u}$ fixed, we have $\beta_{\bar{v}}(\bar{u},\bar{v})=\beta(\bar{v})|_{\rm Ising}$, so that $\bar{v}^*=\bar{v}_I^*$. Indeed, our estimate of $\bar{v}^*$ is in agreement with the estimate obtained with the same method of analysis in Ref. 44: $\bar{v}_I^*=1.416(5)$. We have then compared our estimates $\eta=0.0304(20)$, $\nu=0.708(8)$, and $\gamma=1.396(14)$ with the predictions

TABLE IX. Estimates of the critical exponents at the cubic fixed point for various values of N and order p of the perturbative series. The column ''Final'' reports our final results. The results of Ref. 40, obtained by a Padé-Borel analysis of the four-loop series, are reported in the last column.

N		$p=4$	$p=5$	$p=6$	Final	Ref. 40
3	γ	1.399(46)	1.390(11)	1.390(6)	1.390(12)	1.3775
	ν	0.710(26)	0.706(7)	0.706(3)	0.706(6)	0.6996
	η	0.0338(90)	0.0325(20)	0.0333(13)	0.0333(26)	0.0332
4	γ	1.415(41)	1.403(12)	1.405(5)	1.405(10)	1.4028
	ν	0.718(23)	0.714(7)	0.714(4)	0.714(8)	0.7131
	η	0.0323(73)	0.0305(18)	0.0316(11)	0.0316(22)	0.0327
8	γ	1.401(38)	1.402(12)	1.404(5)	1.404(10)	1.4074
	ν	0.709(21)	0.711(7)	0.712(3)	0.712(6)	0.7153
	η	0.0292(61)	0.0296(18)	0.0306(10)	0.0306(20)	0.0324
∞	γ	1.400(21)	1.395(9)	1.396(7)	1.396(14)	
	ν	0.709(12)	0.707(5)	0.708(4)	0.708(8)	
	η	0.0287(35)	0.0294(11)	0.0304(10)	0.0304(20)	

$$\eta=\eta_I=0.0364(4),$$

$$\nu=\frac{\nu_I}{(1-\alpha_I)}=0.7078(3), \qquad (3.15)$$

$$\gamma=\frac{\gamma_I}{(1-\alpha_I)}=1.3899(7).$$

The Ising model results have been taken from Ref. 8. There is a substantial agreement, although we note a small discrepancy for η. This small difference is not unexpected. Indeed, the estimate of η for the Ising model obtained from the fixed-dimension expansion shows a systematic discrepancy with respect to high-temperature and Monte Carlo results.

Finally, we checked the prediction $\omega_2=-\alpha_I/\nu_I$ at the Ising fixed point. It is immediate to verify that

$$\omega_2=\frac{\partial\beta_{\bar{u}}}{\partial\bar{u}}(0,\bar{v}) \qquad (3.16)$$

is independent of N. Numerically we find

$$\omega_2=-0.174(31)\ \ (4\ \text{loop}),\quad \omega_2=-0.178(7)\ \ (5\ \text{loop}),$$

$$\omega_2=-0.177(3)\ \ (6\ \text{loop}). \qquad (3.17)$$

Our final estimate is therefore $\omega_2=-0.177(6)$, which should be compared with the prediction $\omega_2=-\alpha_I/\nu_I=-0.1745(12)$.

IV. ANALYSIS OF THE ϵ-EXPANSION FIVE-LOOP SERIES

In this section we consider the alternative field-theoretic approach based on the ϵ expansion. The β function and the exponents in the cubic model are known to $O(\epsilon^5)$.[34] These series have already been the object of several analyses using different resummation methods.[34–37,40] Reference 36 studies the stability of the $O(N)$-symmetric and of the cubic fixed points for $N=3$. They rewrite the double expansion in u and v in terms of $g\equiv u+v$ and of the parameter $\delta\equiv v/(u+v)$. Then, each coefficient in the expansion in terms of δ, which is a series in g, is resummed using the known large-order behavior. Since v^*, and hence δ^*, is small, one expects the resulting series to be rapidly convergent near the fixed point, and therefore no additional resummation is applied.[35] For the cubic fixed point, Ref. 36 quotes $\omega_2\simeq0.0049$, 0.0085, 0.0021, obtained, respectively, from the analysis of the three-, four-, and five-loop series. Similar results (with the opposite sign) are found for the $O(N)$-symmetric fixed point. These estimates are compatible with a stable cubic fixed point, but the observed trend towards smaller values of $|\omega_2|$ leaves open the possibility that the estimate of ω_2 will eventually change sign, modifying the conclusions about the stability of the fixed points. The same expansion has been analyzed in Ref. 37 using a Padé-Borel resummation technique. The authors report the estimate $N_c\approx2.86$, but again the uncertainty on this result is not clear. We will present here a new analysis of the five-loop series using the method presented in Sec. III and the value of the singularity of the Borel transform given below.

As before, we will consider expansions in u with z

TABLE X. ϵ-expansion estimates of the critical exponents ω_1 and ω_2 at the cubic and at the $O(N)$-symmetric fixed points for $N=3$. The number p indicates the number of loops included in the analysis, the columns labeled by α and b indicate the intervals of α and b used, ''Final'' indicates our final estimate from the given analysis.

	α	b	$p=3$	$p=4$	$p=5$	Final
			$O(N)$-symmetric fixed point			
ω_1	[−0.5,1.5]	[11,17]	0.799(20)	0.790(8)	0.795(4)	0.795(8)
	[−0.5,1.5]	[8,20]	0.804(36)	0.784(20)	0.801(14)	0.801(28)
	[−1.5,2.5]	[11,17]	0.830(98)	0.784(19)	0.795(10)	0.795(20)
ω_2	[0.0,2.0]	[12,18]	0.004(6)	0.000(3)	−0.003(2)	−0.003(4)
	[0.0,2.0]	[9,21]	0.005(9)	−0.002(5)	−0.003(2)	−0.003(4)
	[−1.0,3.0]	[12,18]	0.022(32)	0.000(7)	−0.003(4)	−0.003(8)
			cubic fixed point			
ω_1	[−0.5,1.5]	[12,18]	0.789(16)	0.797(6)	0.796(2)	0.796(4)
	[−0.5,1.5]	[9,21]	0.793(31)	0.793(13)	0.799(7)	0.799(14)
	[−1.5,2.5]	[12,18]	0.82(10)	0.794(20)	0.793(6)	0.793(12)
ω_2	[−0.5,1.5]	[11,17]	0.005(12)	0.006(4)	0.007(2)	0.007(4)
	[−0.5,1.5]	[8,20]	0.004(13)	0.006(4)	0.006(2)	0.006(4)
	[−1.5,2.5]	[11,17]	−0.004(17)	0.002(7)	0.006(3)	0.006(6)

$=v/u$ fixed. The singularity closest to the origin of the Borel transform is given by

$$\frac{1}{u_b}=-(1+z) \quad \text{for } z>0,$$

$$\frac{1}{u_b}=-\left(1+\frac{1}{N}z\right) \quad \text{for } z<0. \tag{4.1}$$

The fixed-point values of u and z associated to the $O(N)$-symmetric and to the cubic fixed points are, respectively,

$$u^*=\frac{3}{8+N}\epsilon+O(\epsilon^2), \quad z^*=0, \tag{4.2}$$

and

$$u^*=\frac{1}{N}\epsilon+O(\epsilon^2), \quad z^*=\frac{N-4}{3}+O(\epsilon). \tag{4.3}$$

Since only the leading term in the ϵ expansion is relevant for the determination of the singularity ϵ_b,[43,44] using Eqs. (4.2), (4.3), and (4.1), we find

$$\epsilon_b=-\frac{N+8}{3} \tag{4.4}$$

for the $O(N)$-invariant fixed point,[43,44] and

$$\epsilon_b=-\frac{3N^2}{4(N-1)} \quad \text{for } N<4,$$

$$\epsilon_b=-\frac{3N}{N-1} \quad \text{for } N>4, \tag{4.5}$$

for the cubic fixed point.

The method of analysis is the one explained in Sec. III with two simplifications: first, we consider expansions in one variable only; moreover, the value at which we should compute the expansion is known ($\epsilon=1$). Each series is resummed using several different values of α and b belonging to the intervals $\alpha\in[\overline{\alpha}-\Delta\alpha,\overline{\alpha}+\Delta\alpha]$ and $b\in[\overline{b}-\Delta b,\overline{b}+\Delta b]$. The parameters $\overline{\alpha}$ and $\overline{b}$ were chosen as before, while after several trials, we fixed $\Delta\alpha=1$ and $\Delta b=6$. The error in the final results is always two standard deviations. The dependence of the results on the different choices of α and b is shown in Table X. The final results are reported in Tables XI and XII.

The final results are in reasonable agreement with the estimates of Sec. III. However, the instability of the $O(N)$-symmetric fixed point is less clear in the ϵ expansion: indeed we find $\omega_2=-0.003(4)$ at the symmetric fixed point and $\omega_2=0.006(4)$ at the cubic fixed point. This is not surprising: for the $O(N)$-symmetric model the five-loop ϵ expansion gives results that are less precise than the six-loop expansion in fixed dimension. We also mention that for $N=\infty$ we obtain an estimate of η, $\eta=0.0349(22)$, that is in perfect agreement with the theoretical prediction $\eta=0.0364(4)$.

Finally we compute N_c. We will determine it following two different strategies. A first possibility is to consider ω_2 and determine the value of N for which the three-dimensional estimate of ω_2 vanishes, which is exactly what we did in the fixed-dimension expansion. At the symmetric fixed point we find

$$N_c=3.07(9)\ (3\text{ loops}), \quad 2.99(6)\ (4\text{ loops}), \quad 2.98(2)\ (5\text{ loops}), \tag{4.6}$$

while at the cubic fixed point

TABLE XI. Estimates of the critical exponent ω_2 at the $O(N)$-symmetric fixed point for several values of N. The number p indicates the number of loops included in the analysis, the columns labeled by α and b indicate the interval of α and b used, ''final'' reports our final estimate.

N	α	b	$p=3$	$p=4$	$p=5$	Final
2	[0.0,2.0]	[9,21]	0.116(12)	0.113(5)	0.114(2)	0.114(4)
3	[0.0,2.0]	[9,21]	0.005(9)	−0.002(5)	−0.003(2)	−0.003(4)
4	[0.5,2.5]	[8,20]	−0.088(11)	−0.103(7)	−0.105(3)	−0.105(6)
8	[1.5,3.5]	[7,19]	−0.374(24)	−0.395(12)	−0.395(4)	−0.395(8)

$$N_c = 3.04(10)\ (3\text{ loops}), \quad 2.95(6)\ (4\text{ loops}),$$
$$2.96(3)\ (5\text{ loops}). \tag{4.7}$$

Averaging the two results and reporting the error as two standard deviations, we obtain $N_c=2.97(6)$, which is in agreement with the analysis of the same series of Ref. 34, $N_c\approx 2.958$. It is also consistent with the results of the analysis of the eigenvalues ω_2 for $N=3$: indeed we found $\omega_2 > 0$ and $\omega_2<0$ at the cubic and at the symmetric point, respectively, implying $N_c<3$. However, the error bars on ω_2 are such that the opposite inequalities are not excluded. Analogously here, although $N_c<3$, the error is such that it does not exclude $N=3$. In practice, from this analysis alone, it is impossible to conclude safely that $N_c<3$.

A second possibility consists in solving the equation $\omega_2 = 0$ perturbatively, obtaining for N_c an expansion in powers of ϵ. The result is independent of which fixed point is chosen. Unfortunately, the singularity of the Borel transform of $N_c(\epsilon)$ is not known and therefore, we used the Padé-Borel

TABLE XII. Estimates of the critical exponents at the cubic fixed point for several values of N. The number p indicates the number of loops included in the analysis, the columns labeled by α and b indicate the intervals of α and b used, ''final'' reports our final estimate.

N	α	b	$p=3$	$p=4$	$p=5$	Final
			ω_1			
3	[−0.5,1.5]	[9,21]	0.793(31)	0.793(13)	0.799(7)	0.799(14)
4	[−1.0,1.0]	[10,22]	0.790(20)	0.787(8)	0.790(4)	0.790(8)
8	[−1.0,1.0]	[8,20]	0.779(16)	0.782(5)	0.786(3)	0.786(6)
∞	[−1.0,1.0]	[10,22]	0.772(22)	0.789(11)	0.802(9)	0.802(18)
			ω_2			
3	[−0.5,1.5]	[8,20]	0.004(13)	0.006(4)	0.006(2)	0.006(4)
4	[−1.5,0.5]	[11,23]	0.073(15)	0.078(5)	0.078(2)	0.078(4)
8	[−1.5,0.5]	[11,23]	0.154(9)	0.155(4)	0.155(1)	0.155(2)
∞	[−2.5,−0.5]	[11,23]	0.210(17)	0.208(6)	0.202(4)	0.202(8)
			γ			
3	[1.5,3.5]	[3,15]	1.370(29)	1.375(8)	1.377(3)	1.377(6)
4	[1.5,3.5]	[3,15]	1.414(20)	1.421(7)	1.419(3)	1.419(6)
8	[0.5,2.5]	[3,15]	1.420(22)	1.424(8)	1.422(3)	1.422(6)
∞	[0.0,2.0]	[3,15]	1.394(27)	1.399(11)	1.399(4)	1.399(8)
			ν			
3	[1.5,3.5]	[3,15]	0.695(16)	0.699(5)	0.701(2)	0.701(4)
4	[1.5,3.5]	[3,15]	0.717(11)	0.723(5)	0.723(2)	0.723(4)
8	[1.0,3.0]	[3,15]	0.723(7)	0.724(3)	0.723(1)	0.723(2)
∞	[1.0,3.0]	[3,15]	0.713(4)	0.711(2)	0.711(1)	0.711(2)
			η			
3	[1.0,3.0]	[3,15]	0.0319(61)	0.0359(16)	0.0374(11)	0.0374(22)
4	[0.5,2.5]	[4,16]	0.0319(23)	0.0339(10)	0.0357(9)	0.0357(18)
8	[0.5,2.5]	[3,15]	0.0301(31)	0.0336(11)	0.0349(8)	0.0349(16)
∞	[0.5,2.5]	[3,15]	0.0296(36)	0.0332(14)	0.0349(11)	0.0349(22)

method. The analysis of the series already appears in Ref. 40, and will not be repeated here. Instead, we will try to make use of the fact that $N_c=2$ for $d=2$, performing a constrained analysis. The method has already been applied in many instances,[68,69,64,70,71] providing more precise estimates of the critical quantities. The method consists in rewriting

$$N_c(\epsilon)=2+(2-\epsilon)\Delta(\epsilon), \tag{4.8}$$

where

$$\Delta(\epsilon)=\frac{N_c(\epsilon)-2}{(2-\epsilon)}. \tag{4.9}$$

The quantity $\Delta(\epsilon)$ is expanded in powers of ϵ and then resummed. One can verify that the coefficients of the expansion of $\Delta(\epsilon)$ are uniformly smaller than the coefficients of the original series, by a factor of 2 approximately. Therefore, one expects a corresponding improvement in the error estimates. We considered Padé's [2/1], [1/2], [2/2], and [3/1] and several different values of the parameter b between the ''reasonable'' values 0 and 20. Padé's [1/2] and [2/2] have a singularity on the real positive axis for $b\lesssim 7$: these cases are of course excluded from consideration. At four loops, we find that the approximant [2/1] gives estimates varying between 2.82 and 2.87, while the approximant [1/2] is stable giving $N_c\approx 2.82$. At five loops, the approximant [3/1] varies between 2.85 and 2.92, while the approximant [2/2] is more stable and gives $N_c\approx 2.83$. To appreciate the improvement of the results due to the constrained analysis, we report the corresponding variation of the estimates for the original series: for $0\leqslant b\leqslant 20$ the approximants [2/1], [1/2], [2/2], and [3/1] give estimates varying in the intervals $2.80\leqslant N_c \leqslant 2.92$, $2.90\leqslant N_c\leqslant 2.96$, $2.90\leqslant N_c\leqslant 2.95$, $2.86\leqslant N_c\leqslant 3.03$. A conservative final estimate is $N_c=2.87(5)$. This result is lower than that of the previous analysis, but still compatible with it. The constrained analysis is in much better agreement with the results obtained in the fixed-dimension expansion, and clearly supports the claim that $N_c<3$.

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APPENDIX: CUBIC FIXED POINT IN THE ϵ EXPANSION

We report here the ϵ-expansion computation of the four-point renormalized couplings $\overline{u}^*$ and $\overline{v}^*$ defined in Sec. II at the cubic fixed point, following Refs. 64,71. We have calculated the three-loop expansion of these two quantities. The final result can be written as

$$\overline{u}^*(\epsilon)=\frac{(N+8)}{3N}\sum_{n=0}\overline{u}_n\epsilon^n,\quad \overline{v}^*(\epsilon)=\sum_{n=0}\overline{v}_n\epsilon^n, \tag{A1}$$

where

$$\overline{u}_0=1, \tag{A2}$$

$$\overline{u}_1=-\frac{(N-1)(19N-106)}{27N^2}, \tag{A3}$$

$$\overline{u}_2=-\frac{2(-11236+22540\,N-14181\,N^2+2446\,N^3+107\,N^4)}{729\,N^4}-\frac{\lambda(N-1)(86+7\,N)}{81\,N^2}+\frac{4(14-7\,N-6\,N^2+2\,N^3)\zeta(3)}{9\,N^3}, \tag{A4}$$

$$\begin{aligned}\overline{u}_3=&\frac{8\,H(10-5\,N-2\,N^2)}{27\,N^3}+\frac{\lambda(-9116+16328\,N-7863\,N^2+209\,N^3+37\,N^4)}{729\,N^4}\\&-\frac{1}{157464\,N^6}(47640640-134959200\,N+141439956\,N^2-64950380\,N^3+11140557\,N^4-136770\,N^5+11821\,N^6)\\&+\frac{(16-8\,N-8\,N^2+3\,N^3)\pi^4}{405\,N^3}-\frac{(N-1)(86+7\,N)(\gamma_E\,\lambda+Q_1)}{162\,N^2}-\frac{4(-68+54\,N-4\,N^2+3\,N^3)Q_2}{27\,N^3}\\&-\frac{40(18-5\,N-2\,N^2-6\,N^3+2\,N^4)\zeta(5)}{27\,N^4}+\frac{(-59360+93392\,N-27220\,N^2-14140\,N^3+5083\,N^4+346\,N^5)\zeta(3)}{486\,N^5},\end{aligned} \tag{A5}$$

and

$$\bar{v}_0 = \frac{N-4}{N}, \tag{A6}$$

$$\bar{v}_1 = \frac{(N-1)(-424+110N+17N^2)}{27N^3}, \tag{A7}$$

$$\bar{v}_2 = -\frac{25\,\lambda(N-4)(N-1)(2+N)}{81\,N^3} - \frac{4(2+N)(28-32\,N+6\,N^2+N^3)\zeta(3)}{9\,N^4} + \frac{2(-44944+93764\,N-61408\,N^2+10951\,N^3+565\,N^4+100\,N^5)}{729\,N^5}, \tag{A8}$$

$$\bar{v}_3 = \frac{8\,H(N-4)(8-4\,N-N^3)}{27\,N^4} - \frac{\lambda(-21200+31540\,N-6536\,N^2-5953\,N^3+581\,N^4+353\,N^5)}{729\,N^5} + \frac{1}{157464\,N^7}(190562560-555117760\,N+598171440\,N^2-281214884\,N^3+47944832\,N^4 + 267603\,N^5-77234\,N^6+23315\,N^7) - \frac{(68-42\,N-32\,N^2+14\,N^3+N^4)\pi^4}{405\,N^4} - \frac{25(N-4)(N-1)(2+N)(\gamma_E\,\lambda+Q_1)}{162\,N^3} - \frac{4(N-4)(-24+22\,N-5\,N^2+2\,N^3)Q_2}{9\,N^4} + \frac{40(72-30\,N-11\,N^2-18\,N^3+7\,N^4+N^5)\zeta(5)}{27\,N^5} - \frac{(-237440+401536\,N-151152\,N^2-36648\,N^3+15662\,N^4+2277\,N^5+68\,N^6)\zeta(3)}{486\,N^6}. \tag{A9}$$

Here γ_E is Euler's constant. We have also introduced the following numerical constants:[71]

$$\lambda = 1.171\,953\,619\,344\,729\,445\ldots,$$

$$Q_1 = -2.695\,258\,053\,506\,736\,953\ldots,$$

$$Q_2 = 0.400\,685\,634\,386\,531\,428\ldots, \tag{A10}$$

$$H = -2.155\,952\,487\,340\,794\,361\ldots.$$

A standard analysis[64] gives

$$N=3: \quad \bar{u}^* = 1.416(10), \quad \bar{v}^* = -0.03(14); \tag{A11}$$

$$N=4: \quad \bar{u}^* = 0.971(19), \quad \bar{v}^* = 0.58(9); \tag{A12}$$

$$N=8: \quad \bar{u}^* = 0.455(73), \quad \bar{v}^* = 1.14(9). \tag{A13}$$

The estimates of $\bar{v}^*$ are in reasonable agreement with the results of Sec. III, although they are much less precise, as it should be expected since here we are analyzing a shorter series. On the other hand, the estimates of $\bar{u}^*$ strongly disagree with the quoted error bars. The estimate for $N=3$ is the worst one: indeed, the six-loop fixed-dimension expansion predicts $\bar{u}^* = 1.321(18)$. However, the quoted errors (based, as usual, on the stability of the estimates when changing the parameters b and α) seem to be largely underestimated. For instance, the three-loop series for $\bar{u}^*$ at the $O(N)$-symmetric fixed point gives $\bar{u}^* = 1.39(7)$ (Ref. 71): the error is in this case seven times larger. There is no reason to believe the error on the estimate of $\bar{u}^*$ at the cubic point to be much smaller than that at the isotropic one. Moreover, it is difficult to accept that an expansion truncated at three loops might give a more precise result than the six-loop fixed-dimension expansion. Thus, the previous error estimates should not be trusted and the observed stability is probably accidental. If we assume that the correct error is of order ≈ 0.07 as in the $O(3)$ case, then all results are in agreement. Note that the errors on $\bar{v}^*$ are instead of the expected order of magnitude.

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