Effective and asymptotic critical exponents of a weakly diluted quenched Ising model: Three-dimensional approach versus $\sqrt{\varepsilon}$ expansion

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We present a field-theoretical treatment of the critical behavior of a three-dimensional weakly diluted quenched Ising model. To this end we analyze in the replica limit $n \to 0$ the five-loop renormalization-group functions of the ϕ^4 theory with $O(n)$-symmetric and cubic interactions [H. Kleinert and V. Schulte-Frohlinde, Phys. Lett. B **342**, 284 (1995)]. The minimal subtraction scheme allows one to develop either the $\sqrt{\varepsilon}$-expansion series or to proceed within the three-dimensional approach, performing expansions in terms of renormalized couplings. Doing so, we compare both perturbation approaches and discuss their convergence and possible Borel summability. To study the crossover effect we calculate the effective critical exponents. We report resummed numerical values for the effective and asymptotic critical exponents. The results obtained within the three-dimensional approach agree pretty well with recent Monte Carlo simulations. $\sqrt{\varepsilon}$ expansion does not allow reliable estimates for $d=3$.

I. INTRODUCTION

Influence of a weak quenched disorder on magnetic second-order phase transition remains the subject of much interest. One of the central problems here is the answer to the questions: (i) do the critical exponents (of a homogeneous magnet) change under dilution (by a nonmagnetic component)? And, if yes, (ii) how do they change?

Replying to the first question it was argued[1] almost 25 years ago that if the heat capacity critical exponent α of the pure (undiluted) system is positive, i.e., the heat capacity diverges at the critical point, then a quenched disorder causes changes in the critical exponents. This statement is known as Harris criterion. Later, for a large class of d-dimensional disordered systems it was proven that the correlation length critical exponent ν must satisfy the bound $\nu \geqslant 2/d$.[2] Both statements focus attention towards studies of the $d=3$ Ising model, where the typical numerical values of the above exponents, together with the magnetic susceptibility and the order-parameter critical exponents in the pure case read: $\alpha = 0.109 \pm 0.004$, $\nu = 0.6304 \pm 0.0013$, $\gamma = 1.2396 \pm 0.0013$, $\beta = 0.3258 \pm 0.0014$.[3]

The reply on the second question concerning the numerical values of the critical exponents of $d=3$ weakly diluted quenched Ising model [random Ising model (RIM)] is more complicated. Below we briefly review some experimental,[5–21] theoretical,[22–37] and numerical[38–51] results for the critical exponents of $d=3$ magnets described by the RIM. Although the statements of Refs. 1 and 2 should hold in principle for arbitrary weak disorder, the new critical exponents appear only in a region of temperatures controlled by the concentration of the nonmagnetic component. Such a region may not always be reached in practice, where only the effective exponents are observed.[52] Theoretical results are due to the application of the renormalization group approach. These are numerous for the asymptotic values of critical exponents but quite seldom for the effective ones.[37]

This paper has been motivated by recent Monte Carlo calculations of the asymptotic[51] and effective[46,49,50] critical exponents for the three-dimensional Ising systems with quenched disorder. Here, we deal with a renormalization group study of both asymptotic and effective critical behavior. Perturbation theory series, which are known in the five-loop approximation for the $O(n)$ model with cubic anisotropy,[53] enable us to perform our studies in the replica limit $n \to 0$. The minimal subtraction scheme[54] allows us to develop either the $\sqrt{\varepsilon}$ expansion or to proceed in the three-dimensional (3D) approach, performing expansions in terms of renormalized couplings. Doing so, we compare both perturbation approaches and discuss their convergence and possible Borel summability.

The paper is arranged as follows. In Sec. II we give a brief account of some results on critical properties of systems of interest, Sec. III describes the model and the renormalization procedure. In Sec. IV we analyze the series obtained, perform their resummation, and give results for the asymptotic values of critical exponents. Effective critical behavior is discussed in Sec. V. Section VI concludes our study. Details of the resummation procedures exploited in our study are given in the Appendix.

II. REVIEW

Experiments. Crystalline mixtures of two compounds provide a typical experimental realization of a RIM (see Table

TABLE I. The experimentally measured critical exponents of the materials, which correspond to the $d=3$ RIM. The measurement procedures as well as materials specifications are given in the following notations: NMR, nuclear magnetic resonance; LB, linear birefringence; NS, neutron scattering; MS, Mössbauer spectroscopy; SMXS, synchrotron magnetic x-ray scattering; XS, x-ray scattering; DAG, dysprosium aluminium garnet; τ_{min} denotes the minimal value of the reduced temperature reached in an experiment.

Ref.	Year	Material	Method	τ_{min}	β	γ	ν	α
5	1981	$Mn_xZn_{1-x}F_2$, $x=0.864$	NMR	?	0.349 ± 0.008			
6	1983	$Fe_xZn_{1-x}F_2$, $x=0.6,0.5$	NS; LB	2×10^{-3}		1.44 ± 0.06	0.73 ± 0.03	-0.09 ± 0.03
13	1985	DAG +1% Y powder	NS	4×10^{-2}	0.350 ± 0.01		0.73	
12	1986	$Fe_xZn_{1-x}F_2$, $x=0.46$	NS	1.5×10^{-3}		1.31 ± 0.03	0.69 ± 0.01	
11	1986	$Fe_xZn_{1-x}F_2$, $x=0.9925-0.95$	MS	10^{-3}	0.36 ± 0.01			
7	1986	$Mn_xZn_{1-x}F_2$, $x=0.75$	NS	4×10^{-4}		1.364 ± 0.076	0.715 ± 0.035	
8	1988	$Mn_xZn_{1-x}F_2$, $x=0.5$	SMXS	10^{-3}	0.33 ± 0.02			
10	1988	$Fe_xZn_{1-x}F_2$, $x=0.9$	MS	3×10^{-4}	0.350 ± 0.009			
14	1988	$Mn_xZn_{1-x}F_2$, $x=0.40,0.55,0.83$	LB	$<10^{-2}$				-0.09 ± 0.03
15	1991	$Fe_xZn_{1-x}F_2$, $0.31\leqslant x\leqslant0.84$	LB	?				-0.09
16	1995	$Fe_xZn_{1-x}F_2$, $x=0.5$	NS	$<10^{-1}$	0.35			
17	1996	$Fe_xZn_{1-x}F_2$, $x=0.52$	NS	10^{-2}	0.35			
18	1997	$Fe_xZn_{1-x}F_2$, $x=0.5$	XS	10^{-2}	0.36 ± 0.02			
20	1998	$Fe_xZn_{1-x}F_2$, $x=0.93$	LB	2.5×10^{-3}				-0.10 ± 0.02
19	1998	$Fe_xZn_{1-x}F_2$, $x=0.93$	NS	2×10^{-3}		1.35 ± 0.01	0.71 ± 0.01	
21	1999	$Fe_xZn_{1-x}F_2$, $x=0.93$	NS	1.14×10^{-4}		1.34 ± 0.06	0.70 ± 0.02	

I). The first compound is an ''Ising-like'' anisotropic uniaxial antiferromagnet with dominating short-range interaction (e.g., FeF_2, MnF_2), the second one (ZnF_2) is nonmagnetic. Mixed crystals ($Fe_xZn_{1-x}F_2$, $Mn_xZn_{1-x}F_2$) can be grown with high crystalline quality and very small concentration gradients providing an excellent realization of random substitutional disorder of magnetic ions (Fe^{+2}, Mn^{+2}) by nonmagnetic ones (Zn^{+2}). Experimental evidence of the critical behavior at a weak quenched dilution was the nuclear-magnetic-resonance measurement of the magnetization in $Mn_{0.864}Zn_{0.136}F_2$.[5] The value of the magnetization exponent β was found to differ strongly from that in undiluted sample (see Table I for details). In a few years this result was corroborated by nuclear scattering measurements of magnetic susceptibility and correlation length critical exponents in $Fe_xMn_{1-x}F_2$ (Refs. 6 and 12) and $Mn_xZn_{1-x}F_2$ (Ref. 7) at different dilutions $1-x$. The linear birefringence measurements brought about the cusplike behavior of a specific heat at the transition point with $\alpha=-0.09\pm0.03$.[6] In particular this proved that within the error bars the hyperscaling relation $d\nu+\alpha=2$ is satisfied.

Crossover phenomena in diluted systems are governed in addition to the temperature by the concentration of the magnetic component. The experimentally obtained exponents are often reported to be effective ones (i.e., temperature and dilution dependent). However, already in early experiments on the critical behavior of quenched diluted Ising-like antiferromagnets it appeared possible to reach the asymptotic region. Thus studying the critical regime in Ref. 6 the authors found neither a region in reduced temperature τ where one finds ''pure'' Ising exponents nor any evidence of crossover from pure to random exponents. This was explained by the crossover either taking place outside the critical region or being too slow. In Ref. 11 the crossover from pure to diluted critical behavior was studied and a magnetization exponent β which does not change under dilution for $x\leqslant0.05$ was found. The crossover occurs within a very narrow range of τ at relatively large values of τ. In Ref. 7 excellent agreement of

the measured exponents γ and ν with the theoretical values of the RIM was obtained for the temperature range $4\times10^{-4}\leqslant\tau\leqslant10^{-1}$. The value of the critical exponent β was also the subject of a crossover analysis in Refs. 8 and 10. Reference 8 concludes that the experimental errors are too large in order to distinguish between the pure Ising model and the RIM critical behavior. In Ref. 10 no crossover was found after the correction-to-scaling has been taken into account, and the RIM critical behavior was found in the whole temperature range.

It is known that the diluted Ising magnet in a uniform magnetic field H along the uniaxial direction exhibits static critical behavior of the random-field Ising model.[55,56] The random-field Ising model is the subject of intensive recent experimental studies (see Ref. 57 for a review). Such experiments give additional information about the critical behavior of the RIM when performed for $H=0$.[14–21]

Renormalization group results. The change of the Ising-type magnets critical exponents under dilution found its theoretical confirmation in the renormalization group (RG) calculations. There, the change of the universal properties is interpreted as a crossover from the ''pure'' fixed point present in the undiluted model towards the ''random'' one which characterizes the new critical behavior of the RIM. The corresponding RG equations are degenerated on the one loop level, which leads instead of the familiar expansion in $\varepsilon=4-d$ to the expansion in $\sqrt{\varepsilon}$.[22–24] This expansion was recently extended from the three-loop[25,26] to the five-loop level[27,28] for the critical exponents. The $\sqrt{\varepsilon}$-expansion series for the amplitude ratios[58] is available with the three-loop accuracy as well.[59] Although the $\sqrt{\varepsilon}$-expansion series are known with pretty high accuracy they seem to be of no use for the $d=3$ case.[29] Alternatively, the scaling field RG approach lead (via the two fixed points scenario) to the $d=3$ critical exponents values $\nu=0.70$, $\gamma=1.38$,[30] definitively different from their ''pure'' counterparts.

RG equations of the massive $d=3$ field theory[60] appeared to be the most fruitful tool tackling the problem. The resulting expansions, considered as the asymptotic ones (see, however, Refs. 29, 61, and 62 and Sec. IV of this paper) were resummed successively increasing the order of the perturbation theory series from the two-loop (2LA) (Refs. 31 and 32) through three-loop (3LA) (Refs. 33 and 34) to the four-loop (4LA) approximations.[35,36] Depending on the resummation procedure applied, the results for the correlation length and the magnetic susceptibility critical exponents $\{\nu,\gamma\}$ read 2LA: {0.678 13, 1.335 51},[31] {0.679, 1.337};[33] 3LA: {0.670, 1.325};[33] {0.671, 1.328};[34] 4LA: {0.6702, 1.3262};[35] {0.6680, 1.318}, {0.6714, 1.321}.[36] Critical amplitudes universal ratios at $d=3$ were obtained in three-loop[63] and five-loop[64] approximations.

It is worthwhile to mention here that the renormalization at fixed $d=3$ dimensions does not necessarily mean an implementation of the massive scheme[60] in which most of the theoretical results[31–36] were obtained. Indeed, the minimal subtraction scheme[54] is also suited for the $d=3$ renormalization and can be applied without ε expansion.[65] Both schemes serve as complementary tools to obtain information about the critical behavior of the pure Ising model [or, more generally, $O(n)$-symmetric ϕ^4 theory]. However, in the case of the RIM there exists only one study relying on the 3D RG approach in minimal subtraction scheme. In particular the resummed values of the critical exponents obtained there in a three-loop approximation read $\nu=0.666$, $\gamma=1.313$.[37]

Griffiths singularities and replica symmetry breaking. Although the RG approach does not allow us to determine the region of concentration where the scenario with two fixed points, explained above, is applicable, it is generally assumed that it holds at least for weak dilutions. The idea of RG presumes that one studies the influence of (thermal) fluctuations around the spatially homogeneous unique ground state. This holds for the pure system but does not hold for the diluted one. Here, in a disorder dominated region one finds a macroscopic number of spatially inhomogeneous ground states, corresponding to the local minimum solutions of the saddle-point equation for an effective Lagrangian.[66,67] Physically the inhomogenous ground state corresponds to the so-called Griffiths phase[68] caused by the existence of ferromagnetically ordered ''islands'' in the region of temperatures between the critical temperatures of pure and diluted systems. This is described by a replica-symmetry breaking Lagrangian[66] leading to a behavior at the critical point[69,70] different from the pure one. The ''traditional'' RG results[22–37,58,59,63,64] are valid for the replica-symmetric Lagrangians.[71] Since the usual RG approach is based on an integration over the disorder at the beginning of the calculation, it cannot give any information about the region of concentrations where the weak dilution concept holds.[72]

Monte Carlo simulations. Monte Carlo (MC) studies of the $d=3$ RIM systems have lasted for almost two decades.[38–51] One of the first studies of critical behavior of a RIM on a simple cubic lattice[38] revealed the critical exponents in a wide dilution region with no deviations within the numerical error from the corresponding exponents of the pure (undiluted) system (see Table II). However, these data were objected by MC simulations on larger lattices,[39,40] where critical exponents varying continuously with the magnetic sites concentration x were obtained. Indications of change of the order-parameter critical exponent β upon dilution initiated the extension of studies to determine the other critical exponents and to check the scaling in the disordered systems. The application of the Swedsen-Wang algorithm to the $d=3$ RIM (Ref. 42) resulted in critical exponents for the susceptibility and correlation length independent of concentration over a wide range of dilution.

Due to Refs. 42 and 43 and especially Refs. 45 and 46 it became clear that the concentration-dependent critical exponents found in MC simulations are effective ones, characterizing the approach to the asymptotic region. The effective exponents γ,β and $\zeta=1-\beta$ (the last one describes the divergence of the magnetization-energy correlation function) were shown[45] to be concentration dependent in the concentration region $0.5\leqslant x<1$. These data were refined three years later[46] resulting in more accurate estimates for the above-mentioned exponents and the critical exponent ν of the correlation length yielding continuously varying values. The general conclusion of Refs. 45 and 46 was: while a simple crossover between the pure and weakly random fixed point accounts for the behavior of systems above $x\simeq0.8$, in more strongly disordered systems a more refined analysis is needed. Note that this last statement is supported by the con-

TABLE II. The critical exponents of the $d=3$ RIM as they are obtained in MC simulations for different values of magnetic sites concentration x. Maximal number of lattice sites simulated is $N=L^3$. (The asterisk at concentrations denotes that disorder was realized in a grand-canonical manner.)

Ref.	Year	L	x	β	γ	ν
38	1980	30	$0.4<x\leqslant 1$	0.31	1.25	
			1	0.30 ± 0.02		
			0.985	0.31 ± 0.02		
39	1986	40	0.95	0.32 ± 0.03		
			0.9	0.355 ± 0.010		
			0.8	0.385 ± 0.015		
			1	0.29 ± 0.02		
40	1986	90	0.95	0.28 ± 0.02		
			0.90	0.31 ± 0.02		
			0.80	0.37 ± 0.02		
41	1988	40	0.80	0.392 ± 0.03		
42	1989	100	$0.4\leqslant x\leqslant 0.8$		1.52 ± 0.07	0.77 ± 0.04
43	1990	300	0.8		1.36 ± 0.04	
			1			0.629(4)
44	1990	64	0.9			$<2/3$
			0.8			0.688(13)
			1	0.305 ± 0.01	1.24 ± 0.01	
			0.9	0.315 ± 0.01	1.30 ± 0.01	
45	1990	60	0.8	0.330 ± 0.01	1.35 ± 0.01	
			0.6	0.330 ± 0.01	1.48 ± 0.02	
			0.5	0.335 ± 0.01	1.49 ± 0.02	
			1	0.33 ± 0.01	1.22 ± 0.02	0.624 ± 0.010
			0.95	0.31 ± 0.02	1.28 ± 0.03	0.64 ± 0.02
46	1993	60	0.9	0.31 ± 0.02	1.31 ± 0.03	0.65 ± 0.02
			0.8	0.35 ± 0.02	1.35 ± 0.03	0.68 ± 0.02
			0.6	0.33 ± 0.02	1.51 ± 0.03	0.72 ± 0.02
48	1993	90?	0.6	0.42 ± 0.04		0.78 ± 0.01
49	1998	64	0.8	0.344 ± 0.003	1.357 ± 0.008	0.682 ± 0.003
50	1998	90	0.6	0.316 ± 0.013	1.522 ± 0.031	0.722 ± 0.008
		80	0.6*	0.313 ± 0.012	1.508 ± 0.028	0.717 ± 0.007
51	1998	128	$0.4\leqslant x\leqslant 0.9$	0.3546 ± 0.0028	1.342 ± 0.010	0.6837 ± 0.0053

jecture of a ''steplike'' universality of the three-dimensional diluted magnets.[47]

The critical behavior of the $d=3$ RIM was reexamined recently in Refs. 49–51. In particular, the simulations[50] revealed that a disorder realized in a canonical manner (fixing the fraction of magnetic sites) leads to a different result from those obtained from a disorder realized in a grand-canonical manner (see Table II, where the values for the second case are denoted by asterisk). The studies of Ref. 51 were based on the crucial observation that it is important to take into account the leading correction-to-scaling term in the infinite volume extrapolation of the MC data. The simulations confirmed the universality of the critical exponents of the $d=3$ RIM over a wide region of concentrations. In particular, the value of the correction-to-scaling exponent ω was found to be $\omega=0.37\pm 0.06$ which is almost half as large as the corresponding value in the pure $d=3$ Ising model $\omega=0.799\pm 0.011$.[3] The smallness of ω in the dilute case explains its importance for an analysis of the asymptotic critical behavior.

III. THE MODEL AND THE RENORMALIZATION-GROUP PROCEDURE

In this section, we define the RG procedure as well as the main quantities which we are going to calculate. Making use of the replica method[71] and taking the average over different configurations of quenched disorder it is possible to show that the RIM critical behavior in the Euclidean space of $d=4-\varepsilon$ dimensions is governed by an effective Hamiltonian with two coupling constants:[22]

$$\mathcal{H}(\varphi)=\int \mathrm{d}^d R\left[\frac{1}{2}\sum_{\alpha=1}^{n}\left[|\nabla\varphi_\alpha|^2+m_0^2\varphi_\alpha^2\right]+\frac{v_0}{4!}\left(\sum_{\alpha=1}^{n}\varphi_\alpha^2\right)^2+\frac{u_0}{4!}\sum_{\alpha=1}^{n}\varphi_\alpha^4\right], \tag{1}$$

in the limit $n\to 0$. Here, $\varphi_\alpha\equiv\varphi_\alpha(R)$ is the α's replica of a scalar field; $u_0\sim x>0$, $v_0\sim x(x-1)<0$ is the bare coupling constant of the fluctuation's effective interaction due to the presence of impurities; m_0 is a bare mass.

In order to describe the long-distance properties of the model (1) in the vicinity of the phase transition point we shall use the field-theoretical RG approach. The results for the RG functions corresponding to Eq. (1) are obtained on the basis of the dimensional regularization and the minimal subtraction scheme.[54] The renormalized fields, mass, and couplings ϕ, m, u, v are introduced by

$$\varphi = Z_\phi^{1/2}\phi,$$

$$m_0^2 = Z_{m^2} m^2,$$

$$u_0 = \mu^\varepsilon \frac{Z_{4,u}}{Z_\phi^2} u,$$

$$v_0 = \mu^\varepsilon \frac{Z_{4,v}}{Z_\phi^2} v.$$

Here, μ is the external momentum scale and Z_ϕ, Z_{m^2}, $Z_{4,u}$, $Z_{4,v}$ are the renormalization constants. They are determined by the condition that all poles at $\varepsilon = 0$ are removed from the renormalized vertex functions.

The RG equations are written bearing in mind that the bare vertex functions Γ_0^N are calculated with the help of the bare Hamiltonian (1) as a sum of one-particle irreducible (1PI) diagrams:

$$\Gamma_0^N(\{r\}) = \langle \varphi(r_1) \cdots \varphi(r_N) \rangle_{1\mathrm{PI}}. \tag{2}$$

The Γ_0^N do not depend on the scale μ, and therefore their derivatives with respect to μ at fixed bare parameters are equal to zero. So one gets

$$\mu \frac{\partial}{\partial \mu} \Gamma_0^N|_0 = \mu \frac{\partial}{\partial \mu} Z_\phi^{-N/2} \Gamma_R^N|_0 = 0, \tag{3}$$

where the index $|_0$ means a differentiation at fixed bare parameters. Then the RG equation for the renormalized vertex function Γ_R^N reads

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_u \frac{\partial}{\partial u} + \beta_v \frac{\partial}{\partial v} + \gamma_m m \frac{\partial}{\partial m} - \frac{N}{2} \gamma_\phi \right) \Gamma_R^N(m,u,v,\mu) = 0, \tag{4}$$

and the RG functions are given by

$$\beta_u(u,v) = \mu \left. \frac{\partial u}{\partial \mu} \right|_0,$$

$$\beta_v(u,v) = \mu \left. \frac{\partial v}{\partial \mu} \right|_0,$$

$$\gamma_\phi = 2\gamma_2(u,v) = \mu \left. \frac{\partial \ln Z_\phi}{\partial \mu} \right|_0,$$

$$\gamma_m(u,v) = \mu \left. \frac{\partial \ln m}{\partial \mu} \right|_0 = \frac{1}{2} \mu \left. \frac{\partial \ln Z_{m^2}^{-1}}{\partial \mu} \right|_0.$$

Using the method of characteristics the solution of the RG equation may be written formally as

$$\Gamma_R^N(m,u,v,\mu) = X(l)^{N/2} \Gamma_R^N[Y(l)m, u(l), v(l), \mu l], \tag{5}$$

where the characteristics are the solutions of the ordinary differential equations (flow equations):

$$l \frac{d}{dl} \ln X(l) = \gamma_\phi[u(l),v(l)], \quad l \frac{d}{dl} \ln Y(l) = \gamma_m[u(l),v(l)],$$

$$l \frac{d}{dl} u(l) = \beta_u[u(l),v(l)], \quad l \frac{d}{dl} v(l) = \beta_v[u(l),v(l)] \tag{6}$$

with

$$X(1) = Y(1) = 1, \quad u(1) = u, \quad v(1) = v. \tag{7}$$

For small values of l, Eq. (5) maps the large length scales (the critical region) to the noncritical point $l = 1$. In this limit the scale-dependent values of the couplings $u(l)$, $v(l)$ will approach the stable fixed point, provided such a fixed point exists.

The fixed points u^*, v^* of the differential equations (6) are given by the solutions of the system of equations:

$$\beta_u(u^*, v^*) = 0,$$

$$\beta_v(u^*, v^*) = 0. \tag{8}$$

The stable fixed point is defined as the fixed point where the stability matrix

$$B_{ij} = \frac{\partial \beta_{u_i}}{\partial u_j}, \quad u_i = \{u, v\} \tag{9}$$

possess eigenvalues ω_1, ω_2 with positive real parts. The stable fixed point, which is reached starting from the initial values in the limit $l \to 0$, corresponds to the critical point of the system. In the limit $l \to 0$ (corresponding to the limit of an infinite correlation length) the renormalized couplings reach their fixed point values and the critical exponents η and ν of the pair correlation function at T_c and of the correlation length, respectively, are then given by

$$\eta = 2\gamma_2(u^*, v^*),$$

$$1/\nu = 2[1 - \gamma_m(u^*, v^*)]. \tag{10}$$

In the nonasymptotic region deviations from the power laws with the fixed-point values of the critical exponents are governed by the correction-to-scaling exponent $\omega = \min(\omega_1, \omega_2)$ in accordance with the Wegner expansion.[73]

The rest of the critical exponents are obtained by familiar scaling laws. We note that the expression for correlation length exponent may be recast in terms of the renormalization constant Z_{ϕ^2} of the two-point vertex function with ϕ^2 insertion by a substitution $2\gamma_m = \gamma_\phi + \gamma_{\phi^2}$, which follows from the relations $Z_{m^2} = Z_{\phi^2} Z_\phi^{-1}$ and $\gamma_{\phi^2} = \mu \partial \ln Z_{\phi^2}^{-1} / \partial \mu|_0$.

IV. RESUMMATION AND THE RESULTS

In this section we analyze series for the RIM β functions and critical exponents. The RG functions of the corresponding effective Hamiltonian (1) in the replica limit $n = 0$ have been obtained up to five-loop order from the appropriate expressions for RG functions of the ϕ^4 theory with $O(n)$-symmetric and cubic interactions[53] and read

$$\beta_u/u = -\varepsilon + 3u + 4v - 17/3u^2 - 46/3vu - 82/9v^2 + 32.549\,682\,84u^3 + 123.198\,731\,3vu^2 + 158.181\,641\,8v^2u + 60.325\,268\,11v^3 - 271.605\,784\,2u^4 - 1318.116\,311vu^3 - 2452.429\,994v^2u^2 - 2003.560\,971v^3u - 559.714\,385\,4v^4 + 2848.568\,254u^5 + 16\,789.898\,43vu^4 + 40\,367.085\,93v^2u^3 + 48\,971.127\,30v^3u^2 + 29\,091.771\,79v^4u + 6377.751\,189v^5, \quad (11)$$

$$\beta_v/v = -\varepsilon + 8/3v + 2u - 14/3v^2 - 22/3vu - 5/3u^2 + 25.457\,148\,97v^3 + 62.254\,991\,70v^2u + 36.366\,455\,22vu^2 + 7u^3 - 200.926\,369\,0v^4 - 667.376\,189\,5v^3u - 650.564\,181\,6v^2u^2 - 259.258\,689\,1vu^3 - 39.912\,610\,12u^4 + 2003.976\,188v^5 + 8469.158\,907v^4u + 11\,721.608\,76v^3u^2 + 7434.635\,066v^2u^3 + 2344.277\,996vu^4 + 301.511\,097\,6u^5, \quad (12)$$

$$\gamma_2 = 1/18v^2 + 1/6vu + 1/12u^2 - 1/27v^3 - 1/6v^2u - 3/16vu^2 - 1/16u^3 + 125/648v^4 + 125/108v^3u + 145/72v^2u^2 + 65/48vu^3 + 65/192u^4 - 1.005\,978\,154v^5 - 7.544\,836\,154v^4u - 18.048\,546\,21v^3u^2 - 19.078\,389\,90v^2u^3 - 9.627\,924\,878vu^4 - 1.925\,584\,976u^5, \quad (13)$$

$$\gamma_m = 1/3v + 1/2u - 5/18v^2 - 5/6vu - 5/12u^2 + 37/36v^3 + 37/8v^2u + 251/48vu^2 + 7/4u^3 - 5.380\,801\,7v^4 - 32.284\,81v^3u - 57.177\,011v^2u^2 - 39.765\,731vu^3 - 9.978\,152\,53u^4 + 37.850\,485v^5 + 283.878\,638v^4u + 686.375\,317v^3u^2 + 737.493\,196v^2u^3 + 376.177\,633\,9vu^4 + 75.377\,774\,45u^5. \quad (14)$$

In order to obtain the qualitative characteristics of the RIM critical behavior one can proceed in two ways. The historically first scheme is known as ε expansion and consists (i) in expanding the values of the couplings at the stable fixed point in ε, (ii) inserting these expansions into the field-theoretic functions for the exponents (13) and (14), and (iii) expanding again in ε. Due to the degeneracy of the β functions (11) and (12) at one-loop level, an expansion in $\sqrt{\varepsilon}$ has to be performed.[22–24] Then in five-loop order on the basis of Eqs. (11)–(14) one obtains:[27,28]

$$\nu = 1/2 + 0.084\,115\,82\varepsilon^{1/2} - 0.016\,632\,03\varepsilon + 0.047\,753\,51\varepsilon^{3/2} + 0.272\,584\,31\varepsilon^2,$$

$$\eta = -0.009\,433\,96\varepsilon + 0.034\,943\,50\varepsilon^{3/2} - 0.044\,864\,98\varepsilon^2 + 0.021\,573\,21\varepsilon^{5/2}, \quad (15)$$

$$\gamma = 1 + 0.168\,231\,64\varepsilon^{1/2} - 0.028\,547\,08\varepsilon + 0.078\,828\,81\varepsilon^{3/2} + 0.564\,504\,90\varepsilon^2,$$

$$\omega_1 = 2\,\varepsilon + 3.704\,011\,194\,\varepsilon^{3/2} + 11.308\,738\,37\varepsilon^2, \quad (16)$$

$$\omega_2 = 0.672\,926\,585\,0\varepsilon^{1/2} - 1.925\,509\,085\varepsilon - 0.572\,525\,180\,6\varepsilon^{3/2} - 13.931\,259\,52\varepsilon^2.$$

Another method consists in (i) fixing the value of ε, i.e., the lattice dimensionality, (ii) solving the system of equation for the fixed point and (iii) substituting the fixed point values of the couplings into the series for the critical exponents[65] (the so-called 3D approach).

One should note here that often, for the sake of convenience, within the 3D as well as the ε-expansion approach one deals with the expansions of some combinations of the critical exponents instead of working with them directly on the basis of expressions (10). In the present paper the values of critical exponents are calculated from the expansion for $1/\nu - 1 = 1 - 2\gamma_m$ and the inverse exponent for magnetic susceptibility $\gamma^{-1} = (1-\gamma_m)/(1-\gamma_2)$. The numerical values of the other exponents are obtained by the familiar scaling laws.

It is well known that the perturbation-theory series for the RG functions in the weak-coupling limit as well as in ε expansion are asymptotic at best. In order to compare the results obtained on the basis of the $\sqrt{\varepsilon}$ expansion and of the 3D approach we have to refer to resummation procedures in the calculation of critical exponents. Adjusting the resummation procedure we discuss first the one-variable case in both schemes. We start from the ε expansion of the pure Ising model critical exponents which in the five-loop approximation reads:[74]

$$\nu = 1/2 + 1/12\varepsilon + 0.043\,209\,88\varepsilon^2 - 0.019\,043\,37\varepsilon^3 + 0.070\,883\,76\varepsilon^4 - 0.217\,017\,87\varepsilon^5,$$

$$\gamma = 1 + 1/6\varepsilon + 0.077\,160\,49\varepsilon^2 - 0.048\,974\,95\varepsilon^3 + 0.143\,574\,22\varepsilon^4 - 0.446\,624\,83\varepsilon^5. \quad (17)$$

An analysis of the ε-expansion case starts by representing the expressions for the critical exponents ν and γ of the pure Ising model (17) in the form of the Padé approximant: $[M/N](x) = \Sigma_{i=0}^{M} a_i x^i / \Sigma_{j=0}^{N} b_j x^j$ in the variable $x = \varepsilon$. The results are shown in the form of a Padé table (Table III). The number of the row N and of the column M corresponds to the order of the numerator and the denominator of the Padé approximant $[M/N]$, respectively. One can see from this table the expected convergence of the values in the diagonal and

TABLE III. The Padé table for the values of the correlation length (ν) and the susceptibility (γ) critical exponents from the ε expansion of the pure $d=3$ Ising model. Here and in Tables IV, VI, and VII the number of the row and of the column corresponds to the order of numerator and denominator of a Padé approximant, ''o'' means that the corresponding Padé approximant cannot be constructed.

ν	0	1	2	3	4	5
0	0.500	0.571	0.609	0.608	0.648	0.514
1	0.599	0.673	0.608	0.610	0.620	o
2	0.646	0.621	0.633	0.625	o	o
3	0.597	0.631	0.627	o	o	o
4	0.732	0.625	o	o	o	o
5	0.431	o	o	o	o	o
γ	0	1	2	3	4	5
0	1.000	1.167	1.244	1.195	1.339	0.892
1	1.200	1.310	1.213	1.231	1.230	o
2	1.276	1.230	1.243	1.230	o	o
3	1.171	1.242	1.235	o	o	o
4	1.440	1.227	o	o	o	o
5	0.845	o	o	o	o	o

first off diagonal. From this we estimate the values of the critical exponents to be $\nu=0.628, \gamma=1.236$. These values can be compared with the most accurate values $\nu=0.628 \pm 0.001, \gamma=1.234 \pm 0.002$,[75] obtained by means of more sophisticated resummation procedures[76] on the basis of the five-loop ε expansion. We conclude that this good agreement justifies the application of the Padé analysis for the ε-expansion series (17).[77]

Possessing the information on the asymptotic divergence of the ε expansion we apply a more complicated Padé-Borel resummation technique which takes into account the factorial divergence of the series terms. The resummation procedure consists in several steps:

- starting from the initial sum S of L terms one constructs its Borel-Leroy image,

$$S=\sum_{i=0}^{L} a_i x^i \Rightarrow \sum_{i=0}^{L} \frac{a_i (xt)^i}{\Gamma(i+p+1)}, \tag{18}$$

  where $\Gamma(x)$ is the Euler's gamma function and p is an arbitrary non-negative number.
- the Borel-Leroy image (18) is extrapolated by a rational Padé approximant

$$[M/N](xt).$$

- the resummed function S^{res} is obtained in the form

$$S^{res}=\int_0^{\infty} dt \exp(-t) t^p [M/N](xt). \tag{19}$$

The values of the critical exponents ν and γ obtained with the Padé-Borel resummation for different M and N are given in Table IV. As far as the Padé approximant enters the integral, it might happen that the integrand contains poles. If this is the case, the corresponding number in Table IV is calculated by analytic continuation taking the principal values of the integral. For the sake of completeness we include such numbers in Table IV as well as in the forthcoming Table VII. However, the final results will be displayed on the basis of data which did not require such analytic continuation. Except from the [1/1] case all our final results are obtained from approximants with a linear denominator (second columns), which reconstitutes the sign-alternating behavior of the initial series (17). For the Borel resummed Padé approximant [4/1] the estimates for the exponents $\nu=0.629, \gamma=1.236$ are in a good agreement both with the above Padé analysis (Table III) as well as with the data of Ref. 75 given above.

TABLE IV. The results of the Padé-Borel resummation of the correlation length (ν) and the susceptibility (γ) critical exponents from the ε expansion of the pure $d=3$ Ising model. Here and in Tables VII and VIII the ^{c} superscript denotes that the real part of the corresponding value is given.

ν	0	1	2	3	4	5
0	0.500	0.560	0.584	0.592	0.599	0.601
1	0.600	0.699^c	0.604	0.495^c	0.622^c	o
2	0.645	0.623	0.631^c	0.628	o	o
3	0.597	0.629	0.629	o	o	o
4	0.731	0.629	o	o	o	o
5	0.431	o	o	o	o	o
γ	0	1	2	3	4	5
0	1.000	1.147	1.205	1.208	1.232	1.204^c
1	1.200	1.359^c	1.208	1.205^c	1.221	o
2	1.276	1.233	1.238^c	1.234	o	o
3	1.171	1.238	1.236	o	o	o
4	1.440	1.234	o	o	o	o
5	0.845	o	o	o	o	o

In order to complete the study of the pure Ising model we perform an analysis based on the 3D approach. We resum the corresponding RG functions of the pure model [they can be obtained by putting $v=0$ in the diluted model RG functions (11)–(14)] by means of the Padé-Borel resummation technique with a linear denominator approximant (see the Appendix). The results obtained with this method are shown in Table V. One should compare them with the results obtained recently from the RG functions in the 3D massive field-theoretical approach, $\nu=0.6304 \pm 0.0013$, $\gamma=1.2396 \pm 0.0013$,[3] and the results of 3D minimal subtraction scheme

TABLE V. The results of the application of the Padé-Borel resummation ([L/1]) to the RG functions of the pure $d=3$ Ising model. Fixed point coordinate and the critical exponents of the pure $d=3$ Ising model obtained by Padé-Borel resummation in 3D scheme.

Loop	u^*	γ	ν	α	η	ω
2	0.6573	1.269	0.644	0.068	0.031	0.566
3	0.4641	1.231	0.623	0.131	0.024	0.853
4	0.4958	1.239	0.632	0.104	0.040	0.756
5	0.4877	1.246	0.634	0.097	0.036	0.792

TABLE VI. The Padé table for the values of the correlation length (ν) and the susceptibility (γ) critical exponents from the $\sqrt{\varepsilon}$ expansion of the $d=3$ RIM.

ν	0	1	2	3	4
0	0.500	0.572	0.570	0.601	0.727
1	0.601	0.570	0.572	0.564	*o*
2	0.560	0.586	0.565	*o*	*o*
3	0.640	0.541	*o*	*o*	*o*
4	1.828	*o*	*o*	*o*	*o*
γ	0	1	2	3	4
0	1.	1.168	1.140	1.219	1.783
1	1.202	1.144	1.161	1.127	*o*
2	1.125	1.172	1.137	*o*	*o*
3	1.257	1.101	*o*	*o*	*o*
4	3.824	*o*	*o*	*o*	o

in the five-loop approximation $\nu=0.629\pm0.005, \gamma=1.235\pm0.005$.[65] Comparing all the results for the pure Ising model we conclude that (i) the application of Padé-Borel resummation technique yields accurate results and (ii) both the ε expansion and 3D approach lead to reliable values for the exponents.

We now turn to the results of the Padé and the Padé-Borel resummation techniques applied to the $\sqrt{\varepsilon}$ expansions (15) and (16) and the 3D approach RG functions (11)–(14) of the weakly diluted Ising model. We construct Padé approximants and perform the Padé-Borel resummation introducing an auxiliary variable t in the expressions (15) and (16) by the substitution $\varepsilon\rightarrow\varepsilon t^2$ and putting $t=1$ in the final results.

The appropriate values for the $\sqrt{\varepsilon}$ expansions are listed in Tables VI and VII in the same notations as in the Tables III and IV. One can see from the tables that neither the experimental nor the reliable theoretical values listed in Sec. II are obtained. Moreover, considering the expansions for the stability matrix eigenvalues (16) it turns out that no stable fixed point exists in a strict $\sqrt{\varepsilon}$ expansion, even with resummation.

TABLE VII. The results of the Padé-Borel resummation of the correlation length (ν) and the susceptibility (γ) critical exponents from the $\sqrt{\varepsilon}$ expansion of the $d=3$ RIM.

ν	0	1	2	3	4
0	0.500	0.560	0.569	0.577	0.592
1	0.601	0.573	0.560[c]	0.565[c]	*o*
2	0.560	0.584	0.568[c]	*o*	*o*
3	0.639	0.529[c]	*o*	*o*	*o*
4	1.828	*o*	*o*	*o*	*o*
γ	0	1	2	3	4
0	1.000	1.149	1.148	1.176	1.252
1	1.202	1.148	1.149	1.141[c]	*o*
2	1.125	1.168	1.141[c]	*o*	*o*
3	1.257	1.086[c]	*o*	*o*	*o*
4	3.825	*o*	*o*	*o*	*o*

Comparing with the corresponding data for the pure Ising model we conclude that the nature of the $\sqrt{\varepsilon}$-expansion series does not allow to obtain reliable information at $d=3$ by means of the methods mentioned for the case of pure Ising model. This can be considered as an indirect evidence of the nonasymptotic nature of the $\sqrt{\varepsilon}$ expansion. Thus a different kind of analysis for the $\sqrt{\varepsilon}$ expansion has to be developed. The fact that ε expansion will not be able to give information on critical exponents in system with quenched disorder was predicted already in Refs. 61 and 62. There, studying the randomly diluted model in zero dimensions, it was shown that the non-Borel summable properties of the perturbation-theory series are the direct consequence of the existence of Griffiths-like singularities[68] caused by the zeroes of the partition function of the pure system.

In order to treat the theory directly at $d=3$ we need a generalization of the Padé-Borel resummation technique to the case of two variables since the RG functions of the weakly diluted Ising model depend on two couplings. The corresponding Chisholm-Borel resummation technique can be defined as follows:[31]

- constructing the Borel-Leroy image of the initial Lth-order polynomial S in the variables u and v,

$$S=\sum_{0\leq i+j\leq L} a_{i,j}u^i v^j \Rightarrow \sum_{0\leq i+j\leq L} \frac{a_{i,j}(ut)^i(vt)^j}{\Gamma(i+j+p+1)}, \qquad (20)$$

  where $\Gamma(x)$ is the Euler's gamma function and p is an arbitrary non-negative number.
- extrapolating the Borel-Leroy image (20) by a rational Chisholm approximant[78] $[M/N](ut,vt)$ which can be defined as a ratio of two polynomials both in variables u and v, of degree M and N so that the first terms of its expansion are equal to those of the function which is approximated.
- the resummed function S^{res} then reads

$$S^{res}=\int_0^\infty dt\exp(-t)t^p[M/N](ut,vt). \qquad (21)$$

Here, similarly to the pure case, we restrict the approximants to linear denominators and choose the value of the fitting parameter $p=0$. The motivation of such a choice is discussed in detail in the Appendix.

Treating the β functions by means of this resummation technique leads to a random fixed point, $u^*>0, v^*<0$, of the model already in the two-loop approximation. The stability analysis shows that this fixed point is stable proving the crossover to a new critical regime under dilution. In Figs. 1 and 2 we show the curves $\beta_u(u,v)=0$, $\beta_v(u,v)=0$ in the u-v plane. The intersections of these curves (i.e., simultaneous zeros of both β functions) correspond to the fixed points, the stable and unstable points are marked by open circles and filled boxes, respectively. The ''naive'' analysis of the β functions, without applying any resummation procedure leads to the curves, which are shown on the left-hand side of Figs. 1 and 2. Without resummation only in the three-loop approximation one gets stable a random fixed point $u^*\neq0, v^*\neq0$. However, the fixed point disappears in the four-loop approximation. A completely different picture is observed when the resummation procedure is applied (right-hand columns of the figures). In the region of interest for the

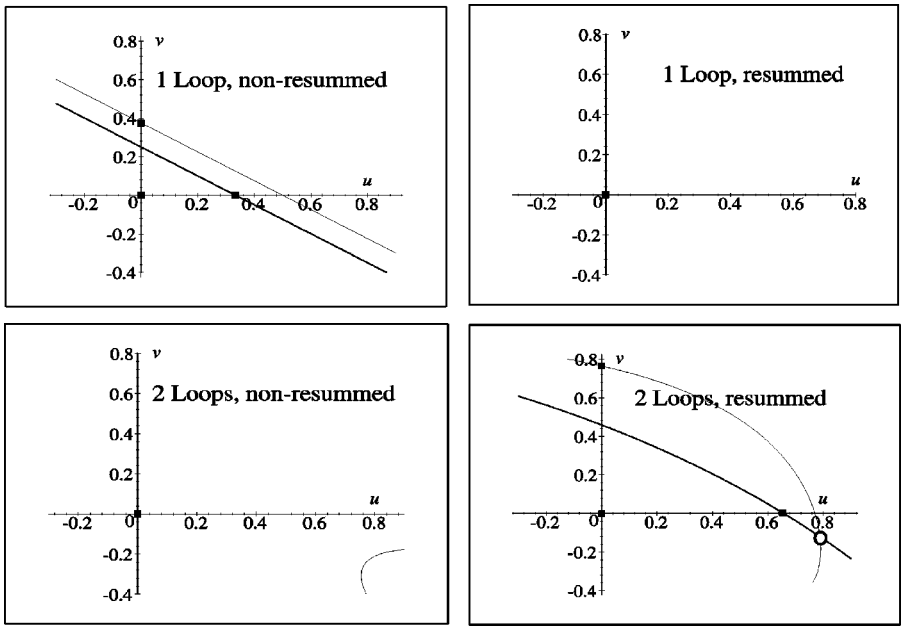

FIG. 1. The lines of zeros of nonresummed (left-hand column) and resummed by the Chisholm-Borel method (right-hand column) β functions in different orders of the perturbation theory: one- and two-loop approximations. Thick line corresponds to $\beta_u=0$, thin line depicts $\beta_v=0$. One can see the appearance of the random fixed point $u^*>0, v^*<0$ in the two-loop approximation for the resummed β functions. Stable fixed point is shown by an open circle, unstable ones are shown by filled boxes.

values of the couplings the topology of the fixed point picture remains stable increasing the order of approximation from the two-loop to the four-loop level; the same behavior has been observed[34] for the β functions obtained in the massive 3D scheme.[35]

The corresponding values of the random fixed point coordinates, critical exponents, and eigenvalues of the stability matrix are listed in Table VIII. One can see that increasing the order of approximation one reaches convergent results, compatible with experimental and theoretical data (see Sec. II).

Considering the estimation of the accuracy of the results we note that setting $v=0$ in the RG functions (11)–(14) they are transformed into the appropriate functions of the pure Ising model. In this case the deviation of our four-loop result (see Table V) from the most accurate one obtained within massive field RG scheme on the basis of six-loop expansion for β functions and seven-loop expansion for γ functions[4] is within parts of a percent. This can be considered as an upper bound for the accuracy of RIM results. Here the comparison of our data with the results of four-loop massive scheme results[35,36] yields an accuracy of several percent. Since the series for β and γ functions (11)–(14) are sign alternating, the unknown exact stable point coordinates and critical exponents must lie between the three- and four-loop values giving the same estimate (see the Appendix about adjusting a free parameter for the fastest convergence of results). Thus we conclude the accuracy of the RIM critical exponents obtained in our study to be of order of several percent.

A peculiarity of the Table VIII is that in five-loop order the applied resummation technique does not lead to a real root for the random fixed point. It is expected that the applications of more sophisticated resummation procedures incorporating the higher-order behavior, still unavailable, will permit an improvement of the estimates for the critical exponents in the two-, three-, and four-loop level as well as

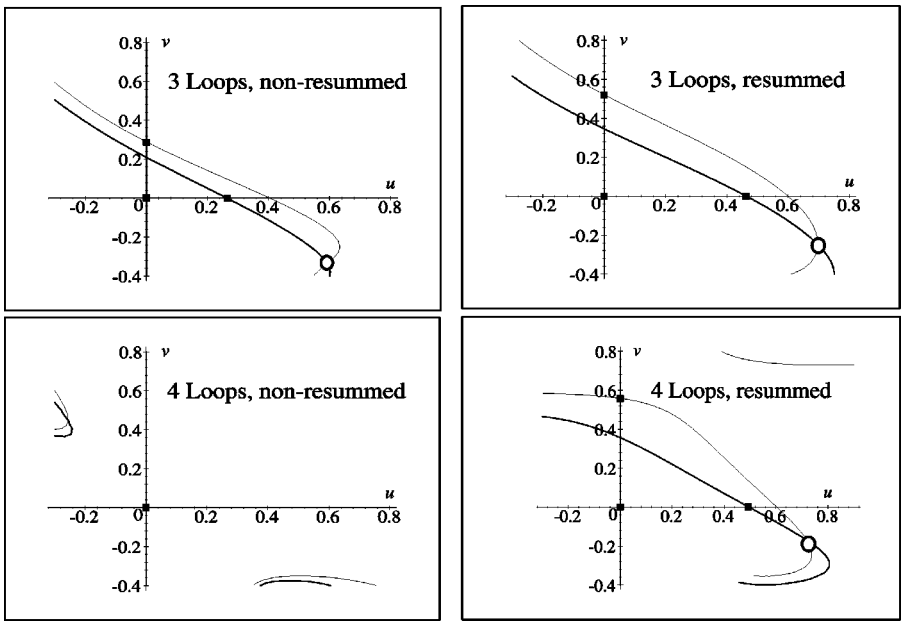

FIG. 2. The lines of zeros of nonresummed (left-hand column) and resummed by the Chisholm-Borel method (right-hand column) β functions in three- and four-loop approximations. The notations are the same as in Fig. 1. Close to the random fixed point the behavior of the resummed functions remains alike with the increase of the order of approximation. This is not the case for nonresummed functions.

TABLE VIII. Fixed-point coordinates, critical exponents, and stability matrix eigenvalues of the $d=3$ RIM obtained by Chisholm-Borel resummation in 3D scheme.

Loop	u^*	v^*	γ	ν	α	η	ω_1	ω_2
2	0.7886	-0.1208	1.308	0.665	0.006	0.032	0.162	0.542
3	0.6968	-0.2484	1.293	0.654	0.039	0.022	0.430	0.974
4	0.7188	-0.1697	1.318	0.675	-0.026	0.049	0.390[c]	0.390[c]

to obtain them in the five-loop level. However, it is not excluded that the absence of a fixed point solution on the five-loop level might be connected with a (possible) Borel nonsummability of the series under consideration. In this case the four-loop approximation will be an ''optimal truncation'' for the resummed perturbation theory series, similar to the nonresummed asymptotic series, e.g., in the ε expansion of $O(n)$-symmetric ϕ^4 model, where ''naive'' interpretation of the series truncated by ε^2 term leads to the best (optimal) result. On the other hand, numerical and analytic studies of a toy model of a disordered system[61,62] revealed two possible regimes of the high-order behavior: the first one corresponds to a Borel-summable series whereas the second one does not correspond to a Borel-summable series. Numerical studies of up to 200 terms of the expansion[61] resulted in the conclusion, that the convergence of the Borel-resummed results depends on the strength of disorder (relation of the couplings u/v). The convergence of the numerical data of Table VIII is evidence of the fact that the fixed point values u^*, v^* in $d=3$ lie in a region, where the resummed series gives reliable information. In any case, on the basis of the above analysis one can definitely say: while for the pure Ising model the ε expansion and the 3D approach analysis of the RG functions are of equal usefulness, an interpretation of these functions in the diluted model can be done only within the framework of the 3D approach. The application of the $\sqrt{\varepsilon}$ expansion remains to be valid only near the upper critical dimension $d=4$. This conclusion holds at least within the discussed resummation schemes.

V. EFFECTIVE CRITICAL BEHAVIOR

If one is not within the asymptotic region of the stable fixed-point power laws for the physical quantities may only apply with effective exponents. The critical behavior is then to be understood as a crossover behavior between the uncritical background behavior and the true asymptotic behavior. As it has been noted in Ref. 37 this has, in principle, nothing to do with crossovers between special fixed points (e.g., the pure one and the random one). However, depending on the nonuniversal parameters entering the nonasymptotic behavior, the crossover may be more or less influenced by the unstable pure fixed point.

The effective exponents are defined by logarithmic derivatives of corresponding thermodynamical quantities with respect to reduced temperature τ.[52] In the RG scheme they are calculated in the region where couplings have not yet reached their asymptotic (fixed point) values and depend on the flow parameter l. For instance, the magnetic susceptibility effective exponent γ_{eff} is defined by

$$\gamma_{eff}(\tau)=\frac{d\ln\chi(\tau)}{d\ln t}=\gamma\{u[l(\tau)],v[l(\tau)]\}+\cdots, \quad (22)$$

where the second part is proportional to the β functions and comes from the change of the amplitude part of the susceptibility. In order to proceed we have to neglect this part since we do not know the amplitude function in the same orders as the field-theoretical functions for the exponents. Moreover, the contribution of the amplitude function to the crossover seems to be small, at least in other cases.[79,80] This approximation has also been used in the earlier work on diluted models.[37]

The flow parameter $l(\tau)$ may be related to the temperature via the matching condition $m(l)=(\xi_0^{-1}l)^2$ and $m_0^2\sim\tau$. However, we discuss the effective exponents as functions of the flow parameter. Then the effective exponents are simply given by the expressions for the asymptotic exponents (10) but with replacing the fixed-point values of the couplings u^* and v^* by the solutions of the flow equations (6):

$$1/\nu_{eff}=2\{1-\gamma_m[u(l),v(l)]\},$$

$$\eta_{eff}=2\gamma_2[u(l),v(l)], \quad \gamma_{eff}=\nu_{eff}(2-\eta_{eff}). \quad (23)$$

These solutions are shown in Fig. 3 for several initial conditions. Shown are the two unstable fixed points: G (Gaussian fixed point) and P (pure Ising fixed point) and the stable random fixed point R, with both couplings nonzero. In the background region the couplings are expected to be small thus all the flows shown start near the Gaussian fixed point

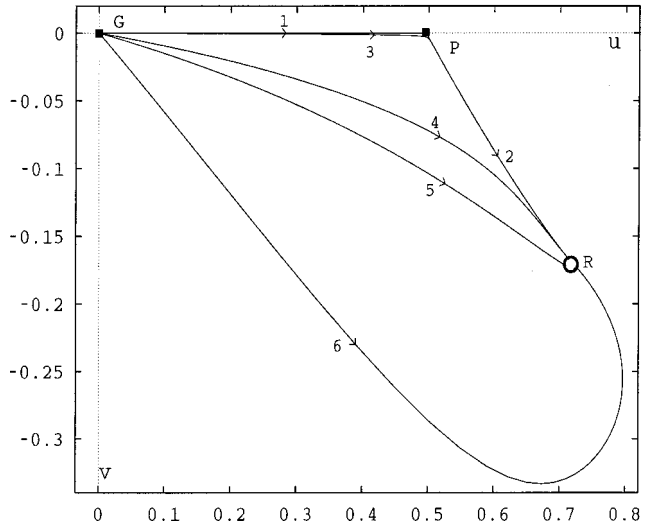

FIG. 3. Flow lines for the $d=3$ RIM. Fixed points G and P are unstable, fixed point R is the stable one.

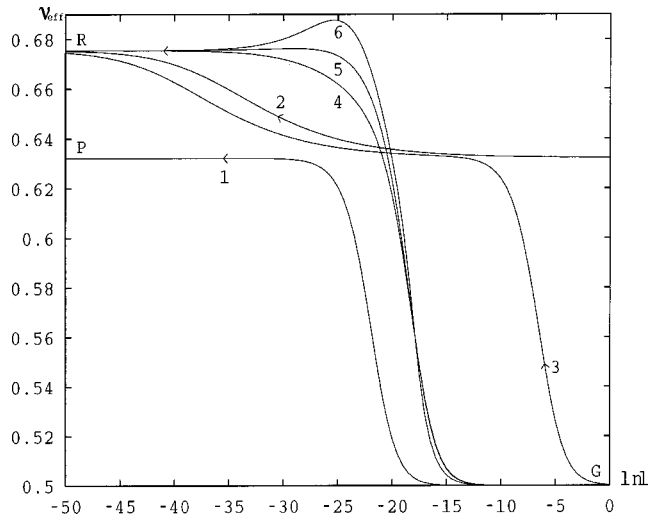

FIG. 4. Effective exponent ν_{eff} versus logarithm of the flow parameter l for the flows shown in Fig. 3.

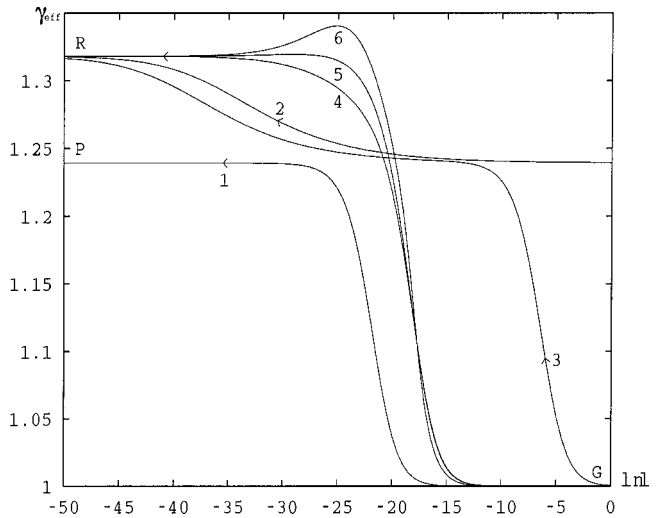

FIG. 5. Effective exponent γ_{eff} versus logarithm of the flow parameter l for the flows shown in Fig. 3.

with different ratios $v(l_0)/u(l_0)$ (curves 1 to 6 except curve 2). Curve 1 is the separatrix connecting the fixed point G with P, and curve 2 is the separatrix connecting P with R. All curves starting with a negative coupling v remain negative but the dependence might be nonmonotonous (see curve 6). Thus several scenarios for the values of the effective critical exponents are possible (see Figs. 4 and 5).

Both in experiment as well as in computer simulations (see Tables I and II) exponents reported differ and even exceed the known asymptotic theoretical values. This nonuniversal behavior might be related to the possible nonasymptotic behavior found in our different flows as has been suggested in Refs. 37 and 46. The difference might be due to (i) the different temperature regions of the experiment and/or (ii) the different concentrations. The initial values for the couplings in the flow equations depend on the value of the concentration, especially for small dilution one expects the coupling v to be proportional to the concentration. If this is the case we expect a monotonous increase of the values of the effective exponent to the asymptotic value. A typical scenario is seen in curve 3 of Figs. 3–5. In this case effective exponents equal to the pure model critical exponents might be found in relatively wide region of temperature. Then as the attraction region of the fixed point P becomes weaker and weaker, an overshooting is possible and effective exponents larger its asymptotic values might be found. This scenario is predicted for larger dilutions and represented by the curve 6 of Figs. 3–5. Curves 5 and 6 correspond to situation when crossover from the mean-field behavior towards the random one is not influenced by the presence of a pure fixed point.

VI. CONCLUSIONS

We studied the critical behavior of the three-dimensional weakly diluted Ising model asymptotically close to the critical point, in the intermediate region, and far from it. To this end we calculated the values of the asymptotical critical exponents, analyzed the behavior of the effective critical exponents and obtained the value of the correction-to-scaling exponent entering the Wegner expression which describes the approach of singular quantities to the critical temperature. Our study is based on the five-loop minimal subtraction field-theoretical renormalization-group functions[53] of the ϕ^4 theory with $O(n)$-symmetric and cubic interactions which in the replica limit $n\to 0$ correspond to the diluted Ising model case. As the minimal subtraction scheme allows to develop either the $\sqrt{\varepsilon}$ expansion or the 3D approach, we adopted the both schemes and compared the results obtained on their basis.

The perturbation theory RG functions series are asymptotic at best. In order to calculate the critical exponents we adopt different resummation procedures within the both schemes, testing them on the well established case of the ϕ^4 model with one coupling (pure Ising model). While after the resummation both ε expansion and 3D approach give reliable results for the numerical values of the pure model critical exponents, in the case of the diluted Ising model the bad convergence properties of the $\sqrt{\varepsilon}$ expansion do not allow to obtain reliable values at $d=3$. Using the direct resummation of the RG functions at $d=3$ in the minimal subtraction scheme we obtained numerical values for the asymptotic critical exponents of the diluted Ising model and estimated their accuracy to be of order of several percent. The results obtained in the 3D approach agree well with other theoretical and recent Monte Carlo simulations. Studying the crossover effect we calculated the effective critical exponents and their flows in the nonasymptotic region. We observed several scenarios of crossover in the RIM including: monotonous crossover from the mean-field values of critical exponents to the random ones; existence of a wide temperature region where the RIM effective exponents coincide with the asymptotic exponents of the pure Ising model; possible values of effective exponents exceeding those of asymptotic ones.

Though the 3D approach of calculation encountered difficulties in the five-loop level we guess that the fixed d approach, both within the massive[60] and minimal subtraction[54,65] schemes, remains the only reliable way to study critical behavior of the model by means of the RG technique.

Note added in proof. During processing of the article several new results appeared in the field. The massive scheme four-loop RG functions of the RIM (Refs. 35,36) were ex-

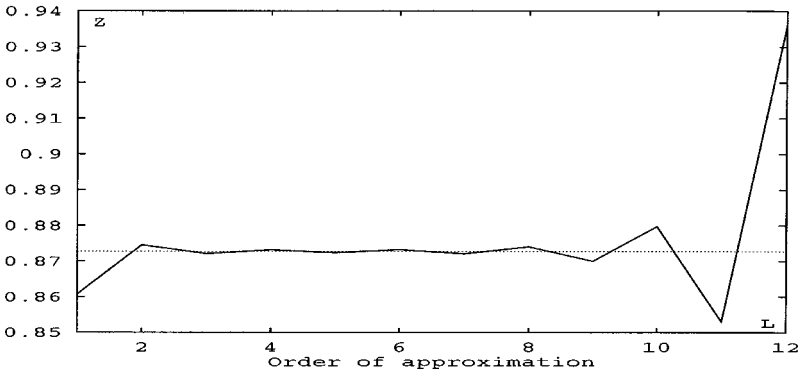

FIG. 6. The estimate of the toy-model partition function Z at $u=1/10$ and $v=-1/100$ in dependence of the order of perturbation theory L in couplings u and v. The application of Padé-Borel resummation with linear denominator approximants of type $[N/1]$ (solid line) provides convergence to the exact value (dotted line) only for some first orders of approximation.

tended to the five-loop[91] and six-loop levels.[92] While the traditional analysis of the first via PB resummation allowed to obtain numerical values for the RIM asymptotic critical exponents,[91] the method failed for the higher-order functions.[93] However, an application to the functions of a refined resummation procedure which treats renormalized couplings of the RG functions asymmetrically[94] reconstituted and improved[95] earlier data for the RIM asymptotic critical exponents.

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APPENDIX: THE CHOICE OF THE RESUMMATION PROCEDURE

In order to obtain reliable quantitative description of the problem under consideration one should adjust precisely resummation procedures necessary for analysis of RG functions. Since no information is available about the high-order behavior of the series for β and γ functions (11)–(14) (compare with the $d=2$ and $d=3$ scalar ϕ^4 model where the Borel summability of the RG functions is proven[81,77] and the large order behavior is calculated[82,83]), we reject all powerful methods implementing such an information (e.g., resummation refined by a conformal mapping, widely used in the models of critical phenomena with one coupling[84,4]) and restrict ourselves to the simplest resummation techniques which are the generalization of the Padé-Borel technique (18) and (19) to the two variable case. Among them we choose that procedure, which, for the different orders of approximation, provides the maximal stability. That means the fastest convergence of the results as well as the maximal similarity of the topological structure of the lines defined by the zeros of the β functions.

1. A $d=0$ theory

Let us start tuning the resummation technique by considering the expressions for a toy model. It is defined by the partition function

$$Z=\frac{1}{\pi^{n/2}}\int_{-\infty}^{+\infty} d\phi_1\cdots d\phi_n \times\exp\left[-\sum_{i=1}^{n}\phi_i^2-u\left(\sum_{i=1}^{n}\phi_i^2\right)^2-v\sum_{i=1}^{n}\phi_i^4\right],$$

which corresponds to the cubic model described by the Hamiltonian (1) in dimension $d=0$. One can easily calculate the sufficient number of terms representing Z in the form of a series in u and v for arbitrary n. However, the simplest case which reproduces the series in two couplings is $n=2$ since the case $n=0$ is trivial and $n=1$ corresponds to a series in a single variable $u+v$.

We write the series for Z in the two variables u and v as a series in one auxiliary variable τ introduced by the substitution $u\to u\tau$, $v\to v\tau$ (the so-called resolvent series[85]) and apply the Padé-Borel (PB) method (18) and (19) to the series in τ. Here we choose two possibilities of the approximants, one with linear denominator and the other one of diagonal type. Moreover, we fix the parameter p (18) to $p=0$ but discuss other choices for the parameter later on. The results of this procedure are shown in Figs. 6 and 7 for the specific values $u=0.1$ and $v=-0.01$. One notes that a stable con-

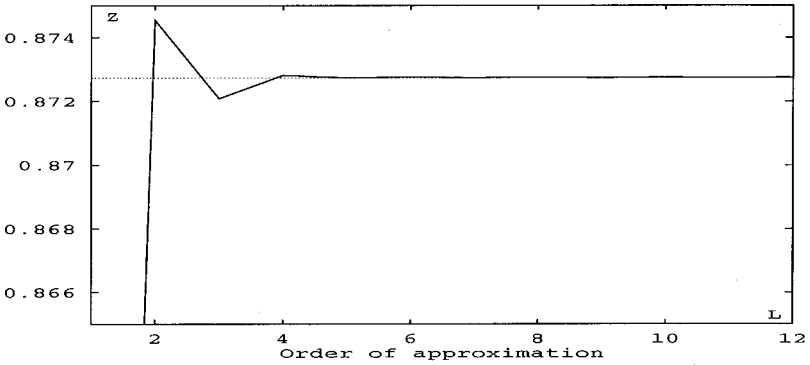

FIG. 7. The estimate of the toy-model partition function Z at $u=1/10$ and $v=-1/100$ in dependence of the order of perturbation theory L in couplings u and v. The application of Padé-Borel resummation with diagonal approximants of type $[N/N]$ or $[(N+1)/N]$ (solid line) provides perfect convergence of the estimate to the exact value (dotted line).

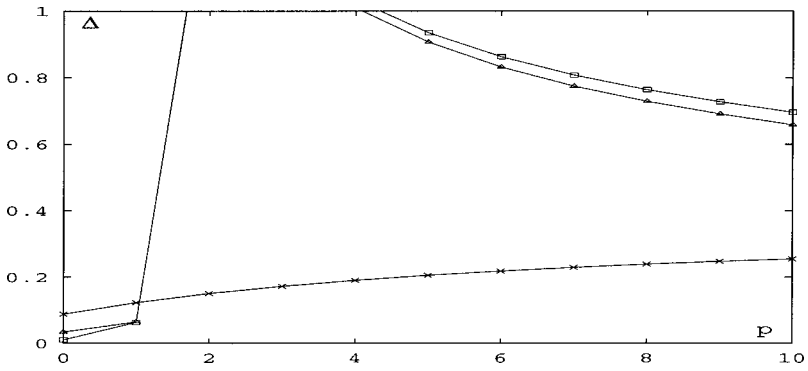

FIG. 8. The dependence of a measure Δ of total deviation between three- and four-loop pure Ising model results (triangles) as well as of three- and four-loop RIM results (crosses) on fitting parameter p. One can see the minimum for $p=0$. The deviation between five- and four-loop results (boxes) for pure Ising model confirms $p=0$, too.

vergence to the exact value takes place only for resummations with diagonal approximants, while the approximants with linear denominator converge only for certain, ''optimally truncated,'' orders of approximation. The larger the value of the variables u and v the less is the order of ''optimal truncation.'' We believe that processing the divergent series (11)–(14) in the same manner one may encounter a similar difficulty.

Analyzing the toy-model series by the Chisholm-Borel (CB) scheme (20) and (21), one notes that the coefficients in the Chisholm approximant are underdetermined. For example, for the L-loop sum an approximant of the linear-denominator type $[L/1]$ being uniquely defined requires two additional equations. This is the minimal number of additional conditions, necessary to determine any approximant for $L=2\cdots5$ except for $L=4$, where the $[3/2]$ approximant is determined uniquely without additional conditions. Since, generally speaking, a near-diagonal Chisholm approximant requires additional conditions the number of which depends on the order of approximation, we reject this type of approximants and consider in the following only the $[L/1]$-type approximants. The two additional equations have to be symmetric in the variables u and v, otherwise the properties of the symmetry related to these variables would depend on the method of calculation. By the substitution $v=0$ all the equations describing the critical behavior of the diluted model are converted into the appropriate equations of the pure model. However, if a pure model is solved independently, the resummation technique uses the Padé approximant. Thus the Chisholm approximant has to be chosen in such a way that, for either $u=0$ or $v=0$ one obtains the Padé approximant of the one-variable case. This also needs a special choice of the additional conditions. This is achieved by choosing the Chisholm approximant $[L/1]$ with the numerator coefficients at u^L and v^L equal to zero.

The analysis of the toy-model series by means of this type of CB technique leads to the existence of an ''optimally truncated'' order of approximation similar to the one shown in Fig. 7.

2. A $d=3$ theory

Let us turn back now to the expressions for the RG functions (11)–(14) at $d=3$. Starting from the PB analysis one finds that diagonal-type approximants lead to poles on the real axis. This may be due to the fact that approximants do not reconstitute the sign-alternating high-order behavior of the general term of the RG functions, which was confirmed in the particular cases $n=2$ and $n=3$.[86] Thus for the further discussions we proceed with approximants of linear-denominator type.

Applying the PB method with linear-type approximants we find that the random fixed point for the $d=3$ RIM cannot be reconstituted already in the three-loop approximation, while the random fixed point exists even for nonresummed β functions in this approximation. The picture does not change

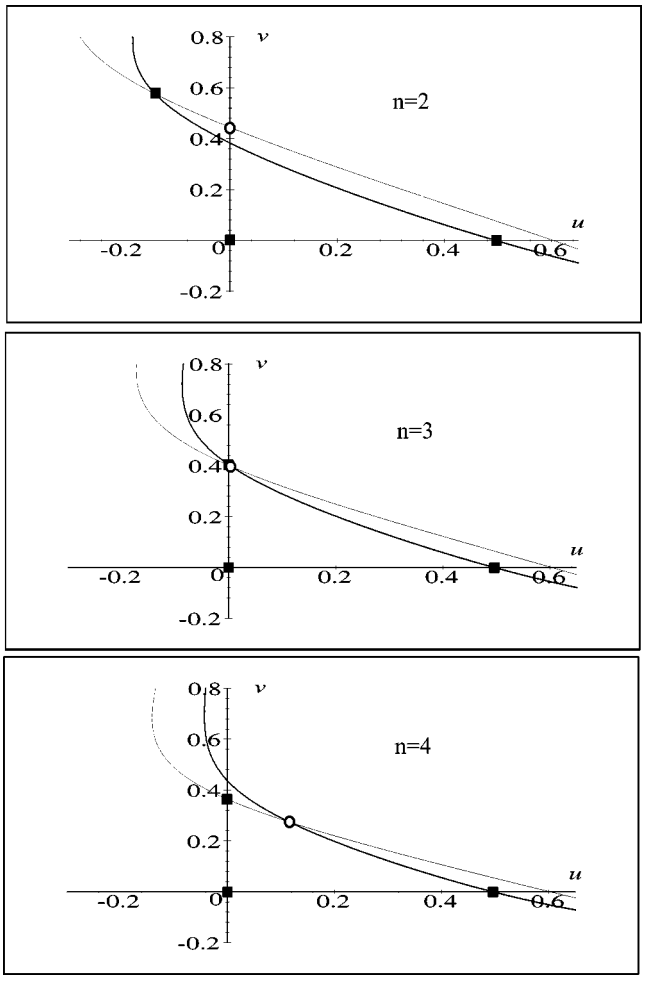

FIG. 9. The lines of zeros of the cubic model β functions resummed by the Chisholm-Borel method in four-loop approximation for different n. Thick lines correspond to $\beta_u=0$, thin lines depict $\beta_v=0$. The filled boxes and open circles show the positions of unstable and stable fixed points, respectively. One can see that the crossover to a new behavior appears for values of n very close to 3. Our estimate yields $n_c=2.950$ (see the text).

qualitatively when we try to increase effectively the order of polynomial representation for the β functions by resumming expressions $[1+r_u(u+v)]\beta_u, [1+r_v(u+v)]\beta_v$ with r_u, r_v taken as fit parameters (compare with Ref. 4). The modified construction of Ref. 37 for the Padé approximant with a linear denominator reconstituting the known large order behavior of the one coupling β functions, $\beta_u(u, v=0)$ and $\beta_v(u=0, v)$, does not lead to the appearance of the random fixed point either. We conclude, although the PB scheme works in two- and four-loop approximations, it appears to be unstable in the three-loop approximation and therefore we eliminate it from our consideration.

The CB method reconstitutes the random fixed point in two-, three-, and four-loop approximation, however, it fails in the five-loop approximation. In order to reestablish the random fixed point we have varied the values of p as well as of r_u, r_v, but even then we find no region for $r_u>0, r_v>0$ so that the random fixed point exists in all orders of perturbation theory. We also predicted the values of the sixth-order contribution for β functions (see Refs. 87 and 88), but the resummation of such pseudo-six-loop functions did not allow to find the random fixed point either. Again a modified construction of the Chisholm approximant with the known linear denominator[37] failed. Comparing the behavior of the toy-model series (see Fig. 6) and the convergence of the results of the RIM (Table VIII) we consider $L=4$ as the ''optimal truncation'' order within the CB scheme of linear-denominator approximant.

Once we have chosen the CB method based on Chisholm approximants of $[L/1]$ type as the tool for analyzing the RIM RG functions we look now for the fastest convergence of the results with increase of the order of approximation in number of loops L. To this end we fit the parameters p entering Borel-Leroy images (20) of RG functions. For the RIM we introduce a measure of total deviation between L- and L'-loop results by a function $\Delta^2=(u^{*,L}-u^{*,L'})^2+(v^{*,L}-v^{*,L'})^2+(\gamma^{*,L}-\gamma^{*,L'})^2+(\nu^{*,L}-\nu^{*,L'})^2$, where the superscripts $L(L')$ indicate a value obtained in $L(L')$-loop approximation. For the pure Ising model an appropriate measure is given by a similar function with $v^*=0$. We adjust now the parameter p to minimize Δ. The behavior of $\Delta(p)$ is shown in Fig. 8. A minimum of Δ for the pure Ising model is achieved for $p=0$ (curves pointed by boxes and triangles in Fig. 8) in both cases $L=4, L'=3$ and $L=5, L'=4$. Similar behavior in p is observed for the RIM (curve pointed by crosses) suggesting the choice $p=0$ as well.

3. A cubic model

Since originally the RG functions under consideration (11)–(14) were obtained in order to study the critical behavior of the cubic model introduced by the effective Hamiltonian (1), we will use it for another test of the resummation procedure. In the case of nonzero values of n the critical behavior of this system is governed either by the $O(n)$-symmetric fixed point for values of n small enough or by the cubic fixed point for $n>n_c$, where n_c is the marginal value of the order-parameter component. The identification of the marginal dimension n_c as well as of the critical exponents governing the phase transition of the model was recently performed within the ε expansion.[53] The same model was studied by means of the massive RG approach.[35] The numerical value of n_c was found to be only slightly less than $n=3$ leading to values of the critical exponents of the cubic model practically indistinguishable from those of the $O(3)$-symmetric model. Recent MC simulations strongly suggest $n_c=3$ although $n_c<3$ cannot be excluded.[89] We present estimations of n_c obtained from the conditions that the $O(n)$ symmetric and the cubic fixed point of the resummed β functions coincide. This calculation has two advantages: (i) one tests once more the resummation methods and, (ii) a new estimate of n_c is obtained from the 3D approach within minimal subtraction scheme. Up to now n_c has been calculated in the frames of ε expansion in the five-loop approximation[53,27] as well as in the massive scheme in four-loop approximation.[35] We perform estimates within the 3D minimal subtraction approach[65] in two-, three-, and four-loop approximation. The corresponding values on the basis of β functions resummed by the Chisholm-Borel method read: $n_c(2\text{Loops})=2.7730$, $n_c(3\text{Loops})=3.1078$, $n_c(4\text{Loops})=2.9500$. This should be compared with the result of the five-loop ε expansion $n_c=2.958$ (Padé analysis),[53] $n_c=2.855$ (Padé-Borel resummation),[27] and the four-loop massive 3D RG scheme $n_c=2.90$ (Chisholm-Borel resummation).[35] In Fig. 9 we show the lines of zeros for the resummed β functions as well as the fixed points for different n. One can see that the change of stability of the fixed points appears for n very close to $n=3$. As our numerical result yields $n_c<3$ the cubic model at $n=3$ is governed by a *new set* of critical exponents which read $\gamma=1.387, \nu=0.709, \alpha=-0.127, \eta=0.044$. The coordinates of the stable cubic fixed point $u^*=0.0064, v^*=0.3950$ and stability matrix eigenvalues $\omega_1=0.0440, \omega_2=0.0055$ should be compared with the corresponding values for the nearby situated unstable $O(n)$-symmetric fixed point: $u^*=0, v^*=0.4009, \omega_1=-0.0056, \omega_2=0.0751$. Of course no experiment can distinguish between the critical exponents of these almost coinciding fixed points. Note however that at $n>n_c$ the stable fixed point can be reached only for $u>0$. Then $n_c<3$ means that all systems described by the phenomenological Landau-Ginsburg Hamiltonian with cubic anisotropy of negative coupling should undergo a weak first-order phase transition at $n=3$.[90]

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