Critical exponents at the superconductor-insulator transition in dirty-boson systems

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I obtain the inverse of the correlation length exponent at the superfluid–Bose-glass quantum critical point as a series in a small parameter $\sqrt{d-1}$, with d being the dimensionality of the system, and compute the two lowest terms. For $d=2$, I find $\nu_s=0.81$ and $\nu_c=1.03$ for short-range and Coulomb interactions between bosons, respectively. When combined with the exact values of the dynamical critical exponents, these results are in quantitative agreement with experiments on onset of superfluidity in ^{4}He in porous glasses and on the superconductor-insulator transition in homogeneous metallic films.

The phenomenon of the superconductor-insulator (SI) transition occurs in a variety of low-dimensional electronic systems, examples ranging from Josephson junction arrays[1] and homogeneous thin films[2,3] to high-temperature superconductors,[4] and is believed to represent a prototypical quantum (zero-temperature) phase transition. At low temperatures, as some controlling parameter like the thickness of a film is varied, the resistivity changes from a sharply decreasing to a continuously increasing function of temperature.[5] The good collapse of the resistivity data under scaling and near universality of the critical value of the conductivity indicate a quantum $(T=0)$ critical point that separates two many-body ground states with different symmetries. A natural question arises: what is the mechanism of destruction of the superfluid ground state in a disordered system? One possibly universal answer is provided by the so-called dirty-boson theory, which postulates that it is the loss of the superconducting phase coherence due to localization of Cooper pairs which is ultimately responsible for the SI transition.[6–8] Since on the scale of the diverging phase-coherence correlation length Cooper pairs will appear as point particles, the SI transition would, under this hypothesis, in general fall into the same universality class as the onset of superfluidity in ^{4}He in disordered media,[9] corrected for the long-rangeness of the Coulomb interaction. In principle, a way to assess the validity of this idea is to compare the measured critical exponents with the calculations for the dirty-boson Hamiltonian. The strongly coupled nature of the dirty-boson critical point, however, poses a fundamental obstacle to this procedure, and makes any but a most qualitative understanding of the superfluid–Bose-glass transition very difficult. The absence of a useful noninteracting starting point for disordered bosons forces one to rely on uncontrollable approximation schemes or turn to numerical calculations.[10] This seems to be a common problem for theories of interacting disordered low-dimensional quantum systems, apparent also in fermionic systems of this type.[11] In fact, the dirty-boson Hamiltonian may be the simplest quantum problem that irreducibly contains the physics of interactions and disorder,[12] and as such has received a lot of attention through the years.[10]

Recently, an approach to the dirty-boson criticality has been suggested,[13] according to which the strongly coupled superfluid–Bose-glass critical point in two dimensions $(d=2)$ could be understood as smoothly evolving from the zero-disorder critical point in $d=1$. The idea is to note that, by preventing the clean superfluid ground state in $d=1$ from exhibiting a true long-range order, the Mermin-Wagner theorem[14] forces the SI fixed point in $d=1$ to lie precisely at *zero* disorder.[15] In $d=1+\epsilon$ a true long-range order becomes possible and the superfluid thus becomes more resilient to disorder, which causes the SI fixed point to shift to a finite, but small, value of disorder, controlled by the parameter ϵ.[13] Although the dirty-boson transition probably lacks the upper critical dimension,[16] it has the lower critical dimension $d_l=1$, and this in principle allows one to compute the universal quantities at the transition perturbatively in the small parameter $\epsilon=d-1$. Within this formalism, a particular symmetry of the low-energy action present in $d=1$ guarantees that the dynamical critical exponent is $z=d$ ($z=1$ for Coulomb interaction) exactly,[13] in agreement with the expectation based on the compressible nature of the Bose glass.[16] The second correlation length exponent ν, however, turns out to be a perturbation series in $\sqrt{\epsilon}$. On the experimental side, a directly measurable quantity is typically the product of the two exponents $z\nu$,[5,9] and a meaningful comparison with experiment requires knowledge of the exponent ν to some accuracy. In this paper I present a field-theoretic method for higher-order calculations within the ϵ expansion for the superfluid–Bose-glass transition, and use it to compute the correlation length exponent to *two* lowest orders in $\sqrt{\epsilon}$. The result for both short-range and Coulomb interactions between bosons (see Table I) leads to values of ν in $d=2$ in a very good agreement with the experiments on the onset of superfluidity in ^{4}He in aerogel[9] and on the SI transition in thin metallic films,[5] as well as with the Monte Carlo calculations on the dirty-boson Hamiltonian.[17] My calculation supports the idea that the SI transition in disordered electronic systems falls into the dirty-boson universality class, and establishes a way for a quantitative understanding of the SI criticality, as presently exists for the thermal critical phenomena.[18] The effort involved in higher-order calculations of the correlation length exponent and of the universal critical conductivity is discussed.

To be specific, consider the effective low-energy $T=0$ action for the disordered superfluid in $d=1$:[19,16]

TABLE I. Comparison between experiment, Monte Carlo, and second-order ϵ-expansion results for the critical exponents, for short-range (s) and Coulomb (c) universality classes.

	Experiment	Monte Carlo [a]	ϵ expansion
ν_s	0.80 ± 0.04 [b]	0.90 ± 0.10	0.81
z_s	2	2.0 ± 0.1	2
ν_c	1.2 ± 0.2 [c]	0.9 ± 0.15	1.03
z_c	1.0 ± 0.1 [d]	1.0	1

[a]Reference 17.
[b]Reference 9.
[c]Reference 24.
[d]Reference 3.

$$S=\frac{K}{\pi}\sum_{i=1}^{N}\int_{-\infty}^{\infty}dx\,d\tau\{c^2[\partial_x\theta_i(x,\tau)]^2+[\partial_\tau\theta_i(x,\tau)]^2\}$$
$$-D\sum_{i,j=1}^{N}\int_{-\infty}^{\infty}dx\,d\tau\,d\tau'\cos 2[\theta_i(x,\tau)-\theta_j(x,\tau')]. \tag{1}$$

The coupling K is inversely proportional to the superfluid density at some microscopic cutoff length Λ^{-1}, c is the velocity of low-energy phonons, and D is proportional to the width of the Gaussian random potential. The interaction between particles in Eq. (1) is taken to be short ranged, and the standard limit on the number of replicas $N\rightarrow 0$ is assumed. The field $\theta(x,\tau)$ is the *dual* phase,[19,10] and the above theory describes the destruction of the sound mode in the superfluid due to unbinding of topological defects (phase slips) at the point of transition in $d=1$. Its role is similar to that of sine-Gordon theory for the Kosterlitz-Thouless transition in the two-dimensional XY model.[20] To determine the macroscopic state of the system, one is in principle interested in the fate of the couplings K, c, and D as the cutoff in the theory is lowered. Under a change $\Lambda\rightarrow\Lambda/s$ the combinations of the coupling constants $u=3-\eta^{-1}$ (where $\eta=Kc$), $\kappa=1/Kc^2$, and $W\sim D$ (to be precisely defined shortly), in $d=1+\epsilon$ dimensions, are expected to renormalize according to the β functions:[13]

$$\dot{u}=\epsilon(u-3)+W+auW+O(W^2), \tag{2}$$

$$\dot{W}=uW+bW^2+O(uW^2), \tag{3}$$

$$\dot{\kappa}=(d-z)\kappa, \tag{4}$$

where $\dot{x}=dx/d\ln(s)$, and a and b are numerical coefficients. The d-dependent terms in the recursion relations (2)–(4) can be understood as deriving from the scaling of the superfluid density, $\rho_{sf}\sim K^{-1}\sim\xi^{2-z-d}$, and the compressibility, $\kappa\sim\xi^{z-d}$,[13,16] where ξ is the diverging correlation length near the critical point and z the dynamical exponent. Adopting the logic of the minimal subtraction scheme,[18] the disorder-dependent terms in the recursion relations should be computed precisely in $d=1$, where one has the dual representation (1) of the low-energy theory on his disposal. The symmetry of the interaction term in Eq. (1) under a transformation $\theta_i(x,\tau)\rightarrow\theta_i(x,\tau)+f(x)$, for arbitrary $f(x)$, implies then that there could be no disorder-dependent terms in Eq. (4),[13] so $z=d$ at the fixed point. The correlation length exponent ν follows from the linearization of the first two equations near the critical point at $W^*\sim u^*\sim\epsilon$. It is then straightforward to check that to second order in $\epsilon^{1/2}$

$$\nu_s^{-1}=3^{1/2}\epsilon^{1/2}+\frac{1+3(a+b)}{2}\epsilon+O(\epsilon^{3/2}) \tag{5}$$

for short-range interactions. The $O(\epsilon^{3/2})$ term follows from the higher-order terms in Eqs. (2) and (3).

While the outlined procedure is conceptually simple, its implementation is made difficult by the fact that the action in Eq. (1) has a compact form only in real space, and the interaction term contains all powers of the dual field. A similar obstacle appears in the calculation of the Kosterlitz recursion relations beyond the lowest order in fugacity.[20] Here I will introduce a field-theoretic approach to the problem, which also enables one to avoid the usual pitfalls of the momentum-shell renormalization group when applied beyond the lowest order. The gist of the method is to recognize that in $d=1$, right at $u=0$, the coupling constant W becomes dimensionless, and the theory (1) appears to be just renormalizable. One then expects that the logarithmic divergences at the renormalizable point $d=1$ and $u=0$ will at small finite u show as poles when $u\rightarrow 0$. Since the coupling W acquires a finite scaling dimension for $u\neq 0$, the coefficients in the β functions are expected to stay finite as $u\rightarrow 0$. This is analogous to the standard procedure of dimensional regularization, commonly used to study thermal critical phenomena near the upper critical dimension,[18] except that here the coupling constant u plays the role of dimension, while the real physical dimension is at first fixed at $d=1$. When finally $d\rightarrow 1+\epsilon$, the β functions are deformed into Eqs. (2)–(4).

With this strategy in mind, consider the self-energy defined by the propagator of the dual phase in $d=1$ as $G^{-1}(k,\omega)=(2K/\pi)(\omega^2+c^2k^2)+\Sigma(\omega)$. It will prove useful to separate the first- and second-order contributions to Σ by writing it as $\Sigma(\omega)=\Sigma_1(\omega)+\Sigma_2(\omega)+O(D^3)$, where $\Sigma_n\sim D^n$. Simple calculation then gives

$$\Sigma_1(\omega)=8D\int_{-\infty}^{\infty}d\tau(1-e^{i\omega\tau})e^{-f(\tau)}, \tag{6}$$

where the two-point correlation function $f(\tau)=4\langle\theta(0,0)[\theta(0,0)-\theta(0,\tau)]\rangle$ is given by the integral

$$f(\tau)=(3-u)\int_0^{c\Lambda|\tau|}\frac{dx}{x}(1-e^{-x}). \tag{7}$$

When $u\rightarrow 0$, it readily follows that at small frequencies

$$\Sigma_1(\omega)=\omega^2\frac{2W}{\pi c}\left(\frac{1}{u}+O(1)\right), \tag{8}$$

where I introduced the frequency-dependent, dimensionless coupling $W=[4\pi D/(c^2\Lambda^3)](c\Lambda/\omega)^u$. After a tedious but straightforward algebra, one similarly finds

$$\Sigma_2(\omega)=\frac{\pi}{2K}\left(\frac{\Sigma_1(\omega)}{\omega}\right)^2+I(\omega), \tag{9}$$

where

$$I(\omega)=8D^2\int_{\infty}^{\infty}dy\,d\tau\,d\tau'\,dv\,(1-e^{i\omega\tau})$$
$$\times e^{-f(\tau)-f(\tau')}\left\{F(y,v,\tau,\tau')\left[1+e^{-i\omega v}\right.\right.$$
$$\left.\left.\times\frac{1}{2}(1-e^{-i\omega\tau'})\right]-1\right\}, \qquad (10)$$

and F denotes a four-point correlation function:

$$F(y,v,\tau,\tau')=e^{-4\langle[\theta(0,0)-\theta(0,\tau)][\theta(y,v)-\theta(y,v+\tau')]\rangle}. \qquad (11)$$

When $\omega\to 0$, after rescaling the imaginary times and the length in the integral as $\omega\tau\to\tau$ and $\omega y/c\to y$, the leading divergence in $I(\omega)$ as $u\to 0$ comes from the integration over small values of τ and τ'. To obtain the leading divergence in $I(\omega)$ it therefore suffices to expand F to the lowest order in τ and τ', to find that at small frequencies

$$I(\omega)=-\omega^2\frac{6}{\pi c}W^2\left[\frac{1}{u^2}+O\left(\frac{1}{u}\right)\right]. \qquad (12)$$

The last equation is the central result of this work. Collecting all the terms back into the self-energy one recognizes the renormalized coupling η_r as the coefficient of ω^2 term in the propagator. In general,

$$\eta_r=\eta+\frac{W}{u}+x\frac{W^2}{u^2}+O\left(\frac{W^2}{u}\right), \qquad (13)$$

where the terms finite when $u\to 0$ have been discarded, and x is a number determined from Eq. (12). After judiciously defining the renormalized disorder from Eq. (13) as $W_r=W+2xW^2/u$, and rescaling it to bring the coefficient of the $O(W)$ term in Eq. (2) to unity as $9W_r\to W_r$, differentiation with respect to $\ln(c\Lambda/\omega)$ leads to Eqs. (2) and (3) for the renormalized couplings u_r and W_r, with the coefficients $a=-2/3$ and $b=2x/9$, with $b=0$. The subleading $\sim W^2/u$ term in Eq. (13) determines the next $O(W^2)$ term in Eq. (2), and the next-order correction to ν_s^{-1}.

A remarkable feature of the perturbation series for ν_s is its independence of the renormalization procedure, i.e., of the nonuniversal finite parts of the self-energy which have been dropped in the last equation. To see this to the order of the present calculation consider the most general redefinition of the coupling constants to the order W^2:[20]

$$u_r'=u_r+\alpha W_r+\beta u_rW_r+\gamma W_r^2, \qquad (14)$$

$$W_r'=W_r+\delta u_rW_r+\sigma W_r^2, \qquad (15)$$

where the coefficients $\{\alpha,\ldots,\sigma\}$ are finite and dependable on the finite parts of the self-energy. It is easy to check that the recursion relations for the new couplings have the same form as Eqs. (2) and (3), but with the coefficients $a'=a+\alpha-\delta$ and $b'=b-\alpha+\delta$. Interestingly, while the coefficients a and b by themselves are nonuniversal, the critical exponent requires only their sum, which is perfectly universal, i.e., scheme independent. An invariant similar to $a+b$ appears also in the β functions for the sine-Gordon model,[20] where it determines the first correction to the Kosterlitz-Thouless scaling.

I expect the presented ϵ expansion to lead to a divergent series; the point $D=0$ in the action (1) is nonanalytic, since for $D<0$ the Gaussian probability distribution for the random potential becomes unbounded. Nevertheless, the hope is that the series in Eq. (5), for example, will be asymptotic, and that the few lowest terms may already lead to useful results. Indeed, estimating ν_s for $d=2$ from the simple sum of the first two terms gives $\nu_s=0.81$, within bounds found in the Monte Carlo calculations.[17] The experimental data of Crowell *et al.* on the onset of superfluidity and on the specific heat of ^{4}He in aerogel at low temperatures[9] on its face value are consistent with the effective dimensionality of $d=2$. Under this assumption the product of the two exponents in their experiment is $z\nu=1.60\pm0.08$. Assuming further that $z_s=2$ at the transition in $d=2$ gives $\nu=0.80\pm0.04$. Although the uncertainty cited here should be taken with some reservation, and the accuracy of the measurement falls short of the standards in thermal critical phenomena, the result appears to be in excellent agreement with my calculation (see Table I). It is worth noting that the inequality[21] $\nu\geqslant 2/d$ seems to be violated both by the experiment and by my estimate. It has been argued recently[22] that the above inequality is an artifact of the particular averaging procedure, and that the true exponent is in fact not bound from below. It would be interesting to see if the higher-order corrections eventually push the value of ν_s above unity in $d=2$, or indeed $\nu_s<1$ as the experiment and the present calculation suggest.

To make a comparison with the experiments on SI transition in homogeneous thin films with thickness as the tuning parameter[2] it is necessary to take into account the long-range Coulomb interaction between the Cooper pairs. As explained in detail elsewhere[13,23] within the present scheme this may be simply accomplished by defining the Coulomb interaction as $V_c(\vec{r})=e^2\int d^d\vec{q}\,\exp(i\vec{q}\cdot\vec{r})/q^{d-1}$, so as to coincide with the δ-function repulsion in $d=1$. The only change in the calculation then is that $z_c=1$ and that $\epsilon\to\epsilon/2$ in Eq. (2), while the disorder-dependent terms in the recursion relations, which follow from $d=1$, remain the same. The first two terms in the series then give $\nu_c=1.03$, in accordance with the Monte Carlo results[17] (see Table I). Experiment finds $z_c=1.0\pm0.1$, by suppressing the transition temperature with the magnetic field or by scaling of resistance with the electric field.[3] Collapsing the resistivity data[24] then gives the experimental value of $\nu_c=1.2\pm0.2$, again in very good agreement with my result.

As mentioned earlier, the next term in the series (5) requires only the computation of the subleading, $O(W^2/u)$ term in Eq. (13). Here, however, it appears that it is no longer enough to know the correlator F (after rescaling the lengths with ω) only at small rescaled τ and τ', as it was for the leading divergence in Eq. (12). In light of the likely asymptotic nature of the expansion, this is left for future work.

Another universal quantity of interest is the critical conductivity in $d=2$,[8] which, aside from the universal unit e_*^2/h, for bosons of charge e_* can be obtained as a Laurent series in ϵ.[23] The lowest-order term was obtained in Ref. 23, and for $d=2$ the result $\sigma_c=0.69e_*^2/h$ agrees with the low-temperature experiment on bismuth films[24] quite well. It would nevertheless be useful to compute the next-order correction and see if it pushes the result towards the self-dual value of e_*^2/h, to which a large number of experimental results seems to converge. Calculating the next-order term in the critical conductivity would in principle proceed along the same lines as here, except that one needs to perform the analytic continuation to real frequencies to obtain the real-time dc response at low temperature. This problem was solved in Ref. 23 to lowest order, but applying the same trick in the next-order term does not seem straightforward. A preliminary analysis also suggests that the second-order contribution to the universal conductivity requires a computation of both leading and subleading divergences in the second-order contribution to the self-energy.

In conclusion, it has been shown that the expansion around the lower critical dimension $d_l=1$ for the superfluid–Bose-glass critical point allows a field-theoretic formulation that facilitates a systematic higher-order calculation of the critical exponents. The computation to two lowest orders yields results for the correlation length critical exponents in $d=2$ in respectable agreement with the experiment and Monte Carlo calculations, both for the short-range and Coulomb interactions between particles. The results suggest that the superconductor-insulator transition in homogeneous thin films is in the universality class of disordered bosons.

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