Susceptibility amplitude ratio near a Lifshitz point

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The susceptibility amplitude ratio in the neighborhood of a uniaxial Lifshitz point is calculated at the one-loop level using field-theoretic and ϵ_L-expansion methods. We use the Schwinger parametrization of the propagator in order to split the quadratic and quartic part of the momenta, as well as a new special symmetry point suitable for renormalization purposes. For a cubic lattice $(d=3)$, we find the result $C_+/C_-=3.85$.

I. INTRODUCTION

Universality is a key concept in the theory of critical phenomena, which states that all critical properties only depend on the number of components of the order parameter characterizing the phase transition and the space dimension of the system. Beside the critical exponents, the amplitude ratios above and below the critical temperature for different thermodynamic potentials are examples of universal quantities.[1] One special type of critical behavior is associated with the Lifshitz point.[2] In magnetic systems, the uniaxial Lifshitz point can be described by an axially nearest-neighbor Ising model (ANNNI),[3] which consists of a spin-$\frac{1}{2}$ Ising model on a cubic lattice with nearest-neighbor ferromagnetic couplings and next-nearest-neighbor competing antiferromagnetic interactions along a single lattice axis. Due to the competition, the system presents a modulated phase (in addition to the ordinary paramagnetic and ferromagnetic ones. Although this model possesses a variety of modulated phases, we are going to concentrate our attention only on the Lifshitz critical region, where a simple field-theoretic setting can be defined; see Sec. II below). Theoretical and experimental studies in MnP (Refs. 4 and 5) showed that this system indeed presents this sort of uniaxial Lifshitz critical behavior.

Renormalization-group and ϵ-expansion techniques are particularly suitable to investigate amplitude ratios of critical systems.[6] However, very little is known about these amplitude ratios for the Lifshitz critical behavior. The specific-heat amplitude ratio for a uniaxial Lifshitz point was measured in MnP by Bindilatti, Becerra, and Oliveira.[5] Recently, some authors obtained this amplitude ratio theoretically at the mean-field level.[7] It turned out that the two results do not agree. This disagreement is not surprising, for the fluctuations must be taken into account in a proper treatment using the ϵ expansion. In order to find an outcome beyond mean-field for this amplitude ratio, one needs the coupling constant at the two-loop level. As it is only known at one-loop for the Lifshitz point, we can then ask ourselves if it is possible to calculate some other amplitude ratio at one-loop order with this restricted knowledge of the coupling constant. If one considers the susceptibility amplitude ratio, such a program can be achieved. Besides, having a theoretical prediction for this amplitude ratio, where the renormalization-group technique can be exploited in its full power, should motivate experiments to test the degree of accuracy of this approach for systems of this type.

In this work we calculate the susceptibility amplitude ratio at a Lifshitz point using $\lambda\phi^4$ field theory and ϵ-expansion methods at first order in the loop expansion. In order to perform the one-loop integrals, we use the Schwinger parametrization for the free propagator, as well as a new special symmetry point. We will show that the result has the same dependence on $\epsilon_L=4.5-d$ for a uniaxial Lifshitz point as that exhibited by the usual Ising-like system, where the loop expansion parameter is $\epsilon=4-d$. We find the numerical value $C_+/C_-=3.85$ for this amplitude ratio in a three-dimensional lattice. To our knowledge, this is the first time that an amplitude ratio for the Lifshitz critical behavior is calculated to first order in ϵ_L. The presentation goes as follows. In Sec. II, we develop our approach to deal with the uniaxial Lifshitz behavior and calculate the susceptibility amplitude ratio. In Sec. III, we discuss our results and compare with other methods existing in the literature.

II. SUSCEPTIBILITY AMPLITUDE RATIO

The most convenient way to formulate the $\lambda\phi^4$ field-theoretic approach to the Lifshitz point is the Lagrangian description, which is equivalent to the usual Landau-Ginzburg-Wilson Hamiltonian formulation. For the uniaxial case, the bare Lagrangian is

$$L=\frac{1}{2}|\nabla_1^2\phi|^2+\frac{1}{2}|\nabla_{(d-1)}\phi|^2+\delta\frac{1}{2}|\nabla_1\phi|^2+\frac{1}{2}t_0\phi^2 + \frac{1}{4!}\lambda\phi^4. \tag{1}$$

We see that the competition along one axis produces the first term in the above expression. Furthermore, at the Lifshitz critical point $\delta=0$. We are going to focus our attention in this case from now on.

The expression for the one-loop renormalized Helmholtz free-energy density at the fixed point associated with the uniaxial Lifshitz critical behavior of the system is

$$F(t,M)=\frac{1}{2}tM^2+\frac{1}{4!}g^*M^4+\frac{1}{4}\left(t^2+g^*tM^2+\frac{1}{4}g^{*2}M^4\right)I_{sp} + \frac{1}{2}\int d^{d-1}q\,dk\{\ln[1+(1/2)g^*M^2/(k^4+q^2+t)] - \tfrac{1}{2}g^*M^2/(k^4+q^2)\}. \tag{2}$$

In the above equation, $t, M(t_0 = Z_{\phi^2}^{-1} t, \phi = Z_\phi^{-1/2} M)$ are the renormalized (bare) reduced temperature and order parameter, respectively, Z_{ϕ^2}, Z_ϕ are renormalization functions, g^* is the renormalized coupling constant at the fixed point, $\vec{q}$ is a $(d-1)$-dimensional wave vector along the direction parallel to the plane where only ferromagnetic nearest-neighbor interactions take place, whereas k is a wave vector parallel to the axis where the antiferromagnetic competition is localized. The integral I_{sp} is defined by

$$I_{sp} = \int \frac{d^{d-1} q dk}{[(k+k')^4 + (q+p)^2](k^4+q^2)}. \tag{3}$$

The symmetry point that simplifies the integral is chosen at external momenta $k'=0$, $p^2=1$. The dimensionful coupling constant is related to the dimensionless one through the formula $g^* = (p^2)^{-\epsilon_L/2} u^*$. Therefore, choosing the scale of external momenta $p^2=1$ has the advantage of transforming the dimensionful coupling constant into the convenient dimensionless u^*. At this point it is convenient to extract for each loop integration a geometric angular factor and absorb it in the coupling constant. In our case, it is $(3\sqrt{2}/8) S_{d-1} S_1$, where $S_d = [2^{d-1} \pi^{d/2} \Gamma(d/2)]^{-1}$. In order to calculate this integral, we use the Schwinger parametrization:

$$\int \frac{d^{d-1} q dk}{[k^4 + (q+p)^2](k^4+q^2)} = \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \left(2 \int_0^\infty dk \exp[-(\alpha_1+\alpha_2) k^4] \right) \times \int d^{d-1} q \exp[-(\alpha_1+\alpha_2) q^2 - 2\alpha_2 q p - \alpha_2 p^2]. \tag{4}$$

The q integral can be easily performed,

$$\int d^{d-1} q \exp[-(\alpha_1+\alpha_2) q^2 - 2\alpha_2 q p - \alpha_2 p^2] = \frac{1}{2} S_{d-1} \Gamma\left(\frac{d-1}{2}\right) (\alpha_1+\alpha_2)^{-d-1/2} \exp\left(-\frac{\alpha_1 \alpha_2 p^2}{\alpha_1+\alpha_2}\right), \tag{5}$$

and the k integration is[8]

$$2\int_0^\infty dk \exp[-(\alpha_1+\alpha_2) k^4] = \tfrac{1}{2} (\alpha_1+\alpha_2)^{-1/4} \Gamma(\tfrac{1}{4}). \tag{6}$$

Inserting Eqs. (5), and (6) into Eq. (4) together with the value $p^2=1$, one finds

$$\left(\int \frac{d^{d-1} q dk}{[k^4+(q+p)^2](k^4+q^2)} \right)_{p^2=1} = \frac{1}{4} S_{d-1} \Gamma\left(\frac{d-1}{2}\right) \Gamma(\tfrac{1}{4}) \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \times \exp\left(-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right) (\alpha_1+\alpha_2)^{-[(d-1)/2+(1/4)]}. \tag{7}$$

We can perform one of the integrals in the Schwinger parameters using a change of variables. Then, after a rescale, the result can be expressed in the following form[9]

$$\int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \exp\left(-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right) (\alpha_1+\alpha_2)^{-[(d-1)/2+(1/4)]} = \Gamma\left[2 - \left(\frac{d-1}{2} + \frac{1}{4}\right)\right] \int_0^1 dv \, (v(1-v))^{[(d-1)/2+(1/4)]-2}. \tag{8}$$

Now we make the continuation $d = 4.5 - \epsilon_L$. We can make use of the identity $\Gamma[1.75 - (\epsilon_L/2)]\Gamma(0.25) = (3\sqrt{2}\pi/4)\Gamma[2-(\epsilon_L/2)]$, to get the following expression for I_{sp}:

$$I_{sp} = \frac{1}{\epsilon_L}\left(1 + \frac{\epsilon_L}{2}\right). \tag{9}$$

We are now in a position to calculate the susceptibility amplitude ratio. Using Eq. (2), we find the following renormalized equation of state:

$$H_R = \frac{\partial F}{\partial M} = tM + \frac{1}{6} u^* M^3 + \frac{1}{2} u^* M \left(t + \frac{1}{2} u^* M^2\right) \times \left[I_{sp} - \int \frac{d^{d-1} q dk}{(k^4+q^2)\left(k^4+q^2+t+\frac{1}{2}u^* M^2\right)} \right]. \tag{10}$$

The one-loop integral is then readily calculated:

$$\int \frac{d^{d-1} q dk}{(k^4+q^2)\left(k^4+q^2+t+\frac{1}{2}u^* M^2\right)} = \frac{1}{2}\Gamma\left(2-\frac{\epsilon_L}{2}\right)\Gamma\left(\frac{\epsilon_L}{2}\right)(t+\tfrac{1}{2}u^* M^2)^{-\epsilon_L/2}. \tag{11}$$

The renormalized two-point vertex part

$$\Gamma_R^{(2,0)} = \frac{\partial}{\partial M} H_R \tag{12}$$

is related to the susceptibility as

$$\chi^{-1} = \Gamma_R^{(2,0)}. \tag{13}$$

We can now apply the following procedure to calculate this amplitude ratio.[10] For $T > T_L$ we can put $M=0$ into Eq. (12) above and use the Lifshitz value at the fixed point $u^* = 2\epsilon_L/3$, to get

$$\chi(T > T_L) = t^{-\gamma_L}\left(1 - \frac{\epsilon_L}{6}\right). \tag{14}$$

For $T < T_L$, we use $u^* M^2 = -6t$ and proceeding along the same lines gives the result

$$\chi(T<T_L)=(-t)^{-\gamma_L}\frac{1}{2}\left(1-\frac{\epsilon_L}{6}(4+\ln 2)\right), \qquad (15)$$

with amplitude ratio

$$\frac{C_+}{C_-}=2^{\gamma_L-1}\frac{\gamma_L}{\beta_L}, \qquad (16)$$

where $\gamma_L=1+(\epsilon_L/6)$ and $\beta_L=\frac{1}{2}-(\epsilon_L/6)$ are the susceptibility and magnetization critical exponents, respectively, associated to the Lifshitz point. This result is then the amplitude ratio for the uniaxial case in the neighborhood of the Lifshitz point ($\delta=0$).

III. DISCUSSION

First, we note that expression (16) has the same dependence on ϵ_L as the usual Ising-like critical behavior, the only difference being the value of $\epsilon_L=1.5$ for a cubic lattice ($d=3$). The numerical value for the amplitude ratio is then $C_+/C_-=3.85$. Compared with the value $(C_+/C_-)_{\text{mean field}}=2$, the correction due to the fluctuations is remarkable. Second, the method developed here might be efficient to calculate the fixed point at the two-loop level, and then to find the specific-heat amplitude ratio at order ϵ_L in order to compare with known experimental data.[5] Alternatively, the result obtained for the susceptibility amplitude ratio should motivate the realization of experiments to check whether the renormalization-group techniques are suitable to understand this sort of system. Indeed, the comparison of the critical exponents β_L and γ_L to first order in ϵ_L with Monte Carlo simulations showed that they are different.[3] It was argued that the Monte Carlo result was more appropriate, because the expansion parameter ϵ_L is not small and, therefore, the perturbative expansion might not be reliable. On the other hand, carrying out the calculation of the critical exponents to second order in ϵ_L might actually bring their values closer to those obtained via Monte Carlo. The definite answer to either possibility has to wait until one can figure out the fixed point at two-loop order. As Monte Carlo methods are not available yet to calculate amplitude ratios, the most direct way to probe the numerical value at order ϵ_L of the susceptibility amplitude ratio shown here is to compare with experiments to be done in systems with uniaxial critical Lifshitz behavior, such as MnP. This comparison should give a clue about the reliability of the ϵ_L expansion methods in this case. Finally, although some authors have recently proposed a different field-theoretic approach to the Lifshitz point,[11] their method does not seem to be suitable for the uniaxial case, for their choice of the symmetry point makes the integral I_{sp} more difficult to be performed. Therefore, our result suggests that the uniaxial critical behavior ($\delta=0$) has almost the same critical properties as the usual Ising model. The main difference is the critical dimension, i.e., the loop expansion parameters, characterized by $\epsilon=4-d$ for the Ising-like system and $\epsilon_L=4.5-d$ for the uniaxial Lifshitz point. The issues of two-loop calculations and crossover are under current investigation.

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[1] C. Bervillier, Phys. Rev. B **14**, 4964 (1976).

[2] R.M. Hornreich, M. Luban, and S. Shtrikman, Phys. Rev. Lett. **35**, 1678 (1975); R.M. Hornreich, J. Magn. Magn. Mater. **15-18**, 387 (1980).

[3] W. Selke, Phys. Rep. **170**, 213 (1988), and references therein.

[4] C.S.O. Yokoi, M.D. Coutinho-Filho, and S.R. Salinas, Phys. Rev. B **24**, 5430 (1981); Y. Shapira, J. Appl. Phys. **53**, 1914 (1982); C.S.O. Yokoi, M.D. Coutinho-Filho, and S.R. Salinas, Phys. Rev. B **29**, 6341 (1984).

[5] Y. Shapira, N. Oliveira, C.C. Becerra, and S. Foner, Phys. Rev. B **29**, 361 (1984); V. Bindilatti, C.C. Becerra, and N. Oliveira, *ibid.* **40**, 9412 (1989).

[6] See, e.g., D.J. Amit, in *Field Theory, the Renormalization Group and Critical Phenomena* (World Scientific, Singapore, 1984).

[7] I. Nasser, A. Abdel-Hady, and R. Folk, Phys. Rev. B **56**, 154 (1997); see also I. Nasser, *ibid.* **60**, 2983 (1999).

[8] I.S. Gradshtein and I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1994).

[9] E. Abdalla, M.C.B. Abdalla, D. Dalmazi, and A. Zadra, in *2D Gravity in Non-Critical Strings*, Lecture Notes in Physics m20 (Springer-Verlag, Berlin, 1994).

[10] M.M. Leite, M. Sardelich, and M.D. Coutinho-Filho, Phys. Rev. E **59**, 2683 (1999).

[11] C. Mergulhão, Jr. and C.E.I. Carneiro, Phys. Rev. B **58**, 6047 (1998); *ibid.* **59**, 13 954 (1999).