

Stability of a cubic fixed point in three dimensions: Critical exponents for generic N

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The detailed analysis of the global structure of the renormalization-group (RG) flow diagram for a model with isotropic and cubic interactions is carried out in the framework of the massive field theory directly in three dimensions (3D) within an assumption of isotropic exchange. Perturbative expansions for RG functions are calculated for arbitrary N up to four-loop order and resummed by means of the generalized Padé-Borel-Leroy technique. Coordinates and stability matrix eigenvalues for the cubic fixed point are found under the optimal value of the transformation parameter. Critical dimensionality of the model is proved to be equal to $N_c = 2.89 \pm 0.02$ that agrees well with the estimate obtained on the basis of the five-loop ε expansion [H. Kleinert and V. Schulte-Frohlinde, Phys. Lett. B **342**, 284 (1995)] resummed by the above method. As a consequence, the cubic fixed point should be stable in 3D for $N \geq 3$, and the critical exponents controlling phase transitions in three-dimensional magnets should belong to the cubic universality class. The critical behavior of the random Ising model being the nontrivial particular case of the cubic model when $N=0$ is also investigated. For all physical quantities of interest the most accurate numerical estimates with their error bounds are obtained. The results achieved in the work are discussed along with the predictions given by other theoretical approaches and experimental data.

I. INTRODUCTION

In the present time the critical behavior of the basic models of phase transitions described by isotropic field theories with a quartic interaction like $\sum_{i=1}^N (\varphi_i^2)^2$ is well studied in the framework of different theoretical approaches. In particular, the critical phenomena in polymers, easy-axis ferromagnets, simple liquids, and binary mixtures, easy-plane ferromagnets, certain superconductors, as well as superfluid helium-4, Heisenberg ferromagnets, and quark-gluon plasma in some models of quantum chromodynamics were proved to be governed by the $O(N)$ -symmetric universality class with $N=0, 1, 2, 3$, and 4 , respectively. The large-order field-theoretical renormalization-group (RG) expansions combined with proper resummation techniques, the high-temperature series method, and the most advanced Monte Carlo (MC) simulations provided high precision and comparable to each other numerical estimates for important physical quantities (critical exponents, universal couplings, and critical amplitude ratios) which nowadays are regarded as canonical numbers.^{1,2}

However, in real crystals due to their complex crystalline structure an anisotropy is always present. That is why, when studying phase transitions in real substances, besides the $O(N)$ -symmetric term one should take into account additional quartic interactions in corresponding fluctuation Landau-Wilson (LW) Hamiltonian. The simplest nontrivial crystalline anisotropy is a cubic one. Proper quartic interaction term is represented as $\sum_{i=1}^N \varphi_i^4$. In this case the vector of magnetization is directed either along the edges or diagonals of a hypercube in N dimensions of the order-parameter field.

The critical thermodynamics of magnetic and structural phase transitions in three-dimensional (3D) cubic crystals was extensively investigated more than 25 years ago in a good number of papers. By using the lower-order RG ap-

proximations, Wilson and Fisher,³ Aharony,⁴ and Ketley and Wallace^{5,6} showed that in the critical region the fluctuation instability of continuous phase transitions may be observed, and that it may lead to the isotropization of the system with a cubic anisotropy. This fact gave rise to a question of what regime of the critical behavior is actually realized in 3D cubic crystal with $N=3$. Much effort of many people have been devoted to answering this question. It was understood that for a given model it is enough to calculate the so-called critical (or marginal) dimensionality N_c of the order-parameter field. Indeed, the critical value N_c separates two different regimes of critical behavior of the system. For $N > N_c$ the cubic rather than isotropic fixed point is stable in 3D. At $N=N_c$ the points interchange their stability so that for $N < N_c$ the stable fixed point is the isotropic one. Therefore the calculation of N_c is the crucial point in studying the critical phenomena in 3D cubic crystals. However, attempts to evaluate the critical dimensionality resulted in the dramatically different estimates.

In fact, the field-theoretical RG analysis of the stability matrix eigenvalues of the cubic and isotropic fixed points fulfilled in the one-loop approximation as well as some symmetry arguments (for details, see Sec. III of the paper) lead to the conclusion that N_c should lie between 2 and 4. Many years ago the three-loop expansion for N_c as a power series in ε was obtained in Ref. 5. Summation of that short series at $\varepsilon=1$ ($D=3$) by means of the Padé approximant $[1/1]$ yielded the value $N_c=3.128$,⁷ while making use of the Padé-Borel resummation method results in the estimate $N_c=3$. In contrast to this, in the work of Ref. 8, by using the variational modification of the Wilson recursion relation method, it has been found that $N_c=2.3$. Later, however, Newman and Riedel by decoupling the infinite system of the recursion relations for the so-called scaling fields and then solving them showed⁹ that for $D=3$ $N_c \sim 3.4$. At the same time, the

classical technique of the high-temperature expansions, under some circumstances, allowed to establish that for $N=3$ the isotropic critical asymptotics in the cubic crystal is unstable,¹⁰ thus implying $N_c < 3$. Further, ten years ago, the analysis of the critical behavior of the (mn) -component field model, which has a good number of interesting applications to the phase transitions in real substances, has been carried out within the three-loop RG approach in three dimensions. The calculation of the stability matrix eigenvalues for the cubic model ($m=1$, $n=3$) provided the stability of the cubic fixed point in 3D, and the critical dimensionality turned out to be equal to 2.91.¹¹ In agreement with this, the estimate $N_c=2.9$ was quoted in Ref. 12. More recently, Kleinert and Schulte-Frohlinde calculated the RG functions for the cubic model in $(4-\varepsilon)$ dimensions up to five-loop order.¹³ Resummation of the critical dimensionality expansion with the help of the Padé approximant $[2/2]$ gave the estimate $N_c=2.958$.¹⁴ The cubic fixed point eigenvalues found by means of a simple resummation algorithm of the Borel type, accounting the large-order behavior of the β functions when the parameter of anisotropy is very small,¹⁶ indicated that the cubic point is stable in 3D.¹⁷ Finally, in the recent work of Ref. 18 by using finite-size scaling techniques and the high-precision MC simulations it has been suggested that N_c coincides with 3 exactly. So, such strong scattering in the estimates of N_c inspired us to study this problem with particular care. Calculation of the critical dimensionality of the order parameter field as well as the eigenvalue exponents for the cubic fixed point by exploiting higher-order RG approach in three dimensions and generalized Padé-Borel-Leroy (PBL) resummation technique is the main goal of the paper. As will be shown, our estimates for N_c and eigenvalues are in excellent agreement with recent results by Kleinert and collaborators¹⁷ obtained on the basis of the five-loop ε expansions.

The paper is organized in the following way. In Sec. II the perturbative expansions for β functions of the hypercubic model are deduced within the RG technique in 3D up to the four-loop order. In Sec. III the structure of the RG flows of the model are investigated and fixed point locations for $N \geq 3$ are calculated using the generalized PBL resummation method. The eigenvalue exponents of the most intriguing $O(N)$ -symmetric and cubic fixed points are evaluated for the physically important case $N=3$ and their stability problem is analyzed in Sec. IV. The numerical estimate of the critical dimensionality N_c , at which the topology of the flow diagram changes, is obtained by resumming both the four-loop RG expansions for the β functions in 3D and the five-loop ε expansion for N_c at $\varepsilon=1$. In Sec. V the four-loop RG expansions of the critical exponents for generic N and their numerical estimates are presented. Section VI is devoted to the study of the critical behavior of the three-dimensional random Ising model (RIM) which is the special case of the cubic model when $N=0$. The coordinates, eigenvalues, and the critical exponents of the RIM fixed point are computed by applying the PBL resummation procedure. The correction-to-scaling exponent ω is estimated therein. The results of the investigation are discussed in the Conclusion, along with the predictions and numerical estimates obtained earlier on the basis of the same or other theoretical approaches and experimental data.

II. MODEL AND β FUNCTIONS

The fluctuation LW Hamiltonian of the model reads

$$H = \int d^3x \left[\frac{1}{2} (m_0^2 \varphi_i^2 + \partial_\mu \varphi_i \partial_\mu \varphi_i) + \frac{1}{4!} (u_0 G_{ijkl}^{(1)} + v_0 G_{ijkl}^{(2)}) \varphi_i \varphi_j \varphi_k \varphi_l \right], \quad (1)$$

where φ_i , $i=1, \dots, N$ is the real vector order-parameter field in 3D, m_0^2 is the linear measure of the temperature, and u_0 and v_0 denote the bare coupling constants. The symmetrized tensors associated with isotropic and cubic interactions are

$$G_{ijkl}^{(1)} = \frac{1}{3} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj}), \quad G_{ijkl}^{(2)} = \delta_{ij} \delta_{ik} \delta_{il}, \quad (2)$$

respectively. When the cubic symmetry is present, the anisotropic exchange has been shown to be negligible within the ε -expansion method.¹⁹ We assume that the anisotropic exchange is an irrelevant variable in our case too, and study the critical behavior of the model (1) with isotropic exchange only.

The model (1) has a number of interesting applications to the phase transitions in simple and complicated systems. In fact, when $N=1$ the Hamiltonian (1) describes the critical phenomena in pure spin systems (the pure Ising model), while for $N=2$ it corresponds to the anisotropic XY model (the model of two coupled scalar fields) describing structural phase transitions in ferroelectrics as ordering the two-component alloys.^{3,20} The magnetic and structural phase transitions in a cubic crystal are governed by model (1) as $N=3$. Further, when $N=0$ Hamiltonian (1) determines the critical properties of weakly disordered quenched systems undergoing second-order phase transitions. The latter is the nontrivial specific case of the hypercubic model, the systematic studying of which was initiated in the classical works by Harris and Lubensky^{21,22} and Khmel'nitskii²³ and then considerably advanced by many authors when employing the conventional field-theoretical RG approach both in 3D and $(4-\varepsilon)$ dimensions. Finally, the case $N \rightarrow \infty$ corresponds to the Ising model with equilibrium magnetic impurities.²⁴ In this limit the Ising critical exponent of specific heat α changes its sign and takes the Fisher renormalization²⁵ as well as ν and γ : $\alpha \rightarrow -\alpha_I/(1-\alpha_I)$, $\nu \rightarrow \nu_I/(1-\alpha_I)$, $\gamma \rightarrow \gamma_I/(1-\alpha_I)$. Since the critical phenomena in the pure Ising and anisotropic XY models are well understood, we will consider below the critical behavior of the cubic and random Ising models only.

To calculate the RG functions the standard normalization conditions of the massive renormalized theory at fixed dimensions are applied.²⁶ For each Feynman graph contributing to the RG functions the corresponding contractions are computed by using the algorithm developed in Ref. 27. The combinatorial factors as well as integral values are known from Ref. 28. After performing simple but cumbersome calculations we obtain the four-loop expansions for the β functions:

$$\beta_u = u \left\{ 1 - u - \frac{6}{N+8}v + \frac{1}{(N+8)^2} [3(2.024\,691N + 9.382\,716)u^2 + 44.444\,444uv + 10.222\,222v^2] \right. \\ - \frac{1}{(N+8)^3} [3(0.449\,648N^2 + 18.313\,459N + 66.546\,806)u^3 + 3(6.646\,878N + 164.613\,849)u^2v \\ + 3(0.621\,889N + 100.955\,929)uv^2 + 65.937\,285v^3] + \frac{1}{(N+8)^4} [-(0.155\,646N^3 - 35.820\,204N^2 \\ - 602.521\,231N - 1832.206\,732)u^4 - 3(1.352\,882N^2 - 182.073\,890N - 2064.170\,701)u^3v + 3(27.250\,336N \\ + 2110.408\,809)u^2v^2 + 9(1.291\,017N + 308.599\,361)uv^3 + 495.005\,747v^4] \left. \right\}, \quad (3)$$

$$\beta_v = v \left\{ 1 - \frac{1}{N+8}(12u + 9v) + \frac{1}{(N+8)^2} [(3.407\,407N + 54.814\,815)u^2 + 92.444\,444uv + 34.222\,222v^2] \right. \\ - \frac{1}{(N+8)^3} [-(1.251\,107N^2 - 41.853\,902N - 469.333\,970)u^3 + 9(0.248\,784N + 136.511\,768)u^2v \\ + 957.781\,662uv^2 + 255.929\,737v^3] + \frac{1}{(N+8)^4} [(0.574\,653N^3 - 0.267\,107N^2 + 584.287\,672N \\ + 5032.692\,260)u^4 + 3(0.057\,375N^2 + 107.641\,680N + 5989.283\,536)u^3v + 3(7321.464\,604 \\ - 16.494\,003N)u^2v^2 + 11\,856.956\,858uv^3 + 2470.392\,521v^4] \left. \right\}. \quad (4)$$

It is reasonable to note that the cubic model (1) possesses some specific symmetry property when $N=2$. Namely, the transformation of the field components

$$\varphi_1 \rightarrow \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2), \quad \varphi_2 \rightarrow \frac{1}{\sqrt{2}}(\varphi_1 - \varphi_2) \quad (5)$$

combined with substitution of the quartic coupling constants

$$u \rightarrow u + \frac{3}{2}v, \quad v \rightarrow -v \quad (6)$$

does not change the structure of the initial Hamiltonian itself. As a result, the β functions (3) and (4) should obey certain symmetry relations²⁹ which may be written down as

$$\beta_u \left(u + \frac{3}{2}v, -v \right) = \beta_u(u, v) + \frac{3}{2}\beta_v(u, v), \\ \beta_v \left(u + \frac{3}{2}v, -v \right) = -\beta_v(u, v). \quad (7)$$

It can be easily verified that equations (7) are really satisfied. The specific symmetry of Hamiltonian (1) will be used be-

low to obtain the lower boundary value of the critical dimensionality N_c .

III. METHOD OF SUMMATION AND FIXED-POINT LOCATIONS

Before analyzing the four-loop β functions let us consider briefly the predictions following from the one-loop approximation. It is easy to see that β functions (3) and (4) in this approximation have four different solutions corresponding to Gaussian (trivial), Ising, isotropic (Heisenberg), and cubic fixed points with the coordinates

$$(i) \quad u_c^G = v_c^G = 0, \\ (ii) \quad u_c^I = 0, \quad v_c^I = \frac{(N+8)}{9}, \\ (iii) \quad u_c^H = 1, \quad v_c^H = 0, \\ (iv) \quad u_c^C = \frac{N+8}{3N}, \quad v_c^C = \frac{(N-4)(N+8)}{9N}, \quad (8)$$

respectively. The most intriguing fixed points are the isotropic and cubic ones. At $N=4$ the coordinates of these two points coincide that leads to the conclusion that the critical dimensionality has the upper boundary value $N_c=4$. On the

other hand, the eigenvalue exponents for the isotropic and cubic fixed points are given by the expressions

$$\begin{aligned}\lambda_1^H &= -1, & \lambda_2^H &= (N-4)/(N+8), \\ \lambda_1^C &= -1, & \lambda_2^C &= (4-N)/3N.\end{aligned}\quad (9)$$

If the real parts of both eigenvalues are negative, the corresponding fixed point is infrared stable; if eigenvalues are of opposite signs, the point is of ‘‘saddle-knot’’ type. It is seen from Eqs. (9) that $N_c=4$ actually separates two different regimes of critical behavior of the model. When $N>N_c$ the cubic rather than the isotropic fixed point is stable in 3D, while for $N<N_c$ the stable fixed point is the isotropic one.

To determine the lower boundary of N_c one should employ the above-mentioned specific symmetry property of the model, when $N=2$.³⁰ As was already pointed out, the rotation (5) of the components of φ_i by $\pi/4$ combined with substitution (6) generates relations (7), but does not change the form of the RG equations. However, for $N=2$ transformations (5) and (6) result in the relocation of the coupling constants values so that the cubic and Ising fixed points are transformed into each other at the 3D RG flow diagram. Since the exact RG equations always have the Ising fixed point, which inevitably is the saddle-knot one, these equations should have also the cubic fixed point, which will be unstable. In this situation, the isotropic fixed point, again always existing in the exact RG equations, should be the stable knot only. Therefore we conclude that the lower boundary of N_c is not less than 2. Of course, the real value of N_c can be obtained only on the basis of the thorough analysis of the structure of the RG flow diagram, provided that the β functions of the model are calculated in sufficiently high-order RG approximations and then processed by means of appropriate resummation techniques.

Let us now concentrate our attention on the analysis of the four-loop β functions. It is well known that field-theoretical RG expansions are divergent. The character of their large-order asymptotic behavior for the case of simple $O(N)$ -symmetric models was established in Refs. 31–33. In particular, it was proved that the coefficients of the series at large k behave as $c(-a)^k k! k^b$, where the asymptotic parameters a , b , and c are assumed to be calculated for each RG function. Knowledge of the exact values of the asymptotic parameters in combination with the most powerful resummation procedure of the Borel transformation with a conformal mapping,³⁴ first proposed in Ref. 35 and then elaborated in Refs. 36–38, made it possible to develop the accomplished quantitative theory of critical behavior of simple systems.^{1,2,35,38–41}

At the same time, the asymptotic nature of RG functions of anisotropic models is unknown. Calculation of the exact values of the asymptotic parameters characterizing the large-order behavior of the series in such models is a very difficult and still unsolved problem. As an exception one should mention the anisotropic quartic quantum oscillator representing one-dimensional φ^4 field theory with a cubic anisotropy. Within an assumption of the weak anisotropy, the transformation parameters for the perturbative expansion of the ground-state energy of this system as well as for the β functions of the cubic model have recently been found.^{42,16} Later

this information was used to solve the stability problem of the cubic fixed point in three dimensions from ε expansion¹⁷.

Usually, in a lack of any information about the high-order behavior of the series either the simple Padé-Borel or Chisholm-Borel resummation procedures are used, for treating the perturbative expansions of anisotropic models. The latter technique, however, possesses at least two inherent drawbacks. First, some ambiguity in the calculation of coefficients of denominators of the Chisholm approximants is unavoidable.⁴³ Second, the Chisholm-Borel procedure does not hold the specific symmetry properties of a model. At the same time, exploiting the Borel transformation in combination with the Padé or Chisholm approximants shows that the results of calculation are very sensitive to the choice of the type of the approximant. This may lead to the estimates, which do not provide reliable predictions even in the higher-loop RG approximations (see, for instance, Sec. VI of the paper). Besides, in the framework of both schemes it is very difficult to determine any error bounds for evaluated quantities.

In the present work we apply for processing RG expansions of the β functions and critical exponents of the model (1) the PBL resummation method generalized for the two coupling constant case. This resummation technique, introduced by Baker, Nickel, and Meiron in Ref. 44, turned out to be highly efficient when used to study the critical behavior of simple $O(N)$ -symmetric models in 3D. The critical exponent estimates obtained within the framework of this technique are regarded nowadays as the most accurate values, as those of Refs. 2, 35, and 39. We motivate our choice of the PBL resummation method by the following reasons:

- 3D RG expansions for the β functions and critical exponents of the cubic model are alternating in signs. Therefore using PBL resummation technique is quite natural.
- It can be expected that for complex models with more than one coupling constant, the asymptotics of RG series at large orders will comprise a factor $k!k^b$. The PBL resummation method removes divergences of this type.
- The PBL resummation method allows one to determine the error bounds for the physical quantities to be calculated, in a natural way.

The generalized PBL resummation procedure consists of the following steps. Let a physical quantity $F(u,v)$ be represented by a double series

$$F(u,v) = \sum_{i,j} f_{ij} u^i v^j, \quad (10)$$

where coefficients $f_{ij} \sim (i+j)!(i+j)^b$ at large orders ($i, j \rightarrow \infty$), the additional parameter b being an arbitrary non-negative number to be defined below. Associated with the initial series (10) is the function

$$\mathcal{F}(u,v;b) = \int_0^\infty e^{-t} t^b B(ut,vt) dt. \quad (11)$$

The Borel-Leroy transform $B(x,y)$ is the analytical continuation of its Taylor series

$$B(x,y) = \sum_{ij} \frac{f_{ij}}{\Gamma(i+j+b+1)} x^i y^j, \quad (12)$$

absolutely convergent in a circle of the nonzero radius. In order to calculate the integral in Eq. (11) one should continue analytically $B(x,y)$ for $0 \leq x < \infty$ and $0 \leq y < \infty$. To this end, the rational Padé approximants $[L/M](x,y)$ are used. The Padé approximant method is determined in a conventional way.¹⁵ Let us consider a “resolvent” series

$$\tilde{B}(x,y,\lambda) = \sum_{k=0}^{\infty} \lambda^k \sum_{l=0}^k \frac{f_{l,k-l} x^l y^{k-l}}{\Gamma(k+b+1)} = \sum_{k=0}^{\infty} A_k \lambda^k, \quad (13)$$

where coefficients A_k are uniform polynomials of k th order in u and v . The sum of the series is then approximated by

$$B(x,y) = [L/M]|_{\lambda=1}. \quad (14)$$

The Padé approximants $[L/M]$ in λ are given by an attitude

$$[L/M] = \frac{P_L(\lambda)}{Q_M(\lambda)}, \quad (15)$$

where $P_L(\lambda)$ and $Q_M(\lambda)$ are polynomials of degrees L and M , respectively, with coefficients depending on x and y which should be determined from the conditions

$$\begin{aligned} Q_M(\lambda) \tilde{B}(x,y;\lambda) - P_L(\lambda) &= O(\lambda^{L+M+1}), \\ Q_M(0) &= 1. \end{aligned} \quad (16)$$

Replacing variables $x=ut$ and $y=vt$ in the Padé approximants and then evaluating the Borel-Leroy integral

$$\mathcal{F}(u,v;b) = \int_0^{\infty} e^{-t} t^b [L/M]|_{\lambda=1} dt, \quad (17)$$

we obtain the approximate expressions for RG functions.

Among the Padé approximants the diagonal ($L=M$) or near-diagonal ones were proved to exhibit the best approximating properties.¹⁵ However, as the degree of the denominator M increases, the number of possible poles of the approximant increases too. If some of the poles belong to the positive real semiaxis, the corresponding approximant should be rejected. Due to this the choice of “working” approximants, which might be used for analytical continuation of the Borel-Leroy image onto the complex cut plane, is largely limited. On the other hand, varying the free parameter b in the Borel-Leroy transformation (11) allows one to optimize the resummation procedure under the condition that the fastest convergence of the iteration process is achieved. So, taking into account the above-mentioned remarks, in order to find the fixed points locations of the model we adopt the following scheme. For the fixed N , the β functions are resummed by virtue of transformation (11) in the highest-loop orders by shifting the transformation parameter b . In order to make an analytical continuation of the Borel-Leroy transforms $B_u(u,v)$, $B_v(u,v)$ over the cut plain the most appropriate Padé approximants $[2/1]$, $[3/1]$, and $[2/2]$ are chosen. The fixed points locations are determined then for each b from the solution of the set of equations: $\beta_u^{res}(u_c, v_c) = 0$, $\beta_v^{res}(u_c, v_c) = 0$. The “true” locations are obtained by averaging over the values given by the approximants under the

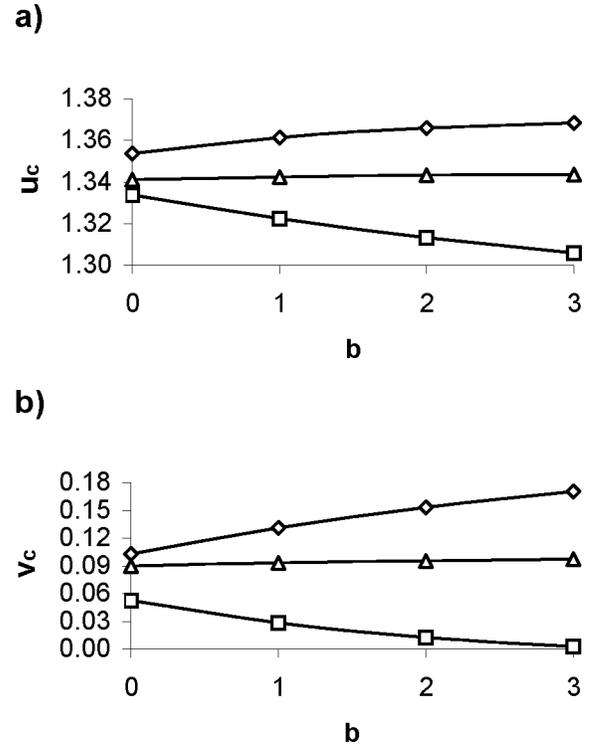


FIG. 1. Curves demonstrating dependence of the results of calculating the cubic fixed point locations (a) u_c component and (b) v_c component on transformation parameter b from the four-loop approximation for $N=3$. The upper curve (\diamond) corresponds to the $[2/1]$ Padé approximant (three-loop approximation), while the middle (\triangle) and lower (\square) curves correspond to the $[2/2]$ and $[3/1]$ Padé approximants (four-loop approximation), respectively.

optimal value of the parameter b , at which the quantity $|1 - \mathcal{F}_L(u,v;b)/\mathcal{F}_{L-1}(u,v;b)|$ reaches its local minima. The quantity $\mathcal{F}_L(u,v;b)$ is evaluated for the L -partial sum of the series in Eq. (17), L stands for the step of truncation of the series.

In Fig. 1 the results of the computation of the cubic fixed-point locations depending on the parameter b are presented for the physically important case $N=3$. Three curves correspond to the three Padé approximants. The parameter b shifts from 0 to 3. As seen from the figure the optimal value of b is zero. At this point the numerical values of the cubic fixed-point locations given by different approximants are the most close to each other. The result of computing the cubic fixed-point locations for $N=3$ are also presented in Table I. In the first three columns of the table the fixed-point locations values found for the Padé approximants $[2/1]$, $[3/1]$, and $[2/2]$ at the point $b=0$ are placed. Averaging the results of processing over the all approximants under the optimal value of b gives the estimates standed in the fourth column of the table. These numbers we adopt as the final estimates of the cubic fixed-point locations found within the four-loop approximation. As an accuracy for these approximate values we take the maximum deviations of the average values of the fixed-point locations from those given by the approximants at $b=0$. One can observe, looking at Fig. 1, that the values of the cubic fixed-point locations given by the symmetric approximant $[2/2]$ weakly depend on the shift parameter b . Averaging over all the values given by this approximant within the

TABLE I. Coordinates of the cubic fixed point of RG equations for $N=3$ found under the optimal value of the transformation parameter $b=0$. “AV” denotes the average value.

	[2/1]	[3/1]	[2/2]	AV	AV over [2/2]	Ref. 11	Ref. 12
u_c	1.3536	1.3338	1.3410	1.3428 ± 0.0200	1.3425	1.348	1.3357
v_c	0.0526	0.1026	0.0894	0.0815 ± 0.0300	0.0937	0.090	0.0906

interval $[0,3]$ results in the cubic fixed-point locations estimates presented in the fifth column of Table I. The coordinates of the cubic fixed point found earlier on the basis of the three- and four-loop approximations with the use of the Chisholm-Borel resummation method are presented in the sixth and seventh columns of the table, for comparison. These numbers include the normalizing multiplier $\frac{11}{9}$ needed to compare our β functions with those obtained in Refs. 11 and 12.

In order to verify the correctness of the chosen approach let us apply the above-considered scheme to estimate the fixed-point locations of the $O(N)$ -symmetric model where the numerical results are well known. Consider, for example, an $O(3)$ -symmetric case relevant to the Heisenberg ferromagnets. The six-loop 3D RG expansion for the β function of this model was reported in Refs. 35 and 44. The PBL resummation of that series with the use of eight types of the Padé approximants [2/1], [3/1], [2/2], [4/1], [3/2], [5/1], [4/2], and [3/3] for analytical continuation of the Borel-Leroy transform yields, after solving the equation $\beta^{res}(g_c) = 0$, the picture displayed in Fig. 2. It is seen that the values of the isotropic fixed point location calculated in the highest

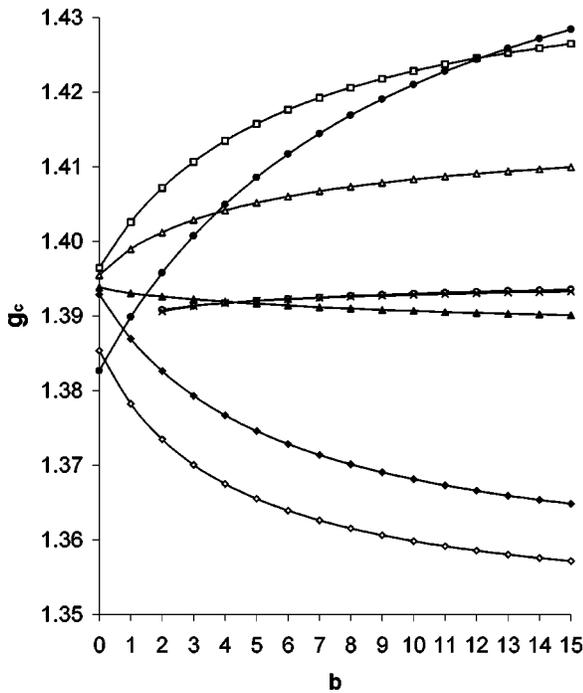


FIG. 2. The results of computation of the $O(3)$ -symmetric fixed-point locations from the three- to the six-loop approximations obtained on the basis of the PBL resummation method when eight types of the Padé approximants used: [2/1], \diamond ; [3/1], \square ; [2/2], \triangle ; [4/1], full \diamond ; [3/2], full \triangle ; [5/1], full \circ ; [4/2], \circ ; [3/3], \times . On the six-loop level under the optimal value of the transformation parameter $b=4.5$ we obtain the estimate $g_c = 1.392$.

RG orders with the help of the approximants [3/3], [4/2], and [3/2] are very weakly dependent on the parameter b varied within the interval $0 \leq b \leq 15$. The curves corresponding to these approximants are intersected at $b=4.5$. Therefore the value $b=4.5$ is the optimal value of the transformation parameter in which the fastest convergence of the iteration procedure is ensured. For $b=4.5$ the central value estimate of the isotropic fixed point is $g_c = 1.392$. The maximum deviation of the central value from the values given by some of the approximants [3/3], [4/2], and [3/2] at the point $b=10$ is adopted approximately as an apparent accuracy of the calculation, $\Delta = 0.0013$. Such a small error can be explained by the small dispersion of the curves within the range $5 \leq b \leq 10$. So, the estimate $g_c = 1.3920 \pm 0.0013$ is in excellent agreement with those found more than 20 years ago in Refs. 35 and 44 as well as with recent results of Ref. 2.

Within the framework of the four-loop approximation there are only three appropriate Padé approximants. Averaging the results of computing the isotropic fixed-point location given by the approximants [2/1], [3/1], and [2/2] under the optimal value of the transformation parameter results in the estimate $g_c = 1.3925 \pm 0.0070$. The error was determined again through the maximum deviation of the central value from those given by each of the approximants at $b=0$. It is seen that the four-loop estimate of the coordinate of the isotropic fixed point is in a good accordance with the best ones followed from the six-loop consideration.

TABLE II. Coordinates of the cubic fixed point of RG equations and critical exponents estimates for some N found under the optimal value of the transformation parameter b within the four-loop approximation. The critical exponents values obtained within the framework of the ε -expansion method (five-loop results) and marked by the symbol (a) are presented for comparison.

N	u_c	v_c	η	ν	γ
3	1.3428	0.0815	0.0332	0.6996	1.3775
			0.0375 ^a	0.6997 ^a	1.3746 ^a
4	0.9055	0.8167	0.0327	0.7131	1.4028
			0.0365 ^a	0.7225 ^a	1.4208 ^a
5	0.6980	1.2361	0.0325	0.7154	1.4076
			0.0358 ^a	0.7290 ^a	1.4305 ^a
6	0.5807	1.5386	0.0324	0.7157	1.4082
			0.0354 ^a	0.7301 ^a	1.4322 ^a
7	0.5060	1.7874	0.0324	0.7155	1.4079
8	0.4544	2.0076	0.0324	0.7153	1.4074
9	0.4168	2.2108	0.0324	0.7149	1.4067
10	0.3881	2.4032	0.0324	0.7147	1.4063
12	0.3473	2.7680	0.0323	0.7142	1.4054

^aQuoted from Ref. 64.

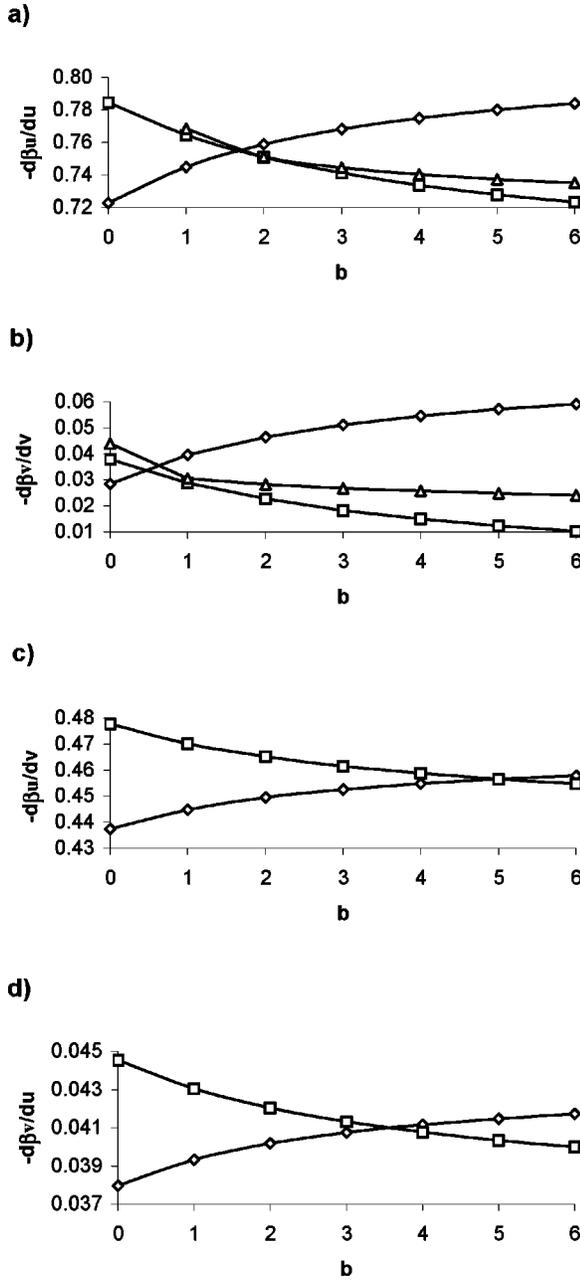


FIG. 3. Graphs of dependence of the results of processing of the series (a) $-\partial\beta_u/\partial u$, (b) $-\partial\beta_v/\partial v$, (c) $-\partial\beta_u/\partial v$, and (d) $-\partial\beta_v/\partial u$ on the parameter b for $N=3$. The curves on the pictures (a) and (b) are given in the same notations as in the previous figures, while for the curves corresponding to the approximants $[1/1]$ and $[2/1]$ on the pictures (c) and (d) the notations \diamond and \square are used, respectively.

Let us note that the coordinate of the $O(3)$ -symmetric fixed point calculated within the five-loop approximation does not approach the exact value. Namely, the PBL resummation procedure leads to the estimate $g_c = 1.3947 \pm 0.0040$. Although the error of the calculation became visibly smaller, the central value of the fixed-point location stepped aside from the four- and six-loop ones. Thus the fulfilled numerical analysis shows that the isotropic fixed-point location estimate obtained in the four-loop level occurs to be close to the “exact” value. Therefore it can be expected that in the case of the cubic model the fixed-point locations u_c

TABLE III. Four-loop eigenvalue exponents estimates for the cubic (CFP) and Heisenberg (HFP) fixed points ($N=3$) found in 3D under the optimal value of the transformation parameter b .

	CFP	CFP, Ref. 17	HFP	HFP, Ref. 17
λ_1	-0.7786	-0.7648	-0.7791	-0.7640
λ_2	-0.0081	-0.0085	0.0077	0.0089

$= 1.3428 \pm 0.0200$, $v_c = 0.0815 \pm 0.0300$ (see fourth column of Table I) will not distinguish strongly from the “exact” values as well. The coordinates of the cubic fixed point for some values N of the order parameter dimensionality are presented in Table II. Our calculations show that for $N=3$ the coordinates of the cubic fixed point practically do not differ from those of the Heisenberg one. However, with increasing N the cubic fixed point runs away from the isotropic point moving towards the Ising one. In the large N limit these two fixed points become close to each other so much that the influence of the $O(N)$ -symmetric invariant on the critical thermodynamics of the cubic model vanishes. This can be easily seen by applying the $1/N$ consideration to the one-loop solutions of the RG equations of the model (1). Indeed, rescaling the coupling constants $u \rightarrow u/N$, $v \rightarrow v/N$ in the initial Hamiltonian and taking then the limit $N \rightarrow \infty$ in Eqs. (8) one can see that the cubic fixed point approaches the Ising one asymptotically. So, the cubic model turns out to be split into N noninteracting Ising models, the critical behavior of each of them is determined by a set of the critical exponents renormalized according to Fisher²⁵. The data listed in Table II will be used further for calculating the stability matrix eigenvalues as well as critical exponents of the hypercubic fixed point.

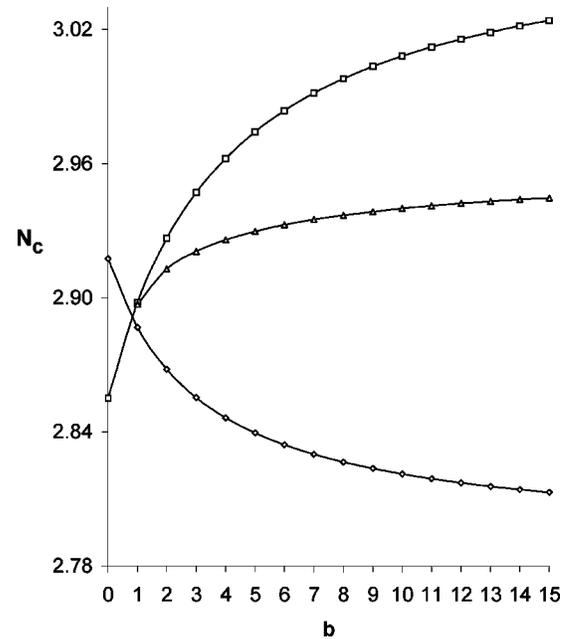


FIG. 4. Dependence of the results of processing of the ε series for the critical dimensionality N_c on transformation parameter b . The crossing of the curves corresponding to the three Padé approximants gives the optimal value of the parameter b at which the estimate $N_c = 2.894$ is obtained.

IV. STABILITY AND CRITICAL DIMENSIONALITY

One of the independent ways to determine the fixed-point locations in fixed D is to construct the RG flow phase diagram of the model. If at the flow diagram there exists a fixed point of stable knot type, the trajectories originated from some point within the range of stability of the initial Hamiltonian would flow towards the knot. The region at the flow diagram where the trajectories are intersected provides the coordinates of this stable fixed point. Thoroughly investigating the 3D RG flow diagram of the cubic model (1) in the four-loop approximation we arrive at the conclusion that the cubic rather than isotropic fixed point is absolutely stable for all $N \geq 3$.

On the other hand, the reliable conclusion about the stability of the cubic fixed point for $N \geq 3$ can be given on the basis of calculating the eigenvalue exponents λ 's of the stability matrix

$$M_{ij} = \begin{pmatrix} \frac{\partial \beta_u}{\partial u} & \frac{\partial \beta_u}{\partial v} \\ \frac{\partial \beta_v}{\partial u} & \frac{\partial \beta_v}{\partial v} \end{pmatrix}$$

taken at $u = u_c$ and $v = v_c$. If the real parts of both eigenvalues are negative, the fixed point is the stable knot in the (u, v) plane. If λ_1, λ_2 have opposite signs, the point is of the ‘‘saddle-knot’’ type.

To calculate the stability matrix eigenvalues of the cubic and isotropic fixed points we have chosen the following strategy. First, the derivatives of the β functions (3) and (4) are calculated, and the new RG expansions resummed by means of the PBL technique are substituted into the matrix M_{ij} . The eigenvalue exponents of the matrix of derivatives M_{ij} obtained in this way are evaluated then under the optimal value of the transformation parameter b . In Figs. 3(a) and 3(b) we present our numerical results for the series $-\partial \beta_u / \partial u$ and $-\partial \beta_v / \partial v$ for the physically interesting case $N = 3$. The curves correspond to the three types of the Padé approximants used within the four-loop approximation. The crossing of the curves gives the optimal value of b at which we find $\partial \beta_u / \partial u|_{opt} = -0.7536$ and $\partial \beta_v / \partial v|_{opt} = -0.0331$. Because the series $-\partial \beta_u / \partial v$ and $-\partial \beta_v / \partial u$ are turned out to be shorter by one order in comparison with $-\partial \beta_u / \partial u$ and $-\partial \beta_v / \partial v$, their resummation performed with the help of the approximant $[2/1]$ only yields the monotonic dependence of the result of processing on the parameter b . In this unfavorable situation, we take into account an additional Padé approximant $[1/1]$ to optimize the iteration procedure. The results are plotted in Figs. 3(c) and 3(d). For the optimal values of b we obtain $\partial \beta_u / \partial v|_{opt} = -0.4566$ and $\partial \beta_v / \partial u|_{opt} = -0.0409$. Straightforward calculation of the eigenvalues of the stability matrix M_{ij} gives for the cubic fixed point the numbers placed in Table III. The eigenvalues of the isotropic fixed point as well as the analogous numerical estimates ob-

tained recently in Ref. 17 on the basis of using the five-loop ε expansions are presented for comparison therein. These estimates show that the cubic fixed point is absolutely stable in 3D for $N = 3$ while the isotropic fixed point appears to be stable on the u axis only. Our numerical results agree well with those obtained in Ref. 17.

Let us now calculate the critical dimensionality N_c of the order parameter field. The critical dimensionality is defined as a value of N at which the cubic fixed point coincides with the isotropic one. Equivalently, for $N = N_c$ the second eigenvalue of the stability matrix M_{ij} vanishes, $\lambda_2 = 0$.

Studying carefully the 3D RG flow diagram of the model (1) depending on the order of approximation with the use of different Padé approximants we arrive at the conclusion that $N_c = 2.910 \pm 0.035$ and $N_c = 2.890 \pm 0.020$ within the three- and four-loop approximations, respectively. The accuracy of calculation of N_c was determined through the evaluation of the stability matrix eigenvalues for different N from the interval of the above errors. That value of $N = N_c$, above or below of its central number, at which the second eigenvalue λ_2 was becoming nonzero, was taking for the upper or lower boundary of N_c , respectively.

It is worthy to compare the four-loop estimate of N_c just found with that which can be obtained within the ε -expansion method. The five-loop ε expansion for N_c has been calculated in Ref. 13. The series proved to be alternating in signs that allows one to resum it by means of the PBL technique. To this end, we will use again the most appropriate Padé approximants $[2/1]$, $[3/1]$, and $[2/2]$ for analytical continuation of the Borel-Leroy transform for all $0 \leq \varepsilon t \leq \infty$. Dependence of the results of processing of the critical dimensionality N_c on the transformation parameter b is depicted in Fig. 4. The curves corresponding to the approximants are crossed at the point $b \sim 1$. The appropriate value of the critical dimensionality is $N_c = 2.894 \pm 0.040$. As an error of the calculation it is natural to assume the maximum scattering of numerical values given by the approximants at $b = 0$ with respect to the value obtained at the crossing point of the curves. This estimate of N_c is in excellent accordance with the above found within the 3D RG approach. So, both schemes, the RG technique directly in 3D and the ε -expansion method, result in the same estimate of the critical dimensionality $N_c = 2.89$, thus implying that the cubic fixed point is stable in three dimensions for $N \geq 3$. This means that the critical behavior of the model (1) should be governed by the cubic fixed point with a certain set of the critical exponents which will be calculated for generic N in the next section.

V. CRITICAL EXPONENTS FOR GENERIC N

Having the coordinates of the cubic fixed point, the stability of which in 3D for $N \geq 3$ has been proved in the previous section, it is possible to obtain the numerical estimates for the critical exponents. Two of them are known to determine the critical behavior of the system while the others can be found via the famous scaling laws. The four-loop RG expressions for the magnetic susceptibility exponent γ^{-1} as well as the correlation function exponent η for the generic symmetry index N are as follows:

$$\begin{aligned} \gamma^{-1} = & 1 - \frac{1}{N+8} \left[\frac{(N+2)}{2}u + \frac{3}{2}v \right] + \frac{1}{(N+8)^2} [(N+2)u^2 + 6uv + 3v^2] - \frac{1}{(N+8)^3} [(0.879\,559N^2 + 6.485\,477N \\ & + 9.452\,718)u^3 + 9(0.879\,559N + 4.726\,359)u^2v + 9(0.128\,340N + 5.477\,578)uv^2 + 16.817\,754v^3] \\ & + \frac{1}{(N+8)^4} [-(0.128\,332N^3 - 7.966\,741N^2 - 51.844\,213N - 70.794\,806)u^4 - 3(0.513\,328N^2 - 32.893\,620N \\ & - 141.589\,613)u^3v + 9(3.423\,908N + 83.561\,044)u^2v^2 + 27(0.208\,999N + 19.120\,991)uv^3 + 130.477\,428v^4], \end{aligned} \quad (18)$$

$$\begin{aligned} \eta = & \frac{1}{(N+8)^2} [0.296\,296(N+2)u^2 + 1.777\,77uv + 0.888\,888v^2] + \frac{1}{(N+8)^3} [0.024\,684(N+2)(N+8)u^3 + 0.222\,156(N+8) \\ & \times u^2v + 1.999\,40uv^2 + 0.666\,468v^3] + \frac{1}{(N+8)^4} [-(0.004\,299N^3 - 0.667\,985N^2 - 4.609\,22N - 6.512\,10)u^4 \\ & - 3(0.017\,194N^2 - 2.706\,33N - 13.0242)u^3v + 3(0.681\,615N + 22.8884)u^2v^2 + 47.140uv^3 + 11.7850v^4]. \end{aligned} \quad (19)$$

These series, however, are known to be divergent. To extract from them a physical information concerning the critical behavior of the substances of interest we will apply the same resummation procedure as that used above. Note that since the coefficients of the series of η are rapidly diminishing, its values are found by the direct substitution of the coordinates of the cubic fixed point into Eq. (19), whereas the series of the exponent γ^{-1} before the substitution needs to be resummed. Using the generalized PBL resummation technique, we evaluate the magnetic susceptibility exponent γ and then, having the values of η at hand, estimate the correlation length critical exponent ν by the scaling relation. The result of the numerical processing of Eq. (18) depending on the parameter b for $N=3$ is depicted in Fig. 5. The three types of the Padé approximants have been used, and the transformation parameter shifted within the range $[0,15]$. For the optimal value of b we obtain $\gamma=1.3775 \pm 0.0040$. Thus, taking into account that in the four-loop approximation $\eta=0.0332 \pm 0.0030$, we arrive at the estimate of the critical exponent ν : $\nu=0.6996 \pm 0.0037$. The error bounds for the exponents γ and ν have been determined through a maximum deviation of its central values at $b=b_{opt}$ from the corresponding values given by the approximants at $b=0$. In the case of the exponent η , as the uncertainty we take the absolute difference between two successive, three- and four-loop, results.

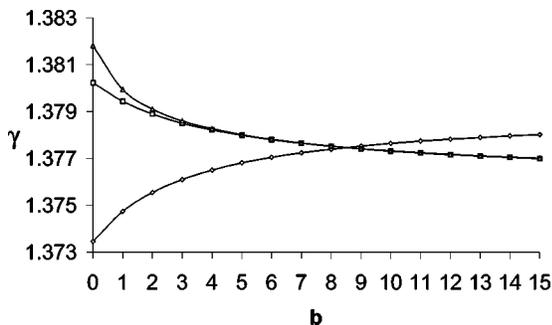


FIG. 5. Curves demonstrating dependence of the result of processing critical exponent γ of the cubic model on the parameter b for $N=3$ in the four-loop approximation.

The critical exponents estimates of the cubic fixed point for other values of N are listed in Table II. The estimates obtained from the ϵ expansions⁶⁴ are placed for comparison therein.

How much do the critical exponent estimates found differ from the exact ones? To answer this we resort again to the analysis of numerical results for the $O(3)$ -symmetric model obtained on the basis of using the PBL procedure. In Fig. 6 the results of processing of the 3D RG series for the susceptibility exponent γ in successive orders of perturbation theory in a number of loops are presented. Within the four-loop approximation under the optimal value of the transformation parameter we find $\gamma=1.3778$. The six-loop calculations under the optimal choice of b (the point of crossing the curves corresponding to the approximants $[4/2]$, $[5/1]$, and $[4/1]$) give $\gamma=1.3867$. The latter estimate seems to be in good agreement with that known from Refs. 35 and 44. Thus the difference between the four-loop results and their six-loop counterparts does not exceed 0.01.

For $N=2$ the numerical results for the cubic model prove to be much better. In this case the cubic fixed point lies in the $v < 0$ half plane and is unstable ($N < N_c$). However, for $N=2$ the Hamiltonian of the model (1) possesses the specific symmetry [see Eqs. (5) and (6)] that transforms the cubic

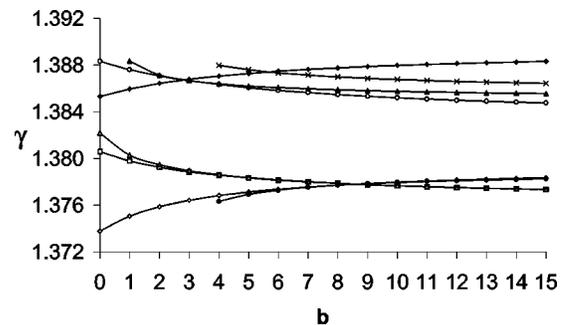


FIG. 6. Graphs of dependence of the results of processing of the 3D RG series for the critical exponent γ of the $O(3)$ -symmetric model on the parameter b for the eight types of Padé approximants in the PBL procedure.

point into the Ising one and vice versa. Due to this symmetry the critical exponents of both fixed points coincide. In the four-loop level for the cubic fixed point at $N=2$ we found the estimates $\gamma=1.2416$, $\eta=0.0323$, and $\nu=0.631$. These numbers agree well with the best estimates followed from the six-loop RG expansions.^{35,44,39,2} One may hope therefore that the critical exponent estimates for the cubic model obtained in the present work within the four-loop approximation will differ from the ‘‘exact’’ values by no more than 1–2%.

Unfortunately, for the most important case $N=3$ having a good number of interesting applications to the critical phenomena in real substances, the critical exponents of the cubic and isotropic fixed points turn out to be practically the same. This is the consequence of the closeness of both points on the 3D RG flow diagram. Thus, although in the course of the investigation the cubic fixed point has been shown to be stable at $N=3$ and therefore the critical behavior of the magnetic phase transitions in crystals with cubic anisotropy should belong to the cubic rather than the isotropic universality class, a certain difficulty arises in trying to identify the cubic fixed point from experimentally determined exponents. Due to this ‘‘near-marginality’’ the calculation of the critical exponents in cubic magnets seems to be of academic interest only.

At the same time, the numerical analysis shows that as N increases the distinction between the critical exponents of the cubic and isotropic fixed points increases as well. In the limit $N \rightarrow \infty$ the critical exponents of the cubic fixed point go over into those of the Ising model with equilibrium magnetic impurities.²⁵

VI. NUMERICAL RESULTS FOR THE RIM

In the critical region the character of critical behavior of the defect crystals such as impure uniaxial ferromagnets $\text{LiTb}_{1-x}\text{Y}_x\text{F}_4$, $\text{Cd}_{1-x}\text{Nd}_x\text{Cl}_3$ or diluted Ising antiferromagnets $\text{Mn}_x\text{Zn}_{1-x}\text{F}_2$, $\text{Fe}_x\text{Zn}_{1-x}\text{F}_2$ are known to be described by the 3D random Ising model with the effective Hamiltonian

$$H = \int d\mathbf{x} \left(\frac{1}{2} [m_0^2 \varphi^2 + (\nabla \varphi)^2] + \frac{1}{4!} v_0 \varphi^4 + \psi(\mathbf{x}) \varphi^2 \right), \quad (20)$$

where $\psi(\mathbf{x})$ is the static random field describing fluctuations of local transition temperature, $m_0^2 - m_c^2 \sim T - T_c$. Averaging over all configurations of $\psi(\mathbf{x})$ with Gaussian weight and employing the replica trick⁴⁵ one can reduce the problem (20) to the analysis of critical behavior of the N -component cubic model (1) in the limit $N \rightarrow 0$.⁴⁶ Moreover the Ising vertex v_0 in Eq. (20) plays a role of the cubic vertex in Eq. (1), while a role of the impure vertex plays the isotropic one u_0 . Obviously, because now $u_0 < 0, v_0 > 0$, the added interaction of critical fluctuations of the order-parameter field through the intermediary of impurities is the attraction.

Studying the magnetic and structural phase transitions in the weakly disordered systems is of considerable interest both from theoretical and experimental point of view. It is well known that the critical exponents of such systems should differ markedly from those of the pure ones, due to

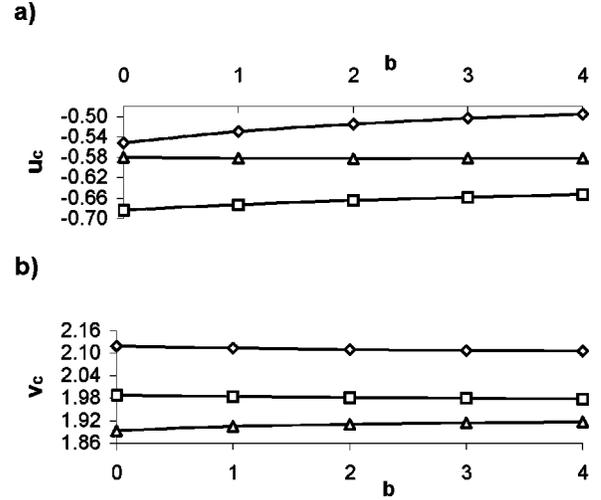


FIG. 7. Graphs of dependence of the results of calculating the RIM fixed point locations (a) u_c component and (b) v_c component on transformation parameter b within the four-loop approximation. The curves marked by \diamond , \triangle , and \square correspond to the Padé approximants $[1/1]$, $[2/2]$, and $[3/1]$, respectively.

the famous Harris criterion.⁴⁷ Indeed, according to Harris, provided the specific-heat exponent α_{pure} of a pure system is positive, i.e., the specific heat is divergent at the critical point ($C \sim |T - T_c|^{-\alpha}$), a new critical behavior under dilution is expected. Otherwise, if $\alpha_{pure} < 0$, the critical behavior of the random system would be similar to that of the pure one. Among the three-dimensional N -vector models, only the Ising model has $\alpha > 0$, and the corresponding new critical behavior has been obtained as an unusual stable fixed point dependent upon $\sqrt{\varepsilon}$ where $\varepsilon = 4 - D$. Furthermore, one (smallest) of the eigenvalues of the stability matrix taken at the RIM fixed point is expressed through the critical exponents α and ν : $\lambda_2 \sim \alpha/\nu$ (for the details see Refs. 48, 49).

The systematic calculation of the RIM critical exponents was historically begun in the seminal works by Harris and Lubensky^{21,22} and Khmel'nitskii.²³ However, the ε -expansion technique could not provide the reliable numerical estimates, because RG equations of the model (1) for $N = 0$ turn out to be degenerate in the one-loop approximation. Such a degeneracy causes powers of $\sqrt{\varepsilon}$ to appear in expansions for the fixed-point locations as well as critical exponents, thus leading to the substantial decrease of accuracy expected within the high-loop approximations.^{50,51}

The following pronounced step to evaluate the RIM exponents was made by G. Jug,⁵² who applied the alternative approach, the RG in fixed dimensions. The reasonable numerical estimates were obtained within the two-loop approximation by making use of the Chisholm-Borel procedure to resum the resulting series. Later the critical exponents series for the 3D RIM were deduced within the three- and four-loop approximations and corresponding numerical estimates were obtained on the basis of the Chisholm-Borel summation method,^{11,12} Padé-Borel procedure and the first confluent form of the ε algorithm of Wynn.⁵³

In this section we will study the critical thermodynamics of the 3D RIM using the generalized PBL summation method. Setting $N=0$ in Eqs. (3) and (4) and solving the system of equations $\beta_u^{res}(u_c, v_c) = 0$, $\beta_v^{res}(u_c, v_c) = 0$ we

TABLE IV. Numerical results for the RIM. Here ‘‘MSA’’ indicates that the fixed-point locations and the critical exponents are found for the most stable approximants in the framework of the PBL resummation technique, ‘‘AV’’ means that the fixed-point locations are calculated as the averages between the values given by the highest approximants [3/1] and [2/2], ‘‘PB’’ denotes a simple Padé-Borel resummation. The results obtained either within the different theoretical approaches, or on the basis of using different resummation techniques, or experimentally (‘‘Expt.’’) and by means of the MC simulations, are presented for comparison.

	u_c	v_c	η	ν	γ	ω
MSA	-0.5816 ± 0.0850	1.9822 ± 0.0740	0.040 ± 0.011	0.681 ± 0.012	1.336 ± 0.020	0.310
AV	-0.6246 ± 0.0600	1.9438 ± 0.0500	0.034 ± 0.010	0.672 ± 0.004	1.323 ± 0.010	0.330
PB [3/1]	-0.6839	1.9877	0.033	0.674	1.326	0.362
PB [2/2]	-0.5800	1.8934	0.034	0.669	1.316	
AW	-0.5874 ^a	1.9362 ^a		0.668 ^a	1.318 ^a	
Ref. 11	-0.728	2.006	0.021	0.671	1.328	0.359
Ref. 55	-0.745	2.011	0.019	0.671	1.328	0.376
Ref. 12	-0.6668	1.9951	0.034	0.670	1.326	
3D MS			0.053 ^b	0.677 ^b	1.319 ^b	0.330 ^b 0.390 ^e
Expt.				0.71 \pm 0.02 ^c 0.70 \pm 0.02 ^{a,d}	1.37 \pm 0.04 ^c 1.37 \pm 0.04 ^{a,d}	
MC, Ref. 58				0.6837 \pm 0.0053	1.342 \pm 0.010	0.37 \pm 0.06

^aQuoted from Ref. 53.

^dQuoted from Ref. 57.

^bQuoted from Ref. 54.

^eQuoted from Ref. 61.

^cQuoted from Ref. 56.

find the RIM fixed-point locations depending on the transformation parameter b . Resulting curves are depicted in Fig. 7. Unfortunately, this picture is not complete, because in the three-loop approximation the Padé approximants [2/1] and [1/2] have the poles for all b . On the other hand, the fixed-point locations values given by the approximant [1/1] seem to be very far from the true ones. So, in order to determine the RIM fixed point locations we cannot apply the optimization algorithm described in Sec. III, at least within the given approximation. In such situation we need a new working criterion. We can, for instance, select the approximants providing the most stable values under the variation of the parameter b . As is seen from Fig. 7, the locations of the RIM fixed point given by the approximants [2/2], for the u_c component, and [3/1], for the v_c component, are practically independent on the parameter b . Indeed, the dispersion of the corresponding curves within the range $0 \leq b \leq 4$ is no more than 1×10^{-4} . The fixed-point locations obtained in such a way may then be used for calculation of the critical exponents γ , η , and ν as well as eigenvalues of the stability matrix when starting the optimization procedure (see Secs. IV and V). Corresponding numerical estimates are summarized in Table IV. For comparison we collected in the table the data obtained earlier either by resumming RG functions within the minimal subtraction (MS) scheme directly at $D = 3$ (3D MS),⁵⁴ or by applying different resummation procedures [Chisholm-Borel technique,^{11,12,55} ϵ algorithm of Wynn, ‘‘AW’’ (Ref. 53)], or found experimentally^{56,57} and through the MC simulations.⁵⁸ To compare those results with our numbers, the fixed-point locations are given by taking the normalizing factor $\frac{8}{9}$ into consideration. As experimental data we consider the averaged values of Ref. 56 obtained in

a course of studying the critical behavior of the site-random Ising system $Mn_xZn_{1-x}F_2$ with $x=0.75$ (or 0.50) by the neutron-scattering method as well as the averaged values of Ref. 57.

In Table IV we present also the numerical results for the fixed-point locations and critical exponents obtained on the basis of the simple Borel summation method combined with the Padé approximants [3/1] and [2/2]. Although the fixed-point coordinates obtained in such a way turned out to be strongly different, the critical exponents estimates differ nevertheless from each other only slightly. From this point of view neither approximant is better.

Another possible way to determine the coordinates of the RIM fixed point is to calculate them as averages between the values given by the highest approximants [3/1] and [2/2] for each of the components (see Fig. 7). The corresponding critical exponents estimates are found to agree with just considered as well as experimental data and MC results within the error bounds (see Table IV).

Unlike the cubic fixed point, the determination of the error bounds for the RIM fixed-point locations is a more difficult problem. In the case of selecting the most stable approximants we have taken the following scheme. First, the values of the fixed-point locations given by the approximants [3/1] and [2/2] are averaged for each of the components separately within the interval $0 \leq b \leq 4$. The discrepancy between averages in these approximants is then adopted as a sought uncertainty in the results. At the same time, the error bounds for the critical exponents are determined in the same way as in Sec. V. If we adopt as the fixed-point locations the averages between the values given by the highest approximants [3/1] and [2/2] separately for each of the components,

the error bounds seem to be even smaller, they are almost three-quarters of the previous ones. Absence of the error bounds in some of the places of Table IV means that the errors either cannot be established or were not established.

We have checked also the stability of the RIM fixed point on the three-dimensional RG flow diagram. In all considered cases the eigenvalue exponents of the stability matrix turned out to be negative, except the calculations based on the simple Borel summation method with the Padé approximant [2/2]. In this case the second eigenvalue λ_2 occurred to be positive and too large. This is in contradiction to the known theoretical and experimental predictions as well as the Monte-Carlo simulations.⁵⁸ Indeed, the second (smallest in modulus) eigenvalue λ_2 of the matrix of derivatives of the β functions is well known to define the so-called correction-to-scaling exponent ω that governs the leading corrections to the universal power laws. Thus the approach of the zero-field susceptibility to the critical temperature, for $T > T_c$, is characterized by the Wegner series⁵⁹

$$\chi \approx \Gamma_0 \tau^{-\gamma} (1 + \Gamma_1 \tau^{\omega\nu} + \dots), \quad (21)$$

with Γ_k being the nonuniversal amplitudes and $\tau = (T - T_c)/T_c$.⁶⁰ Here γ , ν are the asymptotic values of the susceptibility and correlation length critical exponents. As the exponent ω decreases the region increases, where the corrections to scaling laws should be taken into account. So, the smallness of ω in the RIM indicates the importance of its calculation for analysis of the asymptotic critical behavior of dilute systems.⁶¹

Recent MC calculations based on the analysis of the first correction term in Eq. (21) provided the estimate of the correction-to-scaling exponent $\omega = 0.37 \pm 0.06$.⁵⁸ Almost the same number was more recently obtained in the framework of the four-loop 3D RG analysis used for processing divergent series the Borel summation method in combination with the simple rational Chisholm approximants like $[M, M/1, 1]$, $\omega = 0.372 \pm 0.005$.⁶¹ Although the apparent accuracy of this estimate seems to be highly overstated, the central value is in accordance with previous estimates $\omega = 0.366$ (Ref. 62) and $\omega = 0.359$,¹¹ $\omega = 0.376$ (Ref. 55) derived within the three-loop approximation in the framework of the minimal subtraction scheme and the 3D RG, respectively. Our estimate of ω obtained on the basis of the Borel summation method with the Padé approximant [3/1] is close to the above-mentioned one. On the contrary, using of the Padé approximant [2/2] in the Borel transformation leads to the unphysical result for the correction-to-scaling exponent.

At the same time, applying the PBL resummation method to study the asymptotic critical behavior of the systems with impurities results in the correction-to-scaling exponent values which are different from those predicted by either the MC simulations or simple resummation procedures (see Table IV). It is the consequence of that the four-loop approximation is not enough to obtain reliable estimates of the RIM fixed-point locations. Note, however, that our estimates of ω are within the error interval found for the MC result. The critical exponents estimates obtained confirm also the inequality $-\nu\omega < \alpha < 0$ conjectured for the random models.⁶³ Unfortunately, at present we cannot indicate any error bounds in our calculation of the exponent ω .

So, if to assume that the MC simulations⁵⁸ provide the numerical estimate of ω which is close to the exact one, a question is to be put forward: can the estimation of the correction-to-scaling exponent of the dilute systems be used as an additional criterion of selection of the resummation techniques to be employed? Probably the answer will be given in the course of the further investigation of the critical properties of the RIM within the higher-order RG approximations provided a more sophisticated method of the series summation will be used, on a level with the simple techniques. Thus more recently, a different approach to summation of divergent field-theoretical series has been suggested.⁶⁴ The method, based on the Borel transformation combined with a conformal mapping, relies upon the stability of the result of processing on the transformation parameters and therefore does not require knowing the exact asymptotic behavior of the series. This method has been tested on the functions expanded in their asymptotic power series and applied to estimating the ground-state energy of quantum-mechanical systems, including anisotropic oscillator, as well as to calculating the critical exponents for some conformal field theories.^{64,65} The successful testing of the developed technique on simple systems made it possible to apply it for obtaining the critical exponents estimates of anisotropic N -vector field models describing both magnetic and structural phase transitions in cubic and tetragonal crystals^{27,64} from the known five-loop ε expansions.¹³ It would be reasonable therefore to apply this technique in studying the critical behavior of the RIM exactly in three dimensions within the five- or higher-loop approximations.

VII. CONCLUSION

The complete RG analysis of a field model with two quartic coupling constants associated with isotropic and cubic interactions describing magnetic and structural phase transitions in a good number of real substances has been carried out within the four-loop approximation directly in three dimensions. Perturbative expansions for the β functions and critical exponents were deduced for generic N . The fixed-point locations were found for $N \geq 3$ by applying the generalized Padé-Borel-Leroy resummation technique, and the global structure of the 3D RG flow diagram was investigated. The analysis of the eigenvalue exponents of the most intriguing isotropic and cubic fixed points fulfilled for the physically important case $N=3$ has shown that the cubic rather than isotropic fixed point is absolutely stable in 3D. The eigenvalues estimates of both fixed points were found to agree well with those of recently calculated on the basis of exploiting the five-loop ε expansions.¹⁷ The critical dimensionality N_c of the order-parameter field, at which the topology of the flow diagram changes, has been analyzed by the two different methods: (i) by resumming the four-loop RG expansions for the β functions in 3D and (ii) by resumming the five-loop ε expansion for N_c at $\varepsilon=1$. The numerical estimates $N_c = 2.89 \pm 0.02$ and $N_c = 2.894 \pm 0.040$ obtained are in a good agreement with the earlier results^{11,12} and confirm the conclusion about the stability of the cubic fixed point for $N \geq 3$. Consequently, the magnetic and structural phase transitions in three-dimensional anisotropic crystals with cubic symmetry are of second order and their critical

TABLE V. The cubic fixed point locations and critical exponents estimates for some N obtained within the PBL resummation procedure in the six-loop approximation. Here the coupling constants are given in the same notations as those of Ref. 66.

N	u_c	v_c	η	ν	γ	λ_1 , Ref. 69	λ_2 , Ref. 69
3	1.3177 ± 0.0170	0.0964 ± 0.0165	0.0327 ± 0.0020	0.7040 ± 0.0040	1.3850 ± 0.0050	-0.7833 ± 0.0054	-0.0109 ± 0.0032
4	0.8804 ± 0.0080	0.6360 ± 0.0050	0.0316 ± 0.0025	0.7150 ± 0.0050	1.4074 ± 0.0030	-0.7887 ± 0.0090	-0.0740 ± 0.0065
8	0.4410 ± 0.0070	1.1331 ± 0.0160	0.0305 ± 0.0025	0.7143 ± 0.0035	1.4068 ± 0.0030	-0.7955 ± 0.0150	-0.1396 ± 0.0100
∞	0.1751 ± 0.0040	1.4122 ± 0.0090	0.0319 ± 0.0035	0.7094 ± 0.0030	1.3962 ± 0.0040	-0.7986 ± 0.0200	-0.1787 ± 0.0050

thermodynamics should be governed by the cubic fixed point with a specific set of critical exponents. The corresponding four-loop critical exponents estimates were found in the framework of the PBL resummation method. On the basis of comparative numerical analysis with the $O(N)$ -symmetric model, the critical exponents of which are solidly established,¹ it has been shown that in the case of the cubic model the difference between the four-loop estimates and the ‘‘exact’’ values does not exceed 1–2%. Although our results for the most interesting case $N=3$ are in good accordance with earlier theoretical predictions, the cubic universality class is not easy to distinguish experimentally from the isotropic one, due to the obvious marginality of the problem, $N_c \sim 3$.

The critical behavior of weakly quenched disordered systems undergoing second-order phase transitions and described by the three-dimensional random Ising model, which is the nontrivial specific case of the cubic model when $N=0$, has been investigated also. The coordinates, eigenvalues, and critical exponents of the RIM fixed point were computed by using the PBL resummation method and the more simple Padé-Borel procedure. Our numerical results along with the known theoretical and experimental data were summarized in the table. While the RIM fixed point locations found in the framework of the different approximation schemes turned out to be strongly different, the critical exponents estimates differ from each other only slightly, within the error bounds obtained.

Special attention was given to study the stability of the RIM fixed point on the RG flow diagram. The calculation of the stability matrix eigenvalues based on applying different resummation techniques showed that they are negative. Consequently, the RIM fixed point is stable in 3D. As an exception we have indicated the case of using the simple Borel summation method in combination with the Padé approximant $[2/2]$, which led to the unphysical result for the second eigenvalue $\lambda_2 = \omega$. Applying the PBL procedure to calculate the leading correction-to-scaling exponent ω for the three-dimensional impure systems was shown to result in the numerical estimates which are distinguishable markedly from those predicted by recent MC simulations⁵⁸ or followed from applying simple Borel-like summation procedures^{11,55,61} (see Table IV). Note, however, that our estimates of ω were found to be within the error bounds known for the MC result.

At last, it is worth noting, that one can hardly hope to

extract a reliable numerical estimate of ω from the five-loop ε expansion,¹³ even resummed by a proper method. The point is that the ε series for ω turns out to be very short, due to the degeneracy of the random Ising model β functions on the one-loop level. Therefore, until the appreciable discrepancy between the results given by different methods of the series summation on the one hand, and the MC calculations, on the other, does exist, the further investigation of the asymptotic critical behavior of dilute systems will be highly desirable in the framework of the higher-order (five- or six-loop) RG approximations provided a more sophisticated resummation procedure, for instance, the Borel transformation combined with a conformal mapping, will be applied.

More recently the six-loop study of critical behavior of the 3D cubic model appeared.⁶⁶ Perturbative expansions for β functions and critical exponents deduced within the massive field theory in fixed dimension have been resummed by means of the Borel transformation combined with a conformal mapping that takes into account the singularities of the Borel transform. The fixed point locations, stability matrix eigenvalues, and critical exponents estimates obtained turned out to be essentially the same results as those of the present work. In the Appendix we present our numerical estimates for the cubic model obtained on the basis of the six-loop expansions of Ref. 66 when the PBL resummation procedure is applied. The critical behavior of three-dimensional random Ising systems has recently been studied within the five- and six-loop RG approximations in Refs. 67 and 68, respectively.

APPENDIX

In this section we present our numerical estimates for the cubic fixed point locations and critical exponents for some values of N (see Table V) using the six-loop expansions for RG functions recently obtained by J. M. Carmona, A. Pelissetto, and E. Vicari.⁶⁶ We apply the generalized PBL resummation technique. The susceptibility and correlation length critical exponents are estimated through the original series for γ^{-1} and ν^{-1} (or $\eta_4 = 1/\nu - 2$), whereas the critical exponent η is obtained by the known scaling relation: $\eta = 2 - \gamma/\nu$. It is seen that our estimates are in excellent agreement with those of Ref. 66, where the Borel resummation procedure with a conformal mapping, that takes into account the singularities of the Borel transform, has been applied.

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