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Exact ground state for one-dimensional electronic models

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We have found the exact ground state for two electronic models on a linear chain. The first model describes a half-filling electron system at the ferromagnet-antiferromagnet transition point. In the singlet ground state the spin correlators show giant spiral magnetic ordering, with the period of a spiral equal to the system size. The second electronic model describes the point where the ground state has giant spiral off-diagonal long-range order and is, therefore, superconducting. We suggest the formation of a ground state with giant spiral order (ferromagnetic or off-diagonal) as a probable scenario of the subsequent destruction of the ferromagnetism and the superconductivity.

I. INTRODUCTION

In recent years there has been a growing interest in studying systems of strongly correlated electrons in relation with high-T_c superconductivity. Because of the difficulty in dealing with the many-body problems, exact results are rare. It is well known that some one-dimensional (1D) electron models can be exactly solved by the Bethe ansatz. However, many 1D quantum systems do not obey the Yang-Baxter equation, and thus are nonintegrable. Another approach leading to exact results consists of the construction of an exact ground-state wave function for some quantum systems. Recently considerable progress in this problem was achieved by using the so-called matrix product (MP) form of the ground-state wave function. It allowed one to find the exact ground state for various 1D spin models.[1–3] Its origin can be traced back to the $S=1$ spin chain model.[4] For higher-dimensional spin and electronic systems, there are also some methods for the construction of an exact ground-state wave function.[5–8]

There is a class of 1D quantum spin models describing the ferromagnet-antiferromagnet transition point, for which an exact ground state wave function was found in Refs. 9 and 10. The singlet ground-state wave function at this point has a special recurrent form, and for special values of model parameters it can be reduced to the MP form or resonating-valence bond (RVB) form.[9] Spin correlations in the singlet ground state show a giant spiral magnetic structure, with the period of the spiral equal to the system size. On the antiferromagnetic side of this point the ground state can be either gapless, with an algebraic decay of spin correlations,[11] or gapped with the exponential decay of correlations.[12] Thus, this model describes the boundary between the ferromagnetic phase and the singlet phase without long-range order.

In this paper we present the singlet ground-state wave function of this spin model in another form, which can be easily generalized for 1D electronic models. Then we consider two 1D electronic models. The first model describes a half-filling electron system at the point where the singlet and ferromagnetic states are degenerate. The exact calculation of the correlation functions in the singlet ground state shows the same giant spiral magnetic ordering as for the original spin model, while all other correlations vanish in the thermodynamic limit. The second electronic model describes the boundary on the phase diagram between the superconducting phase with off-diagonal long-range order (ODLRO) and the non-superconducting phase. The correlation functions in the ground state of this model show a giant spiral off-diagonal long-range order. We presume that in one-dimensional systems the destruction of the long-range order (ferromagnetic or off-diagonal) can be followed by the appearance of a ground state with giant spiral order.

We generalize this form of the wave function for the electronic ladder model. This model possesses both giant spiral spin order and giant spiral ODLRO in the ground state. Therefore, this electron ladder model describes the boundary points on the phase diagram between four different phases: two singlet phases with and without ODLRO, and two ferromagnetic phases with and without ODLRO. For some special cases the ground-state wave function can be reduced to the usual MP form.

The paper is organized as follows. In Sec. II we construct an exact singlet ground state for the quantum spin model. In Sec. III two electronic models with exact ground states are considered, and the correlation functions are exactly calculated. Section IV gives a brief summary. In the Appendixes a technique for the calculation of correlators is developed.

II. QUANTUM SPIN MODEL

First we consider a $s=\frac{1}{2}$ spin chain model with nearest- and next-nearest neighbor interactions given by the Hamiltonian

$$H=-\sum_{i=1}^{N}\left(\mathbf{S}_i\cdot\mathbf{S}_{i+1}-\frac{1}{4}\right)+\frac{1}{4}\sum_{i=1}^{N}\left(\mathbf{S}_i\cdot\mathbf{S}_{i+2}-\frac{1}{4}\right), \quad (1)$$

with periodic boundary conditions and even N.

This model describes the ferromagnet-antiferromagnet transition point where the ferromagnetic and singlet states are degenerate. Hamiltonian (1) was considered in Refs. 9 and 10 where the singlet ground-state wave function was constructed in two different forms. In this paper we represent another form of this singlet function, which allows us to generalize this function for the electronic model, and to develop a technique to calculate correlators.

The singlet ground state wave function for Hamiltonian (1) can be written as

$$\Psi_0=P_0\Psi, \quad \Psi=\langle 0_b|g_1\otimes g_2\otimes \ldots \otimes g_N|0_b\rangle, \quad (2)$$

where

$$g_i=b^+|\uparrow\rangle_i+b|\downarrow\rangle_i. \quad (3)$$

Here we introduced auxiliary Bose particle b^+ (the Bose operators b^+ and b do not act on spin states $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$) and the Bose vacuum $|0_b\rangle$. Therefore, the direct product $g_1\otimes g_2\otimes \ldots \otimes g_N$ is the superposition of all possible spin configurations multiplied on the corresponding Bose operators, like $b^+bbb^+ \ldots |\uparrow\downarrow\downarrow\uparrow \ldots\rangle$. P_0 is a projector onto the singlet state. This operator can be written as[13]

$$P_0=\frac{1}{8\pi^2}\int_0^{2\pi}d\alpha\int_0^{2\pi}d\beta\int_0^{\pi}\sin\gamma d\gamma \;\; e^{i\alpha S^z}e^{i\gamma S^x}e^{i\beta S^z}, \quad (4)$$

where $S^{x(z)}$ are components of the total spin operator.

This form of wave function resembles the MP form, but with an infinity matrix which is represented by Bose operators. Therefore, we have to pick out the $\langle 0_b| \ldots |0_b\rangle$ element of the matrix product instead of the usual trace in the MP formalism,[1–3] because the trace is undefined in this case. The function Ψ contains components with all possible values of spin S $(0\leqslant S\leqslant N/2)$ and, in fact, a fraction of the singlet is exponentially small at large N. This component is filtered out by the operator P_0.

In order to show that Ψ_0 is the ground-state wave function for Hamiltonian (1), let us represent Hamiltonian (1) as a sum of Hamiltonians h_i of cells containing three sites

$$H=\sum_{i=1}^{N}h_i, \quad (5)$$

where

$$h_i=-\frac{1}{2}\left(\mathbf{S}_i\cdot\mathbf{S}_{i+1}-\frac{1}{4}\right)-\frac{1}{2}\left(\mathbf{S}_{i+1}\cdot\mathbf{S}_{i+2}-\frac{1}{4}\right)+\frac{1}{4}\left(\mathbf{S}_i\cdot\mathbf{S}_{i+2}-\frac{1}{4}\right).$$

One can easily check that each cell Hamiltonian h_i for $i=1,\ldots(N-2)$ gives zero when acting on $g_i\otimes g_{i+1}\otimes g_{i+2}$. Since each h_i is a non-negatively defined operator, then Ψ is the exact ground-state wave function of an open chain:

$$H_{\text{open}}=\sum_{i=1}^{N-2}h_i.$$

As mentioned above, the function Ψ contains components of all possible values of total spin S, and, therefore, the ground state of the open chain is multiply degenerate. However it can be proven (as was done in Ref. 10), that for the cyclic chain (1) only singlet and ferromagnetic components of Ψ have zero energy. Therefore, for cyclic chain (1) Ψ_0 is the singlet ground-state wave function degenerate with the ferromagnetic state.

The exact calculation of the norm and spin correlation function $\langle\mathbf{S}_i\cdot\mathbf{S}_{i+l}\rangle$ (see Appendix A) in the singlet ground state (2) results in the following expressions:

$$\langle\Psi_0|\Psi_0\rangle=\frac{d^N}{d\xi^N}\left(\frac{1}{\cos^2\left(\frac{\xi}{2}\right)}\right)\Bigg|_{\xi=0}, \quad (6)$$

$$\langle\Psi_0|\mathbf{S}_i\cdot\mathbf{S}_{i+l}|\Psi_0\rangle=\frac{\partial^l}{\partial\xi^l}\frac{\partial^{N-l-2}}{\partial\zeta^{N-l-2}}\times\left(-\frac{3}{8}\frac{\cos(\xi-\zeta)}{\cos^4\left(\frac{\xi+\zeta}{2}\right)}\right)\Bigg|_{\xi=\zeta=0}. \quad (7)$$

It can be shown that in the thermodynamic limit, Eqs. (6) and (7) result in

$$\langle\mathbf{S}_i\cdot\mathbf{S}_{i+l}\rangle=\frac{1}{4}\cos\left(\frac{2\pi l}{N}\right). \quad (8)$$

So, we reproduce the result obtained in Refs. 9 and 10, that in the thermodynamic limit a giant spiral spin structure is realized, with the period of the spiral equal to the system size.

III. ELECTRONIC MODELS

Now we will construct electronic models by generalization of wave function (2),

$$\Psi_{\text{e1}}=P_{S=0}\,\text{Tr}_f\langle 0_b|g_1\otimes g_2\otimes\cdots\otimes g_N|0_b\rangle, \quad (9)$$

where

$$g_i=(b^+c^+_{i,\uparrow}+bc^+_{i,\downarrow}+xf^+c^+_{i,\uparrow}c^+_{i,\downarrow}+xf)|0\rangle_i, \quad (10)$$

$c^+_{i,\sigma}$ and $c_{i,\sigma}$ are the Fermi operators and $|0\rangle_i$ is the vacuum state on the ith site of the electronic model; b^+ (b) and f^+ (f) are the auxiliary Bose and Fermi operators, respectively; $|0_b\rangle$ is the Bose vacuum; and x is a parameter of the model. We note that the Fermi operators f^+ and f anticommute with the electronic operators $c^+_{i,\sigma}$ and $c_{i,\sigma}$. So the product $g_1\otimes\cdots g_N$ is the operator in the auxiliary Bose and Fermi spaces. We pick out the $\langle 0_b| \ldots |0_b\rangle$ element in the Bose space, which can be written as

$$\langle 0_b|\, g_1\otimes g_2\otimes\cdots\otimes g_N|0_b\rangle=\phi_0+(f^+f-ff^+)\phi_1, \quad (11)$$

and then we take the trace over the Fermi operators:

$$\mathrm{Tr}_f\langle 0_b|g_1\otimes g_2\otimes\cdots\otimes g_N|0_b\rangle=2\phi_0 .$$

The projector $P_{S=0}$ filters out the singlet component from the function ϕ_0. Thus we obtain a singlet wave function Ψ_{e1} describing the state with one electron per site.

In order to find the Hamiltonian for which the wave function (9) is the exact ground-state wave function, let us consider what states are present on the two nearest sites in the wave function (9). One can easily check that there are only nine states from the total 16 states in the product $g_i\otimes g_{i+1}$. They are

$$\begin{gathered}|\uparrow\uparrow\rangle,\quad |\downarrow\downarrow\rangle,\quad |\uparrow\downarrow+\downarrow\uparrow\rangle,\\ |20-02\rangle,\quad |\uparrow\downarrow-\downarrow\uparrow\rangle-x^2|20+02\rangle,\quad |\uparrow 0-0\uparrow\rangle,\\ |\uparrow 2-2\uparrow\rangle,\quad |\downarrow 0-0\downarrow\rangle,\quad |\downarrow 2-2\downarrow\rangle.\end{gathered} \tag{12}$$

Here we denote an empty site by $|0\rangle$, a site occupied by one electron by $|\uparrow\rangle$ and $|\downarrow\rangle$, and a doubly occupied site by $|2\rangle$.

The elementary Hamiltonian $h_{i,i+1}$ for which all these states are the exact ground states can be written as the sum of the projectors onto the seven missing states $|\varphi_k\rangle$ with arbitrary positive coefficients λ_k:

$$h_{i,i+1}=\sum_{k=1}^{7}\lambda_k|\varphi_k\rangle\langle\varphi_k|.$$

The total Hamiltonian is the sum of the elementary Hamiltonians:

$$H=\sum_{i=1}^{N}h_{i,i+1}. \tag{13}$$

So, for each value of x there is a family of the Hamiltonians depending on seven positive parameters λ_k. The analysis shows that the most simple form of the Hamiltonian for $x>1$ corresponds to the choice of λ_k in the forms

$$\begin{gathered}\lambda_1=4+4/x^4,\quad |\varphi_1\rangle=x^2|\uparrow\downarrow-\downarrow\uparrow\rangle+|20+02\rangle,\\ \lambda_{2,3}=2-2/x^4,\quad |\varphi_2\rangle=|00\rangle,\quad |\varphi_3\rangle=|22\rangle,\\ \lambda_{4,5,6,7}=2,\quad |\varphi_{4,5}\rangle=|0\sigma+\sigma 0\rangle,\quad |\varphi_{6,7}\rangle=|2\sigma+\sigma 2\rangle.\end{gathered}$$

The elementary Hamiltonians $h_{i,i+1}$ in this case depends only on the model parameter x:

$$\begin{aligned}h_{i,i+1}=&1-4\mathbf{S}_i\cdot\mathbf{S}_{i+1}+4\left(1-\frac{3}{x^4}\right)\eta_i^z\eta_{i+1}^z+\frac{4}{x^4}\eta_i\eta_{i+1}\\ &+\sum_\sigma(c_{i,\sigma}^+c_{i+1,\sigma}+c_{i+1,\sigma}^+c_{i,\sigma})(1-n_{i,-\sigma}-n_{i+1,-\sigma})\\ &+\frac{2}{x^2}\sum_\sigma(c_{i,\sigma}^+c_{i+1,\sigma}+c_{i+1,\sigma}^+c_{i,\sigma})(n_{i,-\sigma}-n_{i+1,-\sigma})^2.\end{aligned} \tag{14}$$

Here $n_{i,\sigma}=c_{i,\sigma}^+c_{i,\sigma}$, and the SU(2) spin operators are given by $S_i^+=c_{i,\uparrow}^+c_{i,\downarrow}$, $S_i^-=c_{i,\downarrow}^+c_{i,\uparrow}$, and $S_i^z=\frac{1}{2}(n_{i,\uparrow}-n_{i,\downarrow})$. Here we also use η operators,

$$\eta_i^+=c_{i,\downarrow}^+c_{i,\uparrow}^+,\quad \eta_i^-=c_{i,\uparrow}c_{i,\downarrow},\quad \eta_i^z=\frac{1-n_{i,\uparrow}-n_{i,\downarrow}}{2},$$

which form another SU(2) algebra,[15,16] and $\eta_1\eta_2$ is a scalar product of pseudospins η_1 and η_2.

Hamiltonian (14) does not conserve the total number of empty and doubly occupied sites because of the last term in the elementary Hamiltonians $h_{i,i+1}$, in contrast to the models considered in Ref. 16. Each elementary Hamiltonian $h_{i,i+1}$ $(i=1,\ldots,N-1)$ acting on functions ϕ_0 and ϕ_1 gives zero, since all of states (12) are eigenstates of $h_{i,i+1}$ with zero energy, while the energies of all other states at $x>1$ are positive. Therefore, the functions ϕ_0 and ϕ_1 are ground-state wave functions of the open chain:

$$H_{\mathrm{open}}=\sum_{i=1}^{N-1}h_{i,i+1}. \tag{15}$$

To determine the degeneracy of model (15), we need to classify the functions ϕ_0 and ϕ_1. Analogously to spin model (2), the functions ϕ_0 and ϕ_1 contain components with all possible values of total spin S. Therefore, ϕ_0 contains multiplets with $S=0,\ldots,N/2$, and ϕ_1 contains components with values of the total spin $S=0,\ldots,N/2-1$ (ϕ_1 does not contain a ferromagnetic component, since at least two sites in ϕ_1 are nonmagnetic: $|0\rangle$ and $|2\rangle$). Thus, $N+1$ multiplets are degenerated for the open chain.

However, for cyclic model (14) it can be proved that only three multiplets are the ground states: singlet state (9) with the momentum $p=0$ (singlet component of ϕ_0), the trivial ferromagnetic state with $p=\pi$, and the state with $S=N/2-1$ and $p=\pi$ (which is the component of ϕ_1 with $S=N/2-1$). The last state with $S=N/2-1$ can be written as

$$\Psi_{N/2-1}=\sum_{i<j}(c_{i,\uparrow}^+c_{j,\downarrow}+c_{j,\uparrow}^+c_{i,\downarrow})\prod_{n=1}^{N}c_{n,\downarrow}^+|0\rangle.$$

Thus the ground states of the electronic model [Eq. (14)] with one electron per site are the singlet state, the ferromagnetic state, and the state with $S=S_{\max}-1$.

It is interesting to note that the singlet wave function (9) can be also written in the form (see Appendix B)

$$\Psi_{\mathrm{e1}}=\sum[i,j][k,l][m,n]\ldots\prod_{n=1}^{N}c_{n,\downarrow}^+|0\rangle, \tag{16}$$

where

$$[i,j]=S_i^+-S_j^++x^2(c_{i,\uparrow}^+c_{j,\downarrow}-c_{j,\uparrow}^+c_{i,\downarrow}),$$

and the summation is made for any combination of sites under the condition that $i<j,k<l,m<n\ldots$. This form of the wave function is analogous to the RVB form found in Ref. 9 for spin model (1).

The norm and the correlators of the electronic model (14) in the singlet ground state are calculated in the same way as for the spin model (Appendix A):

$$\langle \Psi_{e1}|\Psi_{e1}\rangle = \frac{d^N}{d\xi^N}\left(2\frac{1+\cosh(x^2\xi)}{\cos^2\left(\frac{\xi}{2}\right)}\right)\Bigg|_{\xi=0}, \tag{17}$$

$$\langle \Psi_{e1}|\mathbf{S}_i\cdot\mathbf{S}_{i+l}|\Psi_{e1}\rangle = \frac{\partial^l}{\partial\xi^l}\frac{\partial^{N-l-2}}{\partial\zeta^{N-l-2}}\left(-\frac{3}{4}\frac{\cos(\xi-\zeta)[1+\cosh(x^2\xi+x^2\zeta)]}{\cos^4\left(\frac{\xi+\zeta}{2}\right)}\right)\Bigg|_{\xi=\zeta=0}, \tag{18}$$

$$\langle \Psi_{e1}c^+_{i,\sigma}c_{i+l,\sigma}|\Psi_{e1}\rangle = \frac{\partial^l}{\partial\xi^l}\frac{\partial^{N-l-2}}{\partial\zeta^{N-l-2}}\left(-x^2\frac{[\cos(\xi)+\cos(\zeta)][\cosh(x^2\xi)+\cosh(x^2\zeta)]}{2\cos^4\left(\frac{\xi+\zeta}{2}\right)}\right)\Bigg|_{\xi=\zeta=0}, \tag{19}$$

$$\langle \Psi_{e1}|\eta^z_i\eta^z_{i+l}|\Psi_{e1}\rangle = \frac{\partial^l}{\partial\xi^l}\frac{\partial^{N-l-2}}{\partial\zeta^{N-l-2}}\left(-\frac{x^4\cosh(x^2\xi-x^2\zeta)}{2\cos^2\left(\frac{\xi+\zeta}{2}\right)}\right)\Bigg|_{\xi=\zeta=0}, \tag{20}$$

$$\langle \Psi_{e1}|\eta^-_i\eta^+_{i+l}|\Psi_{e1}\rangle = \frac{d^{N-2}}{d\xi^{N-2}}\left(\frac{x^4}{\cos^2\left(\frac{\xi}{2}\right)}\right)\Bigg|_{\xi=0}. \tag{21}$$

As can be seen from Eq. (21), the expectation value $\langle\eta^-_1\eta^+_{i+l}\rangle$, which determines the off-diagonal long-range order,[14] does not depend on the distance l. But in the thermodynamic limit ODLRO vanishes:

$$\langle\eta^z_i\eta^z_{i+l}\rangle = O\left(\frac{1}{N^2}\right),\quad \langle\eta^-_i\eta^+_{i+l}\rangle = O\left(\frac{1}{N^2}\right),$$
$$\langle c^+_{i,\sigma}c_{i+l,\sigma}\rangle = O\left(\frac{1}{N}\right),\quad \langle\mathbf{S}_i\cdot\mathbf{S}_{i+l}\rangle = \frac{1}{4}\cos\left(\frac{2\pi l}{N}\right); \tag{22}$$

however for finite systems all correlators (19)–(21) are nonzero.

The second electronic model can be obtained by simply interchanging of the Bose and the Fermi operators in Eq. (10). Thus, the wave function of this model has the form

$$\Psi_{e2} = P_{\eta=0}\,\mathrm{Tr}_f\langle 0_b|g_1\otimes g_2\otimes\cdots\otimes g_N|0_b\rangle, \tag{23}$$

with

$$g_i = (x f^+ c^+_{i,\uparrow} + x f c^+_{i,\downarrow} + b^+ c^+_{i,\uparrow}c^+_{i,\downarrow} + b)|0\rangle_i. \tag{24}$$

The projector $P_{\eta=0}$ filters out the state with total $\eta=\Sigma\eta_i=0$. Therefore, the function Ψ_0 has $S^z=0$, but it is not an eigenfunction of S^2. Instead, it is an eigenfunction of η^2 with $\eta=0$.

Wave function (23) can be also written in a form analogous to the RVB one:

$$\Psi_{e2} = \sum [i,j][k,l][m,n]\ldots|0\rangle,$$

where

$$[i,j] = \eta^+_i - \eta^+_j + x^2(c^+_{i,\uparrow}c^+_{j,\downarrow} + c^+_{i,\downarrow}c^+_{j,\uparrow}),$$

and the summation is also done over any combinations of sites under the condition that $i<j, k<l, m<n\ldots.$

Considering the product $g_i\otimes g_{i+1}$ one can find that there are only the following nine states on two nearest sites in wave function (23):

$$|22\rangle,\quad |00\rangle,\quad |20+02\rangle,\quad |\uparrow\downarrow-\downarrow\uparrow\rangle,$$
$$|20-02\rangle + x^2|\uparrow\downarrow+\downarrow\uparrow\rangle,\quad |\uparrow 0+0\uparrow\rangle, \tag{25}$$
$$|\uparrow 2+2\uparrow\rangle,\quad |\downarrow 0+0\downarrow\rangle,\quad |\downarrow 2+2\downarrow\rangle.$$

The most simple Hamiltonian for this model has a form which is similar to the previous one given by Eq. (14):

$$H = \sum_{i=1}^{N} h_{i,i+1},$$

$$h_{i,i+1} = 1 - 4\eta_i\eta_{i+1} + 4\left(1-\frac{3}{x^4}\right)S^z_iS^z_{i+1} + \frac{4}{x^4}\mathbf{S}_i\cdot\mathbf{S}_{i+1}$$
$$-\sum_\sigma (c^+_{i,\sigma}c_{i+1,\sigma} + c^+_{i+1,\sigma}c_{i,\sigma})(1-n_{i,-\sigma}-n_{i+1,-\sigma})$$
$$+\frac{2}{x^2}\sum_\sigma \sigma(c^+_{i,\sigma}c_{i+1,\sigma} + c^+_{i+1,\sigma}c_{i,\sigma})$$
$$\times(n_{i,-\sigma}-n_{i+1,-\sigma})^2. \tag{26}$$

This Hamiltonian for $x>1$ is also a non-negatively defined operator, and Ψ_{e2} is the exact ground-state wave function with zero energy. This Hamiltonian commutes with η^2, but does not commute with S^2. Therefore, the eigenfunctions of Hamiltonian (26) can be described by quantum numbers η

and η^z. Making the same analysis as for the previous model we find that for the cyclic model (26), states with three different values of η have zero energy [as it was for model (14)]. These are states with $\eta=0$ and momentum $p=\pi$ [Eq. (23)], all states with $\eta=N/2$ and $p=0$,

$$\Psi_{N/2,\eta^z}=(\eta^+)^{N/2-\eta^z}|0\rangle, \tag{27}$$

and states with $\eta=N/2-1$ and $p=0$:

$$\Psi_{N/2-1,\eta^z}=(\eta^+)^{N/2-1-\eta^z}\sum_{i<j}(c_{i,\uparrow}^+c_{j,\downarrow}^+-c_{i,\downarrow}^+c_{j,\uparrow}^+)|0\rangle. \tag{28}$$

Therefore, for the case of one electron per site ($\eta^z=0$) the ground state of model (26) is threefold degenerate.

The correlation functions in ground states (27) and (28) obviously coincide with each other in the thermodynamic limit, and for the half-filling case ($\eta^z=0$) they are

$$\langle c_{i,\sigma}^+c_{i+l,\sigma}\rangle=O\left(\frac{1}{N}\right),\quad \langle \mathbf{S}_i\cdot\mathbf{S}_{i+l}\rangle=O\left(\frac{1}{N^2}\right),$$
$$\langle \eta_i^z\eta_{i+l}^z\rangle=O\left(\frac{1}{N}\right),\quad \langle \eta_i^-\eta_{i+l}^+\rangle=\frac{1}{4}+O\left(\frac{1}{N}\right). \tag{29}$$

The existence of ODLRO immediately follows from the form of the wave functions (27) and (28).

The correlation functions in the ground state (23) have similar forms to that in Eqs. (18)–(21), and in the thermodynamic limit they reduce to

$$\langle c_{i,\sigma}^+c_{i+l,\sigma}\rangle=O\left(\frac{1}{N}\right),\quad \langle \mathbf{S}_i\cdot\mathbf{S}_{i+l}\rangle=O\left(\frac{1}{N^2}\right),$$
$$\langle \eta_i^-\eta_{i+l}^+\rangle=2\langle \eta_i^z\eta_{i+l}^z\rangle=\frac{1}{6}\cos\left(\frac{2\pi l}{N}\right). \tag{30}$$

The giant spiral ordering in the last equation implies the existence of ODLRO and, therefore, superconductivity[14] in the ground state [Eq. (23)]. We note that though all three ground states of model (26) are superconducting, the properties of these wave functions are essentially different. Let us consider the density-density correlator $\langle n_in_{i+l}\rangle$. For wave functions (27) and (28) in the thermodynamic limit this correlator decouples: $\langle n_in_{i+l}\rangle=\langle n_i\rangle\langle n_{i+l}\rangle=1$. However for wave function (23) it is equal to $\langle n_in_{i+l}\rangle=1+\frac{1}{3}\cos(2\pi l/N)$.

It is interesting to note that another model having the ground-state wave function (23) with $x=0$ and the same spiral ODLRO [Eq. (30)] can be obtained from model (1) by simply replacing operators S with η:

$$H=-\sum_{i=1}^{N}\eta_i\eta_{i+1}+\frac{1}{4}\sum_{i=1}^{N}\eta_i\eta_{i+2}. \tag{31}$$

The direct analogy of this model to spin model (1) results in the conclusion that model (31) describes the boundary point on the phase diagram between superconducting and nonsuperconducting phases, where the off-diagonal long-range order is destroyed. We suppose that model (26) also describes such a point. Thus wave functions (9) and (23) are the ground states for 1D electronic systems in the boundary points between the phases with and without long-range order [ferromagnetic for Eq. (14) and off-diagonal for Eq. (26)]. We suggest the formation of a ground state with long-range spiral order like Eqs. (22) and (30) as a probable scenario to the subsequent destruction of the ferromagnetism and superconductivity.

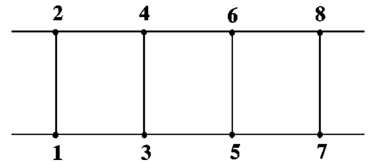

FIG. 1. The two-leg ladder.

This proposed form of wave function can be further generalized for the electronic ladder model (Fig. 1). The wave function for a cyclic ladder model containing $2N$ sites has the form

$$\Psi_{\text{ladder}}=P_{S=\eta=0}\langle 0|g_1\otimes g_2\otimes\cdots\otimes g_N|0\rangle, \tag{32}$$

where each g_i corresponds to the ith rung of the ladder:

$$\begin{aligned}g_i=&c_1[a^+(2x-a^+a)|\uparrow\uparrow\rangle_i-a|\downarrow\downarrow\rangle_i+(a^+a-x)|\uparrow\downarrow+\downarrow\uparrow\rangle_i]\\&+c_2|\uparrow\downarrow-\downarrow\uparrow\rangle_i+c_3[b^+(2y-b^+b)|22\rangle_i-b|00\rangle_i\\&+(b^+b-y)|20+02\rangle_i]+c_4|20-02\rangle_i,\end{aligned} \tag{33}$$

where a^+ and b^+ are Bose operators, $|0\rangle$ in Eq. (32) is the Bose vacuum of a^+ and b^+ particles, and c_i, x, and y are the parameters of the model. $P_{\eta=S=0}$ is the projector onto the state, with $S=\eta=0$.

Let us first consider the case $c_3=c_4=0$. In this case the wave function (32) describes a spin ladder model depending on two parameters c_2/c_1 and x. It can be shown that this model coincides with that considered in Ref. 12. It has a singlet ground state [Eq. (32)] degenerate with the ferromagnetic state. The spin correlators in the singlet ground state show double-spiral ordering with a small shift angle $\triangle\varphi=(2\pi/N)(2c_2/c_1)$ between two giant spirals formed on two legs of the ladder:

$$\langle \mathbf{S}_n\cdot\mathbf{S}_{n+2l}\rangle=\frac{1}{4}\cos\left(\frac{2\pi l}{N}\right),$$
$$\langle \mathbf{S}_n\cdot\mathbf{S}_{n+2l+1}\rangle=\frac{1}{4}\cos\left(\frac{2\pi l}{N}+(-1)^n\triangle\varphi\right). \tag{34}$$

For cases of integer or half-integer $x=j$, which correspond to the special cases of the model,[12] in Eq. (33) one can easily recognize Maleev's boson representation of spin $S=j$ operators:

$$S^+=a^+(2j-a^+a),\quad S^-=a,\quad S^z=a^+a-j.$$

Therefore, in these special cases the infinite matrices formed by the Bose operators a^+ and a can be broken off to the size $n=2j+1$ and wave function (32) is reduced to the usual MP form. The spin correlators in the special cases have an exponential decay.

Now let us return to the general case of the electronic ladder model Eq. (32). In order to find the Hamiltonian for which Eq. (32) is the exact ground-state wave function, one should consider what states are present in Ψ_{ladder} on the two nearest rungs of the ladder. There are only 26 states from the total 256 states in the product $g_i \otimes g_{i+1}$. Therefore, the Hamiltonian of the ladder model can be written as the sum of the projectors onto the 230 missing states $|\varphi_k\rangle$ with arbitrary positive coefficients λ_k:

$$H=\sum_{i=1}^{N} h_{i,i+1}, \quad h_{i,i+1}=\sum_{k=1}^{230} \lambda_k |\varphi_k\rangle\langle\varphi_k|. \tag{35}$$

Unfortunately, we cannot give an explicit form like Eq. (14) or Eq. (26) for this Hamiltonian, because it has a very cumbersome form. But we are able to determine some properties of Hamiltonian (35). This Hamiltonian commutes with both S^2 and η^2. It has a multiply degenerated ground state: the state with $S=\eta=0$ [Eq. (32)], and all the states with $S+\eta=N$ have zero energy. Hence the electronic ladder model [Eq. (35)] describes the boundary point between phases with and without ferromagnetic and off-diagonal long-range order. The correlation functions in ground state (32) can be calculated with the use of the technique developed in the Appendixes. For the case $c_1 > c_3$ there is the same double-spiral spin ordering [Eq. (34)] as for the spin ladder model, while all other correlations are exponentially small. For the case $c_1 < c_3$ the double-spiral ODLRO is realized. In the most interesting symmetric cases $c_1=c_3$, $c_2=c_4$, and $x=y$, the system possesses both giant spiral spin order and giant spiral ODLRO:

$$\langle \mathbf{S}_n \cdot \mathbf{S}_{n+2l}\rangle = \langle \eta_n \eta_{n+2l}\rangle = \frac{1}{8}\cos\left(\frac{2\pi l}{N}\right),$$
$$\langle \mathbf{S}_n \cdot \mathbf{S}_{n+2l+1}\rangle = \langle \eta_n \eta_{n+2l+1}\rangle = \frac{1}{8}\cos\left(\frac{2\pi l}{N}+(-1)^n \Delta\varphi\right), \tag{36}$$

where $\Delta\varphi=(2\pi/N)(2c_2/c_1)$.

Therefore, in this case the wave function (32) describes the boundary points on the phase diagram between the four different phases: singlet phases with and without ODLRO, and ferromagnetic phases with and without ODLRO. In the special cases when $x(y)$ is an integer or half-integer, the spin (off-diagonal) correlations decay exponentially and the wave function in the corresponding Bose space can be represented in MP form with finite matrices of size $n=2x+1$ or $n=2y+1$. When both x and y are integers or half-integers, wave function (32) can be written in the usual MP form with the size of matrices $n=(2x+1)(2y+1)$.

IV. SUMMARY

We have found another form of the singlet ground-state wave function for the quantum spin model considered previously in Refs. 9 and 10. The special technique was developed for an exact calculation of the norm and the correlation functions. This form of the wave function allowed us to generalize it for two 1D electronic models.

The first model describes a half-filling electronic system at the ferromagnet-antiferromagnet transition point when the singlet and ferromagnetic states are degenerate. In the singlet ground state the spin correlators show giant spiral magnetic ordering with the period of the spiral equal to the system size, while all other correlations vanish in the thermodynamic limit.

The second electronic model in the half-filling case has a threefold-degenerate ground state. All ground states have off-diagonal long-range order and, therefore, are superconducting. The calculation of the correlation functions shows that one of the ground states has giant spiral ODLRO.

The comparison of these electronic models with the original spin model[11,12] leads us to the conclusion that these two electronic models describe the boundary points on the phase diagram between the phases with and without long-range order (ferromagnetic for the first model and off-diagonal for the second model). Therefore, we presume that if the Hamiltonian of the 1D quantum system commutes with operators forming the SU(2) algebra (it can be the spin S operator or the pseudo-spin η operator), then the appearance of a ground state with giant spiral order predicts the ensuing destruction of ferromagnetism or superconductivity.

We have briefly considered the generalization of the proposed form of the wave function for the electronic ladder model. The general case of this model has a much richer phase diagram than the two first models. In some particular cases this model describes boundary points on the phase diagram between four different phases: with and without ferromagnetic and off-diagonal long-range order. There are also some special cases of the electronic ladder model when the ground-state wave function is reduced to the usual MP form. In addition, the proposed form of the wave function can be also generalized for the 2D case and different types of lattices.

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APPENDIX A

First we calculate the norm and correlation function of the wave function Ψ_0 [Eq. (2)]. The norm of the singlet wave function Ψ_0 is

$$\langle\Psi_0|\Psi_0\rangle = \langle\Psi|P_0|\Psi\rangle. \tag{A1}$$

Since the function Ψ has $S^z=0$, then the projector P_0 in Eq. (4) takes the form[10]

$$P_0=\frac{1}{2}\int_0^{\pi} \sin\gamma d\gamma e^{izS^-} e^{iz'S^+}, \tag{A2}$$

where $z=\tan(\gamma/2), z'=\sin(\gamma/2)\cos(\gamma/2)$ and $S^{+(-)}$ are the operators of the total spin.

Therefore, the norm takes the form

$$\langle\Psi_0|\Psi_0\rangle = \frac{1}{2}\int_0^{\pi} \sin\gamma d\gamma \times \langle 0_a,0_b|\prod_{i=1}^{N} (g_i^+ e^{izS_i^-} e^{iz'S_i^+} g_i)|0_a,0_b\rangle,$$

where

$$g_i^+ e^{izS_i^-} e^{iz'S_i^+} g_i$$
$$= (a^+\langle\uparrow_i| + a\langle\downarrow_i|) e^{izS_i^-} e^{iz'S_i^+} (b^+|\uparrow\rangle_i + b|\downarrow\rangle_i)$$
$$= a^+b^+ + (1-z'z)ab + izab^+ + iz'a^+b,$$

and a^+ and a are the Bose operators. Thus the norm can be rewritten as

$$\langle\Psi_0|\Psi_0\rangle = \frac{1}{2}\int_0^\pi \sin\gamma d\gamma \langle 0|G^N|0\rangle, \tag{A3}$$

where $|0\rangle = |0_a, 0_b\rangle$ is the Bose vacuum of a^+ and b^+ particles and

$$G = u(a^+b^+ + ab) + iv(ab^+ + a^+b),$$

where $u = \cos(\gamma/2), v = \sin(\gamma/2)$.

Let us introduce the auxiliary function $P(\xi)$:

$$P(\xi) = \langle 0|e^{\xi G}|0\rangle; \tag{A4}$$

then

$$\langle 0|G^N|0\rangle = \left.\frac{d^N P}{d\xi^N}\right|_{\xi=0}.$$

In order to find $P(\xi)$ we perform the following manipulations. First, we take the derivative of $P(\xi)$:

$$\frac{dP}{d\xi} = \langle 0|Ge^{\xi G}|0\rangle = \langle 0|e^{\xi G}G|0\rangle = u\langle 0|abe^{\xi G}|0\rangle$$
$$= u\langle 0|e^{\xi G}a^+b^+|0\rangle. \tag{A5}$$

Now we need to carry a^+b^+ over G in the last expression:

$$e^{\xi G}a^+b^+ = e^{\xi G}a^+e^{-\xi G}e^{\xi G}b^+e^{-\xi G}e^{\xi G}. \tag{A6}$$

This can be done by using the following equations:

$$\begin{aligned} e^{\xi G}a^+e^{-\xi G} &= a^+\cos\xi + (ub + ivb^+)\sin\xi, \\ e^{\xi G}ae^{-\xi G} &= a\cos\xi - (ub^+ + ivb)\sin\xi, \\ e^{\xi G}b^+e^{-\xi G} &= b^+\cos\xi + (ua + iva^+)\sin\xi, \\ e^{\xi G}be^{-\xi G} &= b\cos\xi - (ua^+ + iva)\sin\xi. \end{aligned} \tag{A7}$$

Substituting Eqs. (A6) and (A7) into Eq. (A5), we find

$$\langle 0|e^{\xi G}a^+b^+|0\rangle = u\sin\xi\cos\xi\langle 0|e^{\xi G}|0\rangle$$
$$+ u^2\sin^2\xi\langle 0|abe^{\xi G}|0\rangle.$$

The last equation can be rewritten as the differential equation on $P(\xi)$,

$$\frac{dP}{d\xi} = u^2\sin\xi\cos\xi P(\xi) + u^2\sin^2\xi\,\frac{dP}{d\xi}, \tag{A8}$$

with boundary condition $P(0)=1$.

The solution of Eq. (A8) is

$$P(\xi) = \frac{1}{\sqrt{1-u^2\sin^2\xi}}. \tag{A9}$$

Integrating Eq. (A3) over γ, we obtain

$$\langle\Psi_0|\Psi_0\rangle = \frac{1}{2}\int_0^\pi \sin\gamma d\gamma \left.\frac{d^N P}{d\xi^N}\right|_{\xi=0} = \left.\frac{d^N}{d\xi^N}\left(\frac{1}{\cos^2\left(\frac{\xi}{2}\right)}\right)\right|_{\xi=0}. \tag{A10}$$

Thus, finally, we arrive at

$$\langle\Psi_0|\Psi_0\rangle = 2\left.\frac{d^{N+1}}{d\xi^{N+1}}\left(\tan\frac{\xi}{2}\right)\right|_{\xi=0} = \frac{4(2^{N+2}-1)}{N+2}|B_{N+2}|. \tag{A11}$$

Here B_N are the Bernoulli numbers.

To calculate the spin correlators we need to introduce operators:

$$G_z = g_i^+ e^{izS_i^-} e^{iz'S_i^+} 2S_i^z g_i = u(a^+b^+ - ab) + iv(ab^+ - a^+b),$$

$$G_+ = g_i^+ e^{izS_i^-} e^{iz'S_i^+} S_i^+ g_i = ua^+b + ivab,$$

$$G_- = g_i^+ e^{izS_i^-} e^{iz'S_i^+} S_i^- g_i = uab^+ + iva^+b^+.$$

Then, the correlator $\langle \mathbf{S}_1\mathbf{S}_{l+1}\rangle$ will be defined by

$$\langle\Psi_0|\mathbf{S}_1\mathbf{S}_{l+1}|\Psi_0\rangle = \frac{1}{2}\int_0^\pi \sin\gamma d\gamma \langle 0|\ \frac{1}{4}G_zG^lG_zG^{N-l-2}$$
$$+\frac{1}{2}G_+G^lG_-G^{N-l-2}|0\rangle \tag{A12}$$

(since $\langle 0|G_-\ldots|0\rangle = 0$).

The expectation values in Eq. (A12) can be represented as

$$\begin{aligned} &\langle 0|G_zG^lG_zG^{N-l-2}|0\rangle \\ &\quad= \frac{\partial^l}{\partial\xi^l}\frac{\partial^{N-l-2}}{\partial\zeta^{N-l-2}}\langle 0|G_ze^{\xi G}G_ze^{\zeta G}|0\rangle|_{\xi=\zeta=0}, \\ &\langle 0|G_+G^lG_-G^{N-l-2}|0\rangle \\ &\quad= \frac{\partial^l}{\partial\xi^l}\frac{\partial^{N-l-2}}{\partial\zeta^{N-l-2}}\langle 0|G_+e^{\xi G}G_-e^{\zeta G}|0\rangle|_{\xi=\zeta=0}. \end{aligned} \tag{A13}$$

After a procedure similar to that for the norm and the integration over γ, we obtain

$$\langle\Psi_0|\mathbf{S}_1\mathbf{S}_{l+1}|\Psi_0\rangle = \frac{\partial^l}{\partial\xi^l}\frac{\partial^{N-l-2}}{\partial\zeta^{N-l-2}} \times\left.\left(-\frac{3}{8}\frac{\cos(\xi-\zeta)}{\cos^4\left(\frac{\xi+\zeta}{2}\right)}\right)\right|_{\xi=\zeta=0}. \tag{A14}$$

Now we generalize these calculations for the electronic wave function Ψ_{e1} [Eq. (9)]. The norm of Ψ_{e1} has a form similar to (A3),

$$\langle\Psi_{e1}|\Psi_{e1}\rangle=\frac{1}{2}\int_0^{\pi}\sin\gamma d\gamma\,\mathrm{Tr}_{f_1,f_2}\langle 0|G^N|0\rangle, \quad (A15)$$

where $|0\rangle=|0_a,0_b\rangle$ is the Bose vacuum of the a^+ and b^+ particles, and

$$G=G_b+G_f,$$

$$G_b=u(a^+b^++ab)+iv(ab^++a^+b),$$

$$G_f=x^2(f_2^+f_1^++f_1f_2),$$

where f_1^+ and f_2^+ are the Fermi operators, anticommuting with each other.

The integrating function in Eq. (A15) can be written as

$$\mathrm{Tr}_{f_1,f_2}\langle 0|G^N|0\rangle=\frac{d^N}{d\xi^N}[\mathrm{Tr}_{f_1,f_2}(e^{\xi G_f})\langle 0|e^{\xi G_b}|0\rangle]|_{\xi=0}. \quad (A16)$$

The trace over Fermi operators is calculated easily:

$$\mathrm{Tr}_{f_1,f_2}(e^{\xi G_f})=\mathrm{Tr}_{f_1,f_2}\{1+[\cosh(x^2\xi)-1](f_1^+f_1f_2^+f_2+f_1f_1^+f_2f_2^+)\}=2+2\cosh(x^2\xi).$$

Noting that the quantity

$$\frac{1}{2}\int_0^{\pi}\sin\gamma d\gamma\langle 0|e^{\xi G_b}|0\rangle=\frac{1}{\cos^2\left(\frac{\xi}{2}\right)}$$

has been already calculated in Eq. (A10), we finally obtain:

$$\left\langle\Psi_{e1}|\Psi_{e1}\right\rangle=\frac{d^N}{d\xi^N}\left(2\frac{1+\cosh(x^2\xi)}{\cos^2\left(\frac{\xi}{2}\right)}\right)\Bigg|_{\xi=0}. \quad (A17)$$

The correlation functions over the function Ψ_{e1} are calculated by a similar procedure to that as in the spin model case [Eqs. (A12)–(A14)] leading to expressions (18)–(21).

APPENDIX B

In this appendix we show the equivalence of the two forms (9) and (16) of the wave function Ψ_{e1}. First, let us calculate the anticommutator of two operators g_i and g_j defined by Eq. (10):

$$\{g_i,g_j\}=(-c_{i,\uparrow}^+c_{j,\downarrow}^++c_{i,\downarrow}^+c_{j,\uparrow}^++x^2c_{i,\uparrow}^+c_{i,\downarrow}^++x^2c_{j,\uparrow}^+c_{j,\downarrow}^+)|0\rangle_i|0\rangle_j. \quad (B1)$$

Thus the anticommutator $\{g_i,g_j\}$ does not contain auxiliary operators b^+, b, f^+ and f and presents singlet function located on sites i and j.

Now we can write

$$g_1\otimes g_2\ldots g_N=-g_2\otimes g_1\ldots g_N+\{g_1,g_2\}\otimes g_3\ldots g_N$$
$$=-g_2\otimes g_3\ldots g_N\otimes g_1+\sum_{n=2}^{N}(-1)^n\{g_1,g_n\}\otimes g_2\otimes g_3\ldots g_{n-1}\otimes g_{n+1}\ldots g_N. \quad (B2)$$

Averaging the latter equation by the Bose vacuum $\langle 0_b|\ldots|0_b\rangle$, making Trace over the Fermi operators f^+ and f and filtering out the singlet component, we obtain

$$\Psi_{e1}=-\Psi_{e1}+\sum_{n=2}^{N}(-1)^n\{g_1,g_n\}\Psi_{e1}^{(1,n)}, \quad (B3)$$

where

$$\Psi_{e1}^{(1,n)}=P_{S=0}\,\mathrm{Tr}_f\langle 0_b|g_2\otimes g_3\otimes\cdots\otimes g_{n-1}\otimes g_{n+1}\otimes\cdots\otimes g_N|0_b\rangle.$$

Here we used the fact that the function Ψ_{e1} has a momentum $p=0$.

Repeating this procedure for function $\Psi_{e1}^{(1,n)}$ and so on, we arrive finally at

$$\Psi_{e1}=\frac{1}{2}\sum_{n=2}^{N}(-1)^n\{g_1,g_n\}\Psi_{e1}^{(1,n)}$$
$$=\frac{1}{2^{N/2}}\sum_{i<j\ldots}(-1)^P\{g_i,g_j\}\{g_k,g_l\}\{g_m,g_n\}\ldots, \quad (B4)$$

where $P=(i,j,k,l,\ldots)$ is the permutation of numbers $(1,2,\ldots,N)$, and the summation is done over all combinations of sites under the condition that $i<j,k<l,m<n\ldots$. Expressions for Ψ_{e1} [Eq. (B4) and (16)] coincide with each other up to a constant factor.

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