Classical character of turbulence in a quantum liquid

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A theoretical discussion is presented of recent experiments in which it has been shown that turbulence in the superfluid phase of liquid ^{4}He can be very similar in its characteristics to that in a conventional fluid; particular attention is focused on the work of Stalp, Skrbek, and Donnelly [Phys Rev. Lett. **82**, 4831 (1999)]. It is argued that on length scales significantly greater than both the spacing between the quantized vortex lines in the turbulent superfluid component and the scale on which viscous dissipation occurs in the normal fluid, the two fluids are likely to be coupled together and to behave like a conventional fluid. On smaller length scales account must be taken of dissipation, due to viscosity in the normal fluid, frictional interaction between the vortex lines and the normal fluid, and the radiation of sound from the vortex lines; it is shown that the rate of dissipation is likely to be given by an expression that is similar to that in a conventional fluid but with some important differences. Emphasis is placed on the need for experiments at a very low temperature in which the complicating effects of the normal fluid are either absent or easily taken into account, and the paper includes some theoretical discussion of the decay of superfluid turbulence in the absence of normal fluid and when the normal fluid density is very low. This last discussion is inspired by the ideas of Svistunov [Phys. Rev. B **52**, 3647 (1995)].

I. INTRODUCTION

During the past two or three years it has been observed experimentally that certain types of turbulent flow in the superfluid phase of liquid ^{4}He (helium II) are surprisingly similar to analogous types of flow in a classical fluid at high Reynolds number. This similarity exists in spite of the fact that helium II is described by a two-fluid model, where rotational flow in the superfluid component must take the form of discrete quantized vortex lines,[1] and that there can be no conventional viscous dissipation in the superfluid component. The similarity has also been observed in the results of computer simulations of turbulent flow in superfluid helium at very low temperatures, based on the Gross-Pitaevskii equation (nonlinear Schrodinger wave equation).[2]

We have two aims in this paper. First, we consider the results of one particular experiment carried out and analyzed recently by Stalp, Skrbek, and Donnelly,[3] following earlier work by Smith *et al.*[4] This experiment was concerned with flow behind a moving grid, and the authors showed that there was indeed a remarkable similarity between the observed behavior of the helium II and that expected in turbulent flow behind a similar grid in a classical fluid. We aim to discuss this similarity, to understand some aspects of it, and to focus attention on some remaining problems. Although our analysis is confined to the results of this particular experiment, we believe that with suitable modifications it will prove to be applicable to other experiments. The experiments of Stalp, Skrbek, and Donnelly were carried out only at relatively high temperatures, above about 1.4 K. Our second aim is to discuss the evolution of superfluid grid turbulence at much lower temperatures, where the effect of the normal fluid is either absent or much smaller than it is at higher temperatures. Different and interesting features should then be evident, indicating that experiments at these lower temperatures should be of great interest. It is pleasing to note that some very preliminary experiments at these lower temperatures are now being carried out by McClintock and his collaborators. Our discussion of the low-temperature regime proves useful also in understanding some recent and as yet unpublished results of Stalp on the temperature dependence of a parameter ν' above 1.2 K, which we introduce at the end of Sec. II.

It should be emphasized that throughout this paper we shall take a rather simple-minded view of classical fully developed turbulence: one that involves a dynamical cascade by which energy in the large scale motion is transferred in a random manner to motion on smaller and smaller length scales until viscous dissipation becomes important, but one in which we ignore intermittency and the associated locally ordered flow patterns on a small scale.[5] It should also be emphasized that our discussion is often quite speculative; an important aim is to stimulate further work.

The plan of the paper is as follows. In Sec. II we describe the experiments of Stalp, Skrbek, and Donnelly and pose questions that arise from them. In Sec. III we discuss how to describe the vorticity in the superfluid component, and we consider in general terms how quantized vortex lines are arranged in the superfluid grid turbulence. Section IV takes a look at the length scale on which viscous dissipation occurs in superfluid turbulence and how this length compares with the spacing between vortex lines. Section V addresses the extent to which the two fluids are coupled in grid turbulence, and in Sec. VI we suggest that on length scales at which the two fluids are coupled the turbulence should behave classically. Finally, in Sec. VII we discuss in some detail how dissipation occurs in superfluid grid turbulence, especially at low temperatures, and we compare the results with those applicable to classical turbulence.

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II. THE EXPERIMENTS OF STALP, SKRBEK, AND DONNELLY

In these experiments a grid was moved at constant speed along a tube containing helium II, and a measurement was made as a function of time of the excess attenuation of second sound in a small fixed region through which the grid had moved. The grid creates turbulence in both the normal and the superfluid components. The excess attenuation of the second sound is due to the mutual friction caused by the quantized vortex lines that are associated with turbulence in the superfluid component. The length of vortex line per unit volume L is simply related to the excess attenuation, so that the experiment consists in essence of a measurement of L as a function of time at a fixed point in space after the grid has passed through it.

If helium II is in equilibrium in a vessel that rotates at a steady angular velocity Ω, the superfluid component contains a uniform array of vortex lines,[1] aligned parallel to the axis of rotation, the length of line per unit volume being given by $L=2\Omega/\kappa$, where $\kappa=h/m_4$ is the quantum of circulation. The mean vorticity in the superfluid component is, therefore, given by $\langle\omega\rangle=\kappa L$. Stalp, Skrbek, and Donnelly assume that the mean-square vorticity in the turbulent superfluid is given by the analogous expression

$$\langle\omega^2\rangle=(\kappa L)^2. \tag{2.1}$$

Stalp, Skrbek, and Donnelly show that their experimental results are consistent with the following picture; i.e., the assumptions underlying this picture lead to a predicted time dependence of L that is in agreement with experiment. The turbulence is homogeneous and isotropic. The two fluids are ''coupled'' (locked together with the same velocity fields) on all relevant length scales, and therefore the root-mean-square vorticity in both fluids is equal to κL. The turbulent energy spectrum $E(k)$ has the Kolmogorov form[6]

$$E(k)=C\varepsilon^{2/3}k^{-5/3} \tag{2.2}$$

for wave numbers k in an inertial range defined by the inequality $k_e<k<k_{\eta\rho}$, where k_e^{-1} is the size of the energy-containing eddies and $k_{\eta\rho}$ is the Kolmogorov wave number at which viscous dissipation sets in for a classical turbulent fluid with kinematic viscosity $\nu=\eta_n/\rho$; η_n is the viscosity of the normal component and ρ is the total density of the helium. Immediately behind the grid, the size of the energy-containing eddies is of the order of the grid spacing, but at greater distances from the grid this size increases until it saturates at a value of order the tube size. C is a constant of order unity, and ϵ is the rate of energy dissipation per unit mass of helium. $k_{\eta\rho}=(\epsilon/\nu^3)^{1/4}$. In a classical fluid in which there is homogeneous turbulence this energy dissipation is equal to the product of the kinematic viscosity and the mean-square vorticity;[7] for the turbulent helium II, Stalp, Skrbek, and Donnelly take it as given by

$$\varepsilon=\nu'\langle\omega^2\rangle=\nu'\kappa^2L^2, \tag{2.3}$$

where ν' is a parameter with the dimensions of kinematic viscosity that they suggest ought to be of order ν.

This work by Stalp, Skrbek, and Donnelly, provides striking evidence for a similarity between quantum and classical turbulence. We discuss the origin of this similarity, although we find that in some respects there must be significant differences that are not exposed by the analysis of Stalp, Skrbek, and Donnelly.

It is useful to have in mind a number of questions raised by the analysis of Stalp, Skrbek, and Donnelly, which we must address in our discussion.

(1) Can we justify the use of Eq. (2.1) for the mean-square vorticity in the turbulent superfluid?

(2) Why are the two fluids coupled? Are they coupled at all relevant length scales, which means presumably at length scales within the whole range between k_e^{-1} and $k_{\eta\rho}^{-1}$?

(3) Why do the coupled fluids have a turbulent energy spectrum of the Kolmogorov form?

(4) What is the justification for the use of Eq. (2.3)? Is the parameter ν' necessarily of order η_n/ρ? The experimental results of Stalp, Skrbek, and Donnelly do not allow a precise determination of ν', but they show that it is certainly of order η_n/ρ. In more recent, as yet unpublished, experiments by Stalp the temperature dependence of ν' has been determined, and there is evidence that this dependence accords with that of η_n/ρ only close to the λ-point, and that, contrary to the behavior of η_n/ρ, the value of ν' falls as the temperature is reduced to below about 1.6 K. We need to understand this behavior.

III. THE MEAN-SQUARE VORTICITY AND VORTEX-LINE CONFIGURATIONS IN THE TURBULENT SUPERFLUID

We first discuss the validity of Eq. (2.1), and then make some general observations about the configuration of vortex line in the grid turbulence studied by Stalp, Skrbek, and Donnelly.

Suppose that the superfluid contains a length L per unit volume of quantized vortex line. Strictly speaking, the vorticity is then infinite along lines within the superfluid. In order to deal with this, we suppose first that the vorticity (curl$\mathbf{v}_s$) is concentrated uniformly over a cylinder of radius a_0 surrounding each line, and then let $a_0\to 0$ (note that a_0 must be distinguished from the core radius ξ_0, which determines the line energy). We can show easily that the mean-square vorticity in the superfluid is then equal to $(\kappa^2L)/\pi a_0^2$, which tends to infinity as $a_0\to 0$, in disagreement with Eq. (2.1). There is no corresponding problem with the mean vorticity. Any reference to the mean-square vorticity in the superfluid is therefore meaningless. Instead we ought to refer only to the length of vortex line per unit volume L, which is well defined and is indeed the quantity that is measured experimentally from the attenuation of second sound. We can of course introduce an effective mean-square vorticity in the superfluid component, defined by $\langle\omega^2\rangle_{\text{eff}}=\kappa^2L^2$, and we use this quantity occasionally in this paper. As noted by Stalp (private communication), the quantity $(\kappa L)^2$ can also be regarded as the square of the mean value of the modulus of the vorticity in the superfluid, but it still seems generally best to think in terms of the directly measured quantity L.

Turning to the configuration of vortex line in grid turbulence, we recall first the types of superfluid motion that can, in principle, take place.[1] The motion must be irrotational, except on the cores of the quantized vortex lines. The nearest approximation to uniform rotation of the superfluid component, such as occurs if the helium is in equilibrium in a vessel rotating uniformly at a high angular velocity Ω, is, as we have already mentioned, a structure containing a uniform array of parallel vortex lines, all with the same sense of rotation (completely ‘‘polarized’’), the length of line per unit volume being $\langle\omega\rangle/\kappa$, where $\langle\omega\rangle=2\Omega$ is the magnitude of the spatially averaged superfluid vorticity. However, we can also have less regular arrays of vortex lines such as the random, tangled array that is set up in counterflowing helium[8] carrying a heat current. If the array is completely random, by which we mean that the lines are randomly oriented (no polarization), the spatially averaged vorticity must be zero, although $\langle\omega^2\rangle_{\rm eff}$ is still by definition equal to κ^2L^2. Intermediate situations are also possible, especially at a particular instant of time and within a subvolume of the helium, in which the array of lines is partly polarized. The instantaneous mean vorticity (within, say, the subvolume) is then nonzero, but it is smaller in magnitude than κL.

If the turbulent energy spectrum in the inertial range has the form (2.2), the spectrum of the mean-square vorticity, which must be something like $k^2E(k)$, must have the form $D\varepsilon^{2/3}k^{1/3}$. It follows that in the experiments of Stalp, Skrbek, and Donnelly the mean-square vorticity in the coupled fluids is concentrated in the region of high wave numbers within the inertial range, i.e., around the wave number $k_{\eta\rho}$, as in grid turbulence in a conventional liquid. Therefore, at any instant, the square of the mean vorticity associated with eddies larger than the Kolmogorov length $k_{\eta\rho}^{-1}$, must be small compared with $\langle\omega^2\rangle_{\rm eff}$, so that the vortex lines within the superfluid component must be oriented in a largely random manner, a relatively small polarization being sufficient to generate the eddies that are significantly larger than $k_{\eta\rho}^{-1}$, including the energy-containing eddies.

It should be explained at this point it is not our intention to discuss the very early stages of the evolution of the turbulence, when the necessary high density of vortex lines is being created by the moving grid. We cannot expect to understand these early stages without a detailed understanding of the interaction between the grid and the superfluid, and also, very probably, between the grid and the residual vortex lines that are believed to be important in vortex nucleation at relatively low velocities.[1]

IV. THE KOLMOGOROV LENGTH $k_{\eta\rho}^{-1}$ COMPARED WITH THE INTERVORTEX SPACING

Let us follow Stalp, Skrbek, and Donnelly and assume first that the two fluids are fully coupled, and second that the mean-square vorticity in the normal fluid is equal to κ^2L^2. We shall show that the Kolmogorov length, $k_{\eta\rho}^{-1}$, must then be comparable with the intervortex spacing $l=L^{-1/2}$, which means that in reality the two velocity fields cannot be the same on a length scale comparable with $k_{\eta\rho}^{-1}$. The assumption that the two fluids are fully coupled, even on length scales comparable with $k_{\eta\rho}^{-1}$, cannot therefore be true. It is only on length scales significantly larger than $k_{\eta\rho}^{-1}$ that the two fluids can be coupled, in the sense that the two velocity fields can then be the same. This result further underlines the need to understand why an equation of the form (2.3) can hold.

Motion in a classical fluid on a length scale equal to the Kolmogorov length k_η^{-1}, is characterized by a Reynolds number equal to unity, in the sense that

$$k_\eta^{-1}=\frac{\nu}{u}, \tag{4.1}$$

ν being the appropriate kinematic viscosity and u being a typical fluid velocity on this length scale. Applying this to the two fluids, and assuming complete coupling, we put the mean-square vorticity in the coupled fluids equal to κ^2L^2. This vorticity is concentrated at wave numbers of order $k_{\eta\rho}$, and therefore a typical coupled velocity on the length scale $k_{\eta\rho}^{-1}$ is given by $k_{\eta\rho}u\sim\kappa L$. It follows that

$$k_{\eta\rho}^{-1}\sim\left(\frac{\nu}{\kappa L}\right)^{1/2}=\left(\frac{\nu}{\kappa}\right)^{1/2}l, \tag{4.2}$$

where $\nu=\eta_n/\rho$. Substituting numerical values for ν and κ, we find that the Kolmogorov scale, $k_{\eta\rho}^{-1}$, and the intervortex line spacing are indeed similar in magnitude.

V. THE COUPLING OF THE TWO FLUIDS

There remains the possibility that the two fluids are coupled together on length scales significantly greater than $k_{\eta\rho}^{-1}$. In this section we shall examine the possibility that mutual friction is sufficient to ensure this coupling.

The presence within the superfluid of vortex lines gives rise to a force of mutual friction between the two fluids, as a result of the scattering of the normal-fluid excitations by the lines. We write the force per unit length of line as $\gamma(v_L-v_n)$, where (v_L-v_n) is the velocity of the vortex line relative to the normal fluid, assumed to be flowing at right angles to the line. The force per unit volume containing a randomly arranged length of line L is then given roughly by[1,9]

$$F_{sn}=\frac{2}{3}\left(\frac{\gamma}{\rho\kappa}\right)\rho\kappa L(v_s-v_n), \tag{5.1}$$

where we have assumed that on average the vortex line velocity is equal to the superfluid velocity v_s. We have introduced the dimensionless parameter $\gamma/\rho\kappa$, which will appear again in our discussion in Sec. VII. We argue that the extent to which this force ensures coupling of the two velocity fields depends on the relative value of two characteristic times.

Consider eddy motion on a length scale R, greater than $k_{\eta\rho}^{-1}$. We use as a model two coincident rigid spheres, each of radius R, one composed of the normal fluid, the other of the superfluid. Initially the spheres rotate about a common axis, with different angular velocities. It is easy to show that as a result of the mutual friction (5.1) the two spheres acquire a common angular velocity with a time constant that is independent of R and given by

$$\tau_c=\frac{3}{2\kappa L}\left(\frac{\rho\kappa}{\gamma}\right)\frac{\rho_s\rho_n}{\rho^2}; \tag{5.2}$$

ρ_s and ρ_n are the densities of the superfluid and normal fluid components. This is the first relevant characteristic time. The other time is the ‘‘lifetime’’ of an eddy of size R; i.e., the time associated with the inertial transfer of energy from this eddy to eddies of other sizes. This is given by

$$\tau_i(R)=\frac{R}{U}, \tag{5.3}$$

where U is the fluid speed associated with the eddy. We make the reasonable assertion that the motion associated with the two fluids will be coupled if

$$\tau_c \ll \tau_i(R). \tag{5.4}$$

Let $L_{\min}(R)$ be the minimum length of vortex line that would allow the superfluid eddy of size R to rotate with the required angular velocity, U/R; i.e., $L_{\min}(R)=2U/\kappa R$. The condition (5.4) can then be written

$$\left(\frac{\gamma}{\rho\kappa}\right)\frac{\rho^2}{\rho_s\rho_n}\frac{L}{L_{\min}(R)}\gg 1. \tag{5.5}$$

We noted in Sec. III that in the experiments of Stalp, Skrbek, and Donnelly the vortex lines in the turbulent superfluid must be oriented in a largely random manner. Eddies of size $R(\gg k_{\eta\rho}^{-1})$ are generated by a small degree of polarization. It follows that $L \gg L_{\min}(R)$. In the whole of the temperature range studied by Stalp, Skrbek, and Donnelly the parameter $(\gamma/\rho\kappa)(\rho^2/\rho_s\rho_n)$ is close to unity.[15] Therefore condition (5.5) must be satisfied. We conclude that the eddies large compared with $k_{\eta\rho}^{-1}$ in the two fluids are indeed likely to be strongly coupled. If the passage of the grid induces different velocity fields on a length scale R, the difference will disappear in a time that is small compared with $\tau_i(R)$. This result alone is sufficient to ensure that the two fluids can be expected to behave as a single fluid with density ρ on length scales significantly greater than the Kolmogorov length $k_{\eta\rho}^{-1}$.

VI. TIME EVOLUTION OF THE VELOCITY FIELDS IN THE INERTIAL RANGE

In the case of a classical turbulent fluid, dissipation (due to viscosity) can be neglected on length scales significantly greater than the Kolmogorov length, k_η^{-1}. At high Reynolds numbers there is associated with the turbulence a significant *inertial range* in which the Fourier components of the velocity field have wave numbers less than k_η. Within this inertial range the turbulence evolves as follows: we consider grid turbulence in a channel of finite width d. There is an inertial transfer of energy between different wave numbers, the transfer being most effective if the different wave numbers are not too different in magnitude (the ‘‘independence of Fourier components for distant wave numbers’’ [6]). The rate of transfer is determined by the characteristic times $\tau_i(R)$ for the different length scales R (Fourier components $1/R$). Viscosity has very little effect; the inertial transfer is due to the nonlinear terms in the hydrodynamic equations. Turbulence is generated initially on a length scale comparable with the grid spacing M. Inertial transfer of energy takes place towards eddies that are both larger and smaller than the grid spacing, but the former process saturates because eddies larger than the channel size cannot exist. After a relatively short time, the turbulence reaches a state of ‘‘universal statistical equilibrium,[6]’’ at least within an inertial subrange[6] defined by the inequality $M^{-1} \ll k \ll k_\eta$; a state approximating to universal equilibrium probably exists over the wider range of wave numbers $d^{-1} \leqslant k \leqslant k_\eta$, as assumed by Stalp, Skrbek, and Donnelly. The wave number k_e, at which $E(k)$ is a maximum (defining the energy-containing eddies), increases until it saturates at $k_e = d^{-1}$ (Stalp, Skrbek, and Donnelly). After this saturation has occurred $E(k)$ takes the form (2.2), at least approximately, for all wave numbers in the range $d^{-1} \leqslant k \leqslant k_\eta$, and there is a simple energy cascade, energy being transferred to higher and higher wave numbers until it is dissipated by viscosity at $k \sim k_\eta$.

Turning now to two-fluid turbulence, we note first that when the two fluids are coupled there can be negligible dissipation due to mutual friction. We shall argue in the next section that dissipation in the superfluid component (due in essence to vortex reconnections and the damping of high-wave-number Kelvin waves by mutual friction) can be neglected on length scales significantly greater than the intervortex spacing $l = L^{-1/2}$. Since, as we demonstrated in Sec. IV, $k_{\eta\rho}^{-1} \sim l$, we can assume that dissipative processes have a negligible effect in both fluids for wave numbers less than $k_{\eta\rho}$.

We have seen in Sec. V that mutual friction leads to strong coupling of the two fluids on length scales greater than $k_{\eta\rho}^{-1}$. It is plausible to suppose, therefore, that on these length scales the system will behave as a single inviscid fluid of density ρ, so that a classical cascade will be set up, and with it an energy spectrum of the Kolmogorov form. Indeed we can go further: we suggest that once mutual friction has brought the two velocity fields into coincidence, these fields will evolve in essentially identical ways, without the further need for mutual friction. This is because (a) the two velocity fields are forced by quantum effects (the presence of discrete quantized vortex lines) to be different only on length scales comparable with or less than $k_{\eta\rho}^{-1} \sim l$, and (b) the nonlinear terms in the hydrodynamic equations couple only wave numbers that are not too different in magnitude, so that the behavior at wave numbers significantly less than $k_{\eta\rho}$ is unaffected by these quantum effects. If flow through the grid in the experiment of Stalp, Skrbek, and Donnelly were to produce *initially* the same velocity fields in the two fluids, mutual friction would need to play hardly any role. Furthermore, we would expect the Kolmogorov energy spectrum to be set up even at zero temperature, when there is no normal fluid, for $k < l^{-1}$.

VII. DISSIPATION AT HIGH WAVE NUMBERS

There remain questions associated with the dissipation of turbulent energy at high-wave-number components of the velocity fields. In a classical fluid such dissipation occurs as a result of viscosity in the region of the Kolmogorov wave number k_η, the rate of dissipation per unit mass of fluid being given, as we saw in Sec. II, by the product of the kinematic viscosity and the mean-square vorticity

$$\varepsilon = \nu\langle\omega^2\rangle. \tag{7.1}$$

The situation in a two-fluid system is generally much more complicated, since it involves dissipative processes in both fluids, the two fluids being also coupled together. We consider first the situation at zero temperature where there is only a superfluid component, and then go on to more general cases.

A. Dissipation in a random distribution of vortex lines at zero temperature

At zero temperature in isotopically pure ^{4}He there is no normal fluid, so that the frictional force acting on a moving vortex is zero. We shall consider initially the case of a random distribution of vortex lines, corresponding to a situation where there are no eddies within the superfluid with size significantly larger than the vortex spacing $l = L^{-1/2}$. We also assume initially that the vortex-line configuration has no structure (no Fourier components) on length scales much smaller than l. (We emphasize that this statement does not imply that the superfluid *velocity field* has no structure on a length scale much smaller than l. The structure of an individual vortex leads to a velocity field that has structure on length scales down to the core size ξ_0. We do mean, however, that there is, for example, no wavelike structure on the vortex lines with wave numbers greater than l^{-1}.) If we ignore logarithmic terms, there is only one characteristic velocity associated with this configuration: roughly, κ/l. Noting that a vortex line with radius of curvature R has a self-induced velocity of order $(\kappa/R)\ln(R/\xi_0)$, we see that, more accurately, the characteristic velocity ought to be written

$$v_l = (\kappa/l) f_1[\ln(l/\xi_0)], \tag{7.2}$$

where f_1 is a function that depends on the detailed form of the vortex configuration. Therefore, there is only one characteristic time associated with the vortex configuration, which can be written as

$$\tau_l = \frac{l^2}{\kappa} f_2\left[\ln\left(\frac{l}{\xi_0}\right)\right], \tag{7.3}$$

where f_2 is another function that depends on the detailed form of the configuration.

Dissipation can occur only as a result of the emission of thermal excitations (phonons and rotons) from the vortex lines. Such emission can take place, in principle, as the result of three processes: the movement of an element of a line with velocity exceeding the Landau critical velocity; the emission of sound waves (phonons) by an oscillating vortex; and vortex reconnections. In practice the characteristic velocity κ/l is always much smaller than the minimum Landau critical velocity (for roton creation), so that we can rule out the first of these processes. However, any oscillatory motion of an element of the vortex line will have a characteristic frequency of $1/\tau_l$, and this can lead, in principle, to the radiation of sound; i.e., to the generation of phonons.

We know of no existing calculation of the rate of radiation of sound from an oscillating vortex. We have made an estimate ourselves of this rate, based on the physical picture of sound radiation from a turbulent velocity field, as described by Morse and Ingard,[10] and we find the following results. We use as a model the radiation from a Kelvin wave of amplitude ϕ, wave number $\tilde{k}$, and angular frequency $\tilde{\omega}$, on a rectilinear vortex; $\tilde{\omega}$ and $\tilde{k}$ are related by the approximate dispersion relation

$$\tilde{\omega} = \frac{\kappa \tilde{k}^2}{4\pi} \ln\left(\frac{1}{\tilde{k}\xi_0}\right). \tag{7.4}$$

At distances from the vortex line that are small compared with $\tilde{k}^{-1}$ the radiation is dipolar and the power radiated per unit length of line is given roughly by

$$\Pi = \frac{\pi^2 \rho \kappa^2 \tilde{\omega}^3 \phi^2}{c^2}, \tag{7.5}$$

where c is the speed of sound. In the opposite limit the radiation is quadrupolar, and the power (7.5) is multiplied by a factor $(\tilde{\omega}/c\tilde{k})^2$, which is, in practice, significantly smaller than unity. It is easy to show from Eq. (7.5) that energy will be lost from the Kelvin wave at a rate that is characterized by a time constant τ_s, given by roughly

$$\tau_s^{(d)} = \frac{c^2}{\kappa^3 \tilde{k}^4} \sim \frac{c^2}{\kappa \tilde{\omega}^2}, \tag{7.6a}$$

for the case of dipole radiation, and by

$$\tau_s^{(q)} = \frac{4\pi c^4}{\kappa^5 \tilde{k}^6}, \tag{7.6b}$$

for the case of quadrupole radiation. We are interested in the rate of radiation of sound energy from a vortex tangle with the characteristic frequency $1/\tau_l$. We make the reasonable assumption that the time constant associated with this radiation will be obtained roughly by putting $\tilde{\omega} = 1/\tau_l$ into Eq. (7.6). We find that the resulting time constant is much greater than τ_l. We conclude that the radiation of sound from the vortex tangle at the frequency $1/\tau_l$ can cause decay of the turbulence only on a time scale much greater than τ_l.

It is known, for example from the work of Schwarz[11] on superfluid turbulence in a heat current, in which there is a counterflow of the two fluids, that vortex reconnections are likely to play an important role in our present problem. Pictures of the way in which such reconnections might occur were given by Schwarz[12] without discussion of the microscopic processes, and Koplik and Levine[13] have provided a more microscopic picture with numerical simulations based on the Ginsburg-Pitaevski (or nonlinear Schrodinger) equation. Similar numerical simulations of a more ambitious type, applying to a whole superfluid turbulent field, have been developed by Nore, Abid, and Brachet[2] and have been applied to the Taylor-Green flow. As we have already noted, the results of this latter type of simulation display a remarkable similarity to the results for a corresponding classical flow, an energy spectrum of the Kolmogorov form being seen in appropriate circumstances. Dissipation of turbulent energy in the Taylor-Green flow leads ultimately to the production of sound waves (phonons); the precise mechanism is not yet clear, although the authors believe that reconnection processes may be involved.

We now argue that reconnections must lead to a violation of our assumption that the vortex-line configuration has no structure on length scales significantly smaller than l. Reconnections leave sharp kinks on the reconnected vortex lines,[12,13] and the evolution of these kinks can be described in terms of Kelvin waves (Svistunov[14]). The wavelengths of some of these waves must be significantly smaller than l, so that the vortex-line configuration must indeed have structure on a length scale significantly smaller than l, unless the Kelvin waves of short wavelength are strongly damped. Indeed, the reconnection process itself must involve very small-scale structures, at least for short periods of time. The Kelvin waves will lose energy by radiation of sound, as described by Eq. (7.5). As we have seen, the resulting damping of the waves is small for wavelengths comparable with l; indeed we can go further and state that damping is not large enough to eliminate Kelvin waves unless their wavelengths are significantly smaller than l, so that there must indeed be structure on length scales less than l. Ultimately, however, the Kelvin waves will be dissipated by the radiation of sound, and we must conclude that this dissipation is at least a major contributor to the energy loss from the turbulent superfluid. Whether the reconnection process itself involves the generation of sound is not clear; probably it does, but such a process may not be distinct from the radiation of sound in the early stages of the evolution of the Kelvin waves. The generation of sound (phonons), either by Kelvin waves or during reconnection processes, seems to be the only dissipative mechanism that can operate on superfluid turbulence at the lowest temperatures. In the absence of any theory describing phonon emission during reconnections, we shall ignore such processes, allowing only for phonon emission from the Kelvin waves. It should be added that reconnections may lead to the generation of vortex rings. We follow Svistunov[14] in assuming that these rings play no special role in the loss of energy from the vortex tangle; any rings produced will be reabsorbed by the tangle, leading only to new reconnections.

For our present purposes we would like to know the rate of loss of turbulent energy in the case of superfluid grid turbulence, or, equivalently, the rate of decrease in L due to this sound (or phonon) emission. As we have noted, a loss of turbulent energy takes place in the simulations of Nore, Abid, and Brachet and this loss must be due to sound emission. But the rate of dissipation of turbulent energy in a form useful in the present context is hard to extract, although it appears to be comparable in magnitude with that expected in an analogous flow in helium I.

In homogeneous grid turbulence in a classical fluid there is, as we have explained, a flow of energy from components of the velocity field with small wave numbers to components with large wave numbers, energy being dissipated by viscosity near the Kolmogorov wave number $k_{\eta\rho}$. We have suggested that in superfluid grid turbulence there is a similar flow of energy towards higher wave numbers, by a similar mechanism, at least as far as wave numbers of order l^{-1}. However, we see now that at the lowest temperatures there can be no significant dissipation at these latter wave numbers. Instead the energy is transferred to Kelvin waves with wave numbers greater than l^{-1}, and it can be dissipated only at Kelvin-wave wave numbers greater than, say, $\tilde{k}_2$ ($\gg l^{-1}$), where there can be rapid energy loss by radiation of sound. This type of picture seems first to have been discussed by Svistunov,[14] who suggested that the Kelvin waves achieve a state of universal statistical equilibrium similar, in principle, to that associated with eddies of different sizes that leads to the Kolmogorov energy spectrum. Energy flow between Kelvin waves with wave numbers in the range l^{-1} to $\tilde{k}_2$ occurs as the result of nonlinear interactions, reconnections and perhaps other types of processes mentioned by Svistunov. The superfluid grid turbulence at a very low temperature then involves two cascades: one operating over the range of wave numbers less than l^{-1}; and another operating in the range of wave numbers greater than l^{-1} and less than $\tilde{k}_2$, energy being ultimately dissipated by radiation of sound near the wave number $\tilde{k}_2$.

We shall develop this idea in a way that is simpler, but perhaps less general, than that given by Svistunov. Following Svistunov, we shall introduce a ''smoothed'' length of vortex line per unit volume L_0, obtained after all the Kelvin waves have been removed. For the present we shall continue to assume that the superfluid contains no eddies with size significantly larger than l, which we now take to be the average line spacing of the smoothed lines, i.e., $l = L_0^{-1/2}$. With this simplification only the Kelvin wave cascade operates. On the length scale l the only characteristic velocity is Eq. (7.2) and the only time scale is Eq. (7.3). Therefore, it seems reasonable to assume that energy is fed into the cascade at wave number l^{-1} at a rate per unit volume of helium equal to the energy per unit volume $\rho\kappa^2/l^2$ divided by the time l^2/κ, i.e., at the rate $\rho\kappa^3 l^{-4} = \rho\kappa^3 L_0^2$, where we have made the rough approximation of setting $f_{1,2} = 1$ in Eqs. (7.2) and (7.3) and have ignored factors of order unity. Therefore, the rate at which energy is injected per unit length of smoothed line is $\rho\kappa^3 L_0$. It follows also that the rate of decay of the smoothed length of line L_0, is given by

$$\frac{dL_0}{dt} = -\zeta\kappa L_0^2, \tag{7.7}$$

where ζ may, strictly speaking, be a function of $\ln(\xi_0 L_0^{1/2})$, as we see if we take into account the presence of the functions $f_{1,2}$ in Eqs. (7.2) and (7.3).

Let $E_k(\tilde{k})d\tilde{k}$ be the energy per unit length of smoothed vortex line associated with Kelvin waves with wave numbers in the range $\tilde{k}$ to $\tilde{k} + d\tilde{k}$. The Kelvin waves do not exist for wave numbers less than l^{-1}. Energy is dissipated from the Kelvin waves when $\tilde{k} > \tilde{k}_2$, where $\tilde{k}_2$ is determined by the rate of phonon radiation (7.5); since Eq. (7.5) depends strongly on the Kelvin wave frequency and even more strongly on its wave number, the wave number $\tilde{k}_2$ is likely to be rather well defined. We make an estimate of $\tilde{k}_2$ below. In the range of wave numbers between $1/l$ and $\tilde{k}_2$ dissipation can be neglected, and the energy spectrum, assumed universal, can depend on only the energy per unit length of the vortex line, the wave number $\tilde{k}$, and the rate of dissipation, ϵ_K at wave numbers of the order $\tilde{k}_2$. We shall take the energy per unit length of the vortex line to be $\rho\kappa^2$, where we neglect a weak (logarithmic) dependence on an appropriate upper limit in the integration of the kinetic energy of fluid flow per unit volume. A dimensional argument then shows that

$$E_K(\tilde{k}) = A\rho\kappa^2\tilde{k}^{-1}, \tag{7.8}$$

independent of ε_K, where A is a constant [possibly a function of $\ln(\tilde{k}\xi_0)$].

We emphasize at this point that the total length of vortex line per unit volume L is greater than the length L_0 by an amount that depends on the extent to which the Kelvin waves have been excited. It is the length L that is likely to be measured in any experiment, as was indeed the case in the work of Stalp, Skrbek, and Donnelly. We emphasize that $L \neq l^{-2}$. We can calculate L as follows.

The total energy contained in the Kelvin waves is given by

$$\int_{1/l}^{\tilde{k}_2} E_K(\tilde{k})\,d\tilde{k} = A\rho\kappa^2 \ln(\tilde{k}_2 l). \tag{7.9}$$

Therefore, the extra length of line per unit length of smoothed line resulting from the presence of the Kelvin waves must be given by

$$\Delta L = A' \ln(\tilde{k}_2 l), \tag{7.10}$$

where A' is another constant. Therefore,

$$L = L_0[1 + A' \ln(\tilde{k}_2 l)]. \tag{7.11}$$

It now follows from Eq. (7.7) that

$$\frac{dL}{dt} \approx -\frac{\zeta\kappa L^2}{1 + A' \ln(\tilde{k}_2 L_0^{-1/2})}. \tag{7.12}$$

Now we estimate the value of $\tilde{k}_2$. The energy cascade described by Eq. (7.8) involves the injection of energy at the wave number l^{-1} and dissipation at the wave number $\tilde{k}_2$. We have suggested that the rate of injection of energy per unit length of smoothed line is $\rho\kappa^3 L_0$ [again ignoring corrections of order $\ln(\xi_0 L_0^{1/2})$]. Sound radiation leads to a rate of loss of energy from Kelvin waves with wave numbers in the range $\tilde{k}$ to $\tilde{k} + d\tilde{k}$ equal to $\tau_s^{-1} E_K'(\tilde{k})\,d\tilde{k}$, where $E_K'(\tilde{k})$ is the actual energy spectrum existing after the radiation is taken into account. We see that $\tilde{k}_2$ will be given roughly by

$$\int_{l^{-1}}^{\tilde{k}_2} \tau_s^{-1} E_K(\tilde{k})\,d\tilde{k} = \rho\kappa^3 l^{-2}. \tag{7.13}$$

Hence, we find

$$\tilde{k}_2 l = \left(\frac{cl}{A^{1/2}\kappa}\right)^{1/2}, \tag{7.14a}$$

for the case of dipole radiation, where we have assumed, justifiably, that $\tilde{k}_2 l \gg 1$. For the case of quadrupole radiation Eq. (7.14a) is replaced by

$$\tilde{k}_2 l = \left(\frac{24\pi c^4 l^4}{A\kappa^4}\right)^{1/6}. \tag{7.14b}$$

The result (7.12) is similar to one given by Svistunov, except in two respects. His factor $[1 + A' \ln(\tilde{k}_2 L_0^{-1/2})]$ in Eq. (7.12) is raised to a power ν, which is, however, of order unity. And he ignores the possibility that energy loss can take place by the radiation of sound. He has to assume that the temperature is not accurately zero, so that energy is dissipated by a small frictional interaction between the vortices and the normal fluid, as we discuss in the next section. We emphasize that the result (7.12) depends on the existence of the analog of the Kolmogorov energy spectrum in the Kelvin waves for $\tilde{k} > l^{-1}$, for which, as we see it, there is as yet no real proof, and on our ignoring phonon production during reconnections. Experiments at very low temperatures are required to test the validity of Eq. (7.12).

B. Dissipation in a random distribution of vortex lines at a very low temperature

So far we have assumed that the drag force on a vortex is accurately zero. If there is a small density of normal fluid, there will be a small frictional force on a moving vortex, and it becomes possible that $\tilde{k}_2$ is determined by this force rather than by the phonon radiation. If the force of mutual friction per unit length of line is written as γv_L, where v_L is the line velocity, the normal fluid being assumed to be at rest, the resulting dissipation of energy in a Kelvin wave is described by the time constant τ_{MF}, given approximately by

$$\tau_{\mathrm{MF}} \sim \frac{\rho}{\gamma\tilde{k}^2}. \tag{7.15}$$

A straightforward modification of the argument leading to Eq. (7.14) then shows that this frictional force will lead to significant dissipation at wave numbers greater than that given by

$$\tilde{k}_{2f}\, l = \left(\frac{\rho\kappa}{A\gamma}\right)^{1/2}, \tag{7.16}$$

in essential agreement with a result given by Svistunov. We see then that mutual friction becomes more important than phonon radiation if the friction parameter exceeds the value given by

$$\frac{\gamma_c}{\rho\kappa} = \frac{\kappa}{cl} = \frac{\kappa L_0^{1/2}}{c}, \tag{7.17a}$$

for the case of dipole radiation, or

$$\frac{\gamma_c}{\rho\kappa} = \left(\frac{\kappa^4 L_0^2}{24\pi c^4}\right)^{1/3}, \tag{7.17b}$$

for the case of quadrupole radiation. The temperature dependence of γ is given in Ref. 15. For a value of L_0 of, say, $10^{10}\ \mathrm{m}^{-2}$ ($\kappa L = 10^3\ \mathrm{s}^{-1}$), $\gamma_c = 6\times 10^{-10}\ \mathrm{kg\,m^{-1}\,s^{-1}}$ for the case of dipole radiation, which corresponds to a temperature of about 0.65 K; for the case of quadrupole radiation $\gamma_c = 4.9\times 10^{-12}\ \mathrm{kg\,m^{-1}\,s^{-1}}$, which corresponds to a temperature of about 0.47 K. Below this temperature, dissipation by phonon emission dominates; above it, mutual friction dominates. We emphasize, however, that this result depends on the validity of our rough analysis of sound emission by a Kelvin wave, which awaits an independent check.

We note also that high-wave-number Kelvin waves remain important ($\tilde{k}_{2f}\, l \gg 1$) as long as the dimensionless parameter $\gamma/\rho\kappa$ is small compared with unity; i.e., the excitation of these waves plays an important role in the dissipation of superfluid turbulent energy if $\gamma/\rho\kappa \ll 1$. Whether this condition is satisfied at a given temperature depends on the precise value of the parameter A. If, for example, $A=1$, we find that $\tilde{k}_{2f} l > 10$ for all temperatures less than 1.7 K. It could well be the case, therefore, that high-wave-number Kelvin waves are important even at quite high temperatures.

We emphasize that in this analysis we have assumed that the normal fluid is at rest. We shall return to the validity of this assumption in Sec. VII E, where we shall argue that probably it is effectively valid at all low temperatures.

C. Relationship to earlier work and to dissipation in a conventional fluid

An equation of the form (7.7) was written down by Vinen[8] in his early work on the theory of superfluid turbulence in counterflowing helium at temperatures above 1 K, with the suggestion, supported by experiment, that $\zeta \sim 1$. The numerical simulations of Schwarz,[11] based on a classical treatment of vortex motion with the added possibility of vortex reconnections, confirmed this form, although they suggested that ζ should depend on the mutual friction constant γ and should vanish at very low temperatures. Loss of vortex line could then occur only as the result of mutual friction. We see now that this work of Schwarz failed to take into account the possibility of energy loss by phonon radiation from a Kelvin wave. The recent work of Nore, Abid, and Brachet,[2] to which we have already referred, also shows that this conclusion by Schwarz is incorrect.

We shall now show that the energy dissipation rate in our low-temperature superfluid turbulence has a form that is closely similar to Eq. (7.1). We have already suggested that the rate at which energy associated with a random arrangement of vortex lines is injected into the Kelvin waves is of order $\rho\kappa^3 L_0^2$ per unit volume of helium, and this must be equal to the rate at which energy is ultimately dissipated from the Kelvin waves. The rate of energy dissipation per unit mass of helium is therefore given by

$$\varepsilon' = -\kappa^3 L_0^2 = -\frac{\kappa^3 L^2}{[1 + A' \ln(\tilde{k}_2 L_0^{-1/2})]^2}, \tag{7.18}$$

where we have made use of Eq. (7.11) and ignored corrections involving $\ln(\xi_0 L_0^{1/2})$; $\tilde{k}_2$ is replaced by $\tilde{k}_{2f}$ at the higher temperatures. If we write $(\kappa L)^2 = \langle \omega^2 \rangle_{\rm eff}$, as we did in Sec. III, we see that we can write Eq. (7.18) in the form

$$\varepsilon' = -\nu'' \langle \omega^2 \rangle_{\rm eff} = -\frac{\kappa}{[1 + A' \ln(\tilde{k}_2 L_0^{-1/2})]^2} \langle \omega^2 \rangle_{\rm eff}. \tag{7.19}$$

This has the same form as Eq. (7.1). Furthermore, the numerical factor multiplying $\langle \omega^2 \rangle_{\rm eff}$ is quite close in magnitude to the kinematic viscosity of helium I. We conclude, therefore, that the energy dissipation rate in turbulent helium at very low temperatures is indeed rather similar to that occurring for a similar type of flow in helium I, as suggested by the computations of Nore, Abid, and Brachet. Furthermore, for very low temperatures we can identify the parameter ν'' introduced in Eq. (7.19) with the parameter ν' introduced by Stalp, Skrbek, and Donnelly [Eq. (2.3)].

D. Dissipation in superfluid grid turbulence at a very low temperature

In the preceding section we assumed that there were no eddies within the superfluid with size significantly larger than the (smoothed) vortex spacing $l = L_0^{-1/2}$. The vortex arrays occurring in grid turbulence give rise to energy spectra with wave-number components extending down to values of order the reciprocal of the channel width. There must then be some local polarization of the lines, so that the loss of vortex line described by Eq. (7.7) must be to some extent inhibited (it becomes impossible in a completely polarized array of lines). We take account of this effect by supposing that the length of the line that can take part in the processes underlying Eq. (7.7) is less than the total, by an amount that depends on the degree of polarization; thus we introduce an effective line density $L_1 (< L_0)$, which depends on the form of the Kolmogorov energy spectrum $E(k)$ for $k < l^{-1}$. We speculate later about this dependence, but for the moment we guess that L_1 is not very much less than L_0, since, as we have already emphasized in Sec. III, the degree of polarization in the experiments with which we are concerned is relatively small. The existence of the energy cascade described by $E(k)$ means that there is a mechanism by which L_1 can grow, as energy is transferred to wave numbers of order l^{-1} with a resulting decrease in the degree of polarization of the vortices. Thus we suppose that L_1 obeys an equation of the form

$$\frac{dL_1}{dt} = -\zeta\kappa L_1^2 + G, \tag{7.20}$$

where G is the rate of growth that we have just described.

We emphasize that if L were to obey Eq. (7.7), unmodified, it would decay in time as t^{-1} for large t, as is probably the case in counterflow turbulence.[8,11] With the extra term G in Eq. (7.20) the decay is modified, and is determined by the rate of decay of the large eddies. In fact, as shown by Stalp, Skrbek, and Donnelly L might then be expected to decay as $t^{-3/2}$ for large t. A decay of L as t^{-1} will occur only in the absence of eddies greater in size than the vortex-line spacing. (Strictly speaking the analysis of Stalp, Skrbek, and Donnelly related to temperatures above about 1 K, but their argument leading to a decay as $t^{-3/2}$ applies equally well to grid turbulence at low temperatures.)

We now offer a speculative argument for the factor by which L_1 is less than L. For the completely random vortex tangle the energy spectrum has no Fourier components with wave vectors significantly smaller than $L^{1/2}$, and the same is true for the spectrum of the mean-square vorticity (we take the mean-square vorticity as $\langle \omega^2 \rangle_{\rm av}$, as defined in Sec. II). For an energy spectrum of the Kolmogorov form the spectrum of the mean-square vorticity has the form $D\varepsilon^{2/3} k^{1/3}$, as we noted in Sec. III. The total mean-square vorticity is then given by

$$\langle \omega^2 \rangle_{\rm eff}(k > 0) = \int_0^{L^{1/2}} D\varepsilon^{2/3} k^{1/3} dk = \frac{3}{4} D\varepsilon^{2/3} L^{2/3}, \tag{7.21}$$

there being no Fourier components in the spectrum of $\langle \omega^2 \rangle_{\rm eff}$ with wave numbers greater than $L^{1/2}$. Those Fourier compo-

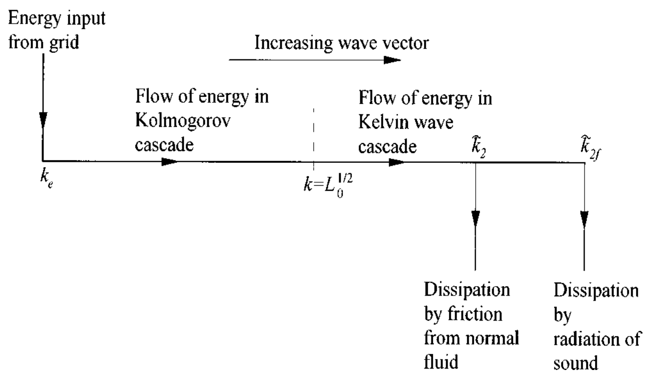

FIG. 1. Illustrating the flow of energy in superfluid grid turbulence at a low temperature. At the very lowest temperatures $\tilde{k}_2 > \tilde{k}_{2f}$; at higher temperatures $\tilde{k}_2 < \tilde{k}_{2f}$, as illustrated.

nents of the mean-square vorticity that are connected with adjacent oppositely directed vortex lines must have wave numbers close to $L^{1/2}$; say between roughly $L^{1/2}$ and $0.5L^{1/2}$. The corresponding contribution to the total mean-square vorticity is given by

$$\langle \omega^2 \rangle_{\text{eff}}(k > 0.5L^{1/2}) = \int_{0.5L^{1/2}}^{L^{1/2}} D\varepsilon^{2/3} k^{1/3} dk = 0.6 \frac{3}{4} D\varepsilon^{2/3} L^{2/3}. \tag{7.22}$$

The ratio of Eq. (7.22) to Eq. (7.21), equal to 0.6, is a measure of the extent to which the vortex lines are arranged in a random unpolarized manner, and we speculate that L_1/L is of the order of the same ratio. Note that the ratio remains the same at all stages of the decay. The fact that the ratio is not very different from unity is a consequence of the fact that even in grid turbulence the vortex tangle is almost random.

We conclude this section by noting that our arguments lead to the conclusion that grid turbulence at a very low temperature ought to behave in a way very similar to that observed by Stalp, Skrbek, and Donnelly. That is, there will be a conventional Kolmogorov energy spectrum at small wave numbers. This spectrum terminates at an upper wave number of order $l^{-1} = L_0^{1/2}$, where the energy is transferred without dissipation to a second cascade formed from Kelvin waves with wave numbers extending from l^{-1} to $\tilde{k}_2$. Energy is ultimately dissipated from these Kelvin waves at wave numbers near $\tilde{k}_2$, either by radiation of sound at the very lowest temperatures or by friction against a residual normal fluid at slightly higher temperatures. The energy flow is illustrated in Fig. 1. The rate of dissipation is given by a formula [Eq. (7.19)] that is very similar to that expected for a conventional viscous fluid with kinematic viscosity of order $\kappa/[1 + A' \ln(\tilde{k}_2 L_0^{-1/2})]^2$, which is close to that for helium I at temperatures near 1 K but falls at lower temperatures owing to an increasing value of $\tilde{k}_2$.

A picture in which vortex motion at wave numbers greater than l^{-1} is described in terms of simple Kelvin waves is probably an oversimplification, but we can hope that it incorporates the essential physics of the problem.

E. Dissipation at higher temperatures

So far we have discussed dissipation of superfluid turbulent energy at high wave numbers only for the case when the normal fluid is either absent or can be regarded as at rest so that its effect is only to introduce a frictional drag on the moving vortices in the superfluid. The experiments of Stalp, Skrbek, and Donnelly were generally carried out at temperatures where the two fluids have comparable densities, and where, as we have explained, turbulence in both fluids must be taken into account. Dissipation must then involve processes inherent in the superfluid component, such as we have discussed in Secs. VII A–VII D, together with conventional viscous dissipation in the normal component. Both processes must be affected by the frictional interaction (mutual friction) between the vortices and the normal fluid.

In general, this is a complicated problem. An understanding of the dissipative process in each fluid acting independently is not enough because the two turbulent velocity fields are coupled through mutual friction, which is itself a dissipative force. We argue, however, that we can usefully obtain results valid in the limits of both low and high temperatures, sensible interpolation between these limits being then possible.

Consider first the *case of low temperatures*. This is the case considered in Secs. VII A–VII D, but with the proviso that motion of the normal fluid can be neglected. We shall now drop that proviso.

We shall take the case of low temperatures to mean one in which the density of the normal fluid is small, so that the turbulent energy in the normal fluid is small compared with that in the superfluid. This condition might hold reasonably well up to temperatures of at least 1.32 K, where the normal fluid fraction is only 5%. Then we need to consider only the dissipation of the turbulent energy in the superfluid component. We have argued that this dissipation is described in the way that we have explained in Secs. VII A–VII D, provided that motion of the normal fluid can be neglected on the length scales relevant to this dissipation. We now argue that such a neglect is indeed justified.

The normal fluid can affect the superfluid only through the frictional interaction described by the parameter γ. One possible process is the transfer of energy from the superfluid to the normal fluid, *via* this frictional interaction, followed by viscous dissipation of this energy in the normal fluid. On length scales significantly larger than $l = L_0^{-1/2}$ the two fluids are strongly coupled, as we saw in Sec. V, so this type of energy transfer can be neglected. However, on the length scale $l = L_0^{-1/2}$ it might be important. The characteristic time associated with the evolution of the turbulence in the superfluid component on this length scale is given by Eq. (7.3). The time associated with the transfer of momentum from the superfluid to the normal fluid by frictional interaction is given in order of magnitude by

$$\tau_{sn} = \frac{\rho_s}{\gamma L_0} \approx \frac{\rho}{\gamma L_0}, \tag{7.23}$$

provided that the velocities of the normal fluid, the superfluid, and the vortex lines are all comparable in magnitude, as we see by noting that $\rho_s \dot{v}_s \sim \gamma(v_n - v_L)L_0$. The ratio τ_{sn}/τ_l is, therefore, given by

$$\frac{\tau_{sn}}{\tau_l} = \frac{\rho\kappa}{\gamma}, \tag{7.24}$$

where we have taken the logarithmic factor in Eq. (7.3) to be unity. We have already seen that this ratio is probably large compared with unity at all low temperatures. It follows that the energy transfer on which we are focusing attention can probably take place only slowly in comparison with the rate at which energy is injected into the Kelvin mode cascade, and it can, therefore, be neglected. It is important to ask also whether the damping of the Kelvin modes by friction [Eq. (7.15)] is likely to be significantly affected by any motion of the normal fluid. This depends on the ratio of the normal fluid velocity in the direction of the smoothed vortex line to the wave velocity of the Kelvin wave. Provided that this ratio is small compared with unity there is very little effect.[16] We assume that the normal fluid velocity on the relevant length scale is not greater than κ/l. Using the dispersion relation (7.4) we find that for Kelvin waves of wave number $\tilde{k}$ the ratio is not greater than about $1/\tilde{k}l$, which is indeed small compared with unity for most of the relevant Kelvin waves.

We conclude that in, say, grid turbulence at temperatures below about 1.3 K, the presence of the normal fluid can be neglected except in its effect in damping the Kelvin waves described in Sec. VII B and therefore in determining the wave number $\tilde{k}_2$ at which dissipation occurs in the Kelvin wave cascade described in Sec. VII A.

Consider now the *case of high temperatures*, by which we mean temperatures at which the superfluid density is so small that the turbulent energy in the superfluid component can be neglected. The superfluid fraction is less than 5% for $T_\lambda - T < 7$ mK. Here the frictional interaction can transfer energy from the normal fluid to the superfluid, but there is no mechanism by which it can be dissipated in the superfluid independently of this frictional interaction. Continuous transfer is not possible. There can, therefore, be very little effect on the turbulence in the normal fluid. We conclude that at these high temperatures the presence of the superfluid component can be neglected, and the dissipation can be taken as being associated almost entirely with the normal fluid viscosity.

F. Comparison with experiment

We suggested in the preceding section that at low temperatures, less than roughly 1.3 K, the presence of the normal fluid plays no role except in determining the parameter $\tilde{k}_2$, and that Eq. (7.19) therefore applies. This is certainly consistent with Stalp's most recent (unpublished) observations, mentioned in Sec. II; the parameter ν'' in Eq. (7.19) has the same order of magnitude as his ν', and it correctly exhibits a value that falls with falling temperature. To determine whether there is quantitative agreement we would need to know the exact value of the parameters A and A' introduced in Sec. VII A; at present we cannot go further than to suggest that they are likely to be of order unity.

We also suggested in the preceding section that we can neglect the presence of the superfluid component when we consider dissipation in grid turbulence at temperatures close to the λ point. Unfortunately, this suggestion cannot be compared in a straightforward way with the recent observations of Stalp. These observations do suggest that the parameter ν' has a temperature dependence at high temperature that is similar to that of $\nu = \eta_n/\rho$. However, this similarity is consistent with our suggestion only if at these high temperatures $\kappa^2 L^2$ is equal to the mean-square vorticity in the normal fluid. We have not found any convincing argument that leads to this equality.

We have no theory of the intermediate range of temperatures. If we accept that at high temperatures $\kappa^2 L^2$ is equal to the mean-square vorticity in the normal fluid, the experimental results seem to be a reasonable interpolation between the high- and low-temperature limits.

VIII. CONCLUSIONS

We have shown that in grid turbulence in helium II there can be an inertial range of wave numbers in which dissipative processes are unimportant, the velocity fields of the two fluids are coupled, and the turbulent energy spectrum has the Kolmogorov form, as in a conventional fluid. Dissipation takes place at wave numbers greater than or of the order of the inverse of the vortex-line spacing. The dissipation is generally complicated, involving viscous dissipation in the normal fluid, frictional interaction between the vortex lines and the normal fluid, and, especially at very low temperatures, radiation of sound from moving vortices. Some simplification is possible at high and low temperatures. At temperatures close to the λ point, the superfluid component plays a minor role, and dissipation is due almost entirely to viscosity in the normal fluid. At very low temperatures, the turbulence in the normal fluid can be neglected; turbulent energy in the superfluid component is transferred to Kelvin waves of high wave number, where it is dissipated either as a result of frictional interaction between the vortex lines and the residual normal fluid or, at the very lowest temperatures, by radiation of sound by the Kelvin waves. Even at low temperatures the rate of energy dissipation is given by a formula similar to that applying in a conventional fluid, but with some significant differences. These conclusions are consistent with the experimental results of Stalp, Skrbek, and Donnelly for grid turbulence in helium II in the temperature range down to about 1.4 K and with the recent unpublished results of Stalp at slightly lower temperatures. There is a need for further experimental exploration of the region of very low temperatures, where the effect of the normal fluid is either absent or takes the form only of a frictional damping of the vortex motion. The theory is still quite speculative and requires further development.

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