Scaling and finite-size scaling in the two-dimensional randomly coupled Ising ferromagnet

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It is shown by the Monte Carlo method that finite-size scaling (FSS) holds in the two-dimensional randomly coupled Ising ferromagnet. It is also demonstrated that the form of the universal FSS function constructed via a FSS scheme depends on the strength of the random coupling in the case of strongly disordered systems. Monte Carlo measurements of thermodynamic (infinite-volume-limit) data of the correlation length (ξ) up to $\xi \simeq 200$ along with measurements of the fourth-order cumulant ratio (Binder cumulant ratio) at criticality are analyzed in light of two competing scenarios of weak universality and the multiplicative logarithmic correction. It is demonstrated that the data are likely to be more consistent with the former scenario than the latter which is the conventional scenario.

I. INTRODUCTION

The two-dimensional (2D) randomly disordered Ising ferromagnet is the simplest nontrivial statistical model that exhibits the effect of another type of fluctuation in addition to the usual thermal fluctuation. By disorder is meant either a random site dilution or random-valued positive coupling in this case. The effect of the combined fluctuations of the thermal and quenched (random) disorder on the critical behavior of the system has been an important subject of the studies. The two-dimensional randomly coupled (or random bond) Ising ferromagnet is defined by the Hamiltonian

$$H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j, \quad J_{ij} > 0, \quad S_i = \pm 1, \tag{1}$$

where the sum is over all the links of the square lattice, and J_{ij} is randomly distributed.

According to a rigorous result by McCoy and Wu,[1] the specific heat (C_v) is nondivergent in a 2D Ising system with one-directional and correlated random bond disorder. A similar feature as in the McCoy-Wu model was obtained for the 2D Ising ferromagnet for uncorrelated disorder as well.[2]

There are currently two main competing scenarios concerning the critical behavior of the 2D uncorrelated randomly disordered Ising ferromagnet: namely, the scenario of the weak universality[3] and that of the logarithmic correction.[4,5] The latter is mainly based on the theoretical prediction of Shalaev, Shankar, and Ludwig (SSL), which can be summerized as

$$\xi \sim t^{-\nu}[1 + C|\ln(t)|]^{\tilde{\nu}}, \quad \nu = 1, \ \tilde{\nu} = 1/2, \tag{2}$$

$$C_v \sim t^{-\alpha} \ln|1 + C|\ln(t)|| + C', \tag{3}$$

$$\chi \sim \xi^{2-\eta}, \quad \eta = 1/4, \tag{4}$$

where t is the reduced temperature [$t \equiv (\beta_c - \beta)/\beta_c$, with β denoting the inverse temperature] and χ is the thermodynamic magnetic susceptibility. The coefficients C and C' as a function of the strength of disorder cannot be determined theoretically, although C is supposed to increase with the strength of disorder.

Equations (2)–(4) reflect the presence of the crossover from the critical behavior of the pure system to that represented by the disorder, namely, e.g., for the ξ, that given by the asymptotic form

$$\xi \sim t^{-\nu} |\ln(t)|^{\tilde{\nu}}. \tag{5}$$

Now it appears to be numerically established[6–10] that the value of η does not depend on the strength of disorder in both 2D randomly coupled and random site diluted Ising ferromagnets. Another generic feature emerging from various numerical studies[6–9,11] is that other critical exponents such as γ and ν increase with the strength of disorder at least effectively. These apparently varying critical exponents were interpreted as originating from the crossover effect of Eqs. (2)–(4) by some authors,[6,11,12] while they were regarded as genuine by others.[8,9]

For later purposes, we here would like to make some nomenclature clear. Suppose we have two singular functions at $t=0$, say, $f(t)$ and $g(t)$. Then we say that $f(t)$ is *more* singular than $g(t)$ over the range between t_1 and t_2 such that $0 < t_1 < t < t_2 \ll 1$ if

$$|f(t_1)/f(t_2)| > |g(t_1)/g(t_2)|. \tag{6}$$

Otherwise, of course, $f(t)$ is *less* than or equally singular to $g(t)$.

$t^{-\rho}$ for any $\rho > 0$ is *essentially* more singular than $|\ln t|$ in the sense that with $f(t) = t^{-\rho}$ and $g(t) = |\ln t|$ Eq. (6) is mathematically valid for sufficiently small values of the t_1 and t_2. However, it is extremely difficult to determine *numerically* whether or not Eq. (6) is indeed satisfied unless precise numerical values of $f(t)$ and $g(t)$ at very small t_1 and t_2 are available. This is a fundamental difficulty encountered in a numerical study no matter whether it is a Monte Carlo or a series expansion. In other words, one needs to get thermodynamic data very close to a critical point or an extremely long series expansion. It should be emphasized that the conventional, finite-size-scaling (FSS) technique that is used to analyze Monte Carlo data obtained at criticality—the sort of studies that mostly claimed evidence for the predictions of SSL (Ref. 11)—cannot overcome this difficulty either, owing to the well-known fact that the correlation length in the scal-

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ing region translates into the linear size of the lattice L at criticality. (For a critique of the most claims made in Refs. 11 and 12, see Ref. 13.)

In this paper we attempt to clarify the controversial issue of the 2D randomly coupled Ising ferromagnet based on different numerical methods from those used previously. First we determine the functional form of an universal FSS function $\mathcal{Q}$ defined by[14,15]

$$A_L(t)=A(t)\mathcal{Q}_A(x(L,t)), \quad x(L,t)\equiv\xi_L(t)/L. \tag{7}$$

To this end it is needed to check the validity of Eq. (7), i.e., the validity of FSS itself in a disordered system. Given the strength of disorder, this can be achieved by numerically calculating the scaling function $\mathcal{Q}_\xi(x)$ at different temperatures close to criticality; if the scaling functions thus calculated turn out to be identical, then the FSS must hold in our system. Accordingly, if the disordered system does indeed belong to the same universality class as the corresponding pure system, the FSS functions calculated for different strength of disorder must be identical to that of the pure system. We will show that this is not the case. Second, We will demonstrate that the thermodynamic data of ξ in a sufficiently deep scaling region tend to be *more* singular than the asymptotic expression of SSL, Eq. (5). Third, we will show that measured value of Binder cumulant ratio at criticality depend on the strength of disorder.

In the sections to follow we give a detailed description of our Monte Carlo simulation, report our results, and finally conclude with some discussions.

II. SIMULATION

We consider a binary distribution of J_{ij}. Namely, the value of J_{ij} at a link $\langle ij\rangle$ is randomly distributed between two positive values J and J' with probabilities p and $1-p$, respectively. For $p=1/2$ the system is self-dual[16] with the self-dual point given by

$$\tanh(J\beta)=\exp(-2J'\beta). \tag{8}$$

A self-dual point equals the critical point of a system, provided that the system has only one critical point. We fix $J=1$ and $p=1/2$ without loss of generality and consider three different values of J', i.e., $J'=0.9$, 0.25, and 0.1. The self-dual points (critical points) are accordingly given by $\beta_c=0.464\,281\,9\ldots, 0.807\,051\,85\ldots$, and $1.103\,895\,23\ldots$ for $J'=0.9$, 0.25, and 0.1, respectively.

Our raw data for each J' are obtained by choosing a realization of random distribution of J', and then running Monte Carlo simulations in the single cluster algorithm[18] with periodic boundary conditions; for each realization, measurements were taken over 10000 configurations, each of which was separated by 2–15 single-cluster updatings according to the autocorrelation times. The procedure is then repeated for different realizations of distribution of J'. The average over all the different realizations converges as the numbers of the random realization increase; basically this means that the value of a physical quantity is something physically interesting. To be more specific, our definitions of ξ_L and U_L are as follows:

$$\xi_L=[\sqrt{G(\mathbf{0})/G(\mathbf{k})-1}/2\sin(\pi/L)]_{J'}, \tag{9}$$

$$U_L=[3-\langle S^4\rangle/\langle S^2\rangle^2]_{J'}, \tag{10}$$

where $\langle\cdots\rangle,[\cdots]_{J'}$, and $G(\mathbf{k})$, respectively, represent the usual thermal average, the average over different realizations of J', and the Fourier transform of the connected two-point Green function with momentum $\mathbf{k}$. (See, for example, Ref. 15 for more details of the definition of ξ_L.)

To achieve the necessary precision for our FSS scheme we used a number of different realizations: approximately 20–40, 150–250, and 300–1000 for $J'=0.9$,0.25, and 0.1, respectively; yet, in general, the fluctuation among different realizations of the random disorder is more significant than the statistical error for a given realization. This was particularly the case for $J'=0.1$. Nevertheless, the average over different realizations clearly converges to a physically meaningful value. Our quoted error bars in our numerics are obtained by the standard jackknife method and taking into consideration only the variation with different realizations. The smallest and largest values of L used for the calculation of the scaling function are 20 and 400, respectively. or $J'=0.05$ it turns out that data of very large ξ are not necessary, but yield compelling results with the data even for $\xi\lesssim 50$. Thus the FSS extrapolation method is not used for this J' and 250–500 different realizations of distribution of J' turn out to be sufficient.

Determination of A_∞ and of the size dependence of A upon L,A_L, is essential to the computation of $\mathcal{Q}_A$. The measurements of the correlation length must be taken into consideration for the computation of the scaling variable x. For simplicity of our presentation, here we mainly focus on the correlation length. For each J', we chose three different inverse temperatures (where the value of thermodynamic ξ is sufficiently large) for the computations of the scaling variable and the scaling function.

III. RESULT AND ANALYSIS

The $\mathcal{Q}_\xi(x)$ is calculated for $J'=0.9$,0.25, and 0.1. To this end, an investigation of the L dependence of ξ is carried out for various sets of the values of (J',β). It is observed that ξ_L is a monotonically increasing function of L and that the L dependence becomes weaker with increasing L and becomes vanishingly small for sufficiently large L, i.e., under the condition $L/\xi_L\gtrsim 10$ (thermodynamic condition). The L independent value within the statistical errors is the corresponding thermodynamic value. Owing to Eq. (7) the thermodynamic condition holds independent of temperature.

The plots of $\mathcal{Q}_\xi(x)$ for the three values of β in the scaling region are shown in Fig.1 for $J'=0.25$. Clearly, each set of data belonging to different value of β superposes onto a single curve that is the universal FSS function for the value of J'. In particular, down to $L=20$ we observe no visible effect that may possibly come from any kind of correction to scaling. Thus the validity of the FSS is verified for the J' down to this L. We repeated the same procedure for the other two values of J' and observed a similar data collapse. For comparison, the FSS function for each J' is shown in Fig. 2. It is observed that the FSS function for $J'=0.9$ is indistinguishable from that of the corresponding pure system that has been calculated in Ref. 15. However, for stronger disorder the FSS scaling function clearly depends on J'.

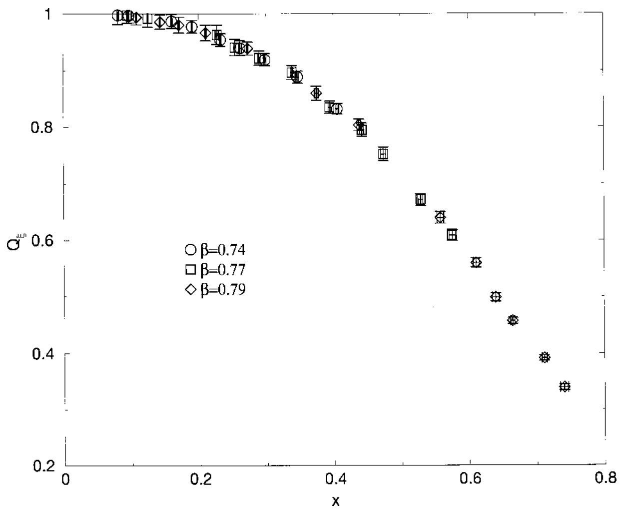

FIG. 1. Numerical calculation of $\mathcal{Q}_\xi$ for $J'=0.25$ and at three arbitrary inverse temperature $\beta=0.74$, 0.77, and 0.79 in the scaling region. Our figure clearly indicates that the scaling function does not have explicit temperature dependence. For each value of β, the smallest value of L used is 20. Note that smaller L means larger x. Down to $L=20$, the effect of the possible logarithmic correction is not observed.

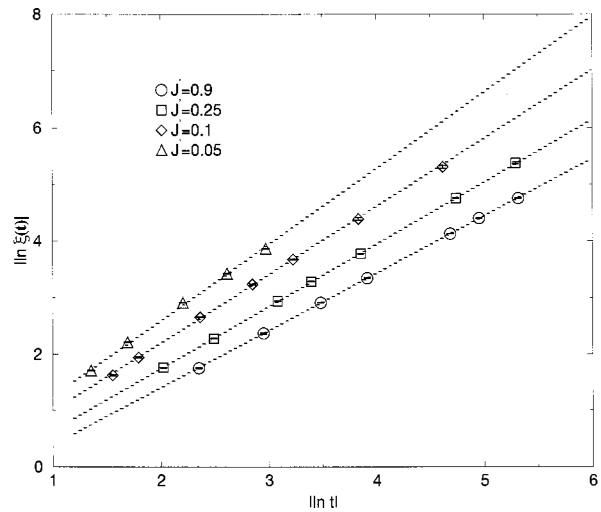

FIG. 3. $\ln\xi$ versus $|\ln t|$. The dotted lines represent the results of the best χ^2 fits assuming a pure power-law-type singularity. The values of the slope, which correspond to the values of ν, are 1.00, 1.10, 1.20, and 1.34, respectively, for $J'=0.90$, 0.25, 0.10, and 0.05.

The ansatz for our scaling function is

$$\mathcal{Q}_\xi(x)=1+c_1x+c_2x^2+c_3x^3+c_4x^4. \tag{11}$$

The values of the coefficients $c_i(i=1,\ldots,4)$ are calculated by fitting our data to the ansatz for each value of J'. The choice of the ansatz for the scaling function is not so important. We tried several different forms of ansatz such as $\mathcal{Q}_\xi(x)=\Sigma_{n=0}^{n=3}b_n\exp(-n/x)$ (Ref. 19) and $\mathcal{Q}_\xi(x)=1+b_1x^{b_2}+b_3x^{b_4}$, but the results were not significantly sensitive to the choice of different forms of ansatz.

The thermodynamic correlation lengths are measured directly (without using any extrapolation method) for the data roughly over the range $\xi\lesssim 40$. With knowledge of the scaling function available, the thermodynamic values closer to criticality can now be easily estimated by the use of the single-step FSS extrapolation method.[14,15] For the extrapolation we always use data with $L\geqslant 50$; we are thus confident that the effect of a possible correction to FSS is negligibly small on the extraction of thermodynamic values. For the each value of J' the thermodynamic values of the correlation length are evaluated over the ranges $5.7(1)\leqslant\xi\leqslant 204(2)$ [5.7(1), 204(2)], [5.8(1), 217(3)], and [5.0(1), 203(5)]. For the $J'=0.05$ the thermodynamic data were obtained by direct measurement under the thermodynamic condition $L/\xi_L\geqslant 12$, which is over the range $5.52(2)\leqslant\xi\leqslant 47.64(29)$.

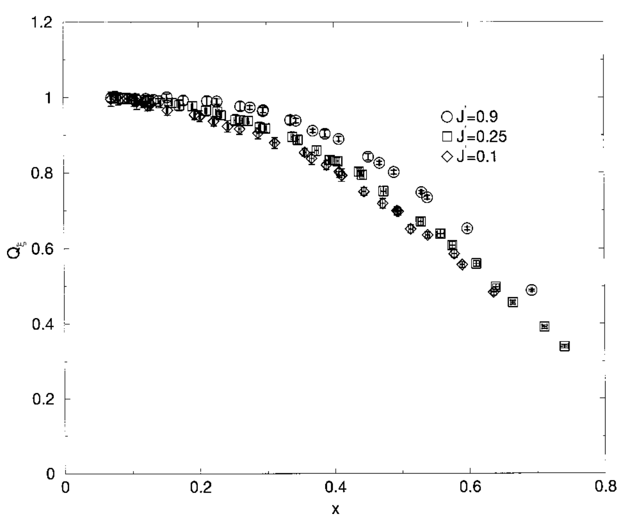

FIG. 2. The $\mathcal{Q}_\xi(x)$ for the three values of J'. The dependence of the scaling function on J' is obvious.

In Fig. 3, $\ln\xi(t)$ is plotted as a function of $|\ln t|$. The slope of each straight line corresponds to the value of ν. It is evident that ν increases with decreasing J', at least, effectively. Fixing the critical points at the self-dual points in the χ^2 fits and assuming a pure power-law-type critical behavior, we obtain $\nu=1.01(1),1.10(2),1.20(3)$, and 1.34(6), for $J'=0.9$, 0.25, 0.1, and 0.05, respectively. Assuming a scaling function with an additive correction term, e.g., $\xi(t)\sim t^{-\nu}(1+at)$, yields the estimate of the critical exponent, e.g., $\nu=1.08(4)$ and $\nu=1.17(5)$ for $J'=0.25$ and $J'=0.1$, respectively. Notice that for $J'=0.9$ the estimated value of ν is virtually the same as that in the pure system.

By fitting our data to Eq. (2), it is found that the data can be fitted with a quite broad range of values of C and $\tilde{\nu}$. With the use of the asymptotic form, Eq. (5), in the fit, we can get more precise estimate of $\tilde{\nu}$ than with Eq. (2). The problem associated with this fit is that one does not know *a priori* whether all the data are beyond the crossover point. For a comparison with the result of Ref. 12 we quote here our estimates of $\tilde{\nu}$ obtained by fitting our data to Eq. (5): that is, $\tilde{\nu}=0.03(3),0.32(3),0.54(3)$, and 0.69(6), respectively, for $J'=0.9$,0.25,0.1, and 0.05. The value of $\tilde{\nu}$ increases monotonically with decreasing J'.

A useful observation is that $[1+C|\ln(t)|]^{\tilde{\nu}}(C,\tilde{\nu}>0)$ is *less* singular than $|\ln t|^{\tilde{\nu}}$ for *any* range of $t>0$. Thus the asymptotic singularity, Eq. (5), is always *more* singular than the mixture of the singularities, Eq. (2), in the sense of Eq. (6). Accordingly, if thermodynamic correlation length data

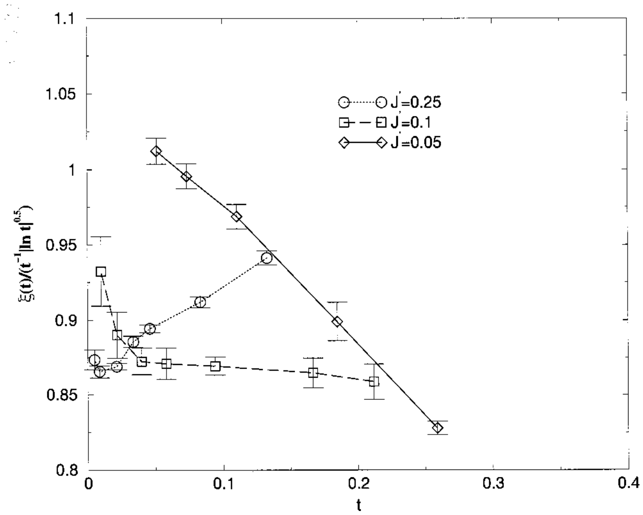

FIG. 4. The ratio $\xi(t)/(t^{-1}|\ln t|^{1/2})$ for $J'=0.25$, 0.1, and 0.05. The data for $J'=0.1$ and 0.05 are uniformly shifted so that the difference in the data points is clearly visible in a single figure. Here the increasing value of the ratio as t becomes smaller is evidence that the data over the region are inconsistent with the prediction of SSL. The tendency of the increasing ratio with t becoming sufficiently small is observed for $J'=0.25$ and 0.1, showing that the apparent consistency with the logarithmic correction of SSL for the larger values of t tends to become invalidated in the sufficiently deep scaling region. In the case of $J'=0.05$, all the data scale *more* steep than the asymptotic scaling, showing that SSL cannot be correct for any data in the scaling region for this J'.

turn out to be *more* singular than the asymptotic singularity *in an arbitrary portion* of the scaling region, then the prediction of SSL must be invalidated.

In Fig. 4 is plotted $\xi(t)/(t^{-1}|\ln t|^{0.5})$ for $J'=0.25$, 0.1, and 0.05. It is observed that the value of the ratio decreases monotonically for $J'=0.25$ until the temperature is very close to criticality, but starts to increase with further approaching to the criticality. This is surprising in view of the picture of SSL, because the figure shows that none of the data are either in the asymptotic region or in the scaling region of the pure system. In the case of $J'=0.1$, we find that the data less close to the criticality are consistent with the prediction of SSL, but start to deviate from it as $t\rightarrow 0$. In the case of $J'=0.05$, we observe that all the data are *more* singular than the asymptotic form. We are thus to led the picture that the apparent consistency with the logarithmic correction for weak disorder starts to become invalidated in the sufficiently deep scaling region.

The binder cumulant ratio at criticality, denoted by $U_L^{(4)}(t=0)$, is another universal quantity.[20,21] For each J' we measured it at the critical point with varying L (Table I). It is observed that $U_L^{(4)}(t=0)$ is invariant with L within the statistical errors for a given J', and that it tends to increase uniformly with decreasing J'. For $J'=0.25$, it is obvious that the value up to $L=400$ is different from the exact value at criticality of the pure system, i.e., $U_L=1.832\,077\,1(47)$.[22] The U_L for $L=20$ is supposed to represent the characteristics of the pure system [a smaller value of L at criticality corresponds to a smaller value of $\xi(t)$ in the scaling region], if the scenario of the crossover is indeed correct. However, $U_L=1.850(3)$ at $L=20$ of $J'=0.25$ is clearly different from the value of the pure system. The value for $J'=0.9$ is indistinguishable from the pure case, as is observed to be the case for the scaling function $\mathcal{Q}_\xi$ for this J'.

TABLE I. Binder cumulant ratio at the self-dual points for three values of J'. Note that $U_L(t=0)$ for each J' does not vary with L within the statistical errors, thus showing that each self-dual point is indeed the critical point. It is also clear that $U_L(t=0)$ increases with decreasing J', although for $J'=0.9$ it is hardly distinguishable from the value of the pure system. For $J'=0.25$ we extended the measurements up to $L=400$, which does not show any sign of crossover.

L	$J'=1.00$	$J'=0.90$	$J'=0.25$	$J'=0.10$
20	1.8324(6)	1.834(1)	1.850(3)	1.864(2)
40	1.8321(6)	1.833(2)	1.846(3)	1.856(3)
60	1.8317(5)	1.832(1)	1.852(3)	1.854(3)
80	1.8318(5)	1.833(1)	1.847(3)	1.860(3)
100	1.8316(6)	1.832(2)	1.849(3)	1.863(2)
200			1.844(3)	
400			1.845(2)	

IV. DISCUSSION AND CONCLUSION

We have obtained unambiguous numerical evidence that FSS holds for a quenched randomly coupled Ising ferromagnet employing a different method from that used in Ref. 23. It was noticed in Ref. 17 that the data collapse in the presence of the logarithmic correction was not so good as in the absence of it. In our system, however, we found excellent data collapse. The universal FSS function is found to be dependent upon the strength of disorder for strongly disordered cases. We also have shown that ξ is *more* singular than the theoretical prediction for the data sufficiently close to criticality or for very strongly disordered cases.

The behavior of the Binder cumulant ratio does not show any sign of crossover. If one speculates that the value of U_L up to L=400 for $J'=0.25$ does not represent the asymptotic scaling region yet, then it would be puzzling why all the previous FSS studies of χ_L at criticality, with the use of more or less similar ranges of L, unambiguously have yielded $\eta=0.25$.[6,8,25] Otherwise, if this value of the η reflects the crossover region, it is still puzzling (i) why the value of U_L does not change over such a broad range of L and (ii) why the value of U_L representing the pure system is not observed. Hence, the only feasible interpretation seems to be that the values of U_L for each J' is already asymptotic and represents a different universality class.

Our result of varying exponent ν combined with the established fact of the invariance of γ/ν supports the scenario of weak universality.[3] The same numerical evidence was also obtained for the 2D randomly coupled three-state Potts ferromagnet.[24]

A very strong claim for the evidence for the SSL made in a recent high-temperature expansion study of the same model[12] is actually misleading. What the authors of the paper observe is the monotonic increase of the effective value of γ with the strength of disorder. On the other hand, they claimed that the same data fitted to the *asymptotic* form of SSL give rise to the same value of the logarithmic exponent as predicted by SSL irrespective of the strength of the disorder. This is unlikely to be mathematically correct because the effective increase of γ leads to the effective increase of the logarithmic exponent, as shown in this work and as can be easily checked by a simple numerical experiment as well.[26] In light of relatively short series terms,[12] it appears to be unlikely that the series expansion analysis is able to make a sharp estimate of $\tilde{\nu}$ using Eq. (2).

One may also suspect that there possibly exists some subtle effect coming from the binary distribution of J_{ij}. Indeed, the $p=1/2$ corresponds to the bond percolation threshold. Although one may not rule out the possibility, it appears to be unfeasible for the following reasons: (i) If there exists such an effect, then it must be independent of the value of J' since our distribution of J' is at the percolation threshold for all the values of J' considered here. For $J'=0.9$, however, we see that the critical behavior is virtually the same as that of the pure system. To make it extreme, one can make the value of J' arbitrarily close to 1 (but different from 1) with a 50% random distribution of J'. The system is still at the bond percolating threshold, but physically the system must be identical to the pure system. (ii) The averaging process is fundamentally different from that of the percolation problem. Moreover, our case does not have any parameters controlling the percolation problem.

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[25] Our $\chi_L(t=0)$ data are also consistent with $\eta=0.25$.
[26] We generated numerical data by using a computer and assuming a pure power-law singularity, $\chi\sim t^{-\gamma}$, with three arbitrary $\gamma=2.1, 2.3$, and 2.5 [corresponding to $J^2/J^1\simeq 5$, 8 and 10 in their paper (Ref. 12)], with the range of t over $0.01\leqslant t\leqslant 0.05$ and with 1% of relative errors. When the data were fitted to the asymptotic form of SSL, $\chi\sim t^{-\gamma}|\ln(t)|^{\tilde{\gamma}}$, the estimates of $\tilde{\gamma}$ were found to be 1.31, 2.06, and 2.44, respectively. The estimates depended on the choice of the range of t, being larger for the value of t smaller. Nevertheless, the feature of its monotonic increase with decreasing J' is invariant to the choice of the range of t.