Effect of symmetry-breaking perturbations in the one-dimensional SU(4) spin-orbital model

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We study the effect of symmetry-breaking perturbations in the one-dimensional SU(4) spin-orbital model. We allow the exchange in spin (J_1) and orbital (J_2) channel to be different and thus reduce the symmetry to $SU(2)\otimes SU(2)$. A magnetic field h along the S^z direction is also applied. Using the formalism developed by Azaria *et al.* [Phys. Rev. Lett. **83**, 624 (1999)] we extend their analysis of the isotropic $J_1=J_2$, $h=0$ case and obtain the low-energy effective theory near the SU(4) point in the generic case $J_1\neq J_2$, $h\neq 0$. In zero magnetic field, we retrieve the same qualitative low-energy physics as in the isotropic case. In particular, the isotropic massless behavior found on the line $J_1=J_2<K/4$ extends in a large anisotropic region. We discover, however, that the anisotropy plays its trick in allowing nontrivial scaling behaviors of the physical quantities. For example, the mass gap M has two different scaling behaviors depending on the anisotropy. In addition, we show that in some regions, the anisotropy is responsible for anomalous finite-size effects and may change qualitatively the shape of the computed critical line in a finite system. When a magnetic field is present the effect of the anisotropy is striking. In addition to the usual commensurate-incommensurate phase transition that occurs in the spin sector of the theory, we find that the field may induce a second transition of the KT type in the remaining degrees of freedom to which it does *not* couple directly. In this sector, we find that the effective theory is that of an SO(4) Gross-Neveu model with an h-dependent coupling that may change its sign as h varies.

I. INTRODUCTION

In past years, there has been an intense interest devoted to one-dimensional spin-orbital models.[1] The main reason stems from the recent discovery of the new quasi-one-dimensional spin gapped materials $Na_2Ti_2Sb_2O$ (Ref. 2) and NaV_2O_5.[3] It is believed that the unusual magnetic properties observed in these compounds should be explained by a simple two-band Hubbard models at *quarter* filling. At this filling, and in the large Coulomb repulsion the effective Hamiltonian simplifies greatly and is equivalent to a model of two interacting spin one-half Heisenberg models:[4,5]

$$\mathcal{H}=\sum_i J_1\vec{S}_i\cdot\vec{S}_{i+1}+J_2\vec{T}_i\cdot\vec{T}_{i+1}+K(\vec{S}_i\cdot\vec{S}_{i+1})(\vec{T}_i\cdot\vec{T}_{i+1}), \tag{1}$$

where $\vec{S}_i$ and $\vec{T}_i$ are spin-1/2 operators that represent spin and orbital degrees of freedom at each site i and $J_{1(2)}$ and K are positive. It is important to notice that Eq. (1) is not only SU(2) invariant in $\vec{S}$ space but also in $\vec{T}$ space. For generic couplings $J_{1(2)}$ and K, the Hamiltonian (1) is $SU(2)_s\otimes SU(2)_t$ symmetric. Two particular cases are of interest. First, when $J_1=J_2$ there is an additional Z_2 symmetry in the exchange between $\vec{S}_i$ and $\vec{T}_i$. The other important case is when $J_1=J_2=K/4$ in which case the Hamiltonian is SU(4) invariant.[6] In fact, the Hamiltonian (1) is a simplified version of the most general case. Indeed, depending on the microscopic couplings, there can be other terms that break the SU(2) symmetry in both spin and orbital sectors. The choice of studying Eq. (1) is dictated by the fact that since it retains some symmetries it is a simple starting point from which one can expect to gain some insight before tackling with the more general case. In this respect, the model (1) describes the simplest physically relevant symmetry breaking pattern $SU(4)\rightarrow SU(2)_s\otimes SU(2)_t$.

The Hamiltonian (1) can be interpreted in terms of a two-leg spin ladder coupled by a four spins interaction. Such an interaction can be generated either by phonons or, in the doped state, by conventional Coulomb repulsion between the holes.[7] Some of the properties of Eq. (1) are well established in the *weak* coupling limit. In the limit $K\ll J_{1(2)}$, the Hamiltonian (1) describes a non-Haldane spin liquid where magnon excitations are incoherent.[7]

The strong coupling regime, $K\sim J_{1(2)}$, has just begun to be investigated.[5,6,8–10] From the theoretical point of view the situation is awkward. Indeed, as stated above, at the special point $J_1=J_2=K/4$, the Hamiltonian (1) has an enlarged SU(4) symmetry[6] and is exactly solvable by the Bethe ansatz.[11] The model is critical with *three* gapless bosonic modes and flows in the infrared towards the Wess-Zumino-Novikov-Witten (WZNW) model $SU(4)_1$.[12] It is conformally invariant with the central charge $c=3$. Therefore, there should be a *qualitative* change in the physical behavior of Eq. (1) when going from small to large K. From the theoretical point of view this situation is striking since this means that one cannot go continuously from weak to strong coupling. This is a manifestation of the Zamolodchikov c-theorem[13] that states that starting at $K=0$ with two decoupled $S=1/2$ chains with the central charge $c=2$ (two gapless bosonic modes) one cannot flow, in the renormalization group (RG) sense, towards the SU(4) point which has $c=3$ (three gapless bosonic modes). Very recently, a new approach to tackle with the strong coupling regime has been developed by Azaria *et al.*[14] The idea is to start from the strong coupling fixed point $\mathrm{SU}(4)_1$ and to perturb around it.

This strategy has been applied to the symmetric line $J=J_1=J_2$. It was shown that when $J<K/4$ a small deviation from the SU(4) point is irrelevant and thus the low energy physics is governed by the $\mathrm{SU}(4)_1$ fixed point. In contrast, when $J>K/4$ the interaction is relevant and a gap opens in the spectrum. The system dimerizes with alternating spin and orbital singlets. In addition it was argued that the SU(4) symmetry was restored at long distance and that the low energy effective Hamiltonian was that of the SO(6) Gross-Neveu (GN) model. The low-lying *coherent* excitations were then shown to be fermions that transform as an antisymmetric tensor of rank two of SU(4). These excitations are coherent with wave vector near $\pi/2$.

The purpose of this work is to extend the analysis to the asymmetric case ($J_1 \neq J_2$) and to inquire how anisotropy modifies the low energy physics described above. This paper is organized as follows. In Sec. II we present the tools that are necessary to explore the vicinity of the $\mathrm{SU}(4)_1$ fixed point and derive the effective low-energy theory associated with Eq. (1). A detailed renormalization group analysis is then presented in Sec. III. We obtain the phase diagram in the plane (J_1, J_2) and discuss the asymptotic behaviors of the mass gap M. A discussion on crossover effects linked to the anisotropy is also presented. In Sec. IV we investigate the effect of a magnetic field on the anisotropic model. Finally in Sec. V we summarize our results and present some technical details relative to the computation of the mass gap in the Appendix.

II. THE LOW ENERGY EFFECTIVE FIELD THEORY

A. The SU(4) Heisenberg chain

Our approach is very similar to the description of the $S=1/2$ Heisenberg spin chain at low energy from the spin sector of the repulsive Hubbard model at half filling.[12] To this end, let us introduce the SU(4) Hubbard model with $U>0$:

$$\mathcal{H}_U = \sum_{ia\sigma} (-t c^\dagger_{i+1a\sigma} c_{ia\sigma} + \mathrm{H.c.}) + \frac{U}{2} \sum_{iab\sigma\sigma'} n_{ia\sigma} n_{ib\sigma'} (1-\delta_{ab}\delta_{\sigma\sigma'}). \tag{2}$$

Here $c^\dagger_{ia\sigma}$ denotes electron creation with channel (orbital) $a=1,2$ and spin $\sigma=\uparrow,\downarrow$ at the ith site. The occupation number is defined by $n_{ia\sigma}=c^\dagger_{ia\sigma}c_{ia\sigma}$. In the limit of large positive U, and at *quarter* filling, it is not difficult to show that Eq. (2) reduces to Eq. (1) with the identification

$$\vec{S}_i = \frac{1}{2} \sum_a c^\dagger_{ia\alpha} \vec{\sigma}_{\alpha\beta} c_{ia\beta},$$

$$\vec{T}_i = \frac{1}{2} \sum_\alpha c^\dagger_{ia\alpha} \vec{\tau}_{ab} c_{ib\alpha}, \tag{3}$$

where $\vec{\sigma}$ (respectively $\vec{\tau}$) are the Pauli matrices acting in the spin (respectively orbital) space. The low energy physics can be described in terms of right movers $R_{a\sigma}$ and left movers $L_{a\sigma}$ fermions which correspond to the lattice fermion $c_{ia\sigma}$ in the continuum limit:

$$\frac{c_{ia\sigma}}{\sqrt{a_0}} \simeq R_{a\sigma}(x)\exp(ik_F x) + L_{a\sigma}(x)\exp(-ik_F x), \quad x=ia_0, \tag{4}$$

where a_0 is the lattice spacing and the Fermi wave vector is defined by $k_F=\pi/4a_0$. At this point we bosonize and introduce four chiral bosonic fields $\Phi_{a\sigma R,L}$ using the Abelian bosonization of Dirac fermions:[15]

$$R_{a\sigma} = \frac{\kappa_{a\sigma}}{\sqrt{2\pi a_0}} \exp(i\sqrt{4\pi}\Phi_{a\sigma R}),$$

$$L_{a\sigma} = \frac{\kappa_{a\sigma}}{\sqrt{2\pi a_0}} \exp(-i\sqrt{4\pi}\Phi_{a\sigma L}), \tag{5}$$

where the bosonic fields satisfy the following commutation relation:

$$[\Phi_{a\sigma R}, \Phi_{b\sigma' L}] = \frac{i}{4}\delta_{ab}\delta_{\sigma\sigma'} \tag{6}$$

so that $\{R_{a\sigma}(x), L_{a\sigma}(y)\}=0$. The anticommutation between fermions with different spin-channel indexes is insured by the presence of Klein factors (here Majorana fermions) $\kappa_{a\sigma}$ with the following anticommutation rule:

$$\{\kappa_{a\sigma}, \kappa_{b\sigma'}\} = 2\delta_{ab}\delta_{\sigma\sigma'}. \tag{7}$$

As in the solution of the two-channel Kondo effect by Abelian bosonization[16] it is suitable to introduce the physically transparent basis:

$$\Phi_c = \frac{1}{2}(\Phi_{1\uparrow}+\Phi_{1\downarrow}+\Phi_{2\uparrow}+\Phi_{2\downarrow}),$$

$$\Phi_s = \frac{1}{2}(\Phi_{1\uparrow}-\Phi_{1\downarrow}+\Phi_{2\uparrow}-\Phi_{2\downarrow}),$$

$$\Phi_f = \frac{1}{2}(\Phi_{1\uparrow}+\Phi_{1\downarrow}-\Phi_{2\uparrow}-\Phi_{2\downarrow}),$$

$$\Phi_{sf} = \frac{1}{2}(\Phi_{1\uparrow}-\Phi_{1\downarrow}-\Phi_{2\uparrow}+\Phi_{2\downarrow}). \tag{8}$$

In this new basis, the total charge degree of freedom is described by the bosonic field Φ_c while the other "spin-orbital" degrees of freedom, are faithfully bosonized by the three bosonic fields $\Phi_s, \Phi_f, \Phi_{sf}$. It is now straightforward to obtain the continuum limit of the Hubbard Hamiltonian (2):

$$\mathcal{H}_U = \mathcal{H}_c + \mathcal{H}_{sf}, \tag{9}$$

where

$$\mathcal{H}_c=\frac{v_F}{2}((\partial_x\Phi_c)^2+(\partial_x\Theta_c)^2)+\frac{3Ua_0}{2\pi}(\partial_x\Phi_c)^2, \quad (10)$$

and

$$\mathcal{H}_{sf}=\sum_{a=s,f,sf}\left(\frac{v_F}{2}((\partial_x\Phi_a)^2+(\partial_x\Theta_a)^2)-\frac{Ua_0}{2\pi}(\partial_x\Phi_a)^2\right)$$
$$+\frac{U}{\pi^2 a_0}(\cos\sqrt{4\pi}\Phi_s\cos\sqrt{4\pi}\Phi_f$$
$$+\cos\sqrt{4\pi}\Phi_f\cos\sqrt{4\pi}\Phi_{sf}$$
$$+\cos\sqrt{4\pi}\Phi_s\cos\sqrt{4\pi}\Phi_{sf}). \quad (11)$$

As in the SU(2) Heisenberg chain, spin and charge degrees of freedom separate. Notice however that at this order in U there are no umklapp terms in the charge sector since we are at quarter filling and the $4k_F$ contribution to the effective Hamiltonian oscillates. Umklapp terms will arise at higher order in perturbation theory and will be responsible for a Mott transition at a finite value of $U=U_c$.[17] Assuming that $U\gg U_c$, we focus now on the spin-orbital sector.

The interaction term in $\mathcal{H}_{sf}$ has scaling dimension 2 and is therefore marginal. This term is nothing but the SU(4) current-current interaction. The 15 $\mathrm{SU}(4)_1$ currents $\mathcal{J}_R^a$ and $\mathcal{J}_L^a$ can be expressed in terms of the three bosonic fields Φ_s, Φ_f and Φ_{sf} and the spin-orbital part of the Hamiltonian (9) takes the form

$$\mathcal{H}_{sf}=\frac{2\pi v_s}{5}\sum_{a=1,15}(\mathcal{J}_R^a\mathcal{J}_R^a+\mathcal{J}_L^a\mathcal{J}_L^a)+2g_s\sum_{a=1,15}\mathcal{J}_R^a\mathcal{J}_L^a, \quad (12)$$

where

$$g_s=-Ua_0,$$
$$v_s=v_F-Ua_0/2\pi. \quad (13)$$

The first term in Eq. (12) is just the Sugawara form of the $\mathrm{SU}(4)_1$ WZNW model while the second term is the marginal current-current interaction. When $U>0$, $g_s<0$, the current-current interaction $\mathcal{J}_R^a\mathcal{J}_L^a$ is thus irrelevant, as a consequence the spin orbital sector is described by the $\mathrm{SU}(4)_1$ WZNW model.[12]

B. Majorana representation and the SO(6) Gross-Neveu model

As first emphasized by Shelton *et al.*[18] in their study of the two-leg spin-1/2 ladders it is very convenient to formulate spin ladder problems in terms of real (Majorana) fermions. This can be done by refermionizing the three bosonic fields Φ_s, Φ_f, and Φ_{sf}. Let us introduce the six Majorana fermions $\xi^a, a=1$–6 as follows:

$$(\xi^1+i\xi^2)_{R(L)}=\frac{\eta_1}{\sqrt{\pi a_0}}\exp(\pm i\sqrt{4\pi}\Phi_{sR(L)}), \quad (14)$$

$$(\xi^3+i\xi^4)_{R(L)}=\frac{\eta_2}{\sqrt{\pi a_0}}\exp(\pm i\sqrt{4\pi}\Phi_{fR(L)}),$$

$$(\xi^5+i\xi^6)_{R(L)}=\frac{\eta_3}{\sqrt{\pi a_0}}\exp(\pm i\sqrt{4\pi}\Phi_{sfR(L)}),$$

$$\partial_x\Phi_s=i\sqrt{\pi}(\xi_R^1\xi_R^2+\xi_L^1\xi_L^2),$$

$$\partial_x\Phi_f=i\sqrt{\pi}(\xi_R^3\xi_R^4+\xi_L^3\xi_L^4),$$

$$\partial_x\Phi_{sf}=i\sqrt{\pi}(\xi_R^5\xi_R^6+\xi_L^5\xi_L^6),$$

where η_i are Klein factors. With all these relations at hand, one can rewrite the Hamiltonian (11) in terms of six Majorana fermions:

$$\mathcal{H}_{sf}=-\frac{iv_s}{2}\sum_{a=1}^{6}(\xi_R^a\partial_x\xi_R^a-\xi_L^a\partial_x\xi_L^a)+g_s\sum_{i<j}\kappa_i\kappa_j, \quad (15)$$

where we have introduced the energy density of the different Ising models: $i\epsilon_a=\kappa_a=\xi_R^a\xi_L^a$. The Hamiltonian (15) is nothing but that of the SO(6) GN model with a marginally *irrelevant* interaction when $U>0$. Thus in the far infrared the six Majorana fermions decouple and remain massless. This is the $\mathrm{SO}(6)_1$ fixed point.

The above result assumes that U is small and the question that naturally arises is whether it can be extended to large values of U where Eq. (2) reduces to Eq. (1). For exactly the same reasons as for the SU(2) Heisenberg chain the answer is positive. We know from the exact solution that the SU(4) model is critical with three massless bosonic modes or equivalently six massless Majorana fermions. We know from conformal field theory that the fixed point Hamiltonian can only be the $\mathrm{SU}(4)_1\sim\mathrm{SO}(6)_1$ WZNW model. The marginal interaction $\Sigma_{i<j}\kappa_i\kappa_j$ is the only one that respects both the SO(6) symmetry as well as translation invariance. Therefore in the vicinity of the fixed point the SU(4) Heisenberg model will be given by

$$\mathcal{H}=-\frac{iu_s}{2}\sum_{a=1}^{6}(\xi_R^a\partial_x\xi_R^a-\xi_L^a\partial_x\xi_L^a)+2G_3\sum_{i<j}\kappa_i\kappa_j, \quad (16)$$

where the spin velocity u_s and the coupling $G_3<0$ are unknown and nonuniversal parameter that could be extracted from the exact solution. The only thing that happens when going from small U to large U is a renormalization of the g_s and v_s.

The Majorana description used here is extremely useful to understand the symmetry properties of our model. Indeed, for example, one can define the spin and orbital triplets:

$$\vec{\xi}_{sR(L)}=(\xi^2,\xi^1,\xi^6)_{R(L)},$$
$$\vec{\xi}_{tR(L)}=(\xi^4,\xi^3,\xi^5)_{R(L)}. \quad (17)$$

These quantities transform like a vector under spin $\mathrm{SO}(3)_s$ and orbital $\mathrm{SO}(3)_t$ rotations. These correspond to the $\mathrm{SU}(2)_s$ and $\mathrm{SU}(2)_t$ transformations acting on the operators $\vec{S}$ and $\vec{T}$, respectively.

To get a complete description of the $\mathrm{SO}(6)_1$ fixed point one needs the continuum expressions for the effective spin and orbital densities in terms of the Majorana fermions:

$$\vec{S}=\vec{J}_{sR}+\vec{J}_{sL}+\exp(i\pi x/2a_0)\vec{\mathcal{N}}_s+\text{H.c.}+(-1)^{x/a_0}\vec{n}_s(x),$$

$$\vec{T}=\vec{J}_{tR}+\vec{J}_{tL}+\exp(i\pi x/2a_0)\vec{\mathcal{N}}_t+\text{H.c.}+(-1)^{x/a_0}\vec{n}_t(x), \tag{18}$$

where $\vec{J}_{s,t}$ is the uniform, $k=0$, part and $\vec{\mathcal{N}}_{s,t}$ and $\vec{n}_{s,t}$ are the $2k_F=\pi/2a_0$ and $4k_F=\pi/a_0$ contributions. Notice that in contrast with the SU(2) Heisenberg chain, the spin density has three oscillating components. The reason for this comes from conformal field theory. Indeed the different oscillating components of the spin density are *primary* fields of the $SU(4)_1$ WZNW model. There are three of them with scaling dimensions (3/4,1,3/4).[19] They all belong to the representations (building blocks) of SU(4) with Young tableau consisting of a $(a=1,2,3)$ boxes and one column. In particular the staggered components at $4k_F=\pi$, $\vec{n}_{s,t}$, are components of an antisymmetric tensor of rank 2.

The uniform components express in terms of the Majorana fermions as follows:

$$\vec{J}_{sR(L)}=-\frac{i}{2}\vec{\xi}_{sR(L)}\wedge\vec{\xi}_{sR(L)},$$

$$\vec{J}_{tR(L)}=-\frac{i}{2}\vec{\xi}_{tR(L)}\wedge\vec{\xi}_{tR(L)}. \tag{19}$$

Notice that in contrast with the SU(2) Heisenberg chain, the uniform part of the spin densities are $SU(2)_2$ currents. The expressions for the $4k_F=\pi/a_0$ densities are given by

$$\vec{n}_s=iB\vec{\xi}_{sR}\wedge\vec{\xi}_{sL},$$

$$\vec{n}_t=iB\vec{\xi}_{tR}\wedge\vec{\xi}_{tL}, \tag{20}$$

where B is a nonuniversal constant. Their scaling dimension at the $SO(6)_1$ fixed point is $\Delta_\pi=1$. Both densities (19) and (20) are rather simple when expressed in terms of the Majorana fermions. This is not the case with the $2k_F=\pi/2a_0$ densities $\vec{\mathcal{N}}_{s,t}$ that are nonlocal in the Majorana fermions ξ^a. Indeed they involve order and disorder operators σ_a and μ_a of the six Ising models that are associated with the six Majorana fermions. The expressions of both $\vec{\mathcal{N}}_{s,t}$ are lengthy and we shall give here only the z component that will be sufficient for our purpose:

$$\mathcal{N}^z_s=A(i\mu_1\mu_2\sigma_3\sigma_4\sigma_5\sigma_6+\sigma_1\sigma_2\mu_3\mu_4\mu_5\mu_6),$$

$$\mathcal{N}^z_t=A(i\sigma_1\sigma_2\mu_3\mu_4\sigma_5\sigma_6+\mu_1\mu_2\sigma_3\sigma_4\mu_5\mu_6), \tag{21}$$

where A is also a nonuniversal constant. Since at the free fermion point, the order and disorder operators have scaling dimension 1/8, the $2k_F$ densities $\vec{\mathcal{N}}_{s,t}$ have the scaling dimension $\Delta_{\pi/2}=3/4$.

This completes the continuum description at the SU(4) point. The theory is critical and flows to the $SO(6)_1$ fixed point. There is a marginally irrelevant correction whose magnitude is nonuniversal and is given by the unknown coupling $G_3<0$. This is in complete agreement with the non-Abelian bosonization approach of Affleck.[12] With the continuum expressions of the spin and orbital operators at the SU(4) point in terms of the six Majorana fermions and the associated Ising models we shall be able to investigate the properties of Eq. (1) close to the SU(4) symmetric model. But before moving to this point, let us stress that the effective theory depends on three unknown parameters A and B and G_3 which can be in principle extracted from numerical studies. In the following, we shall assume that the above description still holds for small deviations of the SU(4) point. The only modification being a renormalization of the nonuniversal constants A, B and G_3.

C. The $SU(2)\otimes SU(2)$ symmetry-breaking perturbation

We shall now derive the effective low energy theory associated with Eq. (1) for arbitrary values of J_1 and J_2 close to the SU(4) invariant point given by $J_1=J_2=K/4$. To this end, let us parametrize the couplings as follows:

$$J_1=\frac{K}{4}+G_1,$$

$$J_2=\frac{K}{4}+G_2, \tag{22}$$

where both G_1 and G_2 are much smaller than K. The Hamiltonian (1) can then be written as

$$\mathcal{H}=\mathcal{H}_{SU(4)}+G_1\sum_i \vec{S}_i\cdot\vec{S}_{i+1}+G_2\sum_i \vec{T}_i\cdot\vec{T}_{i+1}. \tag{23}$$

Using the low energy description of the spin-orbital operators (19), (20), and (21), one can expand Eq. (23) around the $SO(6)_1$ fixed point:

$$\mathcal{H}=-\frac{iu_s}{2}(\vec{\xi}_{sR}\cdot\partial_x\vec{\xi}_{sR}-\vec{\xi}_{sL}\cdot\partial_x\vec{\xi}_{sL})+g_1(\kappa_1+\kappa_2+\kappa_6)^2$$
$$-\frac{iu_t}{2}(\vec{\xi}_{tR}\cdot\partial_x\vec{\xi}_{tR}-\vec{\xi}_{tL}\cdot\partial_x\vec{\xi}_{tL})+g_2(\kappa_3+\kappa_4+\kappa_5)^2$$
$$+2G_3(\kappa_1+\kappa_2+\kappa_6)(\kappa_3+\kappa_4+\kappa_5), \tag{24}$$

where

$$g_1=G_1+G_3,$$

$$g_2=G_2+G_3, \tag{25}$$

and the two renormalized velocities u_s and u_t are given by

$$u_s=v_s+2G_1/\pi,$$

$$u_t=v_s+2G_2/\pi. \tag{26}$$

In the above equations and in the remaining of this paper, we include the effect of the $4k_F$ components of the spin and orbital densities in a redefinition of the couplings: $G_{(1,2)}\rightarrow(1+B^2)G_{(1,2)}$. The Hamiltonian (24) describes two marginally coupled SO(3) Gross-Neveu models: one in the spin channel described by the three Majorana $\vec{\xi}_s$ and one in the orbital channel described by the three Majorana $\vec{\xi}_t$. The situation at hand is to be contrasted another time with the one encountered in the study of spin ladders. In the latter models deviation from criticality leads, in general, to relevant perturbation and a gap always opens independently of the sign of the couplings. There are however notable exceptions where frustration plays its trick. This is the case of some three-leg frustrated ladders where for some particular values of the couplings only marginal interaction remains in the effective action. As a result a nontrivial critical state, the so-called ''chirally stabilized'' liquid, shows off.[20] In the model the situation is even more striking since we find only marginal interactions in a finite region of the couplings. This is mainly due to the fact that the frustation is maximal in the strong coupling region. A direct consequence of the marginality of all the interactions is that we expect the phase diagram to result from a delicate balance between the different terms entering in Eq. (24). This is why it is now worth discussing the effect of the three different terms in Eq. (24).

Consider first the case where $G_3=0$. Then we are left with the two decoupled $\mathrm{SO}(3)_s$ and $\mathrm{SO}(3)_t$ GN models which are exactly solvable. At issue is the sign of G_1 and G_2. When $G_1>0$ and $G_2>0$ a gap opens in both spin and orbital channels. The spectrum consists of kinks and antikinks (there are no fermions).[21] When $G_1<0$ and $G_2<0$, the interaction is irrelevant and the model flows towards the isotropic $\mathrm{SO}(6)_1\sim \mathrm{SU}(4)_1$ fixed point with the central charge $c=3$. When $G_1G_2<0$ one of the SO(3) GN models will become massive while the other one will flow towards the $\mathrm{SO}(3)_1$ fixed point. The Hamiltonian will remain critical with *three* massless Majorana fermions leaving the whole system with the central charge $c=3/2$.

Now the physically relevant question is whether or not this scheme survives a small negative G_3. Indeed, it would be not correct to neglect the last term in Eq. (24). First of all, from the point of view of the lattice Hamiltonian (23), the Hamiltonian (24) has to be thought as the effective Hamiltonian obtained by integrating out high energy modes up to a scale where one sits close enough to the $\mathrm{SO}(6)_1$ fixed point where the continuum limit can be taken. Thus, generically G_3 is not zero. The second reason is that all other interactions are *marginal*. In such a case, it is well known that operators that are naively irrelevant may become dangerous and strongly modify the physics in the infrared. Therefore, even though $G_3<0$, one has to keep it and analyze with care Eq. (24) with all couplings different from zero. As we shall see, the strong tendency to the $\mathrm{SO}(3)_1$ criticality in the regions $G_1G_2<0$ will be spoiled in most of the parameter space. There will be though still a finite region where the Hamiltonian will be critical but with an approximate SO(6) symmetry.

III. RENORMALIZATION GROUP ANALYSIS

A. The renormalization group flow

The RG equations for the couplings entering in Eq. (24) are given at leading order by

$$\dot{g}_1=\frac{1}{\pi u_s}g_1^2+\frac{3}{\pi u_t}G_3^2,$$

$$\dot{g}_2=\frac{1}{\pi u_t}g_2^2+\frac{3}{\pi u_s}G_3^2,$$

$$\dot{G}_3=\frac{2G_3}{\pi}\left(\frac{g_1}{u_s}+\frac{g_2}{u_t}\right). \qquad (27)$$

In the above equation, $\dot{G}$ means $\partial G/\partial t$ where $t=\log(\lambda)$ is the RG scale. It is more convenient to express the set of Eqs. (27) in terms of the couplings that enter in the lattice Hamiltonian (23):

$$\dot{G}_1=G_1^2-2G_2G_3,$$

$$\dot{G}_2=G_2^2-2G_1G_3,$$

$$\dot{G}_3=2G_3(G_1+G_2+2G_3), \qquad (28)$$

where, for simplicity we have made the following redefinition:

$$G_1\rightarrow 1/\pi\sqrt{u_su_t}(\alpha G_1-(1-\alpha)G_3),$$

$$G_2\rightarrow 1/\pi\sqrt{u_su_t}(1/\alpha G_2-(1-1/\alpha)G_3),$$

$$G_3\rightarrow 1/\pi\sqrt{u_su_t}G_3, \qquad (29)$$

with $\alpha=\sqrt{u_t/u_s}$. At the leading order, the velocities u_s and u_t also renormalize:

$$\dot{u}_s=-6u_sG_3^2\mathcal{I}\left(\frac{u_t}{u_s}\right),$$

$$\dot{u}_t=-6u_tG_3^2\mathcal{I}\left(\frac{u_s}{u_t}\right), \qquad (30)$$

where

$$\mathcal{I}(x)=\frac{1-x}{1+x}. \qquad (31)$$

The RG equations (28) and (30) can be exactly solved and reduce to a single differential equation. To show this let us introduce the following variables:

$$\mu=G_1G_2+G_1G_3+G_2G_3,$$

$$d=G_1-G_2,$$

$$s=G_1+G_2+2G_3. \qquad (32)$$

Equation (28) and (30) greatly simplify to

$$\dot{\mu}=\mu s,$$

$$\dot{d}=ds,$$

$$\dot{G}_3=2G_3 s, \tag{33}$$

and

$$\dot{u}_s-\dot{u}_t=-6G_3^2(u_s-u_t),$$

$$u_s u_t=\text{const.} \tag{34}$$

While the meaning of both s and d is clear, the physical interpretation of the variable μ is not straightforward. As we shall see in the following, μ measures the departure from criticality.

The solution of both Eqs. (33) and (34) are then easily obtained:

$$\mu(t)=\mu(0)X(t),$$

$$d(t)=d(0)X(t),$$

$$G_3(t)=G_3(0)X^2(t),$$

$$(u_s-u_t)(t)=(u_s-u_t)(0)\exp\left(-6G_3^2(0)\int_0^t X^4(\tau)d\tau\right),$$

$$(u_s u_t)(t)=(u_s u_t)(0), \tag{35}$$

where

$$X(t)=\exp\int_0^t s(\tau)d\tau \tag{36}$$

is the solution of the differential equation:

$$\left(\frac{\dot{X}}{X}\right)^2=P(X). \tag{37}$$

In the latter equation $P(X)$ is a fourth order polynomial that depends on the initial conditions of the flow:

$$P(X)=4G_3^2(0)X^4+d^2(0)X^2+4\mu(0)X. \tag{38}$$

Once the solution $X(t)$ of Eq. (37) is known, the behavior of the RG flow is completely determined. In particular the time evolution of couplings $G_1(t)$ and $G_2(t)$ is given by

$$G_{(1,2)}(t)=\frac{1}{2}(\epsilon(s)\sqrt{P[X(t)]}-2G_3(0)X^2(t)\pm d(0)X(t)), \tag{39}$$

where $\epsilon(s)=\text{sign}(s)$.

As in the XXZ model we distinguish *three* different phases A, B, and C that are separated by the two surfaces defined by $\mu=0$.

The A phase. There one has $\mu(0)>0$ and $s(0)>0$ and $X(t)\rightarrow\infty$ as $t\rightarrow\infty$ [see Fig. 1(a)]. All couplings are relevant and a gap opens in the spectrum. In addition, the velocities $u_s(t)$ and $u_t(t)$ goes to a fixed value $u_s^*=u_t^*$. Looking in more details in the large t behavior of the couplings, one finds that $G_1(t)\sim G_2(t)\sim-2G_3(t)$ so that the effective interaction in the far infrared takes the suggestive form:

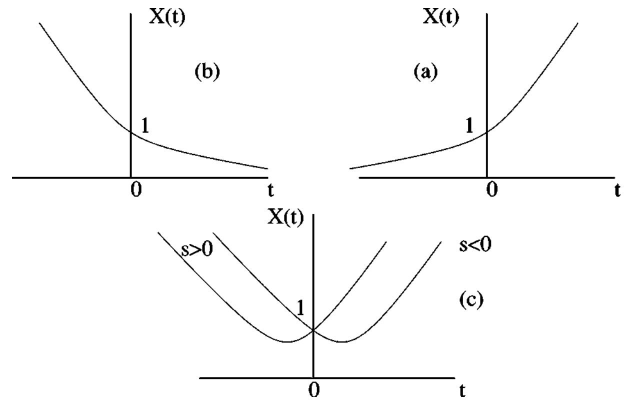

FIG. 1. RG flows for the variable $X(t)$ in the different phases A, B, and C. In the phase C, $X(t)$ reaches a minimum value X^* at a scale t^*. When $s(0)<0$, the minimum is reached for a physical time $t^*>0$.

$\sim[(\kappa_1+\kappa_2+\kappa_6)-(\kappa_3+\kappa_4+\kappa_5)]^2$ which after the transformation $\vec{\xi}_{sR}\rightarrow\vec{\xi}_{sR}$, $\vec{\xi}_{sL}\rightarrow-\vec{\xi}_{sL}$ acquires a manifestely SO(6) invariant form $\sim(\Sigma_1^6\kappa_a)^2$. One may therefore be tempted to conclude that the SO(6) symmetry is restored in the far infrared and that the low energy spectrum of the Hamiltonian would be approximatively that of the SO(6) GN model. However, as pointed out by Azaria, Lecheminant, and Tsvelik,[22] one cannot conclude on the basis of perturbation theory alone. In fact looking at the flow in the infrared is not sufficient and can be misleading. The point is that in the A phase, since $X(t)\rightarrow0$ as $t\rightarrow-\infty$ all couplings go to zero in the ultraviolet and the theory is asymptotically free. From the field theoretical point of view this means that a well defined renormalized theory with *three* different renormalized couplings G_{1R}, G_{2R}, and G_{3R} exists.[24] Therefore, it is most likely that the SO(6) symmetry is *not* restored in the A phase. The situation here is very similar to the XXZ model where in the A phase (the Ising region), though there is an apparent SU(2) symmetry restoration in the far infrared, the *exact* solution tells us that this is not the case.[23]

The B phase. There one has $\mu(0)>0$ and $s(0)<0$ and the flow in the X variable is reversed [Fig. 1(b)]. All couplings goes to zero in the infrared and the interaction is irrelevant. The six Majorana fermions are massless and the model is critical with the central charge $c=6\times1/2=3$. However, for generic values of the initial conditions $[d(0)\neq0]$, the fixed point Hamiltonian does not have the SO(6) symmetry. Indeed, the velocities in both spin and orbital sector have different fixed point values $u_s^*\neq u_t^*$ so that the symmetry at the fixed point is $\text{SO}(3)_s\otimes\text{SO}(3)_t$. It is only on the symmetric line $(G_1=G_2)$ that the fixed point symmetry is SO(6).

The C phase. There one has $\mu(0)<0$ and s can have both signs. As seen in Fig. 1(c), $X(t)\rightarrow\infty$ when both $t\rightarrow\pm\infty$. The theory is not asymptotically free neither in the infrared *nor* in the ultraviolet. Though one certainly expects that a gap opens in the spectrum, perturbation theory is pathological.[24] Indeed, from the field theoretical point of view, the lack of asymptotic freedom in the ultraviolet implies that a well defined renormalized theory with *three* different couplings G_{1R}, G_{2R} and G_{3R} does not exist. The low energy physics in such a situation is a highly nontrivial and essentially nonperturbative problem and one is left to make a sensible conjecture.

As in the A phase discussed above, in the far infrared $G_1(t)\sim G_2(t)\sim -2G_3(t)$ as well as $u_s^* = u_t^*$ so one may wonder again whether the SO(6) symmetry is approximatively restored or not. We stress that the situation at hand here is very different to what happens in the A phase since the theory is *not* asymptotically free in the ultraviolet. This may have important consequences for the low energy physics. Indeed it is well known that the divergency of some couplings at high energy is reminiscent of the fact that some *part* of the interaction is *irrelevant* at low energy. Of course, perturbation theory alone cannot tell us *which* part of the interaction is irrelevant and at present all what we have at hand to make a reasonable hypothesis is our perturbative results. The simplest scenario is that the SO(6) symmetry is approximatively restored provided the initial conditions are not too anisotropic (see however subsection C). As a support of this conjecture let us mention that this is what happens in the C phase of the XXZ model where the Bethe ansatz solution tells us that the SU(2) symmetry is restored up to exponentially small corrections.[23] We are of course aware that our hypothesis is highly questionable but it is the simplest one and could in principle be tested either numerically or experimentally. Indeed, the immediate consequence of the possible SO(6) restoration is that the effective low energy effective theory of the Hamiltonian (1) in the C phase is approximatively that of the SO(6) GN model:

$$\mathcal{H} = -\frac{iu^*}{2}\sum_{a=1}^{6}\left(\xi_R^a\partial_x\xi_R^a - \xi_L^a\partial_x\xi_L^a\right) + G\sum_{i<j}\kappa_i\kappa_j, \quad (40)$$

where the *effective* coupling G is positive. The model (40) is integrable. Its spectrum is known and consists of the fundamental fermion, with mass M, together with a kink of mass $m = M/\sqrt{2}$.[21] At this point the question that naturally arises is how this enlarged SO(6) symmetry reflects in the spin and orbital correlation functions. The answer to this important question requires the computation of the exact dynamical correlation functions in the SO(6) restored massive phase. This could be accomplished in principle by the form factors approach as in the frustrated two leg ladder considered in Ref. 25. However this task is even more difficult for the SO(6) case and goes beyond the scope of this paper.

B. Phase diagram

Let us now sum up our results and present the phase diagram associated with Hamiltonian (24). As even at the SU(4) symmetric point the coupling G_3 is not zero, the best way to visualize the phase diagram is to fix the value of G_3 and to look in the plane (G_1, G_2). As one can see from Fig. 2, there are two curves separating the three regions A, B and C which are given by the equation

$$\mu = G_1G_2 + G_1G_3 + G_2G_3 = 0. \quad (41)$$

In the region B all models are critical with approximate SO(6) symmetry. The spectrum consists into six massless Majorana fermions with different velocities in both spin and orbital sectors. As one crosses the critical line Σ one enters in a fully massive phase (C phase) with an approximate SO(6) symmetry. For large positive values of the couplings G_1 and G_2 one may finally enter the region A. Notice that

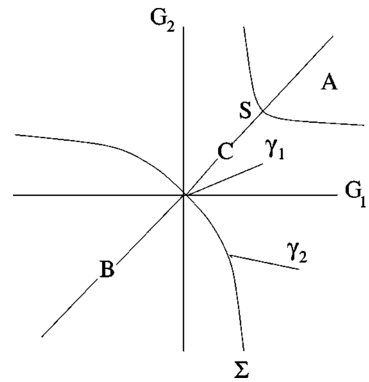

FIG. 2. Phase diagram for anisotropic couplings in the plane (G_1, G_2) at a fixed value of G_3. The massless phase B is separated from the massive C phase by the critical line Σ. The special point S on the symmetric line which is at the border between the C and the A phases is SO(6) symmetric with $G_1 = G_2 = -2G_3$. We plot also the two trajectories labeled γ_1 and γ_2 for which the scaling of the mass gap has two different qualitative behaviors.

there is no phase transition between the A phase and the C phase since both phases are massive but rather a smooth cross over.

The most important feature of our phase diagram is that the two regions with $G_1G_2 < 0$ which would have been massless in the absence of G_3 do not survive an arbitrarily small value of G_3. Therefore the $c = 3/2$ phases discussed in the previous section are not stable. Though the whole region is essentially massive there are still room for criticality since the B phase extends in both quadrants $G_1G_2 < 0$. From Eq. (41) the width of the critical region is of the order $|G_3|$. Therefore the main effect of G_3 is to drive the system either to a fully massive phase (phase C) or to a fully massless phase (phase B), both with an approximate SO(6) symmetry. Of course, the phase diagram as depicted in the Fig. 2 is strictly valid in the small G limit. In particular higher order corrections may modify the shape of the critical curve Σ. We stress however, that its behavior in the vicinity of the SU(4) point at $G_1 = G_2 = 0$ is given by our one loop result. In particular the fact that Σ crosses the symmetric line with a right angle will not change as one includes higher order in perturbation theory. However, as one goes to large deviations from the SU(4) point our effective theory will not apply since for large enough positive (respectively negative) G_1 and negative (respectively positive) G_2 the orbital (respectively spin) degrees of freedom order ferromagnetically.[5,10]

C. Effect of the anisotropy

At this point one may wonder whether the anisotropy has no effect at all in the physical quantities. Apart from the velocity anisotropy in the B phase, one may still expect some nontrivial effect of the anisotropy in the C phase. Indeed, after all even though both $c = 3/2$ phases are unstable there should be some significant signature of the presence of the $SO(3)_1$ fixed point in the scaling of the physical quantities and in the finite size scaling. The very reason for this is that the SO(6) symmetry is restored dynamically with help of a *marginal* operator.

Crossover and finite size scaling. Let us look at the RG flow in more details. The regions of interest are those with

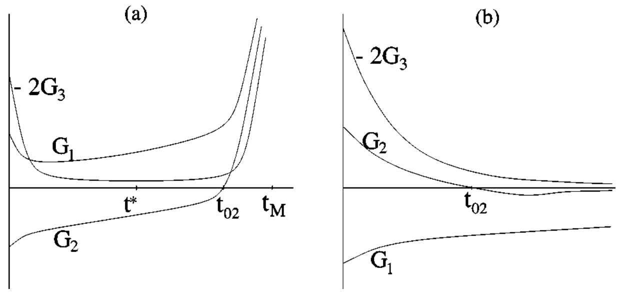

FIG. 3. Qualitative RG evolution of the coupling constants in regions where $G_1(0)G_2(0)<0$ and where the crossover ''time'' defined by $G_2(t_{02})=0$ exists.

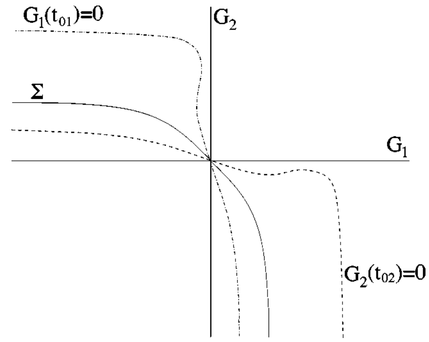

FIG. 4. Iso-t_0 curves in the plane (G_1,G_2); the dashed line corresponds to an iso-t_{02} associated with the orbital degrees of freedom whereas the dashed dotted line is an iso-t_{01} associated with the spin degrees of freedom. Σ denotes the critical line.

$G_1G_2<0$ in which, as discussed in the previous section, if not for G_3 one of the coupling would have been irrelevant. Then either the spin or the orbital degrees of freedom would have remained massless. In the following we shall concentrate on the orbital degrees of freedom. All the results that will be given can be straightforwardly extended to the spin degrees of freedom.

Consider first initial conditions slightly above the phase B in the lower right quadrant with $G_1G_2<0$ (see Fig. 2). We are then in the phase C with $G_1(0)>0$ and $G_2(0)<0$. As seen in the Fig. 3(a), the flow can be splitted into two ''time'' slices.

At first $|G_3|$ strongly decreases and remains almost constant. Then, both spin and orbital degrees of freedom weakly interact since $|G_{(1,2)}|\gg|G_3|$. In the meantime, the coupling G_2 increases and changes its sign at a time t_{02} where it vanishes. At ''times'' $t<t_{02}$ the orbital degrees of freedom do not know yet they shall enter in a massive phase and the system is on the influence of the $SO(3)_1$ fixed point. Finally, as $t\to t_M$ all the couplings blow up and one enters in the strong coupling region. The physical interpretation of this phenomenon is clear: it is the spin degrees of freedom that drive the orbital degrees of freedom away from criticality and it takes roughly $\exp(t_{02})$ RG iterations for the orbital sector to escape from the basin of attraction of the $SO(3)_1$ fixed point.

The other interesting region is the portion of the B phase delimited by the critical curve Σ and the G_2 axis in upper left quadrant of Fig. 2. There $G_1(0)<0$ and $G_2(0)>0$. As seen in Fig. 3(b), it is the spin degrees of freedom that drive the orbital ones to criticality. The coupling G_2 changes its sign and vanishes at a time t_{02} and goes to zero in the limit $t\to\infty$. In both cases, the crossover ''time'' t_{02} is given by the implicit equation:

$$X(t_{02})=\sqrt{\frac{\mu(0)}{d(0)G_3(0)}}. \tag{42}$$

Of course a similar discussion holds for the coupling G_1 and defines another crossover time t_{01} which is defined by the implicit equation:

$$X(t_{01})=\sqrt{\frac{-\mu(0)}{d(0)G_3(0)}}. \tag{43}$$

We plot in Fig. 4 the iso-t_{02} and iso-t_{01} curves in the plane (G_1,G_2).

We stress that this crossover phenomenon may have important practical consequences in numerical simulations. Indeed, in a finite system of size L, the critical region will seem to extend towards the iso-t_{01} with $t_{01}=\ln L$ in the region $G_1<0$, $G_2>0$ and towards the iso-t_{02} with $t_{02}=\ln L$ in the region $G_1>0$, $G_2<0$. The low lying spectrum will look like if either the spin or the orbital degrees of freedom would have been massless. It is important to notice that these two pseudocritical lines bend upwards in contrast with the critical line Σ.

The mass gap. The anisotropy has also nontrivial effect on the mass gap of the system. To see this we have computed the mass gap M with help of the RG equations (28). As well known, the gap M is defined as the scale where perturbation theory breaks down. More precisely it is given by the scale $t_M=\ln(1/Ma_0)$ at which the couplings blow up:

$$X(t_M)=\infty. \tag{44}$$

The above implicit equation can be solved but the resulting analytical expression is too cumbersome to be quoted here. Details are given in the Appendix and we shall content ourselves by its asymptotic behaviors as one approaches the critical surface Σ.

We find two qualitatively different behaviors depending on the way one approaches Σ. When one approaches Σ at the SU(4) point $(G_1=G_2=0)$ we find

$$M\sim\Lambda\,\exp(-C(\gamma_1)/\Delta^{2/3}), \tag{45}$$

where $\Delta=\sqrt{G_1^2+G_2^2}\to 0$ and the constant $C(\gamma_1)$ depends on the trajectory labeled γ_1 in the Fig. 2. On the other hand, as one approaches Σ at any other point the asymptotic behavior of the gap is different:

$$M\sim\Lambda\,\exp(-C(\gamma_2)/\Delta), \tag{46}$$

where now Δ is the Euclidean distance of the critical surface. As above $C(\gamma_2)$ depends on the curve labeled γ_2 in Fig. 2. There are two other marginal cases as one approaches Σ tangentially: the exponents of Δ in both Eqs. (45) and (46) are doubled. We see that the gap is generically *larger* in the regions with *small* anisotropy. To visualize this phenomenon we present in Fig. 5 the isogap curves in the plane (G_1,G_2). We observe that at a given distance Δ from the critical line Σ the gap increases as one moves towards the symmetric re-

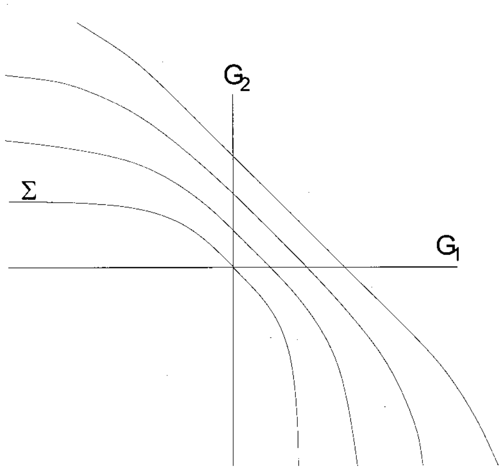

FIG. 5. Isogap curves in the plane (G_1,G_2); Σ denotes the critical line.

gion $(G_1\sim G_2)$ and is maximum on the symmetric line $(G_1=G_2)$. Similarly, to keep the value of the gap to some constant M one has to move away from Σ as one leaves the symmetric region.

Effect of anisotropy on the SO(6) symmetry restoration. We conclude this section by looking at the effect of the anisotropy on the SO(6) symmetry restoration in the C phase. As discussed above, in the C phase the couplings G_1 and G_2 tend to $-2G_3$ in the far infrared (i.e., as $t\rightarrow t_M$) so that the effective Hamiltonian has an apparent SO(6) symmetry. Looking in more detail at the asymptotic behavior of the couplings $G_1(t)$ and $G_2(t)$ we find that

$$G_{(1,2)}(t)\sim -2G_3(0)X^2(t)\pm\frac{d(0)}{2}X(t). \tag{47}$$

Therefore, as when $t\rightarrow t_M$, $X(t)\rightarrow\infty$, there remains a subdominant infrared singularity when $d(0)\neq 0$. We thus expect some corrections to the SO(6) behavior. To get an estimate of these corrections we make the reasonable hypothesis that, when the anisotropy is small enough, the effective theory will be given by an SO(6) GN model with a small symmetry-breaking perturbation proportional to

$$\mathcal{H}\sim -\frac{iu^*}{2}\sum_{a=1}^{6}(\xi_R^a\partial_x\xi_R^a-\xi_L^a\partial_x\xi_L^a)+G\sum_{i<j}\kappa_i\kappa_j +\lambda((\kappa_3+\kappa_4+\kappa_5)^2-(\kappa_1+\kappa_2+\kappa_6)^2), \tag{48}$$

where $\lambda\ll G$. Of course we do not know the value of the effective coupling λ, but it is sufficient for our purpose that it is small enough which will be the case if $d(0)$ is small. From Eq. (48) one can get an estimate of the mass splitting in both the spin and orbital sectors:

$$\frac{M_s}{M_t}\sim 1+\lambda\ln(1/Ma_0). \tag{49}$$

Therefore, we expect the SO(6) symmetry will hold approximatively only for small enough gap and anisotropy. The above argument is of course far from being rigorous but it helps to get an idea of the effect of the anisotropy in the symmetry restoration process. This should serve as a warning that, once again, one should be careful with the conclusions drawn from perturbation theory.

IV. EFFECT OF A MAGNETIC FIELD

In contrast with the SU(2) Heisenberg chain there is no unique way to apply a magnetic field in the SU(4) model. Indeed, while in the SU(2) case there is, apart from the total spin, only one conserved charges S^z in the SU(4) model there are *three* of them. These are the three Cartan generators of the SU(4) Lie algebra. In this work we have chosen for the three commuting generators:

$$H^1=S^z,\quad H^2=T^z,\quad \text{and}\quad H^3=2S^zT^z. \tag{50}$$

Other choices are possible but Eq. (50) is the one that is physically relevant to our problem. We thus see that one is at liberty to apply a magnetic field in any ''direction'' in the Cartan basis (50). However, if one relies on the physical interpretation of both operators $\vec{S}$ and $\vec{T}$ as spin and orbital operators, a magnetic field only couples to $H^1=S^z$. The resulting Hamiltonian thus writes

$$\mathcal{H}_h=\sum_i J_1\vec{S}_i\cdot\vec{S}_{i+1}+J_2\vec{T}_i\cdot\vec{T}_{i+1} +K(\vec{S}_i\cdot\vec{S}_{i+1})(\vec{T}_i\cdot\vec{T}_{i+1})+h\sum_i S_i^z. \tag{51}$$

The magnetic field breaks the SU(4) symmetry down to $SU(2)\otimes U(1)$. For a sufficiently large value of $h>h_o$ the spin degrees of freedom align parallel to the field and the remaining degrees of freedom will decouple. More interesting is the situation when h is small. In this case, one may expand Eq. (51) around the $SU(4)_1$ fixed point, as in previous section, and study its effects in perturbation. For small values of h, the effective low energy Hamiltonian associated with Eq. (51) can be easily derived using the expression of the spin operator S^z in terms of the Majorana fermions as given by Eqs. (19), (20), and (21). Dropping all oscillating terms, the only contribution comes from the uniform part of the spin density and the effect of the magnetic field is just to add to the Hamiltonian (24) the interaction $\mathcal{H}_Z=ih(\xi_R^1\xi_R^2+\xi_L^1\xi_L^2)$ and the low-energy Hamiltonian thus writes

$$\mathcal{H}_h=-\frac{iu_s}{2}(\vec{\xi}_{sR}\cdot\partial_x\vec{\xi}_{sR}-\vec{\xi}_{sL}\cdot\partial_x\vec{\xi}_{sL})+g_1(\kappa_1+\kappa_2+\kappa_6)^2 -\frac{iu_t}{2}(\vec{\xi}_{tR}\cdot\partial_x\vec{\xi}_{tR}-\vec{\xi}_{tL}\cdot\partial_x\vec{\xi}_{tL})+g_2(\kappa_3+\kappa_4+\kappa_5)^2 +2G_3(\kappa_1+\kappa_2+\kappa_6)(\kappa_3+\kappa_4+\kappa_5)+ih(\xi_R^1\xi_R^2+\xi_L^1\xi_L^2), \tag{52}$$

where we recall that

$$g_1=G_1+G_3,$$

$$g_2 = G_2 + G_3. \tag{53}$$

A. Renormalization group analysis

The situation at hand is similar to that of the XXZ model in a field.[26] There exists a magnetic length λ_h below which the magnetic field has no substantial effect on the physics. However as the scale $\lambda \gg \lambda_h$, the field strongly modifies the low energy behavior. Indeed, in this regime, as we shall see below, the two Majorana fermions ξ^1 and ξ^2 completely decouple from the four others. The best way to see this is to bosonize the ''1-2'' sector of the Hamiltonian (52) with help of Eq. (14). The resulting Hamiltonian becomes

$$\begin{aligned}\mathcal{H}_h =& \frac{v_\phi}{2}\left(\frac{1}{K}(\partial_x\Phi_s)^2 + K(\partial_x\Theta_s)^2\right)\\ &- \frac{iv_6}{2}(\xi_R^6\partial_x\xi_R^6 - \xi_L^6\partial_x\xi_L^6) - \frac{iv_t}{2}(\vec{\xi}_{tR}\cdot\partial_x\vec{\xi}_{tR} - \vec{\xi}_{tL}\cdot\partial_x\vec{\xi}_{tL})\\ &+ 2f_2(\kappa_3\kappa_4 + \kappa_3\kappa_5 + \kappa_4\kappa_5) + 2f_3\kappa_6(\kappa_3 + \kappa_4 + \kappa_5)\\ &- \frac{2if_4}{\pi a_0}\cos(\sqrt{4\pi}\Phi_s + 2qx)(\kappa_3 + \kappa_4 + \kappa_5)\\ &- \frac{2if_5}{\pi a_0}\cos(\sqrt{4\pi}\Phi_s + 2qx)\kappa_6,\end{aligned} \tag{54}$$

where

$$\begin{aligned}K &= 1 - \frac{g_1}{\pi u_s},\\ v_\phi &= u_s + \frac{g_1}{\pi},\\ q &= -\frac{Kh}{v_\phi}.\end{aligned} \tag{55}$$

In the above Hamiltonian, we have introduced new velocities and coupling constants with bare values: $v_6 = u_s$, $v_t = u_t$, $f_2 = g_2$, $f_3 = f_4 = G_3$ and $f_5 = g_1$ corresponding to the operators that are stable under renormalization. As readily seen, the cosine terms involving Φ_s will start to oscillate with wave vector $q \sim h$. Therefore, at sufficiently large magnetic field the two last terms in Eq. (54) can be dropped and the bosonic field Φ_s completely decouples. When a gap is present, i.e., when one sits in the C phase at $h = 0$, there exists a critical field $h_c \sim M$ below which the field has little effects on the low energy physics. As h increases and becomes greater than h_c, the spin sector undergoes a commensurate-incommensurate transition of the JNPT type[27] to an incommensurate massless phase with central charge $c_s = 1$. In contrast, in the absence of a gap, i.e., when one sits in the B phase at $h = 0$, the spin sector always decouples at sufficiently low energies. Of course, this decoupling procedure has to be understood in the framework of the renormalization group. To this end we have computed the one loop recursion relation associated with the Hamiltonian (54). Neglecting the velocity renormalization we obtain

$$\begin{aligned}\dot{K} &= -(3f_4^2 + f_5^2)J_0(2q),\\ \dot{q} &= q - (3f_4^2 + f_5^2)J_1(2q),\\ \dot{f}_2 &= f_2^2 + f_3^2 + 2f_4^2 J_0(2q),\\ \dot{f}_3 &= 2f_2f_3 + 2f_4f_5J_0(2q),\\ \dot{f}_4 &= (1-K)f_4 + (2f_2f_4 + f_3f_5)(1 + J_0(2q))/2,\\ \dot{f}_5 &= (1-K)f_5 + 3f_3f_4(1 + J_0(2q))/2,\end{aligned} \tag{56}$$

where the couplings have been rescaled as $f \to f/\pi v^*$, v^* being a velocity scale, and $J_{(0,1)}(x)$ are the Bessel functions of the first and second kinds. Apart from the inherent anisotropy in our problem, the above equations resemble that of the XXZ model in a field.[26] We see on Eqs. (56) that when $q(t) \sim \pi$ the two Bessel functions start to oscillate and the renormalization coming from both the f_4 and f_5 terms is stopped. This defines the magnetic length λ_h as $q(t_h) = \pi$ ($t_h = \ln \lambda_h$) where the spin degrees of freedom decouple from the rest of the interaction (as well known, there is no unique way to define the magnetic length but we have checked that other reasonable choices do not affect qualitatively the physics). At this point, it is worth stressing that in the massive phase, t_h must be much smaller than t_M, the scale at which perturbation theory breaks down. This means, of course, that $h > M$ as discussed above.

At times $t > t_h$, the couplings f_4 and f_5 do not participate in the RG equations of the couplings K, q, f_2 and f_3 and Eqs. (56) reduce to

$$\begin{aligned}\dot{K} &= 0,\\ \dot{q} &= q\end{aligned} \tag{57}$$

and

$$\begin{aligned}\dot{f}_2 &= f_2^2 + f_3^2,\\ \dot{f}_3 &= 2f_2f_3.\end{aligned} \tag{58}$$

This means that in the regime $t > t_h$ the Hamiltonian (54) decouples:

$$\mathcal{H}_h = \mathcal{H}_s + \mathcal{H}_\perp, \tag{59}$$

where

$$\mathcal{H}_s = \frac{v_\phi}{2}\left(\frac{1}{K}(\partial_x\Phi_s)^2 + K(\partial_x\Theta_s)^2\right) \tag{60}$$

and

$$\begin{aligned}\mathcal{H}_\perp =& -\frac{iv_6}{2}(\xi_R^6\partial_x\xi_R^6 - \xi_L^6\partial_x\xi_L^6) - \frac{iv_t}{2}(\vec{\xi}_{tR}\cdot\partial_x\vec{\xi}_{tR} - \vec{\xi}_{tL}\cdot\partial_x\vec{\xi}_{tL})\\ &+ 2f_2(\kappa_3\kappa_4 + \kappa_3\kappa_5 + \kappa_4\kappa_5) + 2f_3\kappa_6(\kappa_3 + \kappa_4 + \kappa_5),\end{aligned} \tag{61}$$

where K, v_ϕ, v_6, v_t, f_2 and f_3 are the *effective* couplings at the magnetic length λ_h.

Let us first concentrate on the spin sector as given by Eq. (60). The Hamiltonian (60) is that of a Luttinger liquid with stiffness K. The spin sector is thus massless and contributes to a central charge $c_s=1$. In addition, the correlation functions involving the field Φ_s will be incommensurate with h-dependent wave vector q. From Eqs. (21) one sees that the incommensurability will reflect in both the spin and orbital correlation functions.

We focus now on the remaining part of the interaction. The Hamiltonian $\mathcal{H}_\perp$ describes the interaction between the orbital sector with the remaining spin-orbital Z_2 degree of freedom (ξ^6) of the spin sector. The low energy physics in this sector is nontrivial and at issue is the behavior of the RG flow associated with Eq. (58) where the initial conditions have to be taken at the magnetic length with $f_2(t_h)$ and $f_3(t_h)$ at $t_h=\ln(\lambda_h)$.

These equations are trivially solved. Indeed, upon introducing the new variables

$$f_\pm=f_2\pm f_3, \tag{62}$$

Eqs. (58) decouple:

$$\dot{f}_\pm=f_\pm^2. \tag{63}$$

As in the previous section, we distinguish between three phases A, B, and C depending on the initial conditions of the flow.

The A phase. This is when $f_+(t_h)>0$ and $f_-(t_h)>0$. Both couplings are relevant and a gap opens in the spectrum. Moreover, since the theory is asymptotically free in the ultraviolet there are two length scales in the problem: $m_\pm \sim \exp-(\pi/f_\pm(t_h))$.

The B phase. There $f_+(t_h)<0$ and $f_-(t_h)<0$. The couplings are irrelevant and the four Majorana fermions become massless leaving the theory with the central charge $c_\perp=2$. The fixed point has only an approximate SO(4) symmetry since there remains a velocity anisotropy. Indeed, as in the zero field case, in general $v_t^*\neq v_6^*$. The generic symmetry of the fixed point is thus rather $SO(3)\otimes Z_2$.

The C phase. Finally is the C phase where $f_-(t_h)>0$ and $f_+(t_h)<0$. Then $f_-(t)$ is relevant and $f_+(t)$ will go to zero in the infrared. Therefore, as in the previous section, one may conjecture that the SO(4) symmetry is approximately restored. In the far infrared the effective Hamiltonian is that of the SO(4) Gross-Neveu model:

$$\mathcal{H}_\perp=-\frac{iu}{2}\sum_{a=3}^{6}(\xi_R^a\partial_x\xi_R^a-\xi_L^a\partial_x\xi_L^a)+f_-(t_h)\sum_{i<j}\kappa_i\kappa_j, \tag{64}$$

which is integrable. Its spectrum consists only on kinks and antikinks with mass $m\sim\exp-(\pi/f_-(t_h))$.[21]

Notice, and this will be important for the discussion that will follow, that in both the phases B and C the effective theories are given (up to a velocity anisotropy) by Eq. (64) with the difference that $f_-(t_h)$ is *negative* in the phase B (so that the interaction is irrelevant) while it is *positive* in the phase C.

B. Phase diagram

It is clear from the discussion given above that the values of the effective couplings at the magnetic length are crucial. Of course, the existence of the commensurate-incommensurate transition in the spin sector depends only on the mass gap M of the zero field Hamiltonian. What is more interesting, is what happens in the remaining orbital and spin-orbital sectors described by the Majorana fermions ξ^3,ξ^4,ξ^5,ξ^6. As we shall see now the anisotropy will play its tricks. Indeed, what is into question is the sign of the coupling $f_-(t_h)$ *at* the magnetic length. Returning to the original variables, one finds that $f_-=G_2$. We saw in the preceding section that in zero magnetic field the time evolution of G_2 was very sensitive to the presence of the $SO(3)_1$ fixed point in the orbital sector and could change its sign at a time t_{02} depending on the the initial conditions [see Fig. 3(a) and Fig. 3(b)]. The presence of a field h does not affect qualitatively this feature and provides for a renormalization of t_{02} which becomes h-dependent: $t_{02}\rightarrow t_{02}(h)$. We consider now two cases.

First is when one sits in the B phase between the G_1 axis and the critical surface Σ, with $G_1<0$ (see Fig. 2). At zero magnetic field the system is critical with the central charge $c=3$. There f_- is positive and decreases as t increases vanishes at some RG time $t_{02}(h)$ and then changes sign. Now if $t_h<t_{02}(h)$, $f_-(t_h)$ will be still positive and a gap will open in the orbital and spin-orbital sector according to Eq. (64). On the other hand, if $t_h>t_{02}(h)$ then $f_-(t_h)<0$ and there is no gap. This means that there exists a critical value of the field h_{c0} above which a gap opens. The portion of critical surface Σ in the region $G_1<0$ is thus unstable. The physical interpretation of this result is clear. In zero field, it was the spin degrees of freedom that drived the orbital degrees of freedom to criticality. When the field is large enough, its effect is to decouple a part of the spin degrees of freedom *before* the remaining fluctuations had a chance to enter the basin of attraction of the $SO(4)_1$ fixed point. Thus, the effect of the magnetic field in this region is to reduce the extension of the phase B.

The other interesting region is when one sits in the C phase just above the critical surface Σ in the lower right quadrant of Fig. 2, i.e., when $G_1>0$ and $G_2<0$. There G_2 is negative but is driven to positive values by the spin degrees of freedom. It changes sign at a time $t_{02}(h)$ where it vanishes. Now if $t_h<t_{02}(h)$, $f_-(t_h)$ will be still negative while if $t_h>t_{02}(h)$, $f_-(t_h)$ will be positive. Therefore there exists a critical field h_{c0} below which the orbital and spin-orbital sectors will be still massive. Above h_{c0}, the gap will close. Again, the physical reason why the gap vanishes above h_{c0} is that, the spin degrees of freedom did not have enough time to drive the orbital sector to strong coupling. We therefore conclude that the B phase has a tendency to extend in the region $G_1>0$, $G_2<0$ when a field is present.

To summarize, we expect two kinds of transition as one varies the magnetic field. When the theory is massive at $h=0$, as one increases h there will be a first transition in the spin sector to an incommensurate phase with the central charge $c_s=1$. The remaining degrees of freedom will be still massive but are described by the SO(4) GN model. The coherent fermionic excitations of the SO(6) GN model disappear from the spectrum and the only massive excitations that remain are the kinks of the SO(4) GN model. Consequently all excitations will be incoherent. What happens as one increases h further strongly depends on the anisotropy. If $G_2>0$ the magnetic field just renormalizes the mass of the SO(4) kinks, the spectrum is still massive. The total central charge of the model is thus $c=c_s+c_\perp=1$. However, when $G_2<0$, there is a *second* phase transition at a field h_{c0} of the Kosterlitz-Thouless (KT) type[28] to a commensurate massless phase with $c_\perp=2$. Both spin and orbital degrees of freedom are massless and the total central charge is $c=c_s+c_\perp=3$.

Notice that there will be three different velocities: $v_6^* \neq v_\phi^* \neq v_t^*$ so that the symmetry at the fixed point is not SO(6) but rather $SO(3)\otimes U(1)\otimes Z_2$.

When there is no gap at zero field the spin sector is always critical with incommensurate correlation functions. What happens for the spin-orbital and orbital degrees of freedom depends again strongly on the anisotropy. When $G_2<0$ they remain massless and the total central charge is thus $c=3$. However, if $G_2>0$, there will be a KT type phase transition at a critical field h_{c0} to a *massive* phase with approximate SO(4) symmetry.

We stress that the mechanism that leads to the KT type phase transition at the magnetic field h_{c0} is highly nontrivial since the magnetic field does *not* couple directly to both the orbital and the spin-orbital degrees of freedom.

V. CONCLUSIONS

In the present work we have studied the effect of symmetry-breaking perturbations in the one-dimensional SU(4) spin-orbital model. Using the low energy effective field theory developed in Ref. 14, we have investigated the phase diagram of the $SU(2)\otimes SU(2)$ model where the exchange in both the spin (J_1) and the orbital (J_2) sectors are different. We found that the different phases of the symmetric $J_1=J_2$ line extend to the case $J_1\neq J_2$. In particular the massless phase, governed by the $SO(6)_1$ fixed point, extends to a finite region in the plane (J_1,J_2) around the SU(4) point $(J_1=K/4, J_2=K/4)$. Similarly, in the vicinity of the critical surface, the massive phase has also an approximate SO(6) symmetry provided the anisotropy is not too large. In this phase, as in the isotropic case, the system spontaneously breaks translational invariance and dimerizes with alternate spin and orbital singlets.[14] Both spin and orbital excitation are coherent at wave vector $k=\pi/2$. All these results remain valid in the vicinity of the SU(4) point. The question that naturally arises is what happens when K decreases. Indeed, in the limit $K\ll J_{(1,2)}$ one enters in the weak coupling limit where magnon excitations are incoherent at wave vector $k=\pi$.[7] In the simplest scenario, as discussed in Ref. 14, one expects that the coherent peak at $k=\pi/2$ in the dynamical susceptibility will disapear at a critical value of $K=K_D$. Such a special point where an oscillating component of the correlation function disappears is a disorder point[29] and therefore, we do not expect a phase transition at K_D but rather a smooth crossover.

Although these results could have been anticipated on the basis of the previous study of the symmetric case,[14] since the interactions are marginal, the anisotropy reveals itself in a nontrivial scaling of the physical quantities. Indeed, we have shown that the anisotropy plays its tricks in two particular regions of the phase diagram with $G_1G_2<0$, where $G_{(1,2)}=J_{(1,2)}-K/4$ measures the departure from the SU(4) point. In these regions, both spin and orbital degrees of freedom compete. For instance, when $G_1>0$ and $G_2<0$, the spin sector tends to open a gap while the orbital one wants to remain massless. Since both sectors interact marginally, at issue is a delicate balance between the strength of the interactions: it is one kind of degree of freedom that drives the other to its favorite behavior. In particular, this is the reason why the massless phase extends in the region with either $G_1>0$ or $G_2>0$. In any case, the whole system ultimately becomes either fully massless or fully massive. The crucial point is that since the interactions are only marginal, it may take a very long time, in the renormalization group sense, to reach the asymptotic low energy regime. This has important consequences.

First of all is the nontrivial behavior of the mass gap of the system. We found that the gap M is generically smaller in the regions with large anisotropy, i.e., in the two quadrants $G_1G_2<0$ above the critical curve Σ. This is due precisely to the strong tendency to massless behavior in these regions. As a consequence the gap M has two qualitatively different scaling behaviors as one approaches Σ either from the symmetric region or the asymmetric one (the trajectories labeled γ_1 and γ_2 in Fig. 2): $M\sim\exp(-C_1/\Delta^{2/3})$ and $M\sim\exp(-C_2/\Delta)$.

Second is the finite size scaling. Since the gap opens exponentially it is very difficult to localize accurately the critical line Σ in a finite system. In the current model the situation is even more askward in the regions $G_1G_2<0$. Indeed, in a finite system of size L, the critical region will seem to extend and the pseudocritical lines will be given by the two iso-t_{01} and iso-t_{02} curves, with $t_{0(1,2)}=\ln L$, that have the opposite curvature than the true critical line Σ (see Fig. 4.). In this pseudocritical region, either the spin or the orbital degrees of freedom will look massless. The phase diagram in zero magnetic field as obtained by very recent DMRG calculations[10,30] is in qualitative agreement with our RG analysis. However, the critical line obtained in these numerical computations has the opposite curvature that the one loop result Σ. Our interpretation of this fact is that what has been observed are the two iso-$t_{0(1,2)}$ curves. This reflects once again the nontrivial finite size scaling induced by the anisotropy.

Finally, is the effect of a magnetic field. The magnetic field affects the spin degrees of freedom in the usual way. In the massless phase it leads to incommensuration in the spin sector while when a gap is present, a commensurate-incommensurate transition can occur at a critical field. However, what happens to the remaining degrees of freedom strongly depends on the anisotropy. In the region, where both degrees of freedom do not compete, i.e., when $G_1G_2>0$, the remaining orbital and spin-orbital sector remains either massless or massive with an approximate SO(4) symmetry. On the other hand, the most stricking effect occurs when the spin and orbital fluctuations compete, i.e., for $G_1G_2<0$. In this region the field reinforces the effect of the orbital degrees of freedom and can induce a second phase transition, of the KT type, for a sufficiently large field, from massive to massless approximate SO(4) behavior. The origin of this nontrivial effect of the magnetic field stems from the interplay of the presence of orbital degeneracy and anisotropy. We hope that this transition will be observed in further experiments on quasi-one-dimensional spin gapped materials with orbital degeneracy.

Note added. After this work was completed, we became aware of a work by Itoi *et al.*[30] who also predict the extension of the massless phase in the anisotropic region.

APPENDIX

In this appendix, we shall compute the mass gap M in the phase C, and obtain the asymptotics of M in the vicinity of the critical surface Σ. As well known, within perturbation theory the gap defines the scale $t_M = \ln(1/Ma_0)$ where all the couplings blow up. Clearly, t_M is given by the equation

$$X(t_M) = \infty. \tag{A1}$$

Integrating Eq. (37) and recalling the dynamic of X in the phase C we find

$$t_M = \left(\int_{X^\star}^{\infty} + \epsilon \int_{1}^{X^\star} \right) \frac{dX}{X\sqrt{P(X)}}, \tag{A2}$$

where $\epsilon = \text{sign}(s)$; $P(X)$ is given by Eq. (38) and has only two reals roots 0 and $X^\star \leqslant 1$. In the following s, μ and G_3 have to be understood as initial conditions.

Performing the integrals we obtain

$$t_M = \sqrt{\frac{p}{|\mu|}} [E(\alpha(0),k) - \epsilon E(\alpha(1),k)] + \frac{1}{2}\sqrt{\frac{p}{|\mu|}} \left(\frac{u^\star}{p} - 1 \right) (F[\alpha(0),k] - \epsilon F[\alpha(1),k]) - \frac{G_3}{\mu} \left[\frac{1}{p+u^\star} - \frac{s}{2|G_3|(p+u^\star-1)} \right], \tag{A3}$$

where F and E the elliptic functions of the first and the second kind, respectively, with parameters

$$\alpha(u) = 2 \arctan \sqrt{\frac{u^\star - u}{p}},$$

$$k = \sqrt{\frac{p + u^\star - G_3^2/2\mu u^{\star 2}}{2p}}, \tag{A4}$$

$u^\star$ and p^2 being given by

$$u^\star = 1/X^\star,$$

$$p^2 = u^{\star 2} - \frac{2G_3^2}{\mu u^\star}. \tag{A5}$$

Developing t_M around a point (G_{1c}, G_{2c}, G_{3c}) belonging to the critical surface Σ between the phases B and C (see Fig. 2), we obtain the asymptotics

$$M \sim \Lambda \exp(-C(\gamma_1)/\Delta^{2/3}) \quad \text{if} \quad G_1^c = G_2^c = 0,$$

$$M \sim \Lambda \exp - C(\gamma_2)/\Delta \quad \text{if} \quad G_1^c \neq G_2^c, \tag{A6}$$

where Δ is the Euclidian distance from Σ. The two constants $C(\gamma_1)$ and $C(\gamma_2)$ are given by

$$C(\gamma_1) = 0.6845 \;\; G_{3c}^{-1/3} (\cos\theta)^{-2/3},$$

$$C(\gamma_2) = ([1 + 2G_{3c}^2/(G_{1c} - G_{2c})^2] \cos^2\theta)^{-1/2}, \tag{A7}$$

where θ is the angle to the normal of Σ.

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