Quasiparticle-phonon downconversion in nonequilibrium superconductors

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We have developed a theory of quasiparticle and phonon energy downconversion in nonequilibrium superconductors following the absorption of an energetic photon. This stage of energy downconversion cascade is important for the production of quasiparticles and is shown to split into two phases. The first is controlled by the evolution of the phonon distribution while the second is dominated by quasiparticle downconversion. The relative durations of the two phases and hence the rates of quasiparticle generation depend on material parameters, and most common superconductors could be classified into three different groups. For typical superconductors used for x-ray detection the downconversion cascade was shown to be fast compared to various time scales in the tunneling regime.

I. INTRODUCTION

The energy downconversion in a superconductor following the absorption of an elementary particle or photon has been the focus of a number of studies. It plays a central role in the process of particle or photon detection by a superconducting tunnel junction (STJ), determining the rate of production of mobile charges (quasielectrons and quasiholes) which tunnel in the biased STJ to produce the measured signal.[1–3]

It is generally accepted that energy downconversion occurs in three distinct stages. The first stage starts as the particle energy E_0 is released in the form of a fast photoelectron. At this point the electrons and holes possess large energy and the downconversion process is dominated by strong electron-electron interactions. Two parallel channels of downconversion are secondary ionization and cascade plasmon emission, which have approximately equal cross sections and are extremely fast. For example, the scale of the emission of a plasmon of typical energy of 15–20 eV can be estimated with the use of the Ferrell formula[4] to be of the order of 0.1 fs. Even a photoelectron of energy ~ 10 keV will decay into plasmons and secondary electrons in less than 0.1 ps. Plasmons are unstable and rapidly decay into electron-hole pairs[5] resulting in strongly interacting electrons and holes which thermalize to a characteristic energy E_1 defining the end of the first stage.

There is considerable disagreement over the actual value of E_1. For instance, Van Vechten and Wood[1] suggested that E_1 was reached when the primary photoelectron has lost enough energy to become indistinguishable from the other electrons in the conduction band, that is, $E_1 \simeq 1$ eV. On the other hand, Ovchinnikov and Kresin[6] recently defined $E_1 \simeq \Omega_D$, where Ω_D is the Debye energy.

The second stage of energy downconversion takes the nonequilibrium distribution of electrons and holes down to a second characteristic energy E_2. Over this stage the electron-phonon scattering becomes stronger than the electron-electron and the energy downconversion process releases a large number of phonons. The definition of the end of this stage also has previously been ambiguous. Van Vechten and Wood define it as the time when the excitation energy ϵ degrades to a few meV, while Ovchinnikov and Kresin take E_2 to be of the order of a few Δ.

Finally, over the third stage, $E_2 > \epsilon > \Delta$, the mixed distribution of quasiparticles and phonons, which remains strongly nonequilibrium, evolves to a quasiparticle distribution centered at the superconducting edge. At the same time phonons may be lost from the superconducting film into the substrate or downconverted in amorphous cap layers. This third stage the system may be regarded as the operational stage of the process. It lasts much longer than the complete duration of all the preceding cascade stages which do not last more than few nanoseconds for a Nb- or Ta-based STJ. During the third stage the nonequilibrium quasiparticles can also take part in various transport processes; they may diffuse, tunnel, recombine, be trapped and detrapped, cooled, or heated. It is essentially this stage that determines the form of the STJ output. The most commonly used approach to modelling the operational stage is via the Rothwarf-Taylor equations.[7–9]

We note that, in spite of broad agreement on the physical picture of all three downconversion stages, the transitional energies E_1 and E_2 have not been uniformly defined by previous workers, and their exact meaning not clearly discussed. As a result published estimates of the durations of the various stages differ by orders of magnitude. In this paper we shall discuss the physical processes that determine E_1 and E_2 and propose clear and consistent definitions.

The main objective of our paper is to develop a full analytical theory of the second stage $E_1 \rightarrow E_2$ of energy downconversion in a nonequilibrium superconductor. The importance of the second stage is that it controls quasiparticle generation. An exact modelling of this stage of energy downconversion has direct implications for the development of existing and of the next generation lower gap STJ detectors.[10] Various groups have attempted to solve the problem of quasiparticle production in superconductors in the

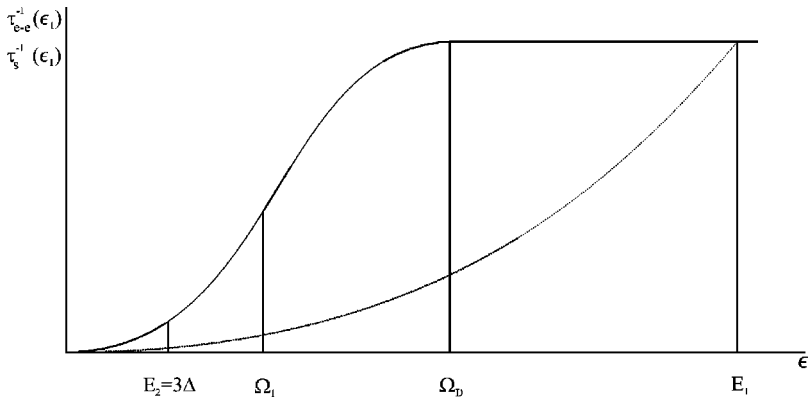

FIG. 1. Electron-electron (solid line) and electron-phonon (dashed line) scattering rates versus quasiparticle energy in typical metal. Arrows indicate the two major spectral intervals of the second stage of the energy downconversion cascade.

course of energy downconversion process using Monte Carlo techniques. The important work[6] represents an attempt at a quantitative description of quasiparticle production during the $E_1 \rightarrow E_2$ stage. The fundamental assumption in this work is that phonons instantly respond to variations in electronic distribution. Having also assumed that the electronic distribution takes the form of a step function the authors then find model solution predicting the growth of quasiparticle number as $t^{1/3}$. Our own work reported in this paper is a more general treatment of the problem over a range of validity which does not have the limitations assumed in Ref. 6.

We will start with a discussion of the physical origin and formal definition of the characteristic energies E_1 and E_2. In Sec. II we will show that the determination of E_1 is achieved by setting the rate of emission of acoustic phonons at this energy to be equal to the rate of electron-electron collisions between the energetic electron and the rest of equilibrium Fermi distribution. We find that this energy is material dependent and is typically much larger than the Debye energy. As a result, the quasiparticles undergo cascade, that is sequential emission of several tens of Debye phonons before reaching energy Ω_D. Our definition of the low transitional energy $E_2=3\Delta$ recognizes the fact that the generation of excess quasiparticles stops when the mean quasiparticle energy reaches the threshold for production of 2Δ phonons. In Sec. III we introduce the general kinetic treatment and discuss the main approximations. We shall find it necessary to introduce another important characteristic energy, Ω_1, which is the energy at which the rate of quasiparticle relaxation with spontaneous emission of phonons becomes equal to the phonon pair breaking rate. As a result in the range $\Omega_D > \epsilon > \Omega_1$ the electron distribution instantly accommodates itself to match the varying phonon distribution so that the downconversion process is controlled by the evolution of the phonon distribution $N(\epsilon,t)$.

The analytic solution of the coupled kinetic equations for interacting quasiparticles and phonons describing energy downconversion in the $\Omega_D \rightarrow \Omega_1$ range is given in Sec. IV. Below Ω_1 the evolution of the whole system enters the regime where all temporal variations of quasiparticle and phonon distributions are controlled by the electronic component. In Sec. V we obtain exact analytic solutions for the integral equation describing the downconversion process in the $\Omega_1 \rightarrow E_2$ phase. The rate of quasiparticle production during this phase calculated with an exact distribution function is found to follow a $t^{1/3}$ law as in Ref. 6. However, the exact distribution function differs significantly from the model steplike solution of Ref. 6. Indeed, we find that in a majority of superconductors this final phase is underdeveloped or absent because Ω_1 falls very close to Δ. Section VI contains a general discussion of the relative importance of the above phases in different superconductors. We show that the calculated duration of the phonon and electronic downconversion phases for all superconductors fall into three distinct classes. Finally, Sec. VII contains a summary of our results.

II. THE ELECTRON-PHONON DOWNCONVERSION PHASE $E_1 \rightarrow \Omega_D$

Van Vechten and Wood[1] define the end of the first stage of downconversion to occur when the electron energy has degraded to ~ 1 eV. In Ref. 6 the transition energy is defined differently as $E_1 \simeq \Omega_D$ with reference to the dominance of electron-phonon scattering. The lack of a formal definition for the transition energy creates ambiguity not only in the classification of downconversion stages but also in separating clearly the different kinetic processes. We propose that the most physical definition can be made on the basis of the relative strengths of electron-electron and electron-phonon scattering. Thus we define stage one of the general cascade as that in which dominant electron-electron interactions establish a strongly nonequilibrium hot electron-hole distribution continuously decaying until a characteristic energy E_1 is reached. Below this energy, with further thermalization the electron-electron scattering rate $\tau_{ee}^{-1}(\epsilon)$ becomes slower and electron-phonon scattering with the rate $\tau_s^{-1}(\epsilon)$ takes over. Thus at E_1 we have $\tau_{ee}^{-1}(E_1)=\tau_s^{-1}(E_1)$. This equation has two different solutions, one in the high-energy range, another very close (much closer than E_2) to the superconducting gap. The reason is that the electron-phonon scattering rate is approximately a cubic function of energy in the region below the Debye energy, and is nearly constant above it when the full phonon spectrum is accessible for phonon emission. On the other hand, the electron-electron scattering rate is a quadratic function of energy, thus crossing the electron-phonon relaxation rate curve twice. These solutions are illustrated in Fig. 1.

Using $\tau_s^{-1}=\frac{1}{3}\lambda\Omega_D(\epsilon/\Omega_D)^3$ for $\epsilon<\Omega_D$ and $\tau_s^{-1}=\frac{1}{3}\lambda\Omega_D$ for $\epsilon>\Omega_D$ and the Landau-Pomeranchuk formula for the electron-electron collision rate $\tau_{ee}^{-1}(\epsilon)=(\epsilon^2/\hbar\epsilon_F)(r_s^{1/2}/7.96)$,[11] we obtain

$$E_1 = 2.82\Omega_D r_s^{-1/4}\left(\frac{\lambda}{3}\frac{\epsilon_F}{\Omega_D}\right)^{1/2}. \tag{1}$$

TABLE I. The numbers in this table were calculated using the characteristic values for τ_0 and τ_{ph} from Ref. 12. For Mo and Hf we calculated these parameters from the data in Ref. 13 assuming the renormalization parameter $Z_1(0)=2$.

Metal	Δ meV	ϵ_F eV	Ω_D meV	E_1/Ω_D	Ω_1/Δ	τ_0 ns	$\tau_{ph}(\tau_{ph,D})$ ps	τ_s^* fs	τ_1 ps
Nb	1.5	6.18	23.7	49	3.5	0.149	4.2 (0.8)	16.7	0.8
Ta	0.7	9.5	20.7	47	4.7	1.78	22. (2.4)	34.6	1.6
Al	0.17	11.63	36.9	67	10.6	110	242 (3.5)	7.1	0.5
Tl	0.37	9.46	6.7	71	2.5	1.76	205 (34.9)	142	10.1
Hg	0.82	8.29	6.2	130	1.0	0.075	135 (52.0)	44.4	5.8
Sn	0.57	10.03	17.2	51	3.2	2.30	110 (11.4)	45.7	2.3
In	0.52	8.60	9.3	68	2.2	0.80	169 (29.1)	75.0	5.1
Pb	1.36	9.37	9.0	45	2.1	0.20	34.0 (14.7)	196	8.8
Zn	0.12	9.39	28.2	32	9.8	780	2310 (30.9)	44.7	1.4
Mo	0.139	9.32	39.6	23	26.8	1.64 μs	420 (4.6)	39	0.9
Hf	0.019	7.32	21.7	27	85.2	217 μs	5200 (14.7)	85	2.3

Here the parameter r_s is the radius of a sphere in atomic units which encloses one electron charge, and λ is the dimensionless electron-phonon coupling strength of the order of unity. In superconductors the electron-phonon scattering strength is very often conveniently expressed in terms of a characteristic parameter τ_0.[12] Since the same characteristic time will later enter the duration of the downconversion cascade and because it has been tabulated for number of traditional superconductors we note the relation between λ and τ_0: $\lambda=(1/\Omega_D\tau_0)(\Omega_D/k_BT_c)^3$. In the formal definition of τ_0 the product $\tau_0(k_BT_c)^3$ does not depend on critical temperature, but only on the electron-phonon coupling strength, and therefore is a material parameter of the normal metal. The parameter under the square root in Eq. (1) is much larger than unity for all normal metals leading to the strong inequality $E_1 \gg \Omega_D$. In all cases $E_1 \ll \epsilon_F$. Another important observation is that at $\epsilon=E_1$ the electron-electron collisions are so fast that there is no need to consider the much slower elastic scattering due to the presence of disorder. Hence the electron-electron collisions can be treated as in an ideal Fermi gas and using the Landau-Pomeranchuk formula is fully justified.

As downconversion enters the second stage the electrons start emitting high-frequency phonons with characteristic energy Ω_D in a long cascade containing several tens of phonons, until they reach into the last spectral interval below Ω_D. Each cascade step takes time $\tau_s^*=\tau_s(E_1)$, and the $E_1 \rightarrow \Omega_D$ cascade takes in all a time $\tau_1 \simeq (3/\lambda\Omega_D)(E_1/\Omega_D)$. Thus at the end of the cascade the energy of the phonon distribution exceeds that of the electronic distribution by a large factor, which is the number of steps in the $E_1 \rightarrow \Omega_D$ phonon emission cascade. Hence the initial state for the next phase is a narrow phonon distribution peaked at around Debye frequency, the so-called ''phonon bubble.'' This occurs because the lifetime of Debye phonon $\tau_{ph,D} \equiv \tau_{ph}(\Omega_D)$ is longer than the duration of $E_1 \rightarrow \Omega_D$ cascade, i.e., $\tau_{ph,D} > \tau_1$.

The characteristic parameters for some important superconductors supporting this picture are given in Table I. Note that the phonon bubble model is valid for most of the metals.

III. KINETICS OF THE QUASIPARTICLE-PHONON $\Omega_D \rightarrow E_2$ CASCADE

To analyze the behavior of our system during the $\Omega_D \rightarrow E_2$ downconversion phase we adopt a kinetic equation approach for interacting quasiparticles and phonons.[14–18] The system of coupled kinetic equations for interacting quasiparticles and phonons has the form

$$\frac{\partial n(\xi,\vec{x},t)}{\partial t} - D\Delta n(\xi,\vec{x},t) = I_{ep}(n), \tag{2}$$

$$\frac{\partial N(\omega,\vec{x},t)}{\partial t} = I_d(N) + I_{pe}(N) + Q(\omega,\vec{x},t). \tag{3}$$

Here $n(\xi,\vec{x},t)$ and $N(\omega,\vec{x},t)$ are distribution functions for quasiparticles and phonons, respectively, depending on quasiparticle energy $\epsilon=\sqrt{\xi^2+\Delta^2}$ and phonon energy ω and time. D is the quasiparticle diffusion coefficient, $I_{ep}(n)$, $I_d(N)$, and $I_{pe}(N)$ are the collision integrals describing, respectively, collisions between quasiparticles and phonons, phonon loss into the substrate, and collisions between phonons and quasiparticles. Q is the source term which following the discussions in the preceding section we chose in the form $Q(\omega,\vec{x},t)=aN(\omega,0)\delta(\vec{x})\delta(t)$ with the constant α normalized to the energy of the incident photon: $a^{-1}E_0=\int_0^{\Omega_D} d\epsilon\rho(\epsilon)\epsilon N(\epsilon,0)$, $\rho(\epsilon)$ being phonon density of states. In writing Eq. (3) we ignored electron-electron collisions, which are not important during the second downconversion stage. We also neglected anharmonic interactions leading to phonon-phonon downconversion. For typical anharmonic potentials the anharmonic decay time for Debye phonons may be comparable to $\tau_{ph,D}$, the lifetime with respect to Cooper pair breaking. However, this quantity scales as ϵ^{-5} while $\tau_{ph}(\epsilon)$ varies as ϵ^{-1}. Thus Cooper pair breaking is the predominant mechanism restricting the energetic phonon lifetime through almost the whole of the phonon spectrum even when anharmonic effects are important for Debye phonons. We also disregard spatial gradient terms in the kinetic equation for phonons because the main expansion

mechanism, via the diffusion of electronic excitations, is much faster than phonon diffusion or quasidiffusion. Finally, we assume that there are no significant temporal and spatial variations of the order parameter induced by the photon absorption. Below we will derive the set of conditions under which this description is fully justified. In what follows we will be mostly interested in the total number of quasiparticles and phonons with fixed energies, rather than in their densities. Thus we will also consider the kinetic equations which have been averaged over spatial variables to eliminate diffusion terms. These equations will be written for distributions of quasiparticle and phonon numbers. The latter will be described, respectively, by functions $n(\xi,t)$ and $N(\omega,t)$ in contrast to $n(\xi,\vec{x},t)$ and $N(\omega,\vec{x},t)$ for density distributions.

To describe the $\Omega_D \rightarrow E_2$ cascade of quasiparticles we disregard modifications of the spectrum for quasiparticles in a superconductor as compared to normal metal, assuming that E_2 is large in comparison with Δ. Thus we take $\xi \simeq \epsilon$ and also replace in expressions for the collision integrals all coherence factors $1 \pm \Delta^2/\epsilon\epsilon'$ by unity. To study the effects of the cascade we need only linearized collision integrals. The nonlinearity of the collision integrals reflects the effects of self-recombination of nonequilibrium quasiparticles, finite occupancy factors for electronic states, and stimulated emission of the phonons in electron-phonon interactions. The latter is of no importance as during a cascade at a sufficiently small initial photon energy E_0 phonon population numbers remain much smaller than unity. We also assume that densities of excess quasiparticles remain small. Self-recombination for $\epsilon > E_2$ is unimportant. Indeed, linear electron-phonon transitions and electron-phonon interactions via recombination involve phonons with approximately equal energies. Hence there is no strong discrimination of these processes with respect to the strength of their cross sections. The exception to that is the third and final cascade stage $E_2 \rightarrow \Delta$ recombination, which involves phonon with $\hbar\omega > 2\Delta$, while scattering occurs with the participation of much less energetic phonons. Under these conditions strong discrimination between the two processes of electron-phonon interaction is present from the very beginning, making it necessary to take into the account the recombination processes.

Using explicit expressions for the collision integrals with all the simplifications described above we obtain the following:

$$\frac{\partial n(\epsilon,t)}{\partial t} = \lambda \left[\int_{\epsilon}^{\Omega_D} \frac{d\epsilon'(\epsilon'-\epsilon)^2}{\Omega_D^2} n(\epsilon',t) - \frac{\epsilon^3}{3\Omega_D^2} n(\epsilon,t) + \int_{\epsilon}^{\Omega_D} \frac{d\epsilon' \epsilon'^2}{\Omega_D^2} N(\epsilon',t) \right], \tag{4}$$

$$\frac{\partial N(\epsilon,t)}{\partial t} = \lambda_1 \left[-\epsilon N(\epsilon,t) + 2\int_{\epsilon}^{\Omega_D} d\epsilon' n(\epsilon',t) \right] + Q(\epsilon,t). \tag{5}$$

We have introduced $\lambda_1 = \lambda \nu_e / 2\nu_{ph}$, the dimensionless electron-phonon coupling constant ($\hbar = 1$) defining the rate of phonon-electron interactions: $1/\tau_{ph}(\epsilon) = \lambda_1 \epsilon$. Here ν_e is the electron density of states at the Fermi level in the normal metal, while ν_{ph} is the phonon density of states at the Debye energy in the Debye approximation $\nu_{ph} = 3\Omega_D^2/2\pi^2 c^3$ (c is the mean sound velocity). In a superconductor, this rate coincides with the phonon pair breaking rate. Very often this expression is written in terms of a characteristic phonon pair breaking time for a given superconductor τ_{ph}.[12] For $\epsilon \gg \Delta$ one obtains $1/\tau_{ph}(\epsilon) = (1/\pi)(1/\tau_{ph})(\epsilon/\Delta)$, relating λ_1 to τ_{ph}. The relation of λ to another material parameter τ_0 was noted earlier. The dependence of the quasiparticle scattering time $\tau_s(\epsilon)$ on energy is much stronger than that of the phonon pair breaking rate, which has very important implications. We define the energy Ω_1 as that at which the relaxation rate of electrons via emission of phonons is equal to that via phonon pair breaking, that is, $\frac{1}{3}\lambda(\epsilon^3/\Omega_D^2) = \lambda_1 \epsilon|_{\epsilon=\Omega_1}$, i.e., $\Omega_1 = \Omega_D \sqrt{3\lambda_1/\lambda} = \Omega_D \sqrt{3\nu_e/2\nu_{ph}} \approx \Omega_D (p_F a_0)^2 (c/v_F) \ll \Omega_D$. Here p_F and a_0 are Fermi momentum and elementary cell length. By definition therefore Ω_1 determines the energy below which phonons create electron-hole pairs instead of undergoing any other quasiparticle scattering transitions. For all superconductors in Table I $\Delta < \Omega_1 \ll \Omega_D$ so that the $\Omega_D \rightarrow E_2$ cascade spectral region splits into two parts: $(\Omega_D \rightarrow \Omega_1)$ and $(\Omega_1 \rightarrow E_2)$. As we will see below, the kinetics of electron-phonon system in the $\Omega_D \rightarrow \Omega_1$ cascade is very different to that in the $\Omega_1 \rightarrow E_2$ cascade, so that they must be consided separately. Thus we treat what has generally been called the second stage of fast electron-phonon cascade $E_1 \rightarrow E_2$ as three separate phases:

(i) the phase $E_1 \rightarrow \Omega_D$ finishes with the formation of a phonon bubble, which sets the initial phonon distribution source term $Q(\epsilon,\vec{x},t)$;

(ii) the phase $\Omega_D \rightarrow \Omega_1$ (the phonon downconversion phase), in which the kinetics of the system of interacting quasiparticles and phonons are fully controlled by the slowly varying phonon distribution, while the quasiparticle distribution readjusts itself swiftly to the local phonon distribution;

(iii) the final phase $\Omega_1 \rightarrow E_2$ (the electronic downconversion phase) in which the quasiparticle distribution slowly changes with the phonons following almost instantly. The importance of the magnitude of the ratio of rates of electronic relaxation to electron-hole pair production was stressed in Refs. 19 and 20.

Figure 1 schematically illustrates our choice of transition energies for the three different stages of the energy downconversion cascade.

IV. PHONON DOWNCONVERSION PHASE: $\Omega_D \rightarrow \Omega_1$

The phonon downconversion phase begins from the phonon bubble, the distribution of energetic phonons $N(\epsilon,0)$ centered at around the Debye energy. Because of very fast electronic transitions $\tau_s(\epsilon) \ll \tau_{ph}(\epsilon)$ the electron distribution rapidly accommodates itself to the slowly varying phonon distribution. Thus to solve Eqs. (5) for this region we may drop the time derivative of the quasiparticle distribution function. As a result the remaining integral equation for the quasiparticle distribution function can be converted by triple differentiation with respect to energy into a third-order linear differential equation. Although cumbersome, the solution of this equation is exact:

$$\int_{\epsilon}^{\Omega_D} d\epsilon' n(\epsilon',t) = \frac{6}{11}\int_{\epsilon}^{\Omega_D} d\epsilon' N(\epsilon',t)\left\{\left(\frac{\epsilon'}{\epsilon}\right)^3 - \mathrm{Re}\left[\left(1-\frac{5i\sqrt{2}}{4}\right)\left(\frac{\epsilon'}{\epsilon}\right)^{-i\sqrt{2}}\right]\right\}. \quad (6)$$

The quasiparticle distribution function can easily be obtained from Eq. (6) by differentiation. In deriving Eq. (6) we neglect the exponentially small terms of the order of $e^{-\lambda_1\Omega_D t}$. In what follows we will consider the evolution of the electron-phonon system at times exceeding the lifetime of the Debye phonon $t > \tau_{ph}(\Omega_D)$. Substitution of this result into the second of Eqs. (5) yields

$$\frac{\partial N(\epsilon,t)}{\partial t} + \lambda_1 \epsilon N(\epsilon,\vec{x},t) - \frac{12}{11}\lambda_1 \int_{\epsilon}^{\Omega_D} d\epsilon' N(\epsilon',t) \quad (7)$$

$$\times\left\{\left(\frac{\epsilon'}{\epsilon}\right)^3 - \mathrm{Re}\left[\left(1-\frac{5i\sqrt{2}}{4}\right)\left(\frac{\epsilon'}{\epsilon}\right)^{-i\sqrt{2}}\right]\right\} \quad (8)$$

$$= Q(\epsilon,\vec{x},t). \quad (9)$$

We begin discussing the properties of the solution of Eq. (9) by going beyond the limits of its formal applicability allowing $\Omega_1 \to 0$. Multiplying this equation by the phonon density of states times phonon energy and integrating over the phonon spectrum, after simple transformations, results in

$$\frac{\partial E_{ph}}{\partial t} = 0, \quad (10)$$

that is, the energy of the phonon system E_{ph} is conserved during the evolution process. The physical meaning of this result is that electron system having absorbed a single quantum of energy from the phonon system instantaneously returns it back in the form of two quanta. The net result is that electronic excitations act as mediating agents modifying the phonon spectral distribution and leaving the energy of the phonon system constant.

This kinetic equation for the phonon distribution function is similar to the integral equations for the phonon distribution discussed in detail for various regimes of phonon downconversion in dielectrics by.[21–23] The important difference is that in superconductors and normal metals splitting of initial phonon into two phonons of lower frequencies takes place through the mediation of quasiparticles, as the initial phonon is absorbed to create a pair of quasiparticles. The quasiparticles then thermalize with the emission of the two phonons. The kernel of our integral equation is determined by the specific features of electron-phonon interaction in metals, and is totally different from the anharmonic downconversion mechanism.

Taking Ω_1 and Δ to be finite involves also qualitative differences in comparison with an anharmonic downconversion cascade. The phonon system becomes open and phonon energy conservation no longer holds true. The reason is that the breaking of a Cooper pair creates two quasiparticles whose residual energy after thermalization cannot be smaller than 2Δ. In addition, thermalization may not take the quasiparticle exactly to the superconducting edge but to some energy below Ω_1 where the quasiparticle lifetime is long. This process contributes to an extra energy loss from the phonon system on a time scale comparable to that for the $\Omega_D \to E_2$ phase. However, for most of the $\Omega_D \to \Omega_1$ phase the energy loss from the phonon system is small, and the energy in the phonon system decreases slowly until the average phonon energy reaches Ω_1.

In order to solve the integral Eq. (9) we shall exploit the slow time variation of the phonon energy, expanding it into a Taylor series, with small parameter being $\lambda_1\Omega_1 t \ll 1$. The strong inequality means that we restrict our treatment of the $\Omega_D \to \Omega_1$ phase to times smaller than $1/\lambda_1\Omega_1$, the lifetime of a phonon at the threshold energy Ω_1. This is exactly the time scale of interest since under any other circumstances there are no excitations left with energies above Ω_1.

We first analyze the energy partition between the electrons and phonons. Using the solution for quasiparticle distribution (6) and keeping only the major terms we find

$$E_{el}^{>}(t) = \nu_e \int_{\Omega_1}^{\Omega_D} d\epsilon \epsilon n(\epsilon,t) \simeq \frac{9}{11}\frac{\nu_e}{\Omega_1^2}\int_{\Omega_1}^{\Omega_D} d\epsilon \epsilon^3 N(\epsilon,t). \quad (11)$$

The expression for the total energy in the phonon system, E_{ph}, in the Debye approximation contains the same integral so that

$$\frac{E_{el}^{>}}{E_{ph}} = \frac{9\nu_e}{11\nu_{ph}}\frac{\Omega_D^2}{\Omega_1^2} = \frac{6}{11}, \quad (12)$$

where $E_{el}^{>}$ is the energy of the quasiparticle distribution above the threshold energy Ω_1. We stress that the singular behavior of the electronic distribution $n(\epsilon) \sim \epsilon^{-4}$ resulting from our solution given by formula (6), does not allow the integration to be taken over the whole energy range, because the integral is divergent at lower integration limit. The meaning of this result is simple: phonons are in control over the electronic distribution only above the threshold energy Ω_1 and here our solution is valid. Below this energy quasiparticle occupation numbers will always remain finite showing no singularity (see Sec. V). The continuity of the electronic distribution across Ω_1 together with the fact that $n(\epsilon)$ remains finite within the range $0 < \epsilon < \Omega_1$ means that the electronic energy is almost equally split between the groups of low (below Ω_1) and high (above Ω_1) energy excitations.

This results suggests the following picture of the phonon control phase. Decay of the phonon bubble on the time scale of the lifetime of a Debye phonon will create a nonequilibrium distribution of phonons and quasiparticles with approximately equal energies. About three quarters of the total energy is in the form of high-energy excitations, both phonon and quasiparticles, above Ω_1. Thus the initial production of the quasiparticles is very rapid and is not described by the equation in which we have dropped all the terms containing the small exponent $e^{-\lambda_1\Omega_D t}$. In the subsequent evolution of the whole system the phonon distribution narrows as higher energy phonons decay into phonons of energy around Ω_1 while approximately conserving phonon energy. The rapid response of the electronic system brings about a corresponding transformation of the electronic distribution towards lower energies. As the mean electronic energy decreases, the total energy can be maintained due to the generation of extra

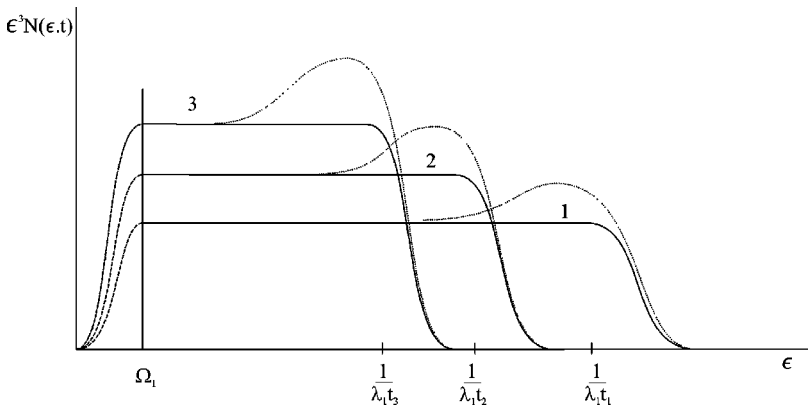

FIG. 2. Phonon energy density distributions $\epsilon^3 N(\epsilon,t)$ at different times during the phonon-control phase $\Omega_D \to \Omega_1$: $t_3 > t_2 > t_1 > \tau_{ph,D}$; 1: $t=t_1$, 2: $t=t_2$, and 3: $t=t_3$. Solid line: solution with $c_1 \neq 0, c_2 = c_3 = 0$, dashed line: extra contribution from solution with $c_2, c_3 \neq 0$.

quasiparticles with lower energies. This is the dominant effect. Next order terms will result in phonons gradually losing their energy to electrons, providing an addition source of energy to the electronic system.

To describe the evolution of the phonon system during the phonon downconversion phase we neglect the variations of its energy. The energy conserving solution can be taken in the form of a scaling solution $N(\epsilon,t) = \varphi(u)/\epsilon^4$ with $u = \lambda_1 \epsilon t = t/\tau_B(\epsilon)$. The $E_{ph} = (\nu_{ph}/\Omega_D^2) \int d\epsilon \epsilon^3 N(\epsilon,t) = (\nu_{ph}/\Omega_D^2) \int du (\varphi(u)/u)$. Substitution of the phonon distribution in this form yields the following integral equation for the unknown function $\varphi(u)$:

$$\varphi + \frac{d\varphi}{du} - \frac{12}{11} \int_u^\infty du' \frac{\varphi(u')}{u'} \left\{ 1 - \mathrm{Re} \left[\left(1 - \frac{5i\sqrt{2}}{4} \right) \times \left(\frac{u}{u'} \right)^{3+i\sqrt{2}} \right] \right\} = 0. \tag{13}$$

This integrodifferential equation can be used to derive the fourth-order linear differential equation for the unknown function $\varphi(u)$. Differentiating Eq. (13) one, two, and three times allows us to exclude the integral terms with three different kernels to obtain

$$\varphi'''' + \left(1 - \frac{3}{u} \right) \varphi''' + 3 \left(-\frac{1}{u} + \frac{2}{u^2} \right) \varphi'' + \frac{12}{u^2} \varphi' + \frac{12}{u^3} \varphi = 0. \tag{14}$$

No approximations can be made in this equation and no analytical solution exists, although the asymptotic behavior of φ can be found. As $u \to 0$ the solution has the following form (c_1, c_2, c_3, c_4 are arbitrary constants):

$$\varphi = c_1 u + c_2 u^{4+i\sqrt{2}} + c_3 u^{4-i\sqrt{2}} + c_4 \left(1 - \frac{12}{11} u \ln u \right). \tag{15}$$

Large u asymptotics can also be found. Using the equation for φ one can show that the asymptotic of any solution is $\varphi(u) = f(u) e^{-u}$, where $f(u)$ is a nondivergent function of u as $u \to \infty$. The solution with $c_4 \neq 0$ should be excluded, as it implies a large number of occupied low-energy phonon states at $t=0$, leading to phonon energy convergence. This is physically impossible after phonon bubble decay. Thus $c_4 = 0$ and depending on initial conditions the dominant term at $u \to 0$ is linear if $c_1 \neq 0$, i.e., $\varphi(u) \sim u$, or $\varphi(u) \sim u^4$ if $c_1 = 0$. Thus

$$N(\epsilon,t) \propto \epsilon^{-3} t \ (c_1 \neq 0) \ \text{ or } \ \epsilon^0 t^4 \ (c_1 = 0) \ \text{ for } \ \Omega_1 \ll \epsilon \ll \frac{1}{\lambda_1 t}, \tag{16}$$

$$N(\epsilon,t) \propto \epsilon^{-4} \exp(-\lambda_1 \epsilon t) \ \text{ for } \ \epsilon \gg \frac{1}{\lambda_1 t}. \tag{17}$$

We may view these dependences as the evolution of the phonon distribution for a given energy interval as a function of time. In this way, at any energy $\Omega_1 \ll \epsilon \ll \Omega_D$ the phonon population numbers rise linearly from zero, reach a maximum around $t \simeq 1/\lambda_1 \epsilon$ and then decay exponentially. The specific feature of the system under consideration is that, in contrast to phonon downconversion in insulators, the ϵ^{-3} ''singularity'' is dragged through to the phonon system. The phonon distribution soon after the initial instance of time starts building up around Ω_1 as formulas (17) suggest. The evolution of the phonon distribution is thus rising the occupation numbers linearly with time at small energies below the threshold energy separating the occupied from the depleted states which sweeps across the spectrum as $\sim 1/\lambda_1 t$.

This type of evolution is shown in Fig. 2. For illustration we plot the phonon energy densities $\epsilon^3 N(\epsilon,t)$ rather than the occupation numbers. The solution with $c_1 \neq 0$ yields a steplike dependence for the phonon energy density with threshold energy $\epsilon = 1/\lambda_1 t$ separating the depleted states from the occupied. While the threshold energy sweeps across the phonon spectrum the occupation numbers below it rise linearly with time so that the area below the $\epsilon^3 N(\epsilon,t)$ remains constant. Solutions with $c_2, c_3 \neq 0$ show a phonon energy density that is concentrated mostly within the group $\epsilon \approx 1/\lambda_1 t$, decreasing as ϵ^3 towards low energies. Again the area below the $\epsilon^3 N(\epsilon,t)$ curve remains constant in accordance with phonon energy conservation. The extent to which the energy in the phonon system scales as prescribed by one or other of the above solutions depends on the magnitude of the coefficients c_1, c_2, and c_3. The latter is determined by the shape of the ''initial'' phonon distribution after the decay of the phonon bubble. Intuitively [and this is confirmed by the shape of the rapidly decaying terms which we ignored while deriving the main Eq. (9)] we expect the steplike solution with $c_1 \neq 0$ to dominate.

Using the scaling solution to calculate the rate at which the average energy of phonons decreases, we obtain

$$\langle\epsilon\rangle=\frac{\int_0^{\Omega_D}d\epsilon\epsilon^3N(\epsilon,t)}{\int_{\Omega_1}^{\Omega_D}d\epsilon\epsilon^2N(\epsilon,t)}$$

$$=\Omega_1\frac{1}{\lambda_1\Omega_1t}\frac{\int_0^{\infty}duu^{-1}\varphi(u)}{\int_{\lambda_1\Omega_1t}^{\infty}duu^{-2}\varphi(u)}\sim\Omega_1\frac{1}{\lambda_1\Omega_1t}\zeta(t), \tag{18}$$

where $\zeta(t)$ is the dimensionless ratio of the two integrals of the order of unity, which may contain weak logarithmic dependence on time. It is seen from Eq. (18) that the phonon mean energy stays above the threshold energy Ω_1 in the course of the $\Omega_D\rightarrow\Omega_1$ phase, that is, for $0<t\ll\tau_B(\Omega_1)$. The duration t_I of the $\Omega_D\rightarrow\Omega_1$ phase can be estimated therefore in terms of the lifetime of a phonon at a threshold energy Ω_1. We will take $t_I\approx2\tau(\Omega_1)$, because the scaling of mean phonon energy as in Eq. (18) suggests that the phonon distribution shrinks down to Ω_1 in a characteristic time $\approx\tau(\Omega_1)$, and it takes another $\tau(\Omega_1)$ to sweep across the Ω_1 boundary into the $\Omega_1\rightarrow E_2$ spectral region.

Finally we consider the rate of quasiparticles production over the $\Omega_D\rightarrow\Omega_1$ phase. We first integrate the first of the main kinetic Eqs. (5) over energy. As a result we have an exact relation

$$\frac{dN_{qp}(t)}{dt}=2\lambda_1E_{ph}(t). \tag{19}$$

Integrating this equation yields

$$N_{qp}(t)=N_{qp}(0)+2\lambda_1\int_0^t dt'E_{ph}(t')$$

$$=N_{qp}(0)+2\frac{E_{ph}(0)}{\Omega_1}\lambda_1\Omega_1t \tag{20}$$

$$\approx\frac{E_{ph}(0)}{\Omega_1}(1+2\lambda_1\Omega_1t). \tag{21}$$

We have arrived in the last expression by estimating $N_{qp}(0)\simeq E_{el}(0)/\Omega_1\simeq E_{ph}(0)/\Omega_1$ in accordance with Eq. (12). The rate of quasiparticle production can be expressed in a form similar to Eq. (19) in the general case of BCS superconductor and arbitrary phonon density of states. This follows directly from the integration of the first of major Eqs. (3) with the most general expressions for the collision integrals in superconductor. The result is

$$\frac{dN_{qp}(t)}{dt}=\frac{2\pi^2\nu_e\Delta}{Z_1(0)}\int_{2\Delta}^{\Omega_D}d\Omega\alpha^2(\Omega)$$

$$\times\nu_{ph}(\Omega)N(\Omega,t)\frac{\tau_{ph}}{\tau_{ph}^{BCS}(\Omega)}. \tag{22}$$

Here $Z_1(0)$ is renormalization parameter, $\alpha^2(\Omega)\nu_{ph}(\Omega)$ is the electron-phonon coupling strength weighted with phonon density of states $\nu_{ph}(\Omega)$, and $\tau_{ph}^{BCS}(\Omega)$ is the phonon pair breaking time in a BCS superconductor.[12] Disregarding the weak deviation of $1/\tau_{ph}^{BCS}(\Omega)$ from linear dependence in a BCS superconductor $1/\tau_{ph}(\epsilon)=\lambda_1\epsilon$ and assuming $\alpha^2(\Omega)$ constant, we see that the quasiparticle production rate in BCS superconductors is determined by the energy in the phonon system. We will use this result in Sec. VI for the estimate of number of generated quasiparticles.

V. ELECTRONIC DOWNCONVERSION PHASE: $\Omega_1\rightarrow E_2$

The characteristic time entering the phonon downconversion phase $\Omega_D\rightarrow\Omega_1$ is the lifetime of Ω_1 phonon. The splitting of the last remaining phonons across the threshold Ω_1 brings the whole system into a totally different relaxation regime. The main feature of this regime is that all temporal variations now are controlled by slower electronic transitions while on that time scale phonons break Cooper pairs instantly. As a result we may ignore the time derivative of the phonon distribution function in Eq. (5) since it instantly accommodates itself to the slowly varying distribution of quasiparticles. Passage across the threshold Ω_1 results predominantly in the population of long-lived electronic excitations, while phonons act as mediators in quasiparticle downconversion resulting in the multiplication of their numbers. During this stage provided $\Omega_1\gg\Delta$ intensive generation of lower energy quasiparticles takes place.

To describe the evolution of the quasiparticle and phonon distributions we solve the second of Eqs. (5) for phonon distribution function and substitute the result into the first equation. The result is

$$\left[-i\omega+\frac{\lambda\epsilon^3}{3\Omega_D^2}\right]n(\epsilon,\omega)-\frac{2\lambda}{\Omega_D^2}\int_{\epsilon}^{\infty}d\epsilon'\epsilon'(\epsilon-\epsilon)n(\epsilon',\omega)$$

$$=\frac{n_0(\epsilon)}{2\pi}. \tag{23}$$

Here we introduced the initial distribution for this stage $n_0(\epsilon)$, normalized to the energy of the absorbed photon, i.e., $E_x=\nu_e\int_0^{\infty}d\epsilon\epsilon n_0(\epsilon)$ which is value of the distribution at a reference time $t=0$, after the system of quasiparticles and phonons has swept across the boundary energy Ω_1. This normalization is correct when $\Omega_1\gg\Delta$, and the energy remaining in the phonon system is small in comparison with that of quasiparticle system. Since all upconversion processes were ignored in the kinetic equations during their linearization, an assumption which is also valid for the phase $\Omega_1\rightarrow E_2$, the upper limit of the integration in Eq. (23) could be set to ∞. As with phonon downconversion phase the actual form of the initial distribution $n_0(\epsilon)$ is of importance only close to $t=0$, while for large t the solution does not significantly depend on it. Due to the special type of kernel in Eq. (23) the equation can be reduced to a second-order linear differential equation. A similar second-order differential equation for stationary tunnel injection of excess quasiparticles was solved in Ref. 24. An equation which is identical to Eq. (23) has also been analyzed in Ref. 6. In this work a model scaling solution was found in the form of a steplike function

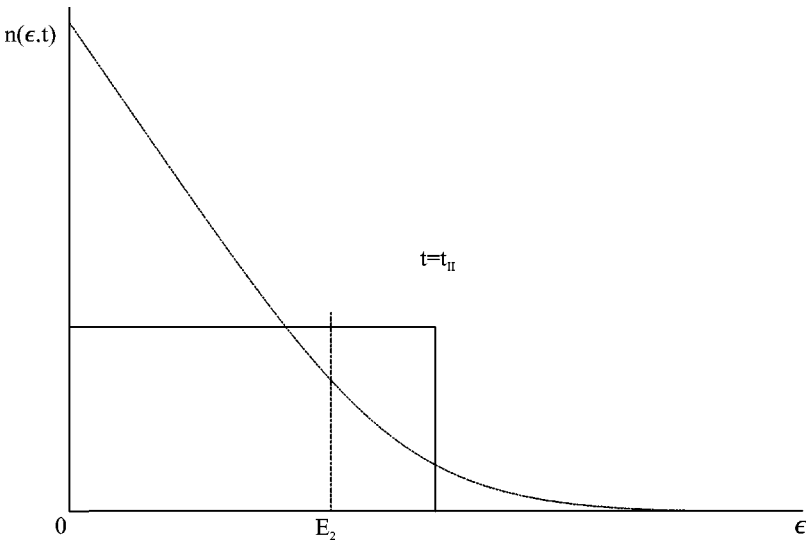

FIG. 3. 1: the exact solution given by formula (26). 2: model steplike solution of Ref. 6. Both solutions were normalized to energy and hence the areas below the curves and therefore the numbers of generated quasiparticles are different.

which gave the correct time dependence of the number of generated quasiparticles. However, the estimates of both the numbers of the generated quasiparticles and the duration of the electronic downconversion cascade cannot be derived until the boundary energy E_2 has been determined. The choice of the characteristic time scale for the second stage[6] of the order of $\Delta^{-1}(\Omega_D/\Delta)^2$ is imprecise.

The solution of Eq. (23) is straightforward:

$$n(\epsilon,t)=n_0(\epsilon)\exp\left[-\frac{\lambda\epsilon^3 t}{3\Omega_D^2}\right]+\frac{2\lambda t}{\Omega_D^2}\int_{\epsilon}^{\Omega_1}d\epsilon'\times\exp\left[-\frac{\lambda\epsilon'^3 t}{3\Omega_D^2}\right]\int_{\epsilon'}^{\Omega_1}d\epsilon''\epsilon'' n_0(\epsilon''). \quad (24)$$

The first term in this solution represents the contribution from the initial distribution of quasiparticles which exponentially dies out, so that the overall solution is not sensitive to the initial distribution at large values of t. For this reason we will not consider this contribution further. We introduce boundary energy ϵ^* separating the populated from the empty states, defined by $\tau_s(\epsilon^*)=t$, so that

$$\epsilon^*(t)=\Omega_1\left[\frac{\tau(\Omega_1)}{t}\right]^{1/3}=\Omega_D\left(\frac{3}{\lambda\Omega_D t}\right)^{1/3}. \quad (25)$$

Here $\tau(\Omega_1)=\tau_s(\Omega_1)=\tau_{ph}(\Omega_1)$. The dominant integration region over $d\epsilon$ in Eq. (24) is $\epsilon\leqslant\epsilon^*(t)$. For large t the lower limit ϵ in the second integral over $d\epsilon$ is below the range of concentration of the initial distribution $n_0(\epsilon)$ which is expected to be centered at higher energies around Ω_1. As a result $\int_{\epsilon}^{\Omega_1}d\epsilon\epsilon n_0(\epsilon)$ hardly depends on ϵ and to high accuracy can be replaced by $\int_0^{\infty}d\epsilon\epsilon n_0(\epsilon)$, which then cancels with the same integral in the denominator. The final result is then

$$n(\epsilon,t)=\frac{2E_x}{\nu_e\Omega_1^2}\left(\frac{t}{\tau(\Omega_1)}\right)^{2/3}\Gamma\left\{\frac{1}{3},[\epsilon/\epsilon^*(t)]^3\right\}, \quad (26)$$

where $\Gamma\{\frac{1}{3},[\epsilon/\epsilon^*(t)]^3\}$ is an incomplete gamma function. Formula (26) allows us to derive the variation of the total numbers of generated quasiparticles with time during the electronic downconversion phase. Thus

$$N_{qp}(t)=\nu_e\int d\epsilon n(\epsilon,t)=\frac{2E_x}{\Omega_1}\left(\frac{t}{\tau(\Omega_1)}\right)^{1/3}\Gamma\left(\frac{2}{3}\right). \quad (27)$$

Note that the reference energy Ω_1 cancels out from the result as $\tau^{-1}(\Omega_1)\sim\Omega_1^3$. The time dependence suggested by Eq. (27) is the same as that derived in Ref. 6. These authors used an approximate model solution of a kinetic equation similar to Eq. (23) with two assumptions: (i) the threshold between the populated and empty states sweeps with time as formula (25) and (ii) the energy is conserved in the electronic system. With this assumptions the correct result was obtained in Ref. 6 even though the model steplike distribution function differs substantially from the exact result, as Fig. 3 illustrates.

Using Eqs. (19) and (27) we obtain

$$E_{ph}=\frac{E_x}{6}\left(\frac{t}{\tau(\Omega_1)}\right)^{-2/3}\Gamma\left(\frac{2}{3}\right), \quad (28)$$

showing that the phonon energy is shrinking rapidly during the electronic downconversion stage: $E_{ph}\ll E_x$ for $t\gg\tau(\Omega_1)$.

Most significant, however, is the consideration of the transition energy E_2 for this phase of a cascade, which we define as $E_2=3\Delta$. This definition has simple physical meaning. The last 2Δ phonons will be emitted in the process of thermalization down to the superconducting edge by generation of quasiparticles residing at $E=3\Delta$. Once the quasiparticle distribution enters the spectral range below E_2 the production of quasiparticles stops since there are no 2Δ phonons left in the system to break extra Cooper pairs. Using this definition of E_2 we can determine the time t_{II} when this occurs. Since the distribution of quasiparticles is given by formula (26) at any later time of the $\Omega_1\rightarrow E_2$ stage, t_{II} is determined by

TABLE II. Calculated duration of phonon and quasiparticle downconversion stages in superconductors.

Metal	Tl	Hg	Sn	In	Pb	Nb	Ta	Al	Zn	Mo	Hf
t_I ps	360	270	170	320	66	6.0	26.0	140	1.4 ns	100	400
t_{II} ps	14.6	0.3	21.8	7.4	1.0	1.15	15.5	1.3 ns	10 ns	16 ns	2.1 μs

$$N_{qp}(\epsilon<E_2,t_{II})=\left(1-\frac{1}{e}\right)[N_{qp}(\epsilon<E_2,t_{II}) + 3N_{qp}(\epsilon>E_2,t_{II})] \quad \text{or} \tag{29}$$

$$N_{qp}(\epsilon<E_2,t_{II})=3(e-1)N_{qp}(\epsilon>E_2,t_{II}). \tag{30}$$

The meaning of this condition is that the number of quasiparticles with energies below E_2 reaches $(1-1/e)$ level of their ultimate number. It has been assumed that each of the quasiparticle above E_2 after thermalization emits a phonon with $\epsilon>2\Delta$, which breaks Cooper pair, thus tripling their contribution to the number of quasiparticles. Using Eq. (26) we rewrite this condition (30) in the form

$$\int_0^{E_2/\epsilon^*(t_{II})} dx\Gamma\left(\frac{1}{3},x^3\right)=3(e-1)\int_{E_2/\epsilon^*(t_{II})}^{\infty} dx\Gamma\left(\frac{1}{3},x^3\right). \tag{31}$$

The solution of Eq. (31) is $t_{II}\cong\tau_s(4.5\Delta)$, yielding the result that the $\Omega_1\rightarrow E_2$ phase takes place in a time two orders of magnitude faster than the estimate of Ref. 6.

Finally, since the phonon distribution function during the electronic downconversion phase instantly adjusts itself to match the electronic distribution we substitute it into Eq. (22) to obtain a convenient result for the general case of BCS superconductors and an arbitrary phonon spectrum:

$$\frac{dN_{qp}(t)}{dt}=2\nu_e\int_{3\Delta}^{\Omega_D} d\epsilon\rho(\epsilon)\left(\frac{1}{\tau_s(\epsilon)}-\frac{1}{\tau_s(3\Delta)}\right)n(\epsilon,t), \tag{32}$$

where $\rho(\epsilon)$ is a dimensionless BCS density of states. This formula which is a result of exact integration of the major system (3) for the electronic downconversion phase is a direct proof of our choice of the stage-2–stage-3 transition energy $E_2=3\Delta$.

VI. DISCUSSION

To analyze the types of phonon and quasiparticle downconversion processes during the second stage of energy downconversion $E_1\rightarrow E_2$ we show in Table II the calculated duration of phonon and quasiparticle downconversion stages t_I and t_{II} for different superconductors. It can be seen that the metals in the Table II fall into three different groups.

The metals of first group Tl, Hg, Sn, In, and Pb are characterized by the fact that the duration of the phonon downconversion stage significantly exceeds the duration of the potential electronic downconversion stage. The characteristic time for the whole cascade is $t_{ch}=t_I$. For these materials during the whole of the second stage the production of quasiparticles varies approximately linearly with time, thus $N_{qp}(t)\propto t$, reflecting the linear growth in phonon numbers. In each elementary event after the absorption of a high-energy phonon by the electron system an electron-hole pair is created. Following its rapid relaxation, a pair of low-energy quasiparticles is created with the remainder of the energy emitted in the form of pair of phonons. The values for t_{II} for these materials have no real meaning since 4.5Δ for all these materials is above the Ω_1 threshold. For metals of this group $\Omega_1/\Delta\sim 1$ and the electronic downconversion phase is essentially absent.

The second group contains predominantly small gap superconductors. The metals in this group are Al, Zn, Mo, and Hf. In these materials the phonon downconversion phase is fast compared to the final quasiparticle downconversion (multiplication) phase. The characteristic time for the cascade is $t_{ch}=t_I+t_{II}$. The quasiparticle numbers increase linearly during the first phase, followed by a much slower rise during the second phase. Thus $n_{qp}(t)\propto t$ for $0<t<t_I$ and $n_{qp}(t)\propto t^{1/3}$ for $t_I<t<t_{II}$. A much slower rate of quasiparticle production over this phase is determined by the corresponding rate of electronic relaxation releasing the energy in the form of mediating phonons to break extra Cooper pairs. This tendency is partly compensated by the fact that in each elementary event the number of quasiparticles triples, that is the energy of an energetic quasiparticle released in the form of a productive ($\hbar\omega>2\Delta$) phonon creates two extra quasiparticles.

It is for this group of metals that the production rate for quasiparticles numbers is described by $t^{1/3}$ law obtained in Ref. 6. However, even for these metals the characteristic time turns out to be much shorter than the estimate given in Ref. 6 (100 ns for Nb based STJ). These estimates show that the profile of the signal front due to the cascade processes is important for smallest gap materials. The whole problem of the quasiparticle production rate as a function of time in the initial stages following the absorption of a photon is crucial for the new generation of STJ's based on the use of small gap materials such as Mo and especially Hf. In these materials tunneling starts long before the quasiparticle relaxation finishes and the ultimate number of quasiparticle has been produced.

It is worth mentioning that a similar material classification with regard to hot-spot formation in superconductors has been proposed in Refs. 19 and 20, in which the figure of merit Ω_D^3/T_c^2 simply measures the ratio $(\Omega_1/\Delta)^3$. Indeed, $(\Omega_1/\Delta)^3=(\Omega_D/\Delta)^3(3\nu_e/2\nu_{ph})=(\Omega_D^3/\Delta^2)(\nu_e/6N)$, where N is number of atoms per cm^3. This ratio strongly depends on $(\Omega_D^3/\Delta^2)(\Omega_D^3/T_c^2)$, while the factor $\nu_e/6N$ is a relatively weak function of the material parameters.

Finally, the materials of the third group of widely used superconductors including Nb and Ta fall into an intermediate range where $E_2\simeq\Omega_1$ and hence the electronic downconversion phase is underdeveloped. Under these conditions the electronic downconversion cascade is short and the time dependence of the quasiparticle numbers does not approach the asymptotic $t^{1/3}$ law, which arises only for long cascades, that is $\Omega_1\gg E_2$. Instead, starting from the later phases of the pho-

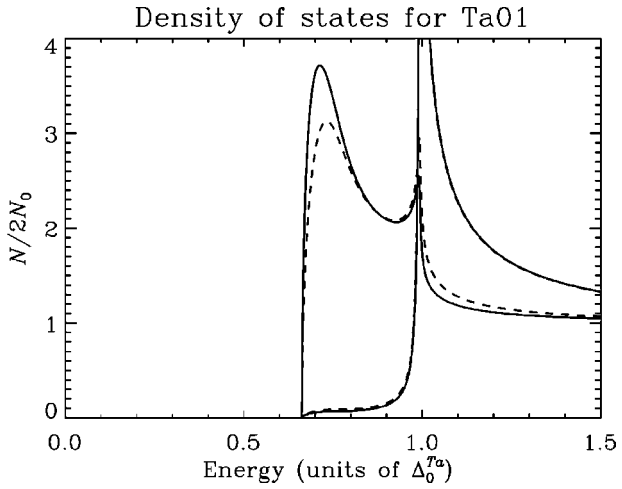

FIG. 4. Density of states versus quasiparticle energy in a Ta/Al bilayer. The solid line in Al is at the barrier, while the dashed line corresponds to the Al-Ta interface. In Ta the density of states is almost position independent.

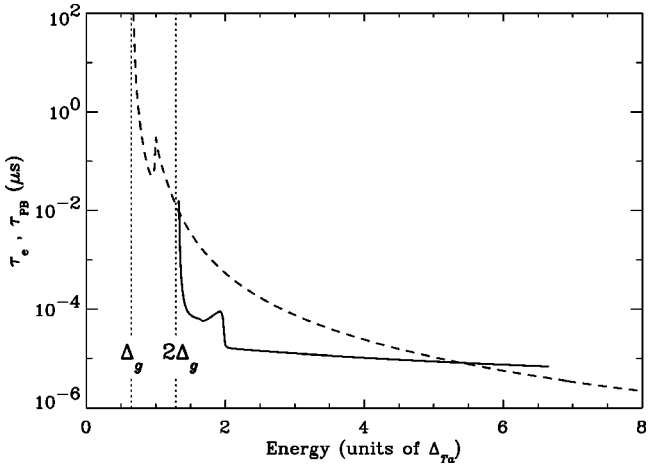

FIG. 5. Quasiparticle relaxation (dashed line) and phonon pair breaking time (solid line) versus energy in Ta01.

non downconversion $\Omega_D \rightarrow \Omega_1$ cascade the initial linear time dependence of quasiparticle numbers on time decreases in the transition range around Ω_1, but fails to reach the $t^{1/3}$ law. This group of materials includes also Nb/Al and Ta/Al proximized structures which are now being extensively studied. While electrons (holes) cascade across the high-energy range $E_1 \rightarrow \Omega_D \rightarrow \Omega_1$ spectral renormalization due to the proximity effect is unimportant. One consequence of the Nb(Ta)/Al bilayers is that part of the cascade occurs in one material and part in another. This happens because at the characteristic electron velocity of $(5-10)\times 10^7$ cm/s and for layer widths of 100 nm it takes only 100–200 fs to pass across one layer and enter the other. Thus although the absorption is more likely to occur in Nb(Ta) the excitation region spreads into the Al layer long before the $E_1 \rightarrow \Omega_D$ thermalization cascade of Debye phonons is completed. Thus the first phase $\Omega_D \rightarrow \Omega_1$ (phonon downconversion) starts from the initial distribution of phonons (phonon bubble) which is present in both layers. In what follows the phonon downconversion phase develops following the scenario which has been discussed. The only difference is that both quasiparticles and phonons can be exchanged between the two materials during this process. This can hardly bring about any qualitative changes as the rates of electron-phonon scattering and pair breaking at the same energy above the Fermi level are similar. With typical thicknesses of the films forming a bilayer of the order of few coherence lengths the superconducting gap stays constant across the bilayer.[25,26] However, densities of states for quasiparticles depend on spatial coordinates and therefore both scattering and pair breaking times must be properly averaged. Figure 4 shows the dependence of densities of states in Ta/Al device, 100 nm Ta and 55 nm Al, $\Delta_g = 0.45$ meV, at four different locations as a typical example, where Δ_g is the value of the gap in the proximized structure.

The variation of scattering and pair breaking times versus an energy for Ta/Al device Ta01 borrowed from Ref. 27 is shown on Fig. 5. As seen from Fig. 4, $\Omega_1 \approx 5.4\Delta_{Ta} \approx 8.3\Delta_g \approx 3.73$ meV, $t_I \approx 20$ ps which is very similar to the value of t_I for an unproximized Ta device, $t_{II} = \tau(4.5\Delta_g) \simeq 80$ ps. Here Δ_{Ta} is superconducting gap in the bulk Ta. This particular Ta/Al device is characterized by a relatively long electron downconversion cascade $t_{II} \approx 4t_I$ and hence it approaches the second group of materials. The Nb/Al devices and Ta/Al with different degrees of proximization all fall between unproximized Nb(Ta) and the above example. There exists one very specific feature of the proximized structures which is absent in the BCS superconductors. The Cooper pair breaking time of Fig. 4 exhibits a dramatic rise on approach to $2\Delta_g$. Above $2\Delta_{Ta}$ it reproduces closely the behavior in bulk Ta. Below $2\Delta_{Ta}$ it is intermediate between bulk Al value (240 ps) and bulk Ta value. However, when the phonon energy approaches $2\Delta_g$ the pair breaking time rapidly rises to large values. The reason for this is the shape of the density of states on Fig. 4, which smoothly goes to zero around the gap, while for BCS superconductors the infinite density of states at the edge causes the pair breaking time discontinuity at 2Δ. This feature contributes to a distinctive signature of the proximized structures with respect to electronic downconversion phase. The lifetimes of the productive phonons (in Fig. 5 below $2\Delta_{Ta}$) turn out to be comparable to or even larger than t_{II}. As a result there appears a significant extra flattening of the time dependence of the numbers of the generated quasiparticles at the very end of the $\Omega_1 \rightarrow E_2$ cascade as compared to the unproximized structure with similar t_I and t_{II}.

We will address now the important question of how much energy ε is needed to produce a single quasiparticle (although of course they are created in pairs). The number of quasiparticles ultimately generated can be expressed in the form

$$N_{qp}(t=\infty) = \frac{E_x}{\varepsilon}. \tag{33}$$

So far the choice of ε has been determined by Monte Carlo simulations giving the value $\varepsilon \simeq 1.75\Delta$[3]. From our considerations it follows that this number is not universal, because for superconductors belonging to different groups in our classification scheme the energy partition between electrons and phonon is different by the time that stage 2 finishes. We will consider the two extreme cases separately. In metals of the third group—low gap superconductors–neither

a steplike solution of Ref. 6 nor the improved formula (26) are adequate, as they are both model solutions obtained in the limit $\Delta \to 0$, disregarding the specific features of a superconductor in comparison to normal metal. Formula (32) provides the answer. Using the approximate distribution (26) in the exact formula (32) is justified as integration starts at 3Δ, where all corrections to normal-metal expressions do not exceed 5%. Using formulas (26) and (32) yields

$$\frac{\varepsilon}{\Delta} = \frac{5}{12}\left(\int_3^\infty dx \frac{(x^3-1)}{x^4\sqrt{x^2-1}} \right)^{-1} \approx 1.65. \tag{34}$$

Now let us consider the superconductors of the first group with $\Omega_1 \sim 1$. In these superconductors the electronic downconversion phase is absent. To make an estimate of ε we approximate the phonon energy density by a steplike function as shown on Fig. 2:

$$\epsilon^3 N(\epsilon,t) = \beta E_x \lambda_1 t \frac{\Omega_D^2}{\nu_{ph}} \Theta\left(\frac{1}{\lambda_1 t} - \epsilon \right), \tag{35}$$

where $\Theta(x)$ is the step function and $\beta < 1$ is the fraction of photon energy immediately after the decay of Debye phonons. Simple calculation with the use of formulas (22) and (35) yields

$$\varepsilon = \frac{1}{2\beta} \frac{\left(\frac{\Omega_1}{\Delta}\right)^2}{\frac{\Omega_1}{\Delta} - 1}. \tag{36}$$

This result shows that ε gets larger as $\Omega_1/\Delta \to 1$ and is significantly larger than for the metals of the third group. For equal energy partition between electrons and phonons at the beginning of the phonon downconversion phase, $\beta = \frac{1}{2}$ and for a typical $\Omega_1/\Delta \simeq 2$ we obtain $\varepsilon \simeq 4$. There are the two complications making this result only qualitatively correct. The first is that the partition parameter β is not a constant, but decreases as phonons lose energy; this is clear from the discussions in Sec. IV. Thus the effective value of β is even smaller making the quasiparticle yield even poorer. Second, all superconductors of the first group are ''soft'' with small values of the Debye energy and relatively small ratios Ω_D/Δ. As a result Ω_1/Ω_D falls between 0.1 and 0.13 (Tl, Hg,In) and reaches even 0.32 for Pb, forcing phonon distributions to evolve across the non-Debye part of spectrum. For all materials of this group, the phonon spectrum extends about twice as far as their Debye energy. Also a Debye approximation underestimates the phonon density of states at small energies. For all materials of the first group this effect has serious consequences as the larger phonon density of states at this energy range causes the electrons to relax faster. This in turn takes Ω_1 towards lower values as compared to the estimates in Table I. The physical meaning of the deterioration of the quasiparticle yield is simple. Indeed, the $1/\epsilon^4$ singularity in the electronic distribution formed during this phase of a cascade (see Sec. IV) is a reflection of the accumulation of quasiparticles in the states with low energy where they are subjected to less phonon scattering and spend effectively a much longer time. This effect does not significantly depend on the electronic spectrum and is equally present in a BCS superconductor and in a normal metal. This singularity in turn drags the $1/\epsilon^3$ singularity in the phonon distribution. As a result immediately after the beginning of the $\Omega_D \to \Omega_1$ phase the phonon distribution builds up around Ω_1, i.e., at low energies, with a large percentage of phonons falling below 2Δ, the smaller Ω_1 being more disastrous for the production of quasiparticles. This feature of the phonon distribution persists during the whole duration of this phase, contributing to the enhanced losses from the system in contrast to the third group materials. The result given by formula (36) is consistent with experimental observations.[19,20] This interpretation of quasiparticle yield in first group materials does not invoke the formation of the hot spot—the highly nonequilibrium phenomenon potentially occurring at strong excitation levels (large photon energy). It will still hold true at low excitation levels beyond the threshold for the formation of a hot spot or any other strongly nonlinear effects.

So far we avoided discussing the expansion of the excitation volume in the course of the downconversion cascade. For the phonon downconversion phase this was justified by referring to slow phonon group velocities. For the second electronic downconversion stage the quasiparticle diffusion can be easily accounted for. The solutions which we obtained for distribution functions (24) and (26) demonstrate that as long as the system of interacting quasiparticles and phonons remains linear the account of the diffusion is straightforward, as all temporal and spatial processes run independently. As a result expressions for the densities can be obtained by multiplying the solutions (24) and (26) by the factor $(1/4\pi^2 r_0^2 w D t) \int d\vec{r} \Theta(r_0 - r) \exp[-(\vec{r}-\vec{r})^2/4Dt]$, where the initial distribution has been taken to occupy a cylindrical volume around the absorption site of radius r_0 height equal to the electrode width w. We may use this model to derive an important condition for the downconversion process in nonequilibrium superconductors. The basic equation of BCS theory for the order parameter for the homogeneous system is

$$1 = \tilde{\lambda} \int_0^{\Omega_D} \frac{d\xi}{\epsilon} [1 - 2n(\xi)], \tag{37}$$

where $\tilde{\lambda}$ is effective electron-phonon coupling constant. When $n(\xi) \to 0$ its solution yields $\Delta \to \Delta(0)$, the value of gap at zero temperature. Excess quasiparticles can significantly suppress the gap or even locally destroy superconductivity. The nonequilibrium state of a superconductor that we consider in this paper differs from the homogeneous case and in general one should derive the equation for the order parameter accounting for the spatial variations. For a simple estimate we assume that the excited spot is confined within the cylinder with the radius $2\sqrt{Dt_{II}}$ and height w and that within this cylinder the quasiparticles distribution is homogeneous. The distribution function $n(\xi)$ can be taken as a solution of Eq. (26) at the end of the electronic downconversion phase $t = t_{ch}$, when the production of quasiparticles stops:

$$n(\xi)\approx \frac{E_x}{\nu_e(k_BT_c)^2}\frac{1}{4\pi Dt_{II}w}\left(1-\frac{1}{e}\right)\times\left(\frac{t_{II}}{3\tau_0}\right)^{2/3}\Gamma\left\{\frac{1}{3},[\epsilon/\epsilon^*(t_{II})]^3\right\}\equiv\frac{\chi}{2}\Gamma\left\{\frac{1}{3},[\epsilon/\epsilon^*(t_{II})]^3\right\} \quad (38)$$

Although $\Gamma\{\frac{1}{3},[\epsilon/\epsilon^*(t_{II})]^3\}$ enters the distribution as a result of solution of the kinetic equations for normal metal it does not contain any singularities at $\epsilon\rightarrow 0$ (see Fig. 2) and indeed might represent an appropriate distribution for the case of a suppressed gap. Numerically $\Gamma\{\frac{1}{3},[\epsilon/\epsilon^*(t_{II})]^3\}<\Gamma(\frac{1}{3})$. Thus we arrive at the conclusion that, depending on the value of a parameter χ, the following possibilities can be encountered:

(i) $\chi\ll 1$: weak excitation level;

(ii) $\chi\approx 1$: moderate excitation level;

(iii) $\chi>\chi_c\geqslant 1$: strong excitation level.

In the regime (i) the excitation level is so weak that the order parameter is homogeneous and the system remains perfectly linear on a time scale of the duration of the second cascade stage $E_1\rightarrow E_2$. In regime (iii) of strong excitation the energy deposition is so large that by the time the production of quasiparticles ceases the superconductivity within the excitation spot has been destroyed. The most interesting is (ii). The main feature of this regime is a deep suppression of the gap within the excited spot. As a result, there are two possible paths for subsequent evolution, depending on the speed of diffusion out of the excited spot compared to the rate of trapping of the quasiparticles from the states above the bulk gap due to thermalization down to the suppressed gap. The first corresponds to faster diffusion. In this case after deep excursion the gap relaxes to its bulk value before significant numbers of quasiparticles have been trapped at the suppressed edge. Although the regime remains quasilinear nonetheless the numbers of the quasiparticles and correspondingly the fraction of photon energy lost for production of quasiparticles can be different from the regime (i). The second scenario is that of slower diffusion and simultaneous faster trapping. If trapping of a significant number of quasiparticles onto the states near the locally suppressed superconducting edge occurs before the quasiparticles with energies above the bulk edge diffuse out of the excitation volume, then the formation of an autolocalized state becomes feasible. The fast initial production of the quasiparticles within the small excitation volume causes deep local suppression of a gap. The subsequent trapping of quasiparticles by this potential well will autolock the system of quasiparticles within the excited spot. The trapped quasiparticles will have no chance to move laterally as they encounter the regions of wider gap and undergo Andreev reflections, keeping them inside the excited spot. Their enhanced numbers will maintain the locally suppressed gap forming the metastable ''hot spot'' which will survive until the self-recombination within the spot destroys the excess quasiparticles and brings about the gap relaxation.

The values of the parameter χ at $E_x=6000$ eV is $\chi_{Nb}^{epi}=0.02$ for an epitaxial base film thickness of 100 nm and diffusion coefficient $D\approx 600$ cm^2/s corresponding to the value of the diffusion coefficient in normal state as measured from residual resistivity ratio (RRR) measurements: RRR=90.[28] For a polycrystalline top film of 200 nm with factor 20 slower diffusion $\chi_{Nb}^{poli}\approx 0.4$, giving an absorption in the top film closer to a moderate excitation level. For a typical Ta film of 100 nm thickness we obtain $\chi_{Ta}^{epi}=0.01$ mainly because of the smaller gap and weaker electron-phonon coupling in Ta. As these values suggest, all Nb- and Ta-based STJ devices in the hard x-ray range up to 6 KeV operate in weak excitation regime.

In superconductors belonging to the first group where the electronic phase is not developed the expansion occurs during the phonon control phase. The phonon diffusion coefficient is extremely small. Nonetheless, as follows from the results given by formula (12) in Sec. IV the amount of energy in the electronic system rises very rapidly to a finite value at the beginning of the $\Omega_D\rightarrow\Omega_1$ phase. Thus the expansion is again due to the electronic excitations. A very rough measure of the possibility of formation of a hot spot will be given by the same sort of formula as Eq. (38) but with the electronic distribution function derived from Eq. (6). The result can be expressed as

$$\chi\simeq\frac{1}{\nu_e\Omega_1}\frac{E_x}{\varepsilon}\frac{1}{4\pi Dt_Iw}. \quad (39)$$

This estimate shows that for superconductors of the first group the threshold for hot spot formation is not easier to achieve as compared to materials of the third group, since the shorter duration of the phase is fully compensated by the production of fewer quasiparticles.

VII. CONCLUSION

In summary, we have demonstrated that the quasiparticle-phonon downconversion cascade in nonequilibrium superconductor starts at hot electron energies E_1 far larger than the Debye energy Ω_D. As a result, the hot quasiparticles undergo a very long cascade down to $\epsilon<\Omega_D$ with extremely fast emission of a large $\sim E_1/\Omega_D\gg 1$ number of Debye phonons. Quasiparticle-phonon downconversion across the remaining spectral range below E_1 starts from a highly nonequilibrium state where most of the energy has been accumulated in the form of Debye phonons. It first proceeds as a spectral transformation of the phonon and electron systems which are controlled by the $\Omega_D\rightarrow\Omega_1$ phonon downconversion phase, followed by the second phase $\Omega_1\rightarrow E_2$ of electronic downconversion. We found the analytical solutions of the coupled kinetic equations for the interacting quasiparticles and phonons describing both phases of $\Omega_D\rightarrow E_2$ cascade and derived characteristic times of the downconversion process together with specific time dependences of the numbers of generated quasiparticles during different cascade phases. We have shown that different superconductors exhibit different cascade patterns depending on specific combination of

their parameters and introduced a classification scheme. The quasiparticle yield has been shown to be different for different groups. For low gap superconductors the mean energy needed to produce one quasiparticle is close to 1.7Δ, while for superconductors of first group such universality is broken, quasiparticle yield is poor and strongly depends on material parameters. The excitation level in the Nb- and Ta-based superconducting tunneling structures widely used for x-ray detection as well as for other superconductors of the Table I remains weak even in the hard x-ray region.

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