Mean-field theory for Josephson junction arrays with charge frustration

G. Grignani, A. Mattoni, P. Sodano, and A. Trombettoni*
Dipartimento di Fisica, Università di Perugia, Via A. Pascoli I-06123 Perugia, Italy
and Unità INFM, Università di Camerino, Camerino, Italy

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We derive, using a finite-temperature path integral approach, the equation for the phase boundary between the insulating and the superconducting phase for quantum Josephson junctions arrays (JJA's) with offset charges and general capacitance matrices. We show that—within the mean field theory approximation—a reentrance in the phase boundary should appear, for systems with a uniform distribution of offset charges, only when the capacitance matrix is nondiagonal. For a model with nearest-neighbor capacitance matrix and uniform offset charge $q/2e=1/2$, we find reentrant superconductivity even if the intergrain interaction is short ranged; for this model, we determine the most relevant contributions to the equations for the phase boundary by explicitly constructing the charge distributions on the lattice corresponding to the lowest-energy states which provide the leading contributions to the partition function at low T_c.

I. INTRODUCTION

Josephson junction arrays (JJA's) and granular superconductors, namely, systems of metallic grains embedded in an insulator, become superconducting in two steps.[1] First, at the bulk critical temperature each grain develops a superconducting gap but the phases of the order parameter on different grains are uncorrelated. Then, at a lower temperature T_c, the Cooper pair tunneling between grains gives rise to a long-range phase coherence and the system as a whole exhibits a phase transition to a superconducting state.

The phase transition is governed by the competition between the Josephson tunneling, characterized by a Josephson coupling energy E_J,[2] and the Coulomb interaction between Cooper pairs, described by a charging energy E_C.[3,4] In classical junction arrays the Josephson coupling E_J is dominant, the transition separates a superconducting low-temperature phase from a normal high-temperature phase. When E_C is comparable to E_J (small grains) charging effects give rise to a quantum dynamics. The grain capacitance is small, so that the energy cost of Cooper pair tunneling may be higher than the energy gained by the formation of a phase-coherent state. Zero point fluctuations of the phase may destroy the long range superconducting order even at zero temperature (see, for example, Ref. 1).

Within the framework of the BCS theory, Efetov[5] derived an effective quantum Hamiltonian in terms of the phases $\varphi_{\mathbf{i}}$ of the superconducting order parameter at the grain $\mathbf{i}$, and their conjugate variables $n_{\mathbf{i}}$ representing the number of Cooper pairs. The Hamiltonian for the quantum phase model reads

$$H=\frac{1}{2}\sum_{\mathbf{ij}} C_{\mathbf{ij}}^{-1}Q_{\mathbf{i}}Q_{\mathbf{j}}-E_J\sum_{\langle\mathbf{ij}\rangle}\cos(\varphi_{\mathbf{i}}-\varphi_{\mathbf{j}}), \qquad (1)$$

$$Q_{\mathbf{i}}=2en_{\mathbf{i}}, \quad [\varphi_{\mathbf{i}},n_{\mathbf{i}}]=i\delta_{\mathbf{ij}},$$

where $Q_{\mathbf{i}}$ is the excess of charge due to Cooper pairs (charge $2e$) on the site $\mathbf{i}$ of a square lattice in D-space dimension and $C_{\mathbf{ij}}$ is the capacitance matrix describing the electrostatic coupling between Cooper pairs. The diagonal elements of the inverse matrix $C_{\mathbf{ij}}^{-1}$ provide the charging energy: $E_C=e^2C_{\mathbf{ii}}^{-1}/2\equiv e^2/2C_0$, where C_0 is the self-capacitance.

The superconductor-insulator transition depends crucially on the spatial dimensionality D. For $D=1$ there may exist also other phases.[6] For $D=2$ the system exhibits a richly structured phase diagram.[7,8] In higher dimensions it is believed that the mean field theory analysis provides qualitatively correct results. It is relevant to clarify how the transition from insulator to superconductor depends on the relevant constitutive parameters—such as the capacitances of, and between, the junctions—as well as on external parameters such as the temperature, offset charges, and external magnetic fields.

Much work has been done to study the phase diagram of quantum JJA's, in the T/E_C-E_C/E_J plane.[1] The analysis uses mean field theory[5,9–15] as well as the renormalization group approach[17,18] or the mapping into a spin system.[16] There is the claim that the phase diagram—under suitable circumstances—may exhibit a reentrant character with the superconducting phase existing between upper and lower critical temperature.[9,10,14] In Refs. 9,19 the influence of the Coulomb energy on the transition temperature was investigated for a model with a diagonal capacitance matrix. The effects of off-diagonal terms in the charging energy were investigated by several authors within the mean field theory approximation:[5,11,12,18,20–22] while it is widely believed that the nearest neighbors interaction enhance the transition temperature T_c by lowering the energy cost for a Cooper pair to tunnel from one neutral grain to another,[12] there is still some dispute on whether there is a reentrance or not for models with nondiagonal capacitance matrix.[12,21,22]

In this paper we shall consider also the effect of a background of external charges on the superconductor-insulator transition of a quantum JJA's.[7,8,23,24] Such an offset of charges might arise in physical systems as a result of charged impurities or by application of a gate voltage between the array and the ground. In the former case offset charges are distributed randomly on the lattice while in the latter case the distribution can be uniform. They might be regarded as effective charges $q_{\mathbf{i}}$ on the sites of the lattice. When $q_{\mathbf{i}}\neq 2e$ the offset charges cannot be eliminated by Cooper pair tunneling.

Offset charges frustrate the attempts of the system to minimize the energy of the charge distribution of the ground state. Consequently the charging energy of any excitations is smaller compared to the unfrustrated case and superconductivity is enhanced: more intuitively, one may view the effect of offset charges as a reduction of the effective value of E_C and thus as an enhancement of superconductivity. For a diagonal capacitance matrix and uniform offset charges of magnitude e the states with zero and one Cooper pair on the islands become degenerate in energy and this allows for superconductivity even for small values of E_J: as evidenced in Appendix A the superconducting order parameter is nonvanishing for $T<T_c$ and for any E_J.

For a nondiagonal capacitance matrix, the phase diagram becomes much more complicated as a function of uniform offset charges.[7] With offset charges the Hamiltonian (1) becomes

$$H=\frac{1}{2}\sum_{\mathbf{ij}} C_{\mathbf{ij}}^{-1}(Q_{\mathbf{i}}+q_{\mathbf{i}})(Q_{\mathbf{j}}+q_{\mathbf{j}})-E_J\sum_{\langle\mathbf{ij}\rangle}\cos(\varphi_{\mathbf{i}}-\varphi_{\mathbf{j}}). \tag{2}$$

To study the effect of charge frustration on the phase diagram of a system described by the above Hamiltonian, it is one of the purposes of this paper; in the following, we shall analyze the equation relating the critical temperature to the ratio E_J/E_C and find a reentrant bulge in the phase boundary between the insulating and superconducting phase, for arrays with a uniform distribution of offset charges $q/2e=1/2$, even if the capacitive interaction is short ranged. For this model, we determine the most relevant contributions to the equations for the phase boundary by explicitly constructing the charge distributions on the lattice corresponding to the lowest-energy states which provide the leading contributions to the partition function at low T_c. The determination of these configurations allows one to simplify the equation for the phase boundary and to evidence analytically the existence of a reentrant superconductivity. For models with a diagonal capacitance matrix, a reentrance is absent.

The approach we follow uses the well known Hubbard-Stratonovich[25] representation for the partition function in terms of coarse-grained classical local variables $\psi_{\mathbf{i}}$ for which the effective action is computed.[20] This method has been used in previous analyses of the phase diagram of JJA's in Refs. 20,7.

In deriving our results on reentrant superconductivity for JJA's, we shall pursue a rather pedagogical approach by revisiting the path integral derivation of the finite temperature mean field theory of a system of JJA's. Our goal is not only to show that the path integral method is much simpler than the more conventional self-consistent mean field theory approach for the analysis of systems of JJA's with offset charges and nondiagonal capacitance matrices, but also to evidence that it allows for a systematic analysis of the effects of the periodicity of the phase variables in the finite temperature theory. The periodicity can be taken into account by introducing a set of integers, so that the partition function factorizes as a product of a topological term, depending only on this set of integers, and a nontopological term which one can explicitly compute. The Poisson resummation formula for the topological part of the partition function turns out to be very useful for the derivation of the low critical temperature expansion. Our revisitation of the path integral derivation of the equation for the phase boundary between the insulating and the superconducting phase of a system of JJA's evidences also that methods used in the analysis of phase transitions in finite temperature gauge theories[26] may be used in the study of condensed matter systems.

In Sec. II we review the self-consistent mean field theory approximation within the Hamiltonian formalism for quantum JJA's. We study the eigenvalue equation of the mean field Hamiltonian with diagonal capacitance, and uniform offset charge $q_{\mathbf{i}}=e$ showing explicitly that at zero temperature there is superconductivity for all values of $\alpha=zE_J/4E_C$.

In Sec. III we use the coarse grained approach to compute the Ginzburg-Landau free energy for quantum JJA's with charge frustration and a general Coulomb interaction matrix. The path integral providing the phase correlator needed to investigate the critical behavior of the system, is explicitly computed.

In Sec. IV, from the Ginzburg-Landau free energy, we derive, within the mean field theory approximation, the analytical form of the critical line equation and analyze the phenomenon of reentrant superconductivity for a variety of systems of JJA's. The phase boundary is drawn for the model with diagonal capacitance matrix for several charge distributions and a reentrance is never found. We then analyze the low-temperature limit of a system with nearest-neighbor interaction matrix. Through the analysis of the charge distribution providing the leading contributions to the low-temperature expansion of the partition function, we are able to establish analytically the existence of a reentrant behavior for a system with a uniform background of external charges $q_{\mathbf{i}}=e$.

Section V is devoted to some concluding remarks. The appendixes contain the derivation of some relevant formulas of the main text and are introduced to keep the text self-contained.

II. SELF-CONSISTENT MEAN FIELD THEORY IN THE HAMILTONIAN APPROACH

Mean field theory for quantum JJA's with diagonal capacitance matrix was first used by Simánek.[9] The approximation consists in replacing the Josephson coupling of the phase on a given island $\mathbf{i}$ to its neighbors by an average coupling so that

$$E_J\sum_{\langle\mathbf{ij}\rangle}\cos(\varphi_{\mathbf{i}}-\varphi_{\mathbf{j}})=zE_J\langle\cos\varphi\rangle\sum_{\mathbf{i}}\cos\varphi_{\mathbf{i}}. \tag{3}$$

In Eq. (3) z is the coordination number; it is assumed also that $\langle\cos\varphi\rangle$ does not depend on the island index $\mathbf{i}$ and the choice $\langle\sin\varphi_{\mathbf{i}}\rangle=0$, which provides a real order parameter, has been made.

Within the mean field approximation the Hamiltonian (1) becomes

$$H_{\rm MF}=\frac{1}{2}\sum_{\bf ij} C_{\bf ij}^{-1}Q_{\bf i}Q_{\bf j}-zE_J\langle\cos\varphi\rangle\sum_{\bf i}\cos\varphi_{\bf i} \tag{4}$$

and the order parameter $\langle\cos\varphi\rangle$ is evaluated self-consistently from Eq. (4). For a diagonal capacitance matrix ($C_{\bf ij}=C_0\delta_{\bf ij}$) mean field theory computation are very simple since Eq. (4) describes on each site a quantum particle in a periodic potential $\cos\varphi_{\bf i}$.[9] The eigenfunction of the array is a product of eigenfunctions $\psi_n(\varphi)$ describing the individual islands and satisfying the Mathieu equation[27]

$$\left(-\frac{d^2}{d\varphi^2}-\frac{zE_J}{4E_C}\langle\cos\varphi\rangle\cos\varphi\right)\psi_n(\varphi)=\frac{E_n}{4E_C}\psi_n(\varphi) \tag{5}$$

with periodic boundary conditions $\psi_n(\varphi)=\psi_n(\varphi+2\pi)$.

It is well known that the Mathieu equation admits also antiperiodic solutions, $\psi_n(\varphi)=-\psi_n(\varphi+2\pi)$ (see Appendix A). If both periodic and antiperiodic solutions are used, the general solution of Eq. (5) does not have a definite periodicity and, consequently, the charges $n_{\bf i}$ take continuous eigenvalues. Such continuous eigenvalues are expected to be relevant in the description of continuous flows of currents due, for example, to ohmic shunt resistances.[28,29] Although the superposition of both periodic and antiperiodic solutions yields to a reentrant behavior even in the unfrustrated dissipationless diagonal model,[10,11,31] this superposition is not allowed in describing physical situations in which the only excitations are Cooper pairs of charge $2e$.[5,12,7] Thus the use of both periodic and antiperiodic solutions does not have physical significance in the models considered in this paper.

The mean field self-consistency condition gives

$$\langle\cos\varphi\rangle=\frac{\sum_n e^{-\beta E_n}\langle\psi_n|\cos\varphi|\psi_n\rangle}{\sum_n e^{-\beta E_n}} \tag{6}$$

with $\beta=1/k_BT$. For high temperatures or low E_J only the solution $\langle\cos\varphi\rangle=0$ exists and there is not superconductivity. For low temperatures or high E_J $\langle\cos\varphi\rangle\neq 0$ and the system as a whole behaves as a superconductor.

From Eq. (6) one gets the equation for the critical temperature[9]

$$\alpha=\frac{\sum_{n=-\infty}^{+\infty}e^{-(4/y)n^2}}{\sum_{n=-\infty}^{+\infty}\frac{1}{1-4n^2}e^{-(4/y)n^2}} \tag{7}$$

with $y=k_BT_c/E_C$ and $\alpha=zE_J/4E_C$. In Fig. 1 we plot T_c as a function of α.

If one considers a diagonal capacitance matrix and uniform offset charges of magnitude e on each site ($q_{\bf i}/2e=1/2$), the Hamiltonian reads

$$H_d=\frac{1}{2C_0}\sum_{\bf i}(Q_{\bf i}+q_{\bf i})(Q_{\bf i}+q_{\bf i})-E_J\sum_{\langle\bf ij\rangle}\cos(\varphi_{\bf i}-\varphi_{\bf j}). \tag{8}$$

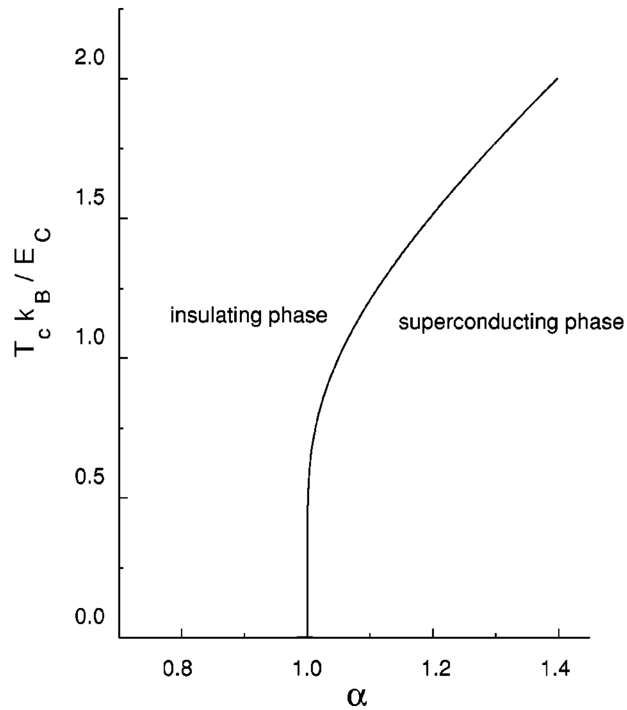

FIG. 1. Phase diagram for the diagonal model without charge frustration.

Mean field theory of this model leads to a Schrödinger equation of the form

$$\left[-\frac{d^2}{d\varphi^2}-2i\frac{q}{2e}\frac{d}{d\varphi}+\left(\frac{q}{2e}\right)^2-\alpha\langle\cos\varphi\rangle\cos\varphi\right]\psi_n(\varphi)=\frac{E_n}{4E_C}\psi_n(\varphi). \tag{9}$$

Redefining the phase of the wave function as

$$\psi_n(\varphi)=e^{-i(q/2e)\varphi}\rho_n(\varphi),$$

Eq. (9) reduces to a Mathieu equation for $\rho_n(\varphi)$

$$\frac{d^2\rho_n}{d\varphi^2}+\left(\frac{\lambda}{4}-\frac{v}{2}\cos\varphi\right)\rho_n=0 \tag{10}$$

with $\lambda_n=E_n/E_C$ and $v=-zE_J\langle\cos\varphi\rangle/2E_C$. Equations (8)–(10) lead to the following modification of Eq. (7) [see Appendix A]:

$$\alpha=\frac{e^{-1/y}+\sum_{n=1}^{+\infty}e^{-(4/y)(n+1/2)^2}}{\frac{4+y}{4y}e^{-1/y}+\sum_{n=1}^{+\infty}\frac{1}{1-4(n+1/2)^2}e^{-(4/y)(n+1/2)^2}} \tag{11}$$

which—at variance with the unfrustrated model—exhibits superconductivity even for infinitesimal values of α. This feature is shown in Fig. 2 which also shows the absence of a reentrant behavior at low T. From Fig. 2 one sees also how the presence of offset charges improves the superconductivity of the array.

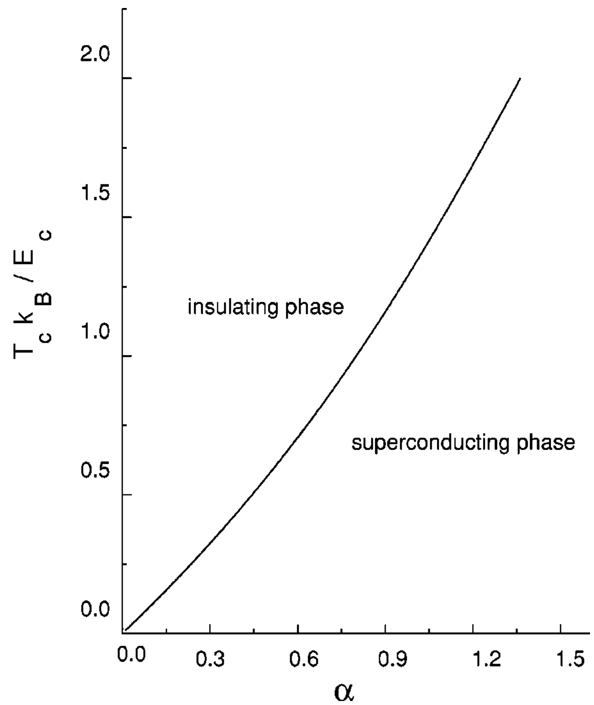

FIG. 2. Phase diagram of the diagonal model with half-integer charge frustration.

For frustrated models with nondiagonal capacitance matrix, the self-consistent mean-field theory approximation becomes very cumbersome and it turns convenient to resort to the more powerful functional approach. Without offset charges, a reentrance at low T_c is expected at least when the interaction between grains is long-ranged.[23,19]

III. GINZBURG-LANDAU FREE ENERGY

The partition function for the frustrated off-diagonal model is given by

$$Z=\mathrm{Tr}\, e^{-\beta H}=\sum_n \langle \psi_n | e^{-\beta H} | \psi_n \rangle, \qquad (12)$$

where H is given in Eq. (2) and the sum is extended only to states of charge $2e$ and thus with definite periodicity. In the functional approach Z reads

$$Z=\int \prod_{\mathbf{i}} D\varphi_{\mathbf{i}} \exp\left\{ -\int_0^\beta d\tau L_E\left(\varphi_{\mathbf{i}}(\tau), \frac{d\varphi_{\mathbf{i}}}{d\tau}(\tau) \right) \right\}, \qquad (13)$$

where the Euclidean Lagrangian L_E can be derived from

$$L=\frac{1}{2}\left(\frac{\hbar}{2e}\right)^2 \sum_{\mathbf{ij}} C_{\mathbf{ij}} \frac{d\varphi_{\mathbf{i}}}{dt}\frac{d\varphi_{\mathbf{j}}}{dt} - \left(\frac{\hbar}{2e}\right)\sum_{\mathbf{i}} \frac{d\varphi_{\mathbf{i}}}{dt} q_{\mathbf{i}} + E_J \sum_{\langle \mathbf{ij} \rangle} \cos(\varphi_{\mathbf{i}}-\varphi_{\mathbf{j}}) \qquad (14)$$

by replacing $it/\hbar \rightarrow \tau$. The path integral that one should compute is then given by

$$Z=\int \prod_{\mathbf{i}} D\varphi_{\mathbf{i}} \exp\left\{ \int_0^\beta d\tau \left[-\frac{1}{2}\sum_{\mathbf{ij}} C_{\mathbf{ij}} \frac{\dot{\varphi}_{\mathbf{i}}}{2e}\frac{\dot{\varphi}_{\mathbf{j}}}{2e} + i\sum_{\mathbf{i}} q_{\mathbf{i}} \frac{\dot{\varphi}_{\mathbf{i}}}{2e} + \frac{E_J}{2}\sum_{\mathbf{ij}} e^{i\varphi_{\mathbf{i}}}\gamma_{\mathbf{ij}} e^{-i\varphi_{\mathbf{j}}} \right] \right\}, \qquad (15)$$

where $-\infty<\varphi_{\mathbf{i}}<+\infty$, $\varphi_{\mathbf{i}}(0)=\varphi_{\mathbf{i}}(\beta)+2\pi n_{\mathbf{i}}$, and $\gamma_{\mathbf{ij}}=1$ if $\mathbf{i},\mathbf{j}$ are nearest neighbors and equals zero otherwise. The integers $n_{\mathbf{i}}$ appearing in this boundary condition take into account the 2π periodicity of the states ψ_n appearing in Eq. (12).

In order to derive the Ginzburg-Landau free energy for the order parameter, it is convenient to carry out the integration over the phase variables by means of the Hubbard-Stratonovich trick:[25] using the identity

$$e^{J^+\Gamma J}=\frac{\det \Gamma^{-1}}{\pi^N}\int \prod_{\mathbf{i}} D^2\psi_{\mathbf{i}} e^{-\psi^+\Gamma^{-1}\psi - J^+\psi - \psi^+ J} \qquad (16)$$

the partition function may be rewritten as

$$Z=\int \prod_{\mathbf{i}} D\psi_{\mathbf{i}} D\psi_{\mathbf{i}}^* e^{\int_0^\beta d\tau [-(2/E_J)\sum_{\mathbf{ij}} \psi_{\mathbf{i}}^* \gamma_{\mathbf{ij}}^{-1}\psi_{\mathbf{j}}]} e^{-S_{\mathrm{eff}}[\psi]}. \qquad (17)$$

where the effective action for the auxiliary Hubbard-Stratonovich field $\psi_{\mathbf{i}}$, $S_{\mathrm{eff}}[\psi]$, is given by

$$S_{\mathrm{eff}}[\psi]=-\ln\left\{ \int \prod_{\mathbf{i}} D\varphi_{\mathbf{i}} \exp\left\{ \int_0^\beta d\tau \left[-\frac{1}{2}\sum_{\mathbf{ij}} C_{\mathbf{ij}} \frac{\dot{\varphi}_{\mathbf{i}}}{2e}\frac{\dot{\varphi}_{\mathbf{j}}}{2e} + i\sum_{\mathbf{i}} \left(q_{\mathbf{i}}\frac{\dot{\varphi}_{\mathbf{i}}}{2e} - \psi_{\mathbf{i}} e^{i\varphi_{\mathbf{i}}} - \psi_{\mathbf{i}}^* e^{-i\varphi_{\mathbf{i}}} \right) \right] \right\} \right\}. \qquad (18)$$

The Hubbard-Stratonovich field $\psi_{\mathbf{i}}$ may be regarded as the order parameter for the insulator-superconductor phase transition since it turns out to be proportional to $\langle e^{i\varphi_{\mathbf{i}}} \rangle$, as it can be easily seen from the classical equations of motion. From Eq. (18) the Ginzburg-Landau free-energy may be derived by integrating out the phase field $\varphi_{\mathbf{i}}$.

Since the phase transition is second order,[30] close to the onset of superconductivity, the order parameter $\psi_{\mathbf{i}}$ is small. One may then expand the effective action up to the second order in $\psi_{\mathbf{i}}$, getting

$$S_{\mathrm{eff}}[\psi]=S_{\mathrm{eff}}[0]+\int_0^\beta d\tau \int_0^\beta d\tau' G_{\mathbf{rs}}(\tau,\tau')\psi_{\mathbf{r}}(\tau)\psi_{\mathbf{s}}^*(\tau') + \cdots, \qquad (19)$$

where $G_{\mathbf{rs}}$ is the phase correlator

$$G_{\mathbf{rs}}(\tau,\tau')=\left.\frac{\delta^2 S_{\mathrm{eff}}[\psi]}{\delta\psi_{\mathbf{r}}(\tau)\delta\psi_{\mathbf{s}}(\tau')}\right|_{\psi,\psi^*=0} = \langle e^{i\varphi_{\mathbf{r}}(\tau)-i\varphi_{\mathbf{s}}(\tau')}\rangle_0. \qquad (20)$$

The partition function (17), can be written as

$$Z=\int \prod_{\mathbf{i}} d\psi_{\mathbf{i}} d\psi_{\mathbf{i}}^* e^{-F[\psi]}, \qquad (21)$$

where $F[\psi]$ is the Ginzburg-Landau free energy; due to Eqs. (18),(19), up to the second order in $\psi_{\mathbf{i}}$, one has

$$F[\psi]=\int_0^\beta d\tau\int_0^\beta d\tau'\sum_{\mathbf{ij}}\ \psi_{\mathbf{i}}^*(\tau)[\gamma_{\mathbf{ij}}^{-1}\delta(\tau-\tau')-G_{\mathbf{ij}}(\tau,\tau')]\psi_{\mathbf{j}}(\tau'). \tag{22}$$

We shall now compute the phase correlator $G_{\mathbf{rs}}$ by evaluating the expectation value in Eq. (21) by means of the path integral over the phase variables $\varphi_{\mathbf{i}}(\tau)$. In performing this integration one should take into account that the field configurations satisfy

$$\varphi_{\mathbf{i}}(\beta)-\varphi_{\mathbf{i}}(0)=2\pi n_{\mathbf{i}}. \tag{23}$$

For this purpose it turns out very convenient to untwist the boundary conditions by decomposing the phase field in terms of a periodic field $\phi_{\mathbf{i}}(\tau)$ and a term linear in τ which takes into account the boundary conditions (23); namely, one sets

$$\varphi_{\mathbf{i}}(\tau)=\phi_{\mathbf{i}}(\tau)+\frac{2\pi}{\beta}n_{\mathbf{i}}\tau, \tag{24}$$

with $\phi_{\mathbf{i}}(\beta)=\phi_{\mathbf{i}}(0)$. Summing over all the phases $\varphi_{\mathbf{i}}(\tau)$ amounts then to integrate over the periodic field $\phi_{\mathbf{i}}$ and to sum over the integers $n_{\mathbf{i}}$. As a result the phase correlator factorizes as the product of a topological term depending on the integers $n_{\mathbf{i}}$ and a nontopological one; namely,

$$G_{\mathbf{rs}}(\tau;\tau')=\frac{\int D\phi_{\mathbf{i}}e^{i\phi_{\mathbf{r}}(\tau)-i\phi_{\mathbf{s}}(\tau')}\exp\left\{\int_0^\beta d\tau\left(-\frac{1}{2}C_{\mathbf{ij}}\frac{\dot\phi_{\mathbf{i}}}{2e}\frac{\dot\phi_{\mathbf{j}}}{2e}\right)\right\}}{\int D\phi_{\mathbf{i}}\exp\left\{\int_0^\beta d\tau\left(-\frac{1}{2}C_{\mathbf{ij}}\frac{\dot\phi_{\mathbf{i}}}{2e}\frac{\dot\phi_{\mathbf{j}}}{2e}\right)\right\}}\times\frac{\sum_{[n_{\mathbf{i}}]}e^{i(2\pi/\beta)(n_{\mathbf{r}}\tau-n_{\mathbf{s}}\tau')}\exp\left\{-\sum_{\mathbf{ij}}\frac{\pi^2}{2\beta e^2}C_{\mathbf{ij}}n_{\mathbf{i}}n_{\mathbf{j}}+\sum_{\mathbf{i}}2i\pi\frac{q_{\mathbf{i}}}{2e}n_{\mathbf{i}}\right\}}{\sum_{[n_{\mathbf{i}}]}\exp\left\{-\sum_{\mathbf{ij}}\frac{\pi^2}{2\beta e^2}C_{\mathbf{ij}}n_{\mathbf{i}}n_{\mathbf{j}}+\sum_{\mathbf{i}}2i\pi\beta\frac{q_{\mathbf{i}}}{2e}n_{\mathbf{i}}\right\}}. \tag{25}$$

After a lengthy computation, the first (nontopological) factor appearing in the left-hand side of equation (25) has the following simple expression [see Appendix B]:

$$\delta_{\mathbf{rs}}\exp\left\{-2e^2C_{\mathbf{rr}}^{-1}\left(|\tau-\tau'|-\frac{(\tau-\tau')^2}{\beta}\right)\right\}. \tag{26}$$

The sum over the integers in the topological factor in Eq. (25) is done by means of the well known Poisson resummation formula

$$|\det G|^{1/2}\sum_{[n_{\mathbf{i}}]}e^{-\pi(n-a)_{\mathbf{i}}G_{\mathbf{ij}}(n-a)_{\mathbf{j}}}=\sum_{[m_{\mathbf{i}}]}e^{-\pi m_{\mathbf{i}}(G^{-1})_{\mathbf{ij}}m_{\mathbf{j}}-2\pi i m_{\mathbf{i}}a_{\mathbf{i}}}.$$

Thus Eq. (25) becomes

$$G_{\mathbf{rs}}(\tau,\tau')=\delta_{\mathbf{rs}}e^{-2e^2C_{\mathbf{rr}}^{-1}|\tau-\tau'|}\frac{\sum_{[n_{\mathbf{i}}]}\exp\left\{-\sum_{\mathbf{ij}}2e^2\beta C_{\mathbf{ij}}^{-1}\left(n_{\mathbf{i}}+\frac{q_{\mathbf{i}}}{2e}\right)\left(n_{\mathbf{j}}+\frac{q_{\mathbf{j}}}{2e}\right)-\sum_{\mathbf{i}}4e^2C_{\mathbf{ri}}^{-1}\left(n_{\mathbf{i}}+\frac{q_{\mathbf{i}}}{2e}\right)(\tau-\tau')\right\}}{\sum_{[n_{\mathbf{i}}]}\exp\left\{-\sum_{\mathbf{ij}}2\beta e^2C_{\mathbf{ij}}^{-1}\left(n_{\mathbf{i}}+\frac{q_{\mathbf{i}}}{2e}\right)\left(n_{\mathbf{j}}+\frac{q_{\mathbf{j}}}{2e}\right)\right\}} \tag{27}$$

with $n_{\mathbf{i}}$ assuming all integer values and $\Sigma_{[n_{\mathbf{i}}]}$ being a sum over all the configurations.

By means of a Euclidean-time Fourier transform, the fields $\psi_{\mathbf{i}}$ are written as

$$\psi_{\mathbf{i}}(\tau)=\frac{1}{\beta}\sum_{\mu}\ \psi_{\mathbf{i}}(\omega_\mu)e^{i\omega_\mu\tau},$$

where ω_μ are the Matsubara frequencies. As a consequence, the phase correlator $G_{\mathbf{ij}}$ can be expressed as

$$G_{\mathbf{ij}}(\tau;\tau')=\frac{1}{\beta}\sum_{\mu\mu'}G_{\mathbf{ij}}(\omega_\mu;\omega_{\mu'})e^{i\omega_\mu\tau}e^{i\omega_{\mu'}\tau'}. \tag{28}$$

From Eq. (27) one can show that $G_{\mathbf{rs}}(\omega_\mu;\omega_\mu')$ is diagonal in the Matsubara frequencies and can be written as

$$G_{\mathbf{rs}}(\omega_\mu;\omega_\mu')=G_{\mathbf{r}}(\omega_\mu)\cdot\delta_{\mathbf{rs}}\cdot\delta(\omega_\mu+\omega_{\mu'}) \tag{29}$$

with

$G_{\mathbf{r}}(\omega_\mu)$

$$=\frac{1}{2E_c}\cdot\sum_{[n_{\mathbf{i}}]}\frac{\exp\left\{-\frac{4}{y}\sum_{\mathbf{ij}}\frac{U_{\mathbf{ij}}}{U_{\mathbf{00}}}\left(n_{\mathbf{i}}+\frac{q_{\mathbf{i}}}{2e}\right)\left(n_{\mathbf{j}}+\frac{q_{\mathbf{j}}}{2e}\right)\right\}}{1-4\left[\sum_{\mathbf{j}}\frac{U_{\mathbf{rj}}}{U_{\mathbf{00}}}\left(n_{\mathbf{i}}+\frac{q_{\mathbf{i}}}{2e}\right)-i\omega_\mu\right]^2}\frac{1}{Z_0}. \tag{30}$$

In Eq. (31) Z_0 is given by

$$Z_0=\sum_{[n_{\mathbf{i}}]}\exp\left\{-\frac{4}{y}\sum_{\mathbf{ij}}\frac{U_{\mathbf{ij}}}{U_{\mathbf{00}}}\left(n_{\mathbf{i}}+\frac{q_{\mathbf{i}}}{2e}\right)\left(n_{\mathbf{j}}+\frac{q_{\mathbf{i}}}{2e}\right)\right\}.$$

with $U_{\mathbf{ij}}=C^{-1}_{\mathbf{ij}}$, $E_C=e^2C^{-1}_{\mathbf{rr}}/2$, and $y=k_BT_c/E_C$. In terms of Matsubara frequencies the Ginzburg-Landau free energy (22) becomes

$$F[\psi]=\frac{1}{\beta}\sum_{\mu\mathbf{ij}}\psi^*_{\mathbf{i}}(\omega_\mu)\left[\frac{2}{E_J}\gamma^{-1}_{\mathbf{ij}}-G_{\mathbf{i}}(\omega_\mu)\delta_{\mathbf{ij}}\right]\psi_{\mathbf{j}}(\omega_\mu). \tag{31}$$

This equation was first derived in Ref. 7 by means of a cumulant expansion and it is the starting point for any analysis of the phase boundary between the insulating and the superconducting phases in JJA's with arbitrary capacitance matrix and with charge frustration.

IV. MEAN FIELD THEORY ANALYSIS OF REENTRANT SUPERCONDUCTIVITY IN JJA'S

In the following we shall derive the equation determining the phase boundary in the plane $(\alpha, K_BT_c/E_C)$, in mean field theory and for a system with arbitrary capacitance matrix and a uniform distribution of off-set charges. For this purpose it is convenient to expand the fields $\psi_{\mathbf{i}}(\omega_\mu)$ and $G_{\mathbf{i}}(\omega_\mu)$ in terms of the vectors of the reciprocal lattice $\mathbf{q}$. One has

$$\psi_{\mathbf{i}}(\omega_\mu)=\frac{1}{N}\sum_{\mathbf{q}}\psi_{\mathbf{q}}(\omega_\mu)e^{i\mathbf{q}\cdot\mathbf{i}}, \tag{32}$$

$$G_{\mathbf{i}}(\omega_\mu)=\frac{1}{N}\sum_{\mathbf{q}}G_{\mathbf{q}}(\omega_\mu)e^{i\mathbf{q}\cdot\mathbf{i}}. \tag{33}$$

Moreover

$$\gamma^{-1}_{\mathbf{ij}}=\frac{1}{N}\sum_{\mathbf{q}}\gamma^{-1}_{\mathbf{q}}e^{i\mathbf{q}\cdot(\mathbf{i}-\mathbf{j})}, \tag{34}$$

where $\gamma^{-1}_{\mathbf{q}}$ is the inverse of the Fourier transform of the Josephson coupling strength $\gamma_{\mathbf{ij}}$ which equals 1 for $\mathbf{i},\mathbf{j}$ nearest neighbors and 0 otherwise. As a consequence

$$\gamma^{-1}_{\mathbf{q}}=\frac{1}{\sum_{\mathbf{p}}e^{-i\mathbf{q}\cdot\mathbf{p}}},$$

where $\mathbf{p}$ is a vector connecting two nearest neighbors sites. Expanding in $\mathbf{q}$ one gets

$$\gamma^{-1}_{\mathbf{q}}=\frac{1}{z}+\frac{\mathbf{q}^2a^2}{z^2}+\cdots, \tag{35}$$

where a is the lattice spacing and z the coordination number. The first term in Eq. (35) provides the mean field theory approximation which, as expected, is exact in the limit of large coordination number.

The Ginzburg-Landau free energy (31), reads

$$F[\psi]=\frac{1}{\beta N}\sum_{\mu\mathbf{qq}'}\psi_{\mathbf{q}}(\omega_\mu)^*\left[\gamma^{-1}_{\mathbf{q}}\delta_{\mathbf{qq}'}-\frac{G_{\mathbf{q}-\mathbf{q}'}(\omega_\mu)}{N}\right]\psi_{\mathbf{q}'}(\omega_\mu).$$

Using Eq. (35) and keeping only terms of zeroth order in ω_μ and $\mathbf{q}$ one obtains the mean field theory approximation to the coefficient of the quadratic term of F

$$\simeq\frac{1}{\beta N}\sum_{\mathbf{q}\mu}\left[\frac{2}{E_Jz}-G_{\mathbf{0}}(0)+\cdots\right]|\psi_{\mathbf{q}}(\omega_\mu)|^2. \tag{36}$$

The equation for the phase boundary line then reads as

$$1=z\frac{E_J}{2}G_{\mathbf{0}}(0) \tag{37}$$

with

$$G_{\mathbf{0}}(0)=\frac{1}{N}\sum_{\mathbf{r}}G_{\mathbf{r}}(0). \tag{38}$$

Equation (37) determines the relation between T_c and α at the phase boundary.

For a uniform distribution of offset charges Eq. (37) simplifies further since in Eq. (38) $G_{\mathbf{r}}$ does not depend on $\mathbf{r}$. As a consequence, the phase boundary equation becomes

$$1=\alpha\cdot\sum_{[n_{\mathbf{i}}]}\frac{\exp\left\{-\frac{4}{y}\sum_{\mathbf{ij}}\frac{U_{\mathbf{ij}}}{U_{\mathbf{00}}}\left(n_{\mathbf{i}}+\frac{q}{2e}\right)\left(n_{\mathbf{j}}+\frac{q}{2e}\right)\right\}}{1-4\left[\sum_{\mathbf{j}}\frac{U_{\mathbf{0j}}}{U_{\mathbf{00}}}\left(n_{\mathbf{j}}+\frac{q}{2e}\right)\right]^2}\frac{1}{Z_0} \tag{39}$$

with

$$\alpha=\frac{zE_J}{4E_c}$$

and

$$Z_0=\sum_{[n_{\mathbf{i}}]}\exp\left\{-\frac{4}{y}\sum_{\mathbf{ij}}\frac{U_{\mathbf{ij}}}{U_{\mathbf{00}}}\left(n_{\mathbf{i}}+\frac{q}{2e}\right)\left(n_{\mathbf{j}}+\frac{q}{2e}\right)\right\}.$$

In the following we shall derive the physical implications of Eq. (39) in a variety of models describing JJA's.

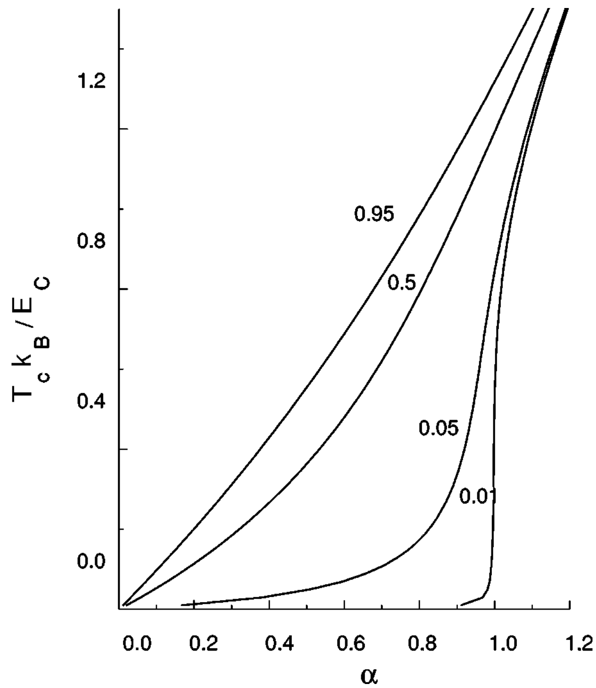

FIG. 3. Phase diagram for several values of $f_{1/2}$.

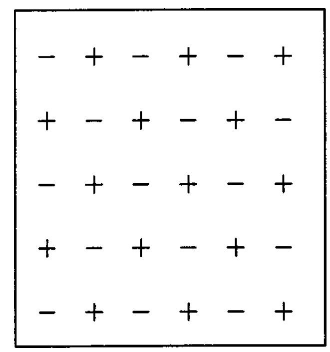

FIG. 4. Ground state.

A. Self-charging model

For a diagonal capacitance matrix, $U_{\mathbf{ij}}=\delta_{\mathbf{ij}}U_{\mathbf{0}}$, one singles out only the self-interaction of plaquettes. This case was already analyzed in Sec. II within the approach of self-consistent mean field theory. As a check of the path integral approach we shall show that one is able to reproduce the same results from Eq. (39).

In the diagonal case Eq. (39) becomes

$$1=\alpha\left(\frac{\sum_{n}\exp\left\{-\frac{4}{y}\left(n+\frac{q}{2e}\right)^{2}\right\}\frac{1}{1-4(n+q/2e)^{2}}}{\sum_{m}\exp\left\{-\frac{4}{y}\left(m+\frac{q}{2e}\right)^{2}\right\}}\right). \tag{40}$$

Since n is an integer (40) is invariant under the shift $q/2e \rightarrow q/2e+1$. For $q=0$ Eq. (40) reduces to Eq. (7). From Fig. 1 one readily sees that there is no superconductivity for $\alpha<1$. Due to the periodicity of Eq. (40) this holds for any integer q. For $q/2e$ equal to $1/2$ one gets Eq. (11). From Fig. 2 one sees that superconductivity is attained for all the values of α, due to the effect of offset charges; the superconducting order parameter at zero temperature is different from zero.

For the self-charging model the system exhibits superconductivity for all the values of α also if the distribution of offset charges is such that integer and half-integer charges coexist on the lattice. If one denotes by f_0 the fraction of integer charges and by $f_{1/2}=1-f_0$ the fraction of half-integer charges, Eq. (40) implies that

$$\alpha=\left(f_0\frac{\sum_{n}e^{-(4/y)n^{2}}\frac{1}{1-4n^{2}}}{\sum_{m}e^{-(4/y)m^{2}}}+f_{1/2}\frac{\sum_{n}e^{-(4/y)(n+1/2)^{2}}\frac{1}{1-4(n+1/2)^{2}}}{\sum_{m}e^{-(4/y)(m+1/2)^{2}}}\right)^{-1}.$$

In Fig. 3 we plot T_c as a function of α for several values of f_0. As expected superconductivity is enhanced as $f_{1/2}$ increases.

B. Models with nondiagonal capacitance matrix

In Ref. 21 Fishman and Stroud, using a low temperature expansion, determined T_c as a function of α for models with nondiagonal interaction matrix without considering the effect of offset charges. They did not find signs of normal state reentrance for nearest-neighbor interaction matrix models in which only the diagonal interaction matrix element $U_{\mathbf{00}}$ and the nearest-neighbor interaction matrix element $U_{\mathbf{0p}}=\theta U_{\mathbf{00}}$ are nonzero. This can be seen from the expansion of the critical line Eq. (39) for $q=0$ and small critical temperatures:

$$\alpha=1+\left[\frac{8}{3}+2z\left(1-\frac{1}{1-4\theta^{2}}\right)\right]e^{-4/y}+\cdots.$$

Reentrant behavior is possible[12] for $\theta>\theta_c=1/\sqrt{4+3z}$ when the coefficient of the exponential $e^{-4/y}$ is negative; in fact, the phase boundary line $\alpha=\alpha(T_c)$ first bends to the left due to the negative coefficient of $e^{-4/y}$ and finally, when the critical temperature is high enough, bends to the right, favoring the insulating phase.

As evidenced by Fishman and Stroud,[21] the regime of physical interest is $\theta<1/z$; namely, when the capacitance matrix is invertible. Reentrance is possible only in one dimension ($\theta_c=1/\sqrt{10}<1/z=1/2$); in higher dimensions reentrance occurs only when the electrostatic interaction is long ranged.[21]

If there are half-integer offset charges on the sites of a square lattice, our analysis shows that the equation for the critical line is

$$\alpha=\sum_{[n_{\mathbf{i}}]} e^{-4/y\,\Sigma_{\mathbf{ij}}(U_{\mathbf{ij}}/U_{\mathbf{00}})(n_{\mathbf{i}}+1/2)(n_{\mathbf{j}}+1/2)} \Bigg/ \sum_{[n_{\mathbf{i}}]} \frac{\exp^{(-4/y)\,\Sigma_{\mathbf{ij}}(U_{\mathbf{ij}}/U_{\mathbf{00}})\,(n_{\mathbf{i}}+1/2)(n_{\mathbf{j}}+1/2)}}{1-4\left[\sum_{\mathbf{j}} \frac{U_{\mathbf{0j}}}{U_{\mathbf{00}}}\left(n_{\mathbf{j}}+\frac{1}{2}\right)\right]^2}, \tag{41}$$

where

$$E[n_{\mathbf{i}}]=\sum_{\mathbf{i}\,\mathbf{j}} \frac{U_{\mathbf{ij}}}{U_{\mathbf{00}}}\left(n_{\mathbf{i}}+\frac{1}{2}\right)\left(n_{\mathbf{j}}+\frac{1}{2}\right) \tag{42}$$

is the electrostatic energy of any charge distribution on the lattice.

Denoting with $n_{\mathbf{i}}^0$ and $n_{\mathbf{i}}^1$ the charge distributions of the two lowest lying energy states and with E^0 and E^1 the corresponding energies, the low temperature expansion of Eq. (41) yields

$$\alpha=\frac{\sum_{[n^0]} e^{-(4/y)\,E^0}+\sum_{[n^1]} e^{-(4/y)\,E^1}+\cdots}{\sum_{[n^0]}\left\{e^{-(4/y)\,E^0}\Bigg/\left\{1-4\left[\sum_{\mathbf{j}}\frac{U_{\mathbf{0j}}}{U_{\mathbf{00}}}\left(n_{\mathbf{j}}^0+\frac{1}{2}\right)\right]^2\right\}\right\}+\sum_{[n^1]}\left\{e^{-(4/y)\,E^1}\Bigg/\left\{1-4\left[\sum_{\mathbf{j}}\frac{U_{\mathbf{0j}}}{U_{\mathbf{00}}}\left(n_{\mathbf{j}}^1+\frac{1}{2}\right)\right]\right\}\right\}^2+\cdots}. \tag{43}$$

Independently on the explicit form of $U_{\mathbf{ij}}$, $E[n_{\mathbf{i}}]$ reaches its minimum value when $(n_{\mathbf{i}}^0+\frac{1}{2})=\pm\frac{1}{2}(-1)^{i_1+i_2+\cdots+i_D}$ with i_j $(j=1,\ldots,D)$ the components of the lattice position vector $\mathbf{i}$ in units of the lattice spacing. This charge configuration is exhibited in Fig. 4. For models with nearest-neighbor interaction, i.e., $U_{\mathbf{ij}}=\delta_{\mathbf{ij}}+\theta\Sigma_{\mathbf{p}}\delta_{\mathbf{i}+\mathbf{p},\mathbf{j}}$ with $\Sigma_{\mathbf{p}}$ denoting summation over nearest neighbors, the charge configuration corresponding to the first excited state is given in Fig. 5. The energy of the charge distribution of Fig. 5 is $E[n_{\mathbf{i}}^1]=E[n_{\mathbf{i}}^0]+z\theta$, where $E[n_{\mathbf{i}}^0]$, the ground state energy, is given by $\Sigma_{\mathbf{i}}\frac{1}{4}(1-z\theta)$.

With the above values of $E[n_{\mathbf{i}}^0]$ and $E[n_{\mathbf{i}}^1]$ and keeping only the leading order term in T_c, Eq. (43) becomes [see Appendix C]

$$\alpha=[1-(1-z\theta)^2]\left\{1+\left[\left(1-\frac{1-(1-z\theta)^2}{1-(1+z\theta)^2}\right)+z\left(1-\frac{1-(1-z\theta)^2}{1-[1-(z-2)\theta]^2}\right)\right]e^{-(4/y)z\theta}+\cdots\right\}. \tag{44}$$

Reentrant behavior at low temperature occurs when the coefficient of the exponential is negative, namely, when

$$a_1\equiv\left(1-\frac{1-(1-z\theta)^2}{1-(1+z\theta)^2}\right)+z\left(1-\frac{1-(1-z\theta)^2}{1-[1-(z-2)\theta]^2}\right)<0. \tag{45}$$

In Appendix C we compute also the coefficients a_2 and a_3 of the higher order exponentials in the expansion (44). In Fig. 6 we plot the coefficients a_1, a_2, and a_3 as a function of θ for $z=6$, i.e., for a three-dimensional array on a square lattice. One sees that the inequality (45) can be satisfied for values of θ consistent with the physical constraint $\theta<1/z=1/6$.

In Fig. 7 we plot T_c versus α for $\theta=0.05$ and $z=6$. In this plot we keep into account also the next two orders of Eq. (44) with coefficients a_2 and a_3. The resulting diagram exhibits reentrance in the insulating phase even for models with nearest neighbors interaction.

In Fig. 8 we plot $\alpha_0=\alpha(T_c=0)$ as a function of θ for q integer and q half-integer and for $z=6$. The plot shows that half-integer offset charges always favor superconductivity and that—at variance with the self-charging model—for non-diagonal interaction matrix there is always a range of α in which the system behaves as an insulator. The plot also shows that for $q/2e=1/2$ and $T=0$ the size of the superconducting region in the phase diagram depends on θ.

V. DISCUSSION

In this paper we investigated—using the path integral approach to finite temperature mean field theory—the phenomenon of reentrant superconductivity in a variety of models of JJA's. For a model with nearest-neighbor capacitance matrix and uniform offset charge $q=1/2$ (in units of $2e$), following the analysis developed in Ref. 21, we determined, in the

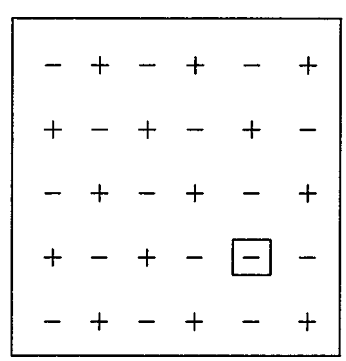

FIG. 5. First excited state.

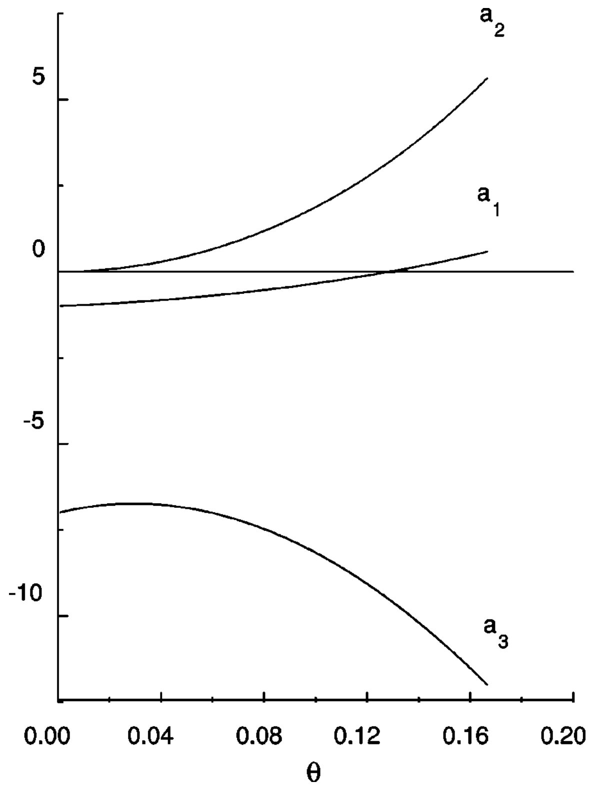

FIG. 6. Expansion coefficients of α as a function of nearest-neighbor interaction θ.

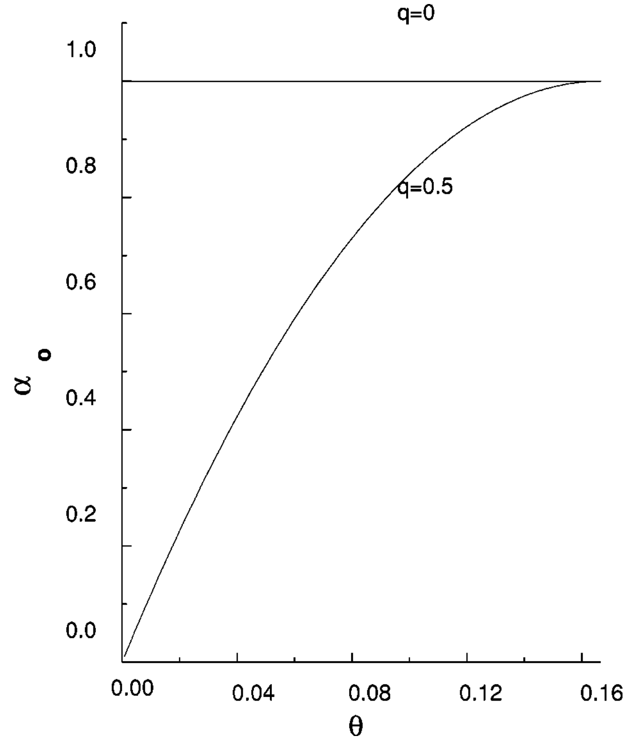

FIG. 8. Broadening of the superconducting phase at $T=0$ with $z=6$ and nearest-neighbor interaction.

low-temperature expansion, the most relevant contributions to the equation for the phase boundary. For this purpose we explicitly constructed the charge distributions on the lattice corresponding to the lowest energies. Confirming the results of the numerical analysis of Ref. 8, we evidenced the appearance of reentrant superconductivity even when the capacitive interaction is short ranged. Our analysis extends the results

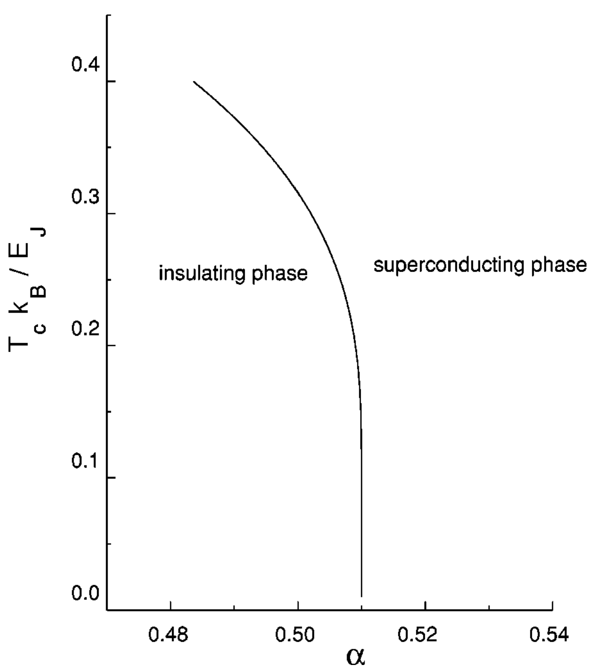

FIG. 7. Phase diagram for small critical temperatures with $z=6$ and $\theta=0.05$.

found in Ref. 21 to the situation in which offset charges are present and provides a physical picture of the states contributing to the reentrant behavior.

For a model with diagonal capacitance matrix our analysis confirms the absence of reentrant behavior for the physical situation where the phase variable is 2π periodic. The diagonal model with offset charge $q=1/2$ exhibits superconductivity for all the values of $\alpha=zE_J/4E_C$, since in this case the superconducting order parameter is different from zero at zero temperature; this is evidenced by Eq. (53) in Appendix A. An offset charge $q=1/2$ tends to decrease the charging energy and thus favors the superconducting behavior even for small Josephson energy E_J.

The search for reliable theoretical approaches to establish the existence of a reentrant bulge in the phase boundary of quantum Josephson junctions arrays is strongly encouraged by the evidence of low temperature instabilities found experimentally in Josephson junctions arrays,[32] ultrathin amorphous films,[33] granular superconductors,[34] and multiphase high-T_c systems.[35] Our analysis not only clarifies issues still open for quantum phase models with a diagonal capacitance matrix and uniform offset charges, but also predicts the appearance of a reentrant bulge—even if the interaction among grains is short ranged—for quantum Josephson junction arrays with non diagonal capacitance matrix and uniform offset charges.

It would be interesting to investigate the superconducting-insulating behavior in quantum JJA's in lower dimensional models, where mean field theory is not expected to provide accurate results, as well as in models in which offset charges are randomly distributed. For $D=1$ there is evidence[6] for a new phase separating the superconducting and the insulating phase. The analysis of the phase diagram for this case should be carried out with different methods such as the renormalization group[15,17] or strong coupling expansion.[6]

Note added. After completion of this paper we became aware of the paper by T. K. Kopec and J. V. José, cond-mat/9903222, analyzing the functional approach for quantum JJA at zero temperature.[36]

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF THE SELF-CONSISTENCY EQUATION (11)

With a uniform charge frustration q the pertinent Mathieu equation is given by

$$\left[-\frac{d^2}{d\varphi^2}-2i\frac{q}{2e}\frac{d}{d\varphi}+\left(\frac{q}{2e}\right)^2-\alpha\langle\cos\varphi\rangle\cos\varphi\right]\psi_n(\varphi)=\frac{E_n}{4E_C}\psi_n(\varphi). \tag{A1}$$

Upon defining

$$\psi_n(\varphi)=e^{-i(q/2e)\varphi}\rho_n(\varphi) \tag{A2}$$

Eq. (A1) becomes

$$\frac{d^2\rho_n}{d\varphi^2}+\left(\frac{\lambda}{4}-\frac{v}{2}\cos\varphi\right)\rho_n=0 \tag{A3}$$

with $\lambda=E_n/E_C$ and $v=-zE_J\langle\cos\varphi\rangle/2E_C$. Equation (A3) yields the canonical form of the Mathieu equations[27]

$$\frac{d^2y}{dx^2}+(\lambda-2v\cos 2x)y=0, \tag{A4}$$

if one puts $\varphi=2x$ e $\psi_n=y$.

The Mathieu equation has the well known periodic solutions[27]

$ce_{2n}(x,v)$ even solutions with period π with eigenvalues $a_{2n}(v)$,

$se_{2n+2}(x,v)$ odd solutions with period π with eigenvalues $b_{2n+2}(v)$,

$ce_{2n+1}(x,v)$ even solutions with period 2π with eigenvalues $a_{2n+1}(v)$,

$se_{2n+1}(x,v)$ odd solutions with period 2π with eigenvalues $b_{2n+1}(v)$,

$n=0,1,2,\ldots$.

If $q/2e$ is integer, the periodic boundary conditions $\psi_n(\varphi=0)=\psi_n(\varphi=2\pi)$ singles out only the 2π-periodic Mathieu eigenfunctions ce_{2n},se_{2n}. With these eigenfunctions one may derive Eq. (7).[10] If $q/2e$ is half-integer, the periodic boundary conditions together with Eq. (A3) single out the π-anti-periodic Mathieu eigenfunctions (i.e., ρ_n is antiperiodic of 2π and periodic of 4π). These are the Mathieu eigenfunctions ce_{2n+1} and se_{2n+1}.

Since, near the critical temperature T_c, the order parameter $\langle\cos\varphi\rangle$ and v are small, apart from the phase factor $e^{-i\varphi/2}$ (important only for the periodicity), to first order in v, Eq. (A1) has the solutions

$$\psi_1^e=\frac{1}{\sqrt{\pi}}\left(\cos\frac{\varphi}{2}-\frac{v}{8}\cos\frac{3\varphi}{2}\right),$$

$$\psi_1^o=\frac{1}{\sqrt{\pi}}\left(\sin\frac{\varphi}{2}-\frac{v}{8}\sin\frac{3\varphi}{2}\right),$$

$$\psi_{2n+1}^e=\frac{1}{\sqrt{\pi}}\left\{\cos\frac{(2n+1)\varphi}{2}-v\left[\frac{\cos\frac{(2n+3)\varphi}{2}}{4(2n+2)}-\frac{\cos\frac{(2n-1)\varphi}{2}}{8n}\right]\right\}, \tag{A5}$$

$$\psi_{2n+1}^o=\frac{1}{\sqrt{\pi}}\left\{\sin\frac{(2n+1)\varphi}{2}-v\left[\frac{\sin\frac{(2n+3)\varphi}{2}}{4(2n+2)}-\frac{\sin\frac{(2n-1)\varphi}{2}}{8n}\right]\right\},$$

$$(n=1,2,\ldots),$$

with the corresponding eigenvalues given by

$$E_1^e=E_C(1+q),$$

$$E_1^o=E_C(1-q), \tag{A6}$$

$$E_{2n+1}^e=E_{2n+1}^o=E_C(2n+1)^2,$$

$$(n=1,2,\ldots).$$

The expectation values of the superconducting order parameter on the eigenfunctions (A5) are given by

$$\langle\psi_n|\cos\varphi|\psi_n\rangle=\int_0^{2\pi}d\varphi\cos\varphi|\psi_n(\varphi)|^2. \tag{A7}$$

Using Eq. (A5), to the first order in v one gets

$$\langle \psi_1^e|\cos\varphi|\psi_1^e\rangle = \frac{1}{2} - \frac{v}{8},$$

$$\langle \psi_1^o|\cos\varphi|\psi_1^o\rangle = -\frac{1}{2} - \frac{v}{8},$$

$$\langle \psi_{2n+1}^e|\cos\varphi|\psi_{2n+1}^e\rangle = \frac{v}{8n(n+1)}, \tag{A8}$$

$$\langle \psi_{2n+1}^o|\cos\varphi|\psi_{2n+1}^o\rangle = \frac{v}{8n(n+1)},$$

$$(n=1,2,\dots).$$

Inserting Eqs. (A6) and (A8) in Eq. (6) and keeping only the terms proportional to $v \sim \langle\cos\varphi\rangle$, one finds

$$1 = \alpha \frac{\left(\frac{2}{y}+\frac{1}{2}\right)e^{-1/y} - \sum_{n=1}^{\infty} \frac{e^{-(1/y)(2n+1)^2}}{2n(n+1)}}{2e^{-1/y} + 2\sum_{n=1}^{\infty} e^{-(1/y)(2n+1)^2}}; \tag{A9}$$

namely, Eq. (11).

APPENDIX B: THE PHASE CORRELATOR

In this appendix to keep the paper self-contained we report the computation of the correlator defined in Eq. (25). One should compute the path integral

$$\frac{\int D\phi_{\mathbf{i}} e^{i\phi_{\mathbf{r}}(\tau) - i\phi_{\mathbf{s}}(\tau')} \exp\left\{\int_0^\beta d\tau\left(-\frac{1}{2}C_{\mathbf{ij}}\frac{\dot{\phi}_{\mathbf{i}}}{2e}\frac{\dot{\phi}_{\mathbf{j}}}{2e}\right)\right\}}{\int D\phi_{\mathbf{i}} \exp\left\{\int_0^\beta d\tau\left(-\frac{1}{2}C_{\mathbf{ij}}\frac{\dot{\phi}_{\mathbf{i}}}{2e}\frac{\dot{\phi}_{\mathbf{j}}}{2e}\right)\right\}}. \tag{B1}$$

Fourier transforming $\phi_{\mathbf{i}}(\tau)$ according to

$$\phi_{\mathbf{i}}(\tau) = \frac{1}{\beta}\sum_{n=-\infty}^{+\infty} \phi_{\mathbf{i},m} e^{i\omega_m \tau} \tag{B2}$$

with $0 \leqslant \tau \leqslant \beta$ and $\omega_m = (2\pi/\beta)m$, the numerator of (B1) becomes

$$\int \prod_{\mathbf{i}} d\phi_{\mathbf{i},0} \prod_{n=1}^{\infty} d\phi_{\mathbf{i},n} d\phi_{\mathbf{i},n}^* \exp\left\{-\frac{1}{4e^2\beta}\right.$$
$$\times \sum_{\mathbf{ij}}\sum_{n=1}^{+\infty} C_{\mathbf{ij}}\omega_n^2 \phi_{\mathbf{i},n}\phi_{\mathbf{j},n}^* + \frac{i}{\beta}\sum_{n=1}^{\infty}(\phi_{\mathbf{r},n}e^{i\omega_n\tau}$$
$$\left. -\phi_{\mathbf{s},n}^* e^{-i\omega_n\tau'}) + \frac{i}{\beta}(\phi_{\mathbf{r},0} - \phi_{\mathbf{s},0}) + \text{c.c.}\right\}. \tag{B3}$$

Upon integrating over the components $\phi_{\mathbf{r},0}, \phi_{\mathbf{s}0}$ one gets a factor $\delta_{\mathbf{rs}}$

$$\left(\prod_{\mathbf{i}\neq\mathbf{r},\mathbf{s}} \int_{-\infty}^{\infty} d\phi_{\mathbf{i},0}\right)\left(\int_{-\infty}^{\infty} d\phi_{\mathbf{r},0}\int_{-\infty}^{\infty} d\phi_{\mathbf{s},0} e^{(i/\beta)(\phi_{\mathbf{r},0}-\phi_{\mathbf{s},0})}\right)$$
$$= \delta_{\mathbf{rs}}\cdot K, \tag{B4}$$

where K is an irrelevant divergent constant which cancels against the denominator. Using Eq. (B4), Eq. (B3) becomes

$$K\delta_{\mathbf{rs}}\prod_{n=1}^{\infty}\int_{-\infty}^{\infty}\prod_{\mathbf{i}} d\phi_{\mathbf{i},n}d\phi_{\mathbf{i},n}^* \exp\left(-\frac{1}{4e^2\beta}\sum_{\mathbf{ij}} C_{\mathbf{ij}}\omega_n^2\phi_{\mathbf{i},n}\phi_{\mathbf{j},n}^*\right.$$
$$+\sum_{\mathbf{i}}\phi_{\mathbf{i},n}\frac{i}{\beta}\delta_{\mathbf{ri}}(e^{i\omega_n\tau} - e^{i\omega_n\tau'})$$
$$\left.-\sum_{\mathbf{i}}\phi_{\mathbf{i},n}^*\delta_{\mathbf{ri}}\frac{i}{\beta}(e^{-i\omega_n\tau'} - e^{-i\omega_n\tau'})\right).$$

The multiple Gaussian integral may be easily computed to give, up to an irrelevant constant which cancels against the denominator,

$$\delta_{\mathbf{rs}}\prod_{\mathbf{i}}\prod_{n=1}^{\infty}\int_{-\infty}^{\infty} d\phi_{\mathbf{i}n}d\phi_{\mathbf{i}n}^* \exp\left\{\sum_{\mathbf{ij}}\frac{i}{\beta}\delta_{\mathbf{ri}}(e^{i\omega_n\tau} - e^{i\omega_n\tau'})\right.$$
$$\left.\times\left(\frac{4e^2\beta C_{\mathbf{ij}}^{-1}}{\omega_n^2}\right)\frac{i}{\beta}\delta_{\mathbf{ri}}(e^{-i\omega_n\tau} - e^{-i\omega_n\tau'})\right\}$$
$$=\delta_{\mathbf{rs}}\exp\left\{\frac{8e^2C_{\mathbf{rr}}^{-1}}{\beta}\sum_{n=1}^{\infty}\left(\frac{1-\cos\omega_n(\tau-\tau')}{\omega_n^2}\right)\right\}$$
$$=\delta_{\mathbf{rs}}\exp\left\{-2e^2C_{\mathbf{rr}}^{-1}\left(|\tau-\tau'| - \frac{(\tau-\tau')^2}{\beta}\right)\right\},$$

where $-\beta \leqslant \tau - \tau' \leqslant \beta$. In the last step, the identity

$$|x| - \frac{x^2}{\beta} = \sum_{n=1}^{\infty}\left(\frac{4}{\beta\omega_n^2} - \frac{4\cos\omega_n x}{\beta\omega_n^2}\right), \quad -\beta\leqslant x\leqslant\beta \tag{B5}$$

has been used. This completes the proof of Eq. (26)

APPENDIX C: LOW T_c EXPANSION

In this appendix we derive Eq. (44) and compute the next two orders whose coefficients are plotted in Fig. 6. Using the notation $(-1)^{\mathbf{i}} = (-1)^{i_1+\cdots+i_D}$, the ground state charge configuration $n_{\mathbf{i}}^0$ can be written as

$$\left(n_{\mathbf{i}}^0 + \frac{1}{2}\right) = \frac{1}{2}(-1)^{\mathbf{i}}.$$

The first excited states read

$$\left(n_{\mathbf{i}}^{1_\mathbf{r}} + \frac{1}{2}\right) = n_{\mathbf{i}}^0(1 - 2\delta_{\mathbf{ri}}),$$

where the apex $1_\mathbf{r}$ means that this first excited state is obtained from the ground state by flipping the sign of the charge at the site $\mathbf{r}$. Higher excitations may be obtained from the ground state by flipping the sign of two charges at sites $\mathbf{r}$ and $\mathbf{s}$ and can be represented as

$$\left(n_{\mathbf{i}}^{2_{\mathbf{rs}}} + \frac{1}{2}\right) = n_{\mathbf{i}}^0(1 - 2\delta_{\mathbf{ri}} - 2\delta_{\mathbf{si}}).$$

The energy shifts are given by

$$\Delta^1 = E^1 - E^0 = \sum_{\substack{\mathbf{i} \\ \mathbf{i}\neq\mathbf{r}}} U_{\mathbf{ir}}(-1)^{\mathbf{r}+\mathbf{i}+1}$$

and

$$\Delta^{2\mathbf{rs}} = E[n_{\mathbf{i}}^{2\mathbf{rs}}] - E^0 = 2\Delta^1 + 2(-1)^{\mathbf{r}-\mathbf{s}} U_{\mathbf{rs}}.$$

Note that, whereas the energy E^1 of the charge configurations $n_{\mathbf{i}}^{1\mathbf{r}}$ does not depend on $\mathbf{r}$, $E[n_{\mathbf{i}}^{2\mathbf{rs}}]$ depends on the relative position $\mathbf{r}-\mathbf{s}$ of the charges whose sign has been flipped.

Defining

$$R^0 = \frac{1}{1 - 4\left[\sum_{\mathbf{j}} U_{\mathbf{0j}}\left(n_{\mathbf{j}}^{0\pm} + \frac{1}{2}\right)\right]^2},$$

$$R^{1\mathbf{r}} = \frac{1}{1 - 4\left[\sum_{\mathbf{j}} U_{\mathbf{0j}}\left(n_{\mathbf{j}}^{1\mathbf{r}} + \frac{1}{2}\right)\right]^2},$$

and

$$R^{2\mathbf{rs}} = \frac{1}{1 - 4\left[\sum_{\mathbf{j}} U_{\mathbf{0j}}\left(n_{\mathbf{j}}^{2\mathbf{rs}} + \frac{1}{2}\right)\right]^2},$$

one may expand Eq. (41) for small critical temperatures ($y \propto T_c \to 0$), according to

$$\alpha = \frac{1 + \sum_{\mathbf{r}} e^{-(4/y)\Delta^1} + \sum^{*}_{\mathbf{r}\neq\mathbf{s}} e^{-(4/y)\Delta^{2\mathbf{rs}}} + \cdots}{R^0 + \sum_{\mathbf{r}} R^{1\mathbf{r}} e^{-(4/y)\Delta^1} + \sum^{*}_{\mathbf{r}\neq\mathbf{s}} R^{2\mathbf{rs}} e^{-(4/y)\Delta^{2\mathbf{rs}}} + \cdots}$$
$$= \frac{1}{R^0}\left[1 + \sum_{\mathbf{r}} \left(1 - \frac{R^{1\mathbf{r}}}{R^0}\right) e^{-(4/y)\Delta^1} + \sum^{*}_{\mathbf{r}\neq\mathbf{s}} \left(1 - \frac{R^{2\mathbf{rs}}}{R^0}\right) e^{-(4/y)\Delta^{2\mathbf{rs}}} + \sum_{\mathbf{rs}} \left(\frac{R^{1\mathbf{r}}}{R^0}\frac{R^{1\mathbf{s}}}{R^0} - \frac{R^{1\mathbf{r}}}{R^0}\right) e^{-(8/y)\Delta^1} + \cdots\right], \qquad \text{(C1)}$$

where $\Sigma^{*}_{\mathbf{r}\neq\mathbf{s}}$ indicates a summation over pairs of different sites $\mathbf{r},\mathbf{s}$, where each pair is counted only once.

For a nearest-neighbor interaction $U_{\mathbf{0j}} = \delta_{\mathbf{0j}} + \theta \Sigma_{\mathbf{p}} \delta_{\mathbf{jp}}$ (where $\mathbf{p}$ denotes the vector connecting two neighboring sites) one has

$$\Delta^1 = z\theta,$$

$$\Delta^{2\mathbf{rs}} = \begin{cases} 2(z-1)\theta, & \mathbf{r}-\mathbf{s}=\mathbf{p}, \\ 2z\theta, & \mathbf{r}-\mathbf{s}\neq\mathbf{p}, \end{cases}$$

$$R^0 = \frac{1}{1-(1-z\theta)^2},$$

$$R^{1\mathbf{r}} = \begin{cases} R^0, & \mathbf{r}\neq\mathbf{0},\mathbf{p}, \\ \dfrac{1}{1-(1+z\theta)^2}, & \mathbf{r}=\mathbf{0}, \\ \dfrac{1}{1-[1-(z-2)\theta]^2}, & \mathbf{r}=\mathbf{p}, \end{cases}$$

$$R^{2\mathbf{rs}} = \begin{cases} R^0, & \mathbf{r},\mathbf{s}\neq\mathbf{0},\mathbf{p}, \\ R^{1\mathbf{s}}, & \mathbf{r}\neq\mathbf{0},\mathbf{p}, \\ \dfrac{1}{1-[1+(z-2)\theta]^2}, & \mathbf{r}=\mathbf{0}\ \mathbf{s}=\mathbf{p}, \\ \dfrac{1}{1-[1-(z-4)\theta]^2}, & \mathbf{r}=\mathbf{p}\ \mathbf{s}=\mathbf{p}'. \end{cases}$$

Substituting these relations in (C1), one obtains the expansion for small temperatures of the critical line equation, up to the first four orders

$$\alpha = [1-(1-z\theta)^2](1 + a_1 e^{-(4/y)z\theta} + a_2 e^{-(8/y)(z-1)\theta} + a_3 e^{-(8/y)z\theta}). \qquad \text{(C2)}$$

a_1 is given in Eq. (45), a_2 is equal to

$$(z-1)z\left(1 - \frac{1-(1-z\theta)^2}{1-[1-(z-2)\theta]^2}\right) + z\left(1 - \frac{1-(1-z\theta)^2}{1-[1+(z-2)\theta]^2}\right)$$

and a_3 is given by

$$\left(\frac{1-(1-z\theta)^2}{1-(1+z\theta)^2}\right)^2 - \left(\frac{1-(1-z\theta)^2}{1-(1+z\theta)^2}\right) + z(z-1) \times \left(\frac{1-(1-z\theta)^2}{1-[1-(z-2)\theta]^2} - 1\right)$$
$$+ z^2 \frac{1-(1-z\theta)^2}{1-[1-(z-2)\theta]^2}\left(\frac{1-(1-z\theta)^2}{1-[1-(z-2)\theta]^2} - 1\right) + z\frac{1-(1-z\theta)^2}{1-(1+z\theta)^2}\left(\frac{1-(1-z\theta)^2}{1-[1-(z-2)\theta]^2} - 1\right)$$
$$+ z\frac{1-(1-z\theta)^2}{1-[1-(z-2)\theta]^2)}\left(\frac{1-(1-z\theta)^2}{1-(1+z\theta)^2} - 1\right) + \frac{z(z-1)}{2}\left(1 - \frac{1-(1-z\theta)^2}{1-[1-(z-4)\theta]^2}\right).$$

The condition for the reentrant behavior is $a_1 < 0$. In Fig. 6 we plot the coefficients a_1, a_2, a_3 as a function of θ. In Fig. 7 we plot the critical equation (C2) with $\theta = 0.05$ and $z = 6$.

*Present address: International School for Advanced Studies, 34014, Trieste, Italy.

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