Quasiperiodicity, bistability, and chaos in the Landau-Lifshitz equation

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The dynamics of an individual magnetic moment is studied through the Landau-Lifshitz equation with a periodic driving in the direction perpendicular to the applied field. For fields lower than the anisotropy field and small values of the perturbation amplitude we have observed the magnetic moment bistability. At intermediate values we have found quasiperiodic bands alternating with periodic motion. At even larger values a chaotic regime is found. When the applied field is larger than the anisotropy one, the behavior is periodic with quasiperiodic regions. Those appear periodically in the amplitude of the oscillating field. Also, even for low values of the driving force, the moment is not parallel to the applied field.

I. INTRODUCTION

Traditionally, the study of the dynamics governed by the Landau-Lifshitz equation is related to the ferromagnetic resonance problems.[1] Recently, the spin dynamics has also become important in other physical phenomena relevant to technological applications, such as, e.g., magnetic recording processes[2,3] due to a continuous increase of the magnetic recording density together with the writing frequency.[3,4] While the writing frequency is approaching the values corresponding to that of the precessional motion, the actual magnetization dynamics becomes more and more important. This complicated dynamics may arise, e.g., during a process of fast magnetization switching.[5] The dynamical micromagnetic calculations have provided a useful tool in studying such important media characteristic as dynamical coercitivity.[6] In spite of the fact that the Landau-Lifshitz equation is widely used in micromagnetic calculations,[5–8] to our knowledge, no systematic study of its dynamics exists in the literature. Let us recall here that this equation is nonlinear, and in some regime one may expect a highly complicated dynamics similar to one arising for an externally driven pendulum. As an example, we can mention the nonlinear stochastic resonance behavior of an individual magnetic moment.[9,10] The purpose of this paper is to present a systematic study of nonlinear dynamics governed by the Landau-Lifshitz equation, including its bifurcation diagram and stability properties. This dynamical behavior is going to be relevant in studying an ensemble of noninteracting Stoner-Wolfarth particles.[11] Also it would be very useful when analyzing results obtained from large simulations of coupled Landau-Lifshitz (LL) equations. There, and for some values of the coupling parameters, the individual characteristics of each magnetic moment may play an important role in the collective behavior. Eventually, when enough of these moments are coupled, a description in terms of magnons would be possible.[12]

Also the subject of chaos in magnetic materials[13] is not new. It has been studied in YIG, both experimentally and theoretically, through spin-wave descriptions.[14,15] In different driving regime, the chaotic behavior can arise in magnetostrictive wires and ribbons due to the magnetoelastic coupling.[16]

The paper is organized as follows. In Sec. II we present the equations and explain the method we use to solve them numerically. The relevant results are presented in Sec. III. In Sec. III we discuss the physical aspects of our results and in Sec. IV outline the conclusions.

II. NUMERICAL PROCEDURE

In the original form the Landau-Lifshitz-Gilbert[17] equation may be written as

$$\frac{d\mathbf{M}}{dt} = -g(\mathbf{M}\times\mathbf{H}_{\mathrm{eff}}) + \frac{\eta}{M_0}\left(\mathbf{M}\times\frac{d\mathbf{M}}{dt}\right), \tag{1}$$

or in the more practical form (Landau-Lifshitz[19])

$$\frac{1+\eta^2}{g}\frac{d\mathbf{M}}{dt} = -(\mathbf{M}\times\mathbf{H}_{\mathrm{eff}}) - \frac{\eta}{M_0}[\mathbf{M}\times(\mathbf{M}\times\mathbf{H}_{\mathrm{eff}})], \tag{2}$$

where g is the local gyromagnetic factor, η is the damping, M_0 the saturation magnetization, $\mathbf{M}$ is the tridimensional local continuous magnetization,[18] whose module is conserved $(\mathbf{M}\cdot\mathbf{M}=M_0)$, and $\mathbf{H}_{\mathrm{eff}}$ the effective field:

$$\mathbf{H}_{\mathrm{eff}} = \mathbf{H}_{\mathrm{ext}} + \beta\mathbf{n}(\mathbf{n}\cdot\mathbf{M}). \tag{3}$$

Here $\mathbf{H}_{\mathrm{ext}}$ is the external magnetic field, β is the anisotropy coefficient and $\mathbf{n}$ is the unitary vector pointing in the anisotropy direction.

As we will deal with only one local magnetic moment we will not consider exchange and dipolar interactions. This description could be relevant to dynamics of bistable magnetic microwires and ribbons,[16] as well as noninteracting Stoner-Wolfarth particles.

The external field will be decomposed in two parts: $\mathbf{H}_n$, constant and parallel to the anisotropy direction, and $\mathbf{h}=\mathbf{h}_0 \cdot \sin(\omega t)$, perpendicular to $\mathbf{n}$ and oscillating in time with frequency ω.

For practical purposes we rewrite the equation in dimensionless form and expanded notation:

$$\kappa \dot{m}_x = -[m_y(h_z+m_z) - m_z h_y] - \eta[m_x m_z(h_z+m_z) + m_x m_y h_y + (m_x^2-1)h_x], \tag{4a}$$

$$\kappa \dot{m}_y = [m_x(h_z+m_z) - m_z h_x] - \eta[m_y m_z(h_z+m_z) + m_x m_y h_x + (m_y^2-1)h_y], \tag{4b}$$

$$\kappa \dot{m}_z = -[m_x(h_y - m_y h_x) - \eta m_x m_z h_x + m_y m_z h_y + (m_z^2-1) \times (h_z+m_z)], \tag{4c}$$

where $\mathbf{m}=\mathbf{M}/M_0$, $\mathbf{h}=\mathbf{H}_{\rm ext}/(\beta M_0)$, $\kappa=1+\eta^2$, and the dot represents the derivative with respect to dimensionless time $\tau=g\beta M_0 t$. We take the z axis in the direction of $\mathbf{n}$ and the x axis in the direction of $\mathbf{h}_0$, which makes $h_y=0$.

These equations with appropriate initial conditions (over 400) have been solved by using a fourth order Runge-Kutta scheme. In Eqs. (4a)–(4c) only two of them are independent because of the constant magnetization constraint. If we choose polar coordinates, we reduce the degrees of freedom to 2 and the constant magnetization constraint is automatically fulfilled. The problem arises when the polar angle becomes zero or π, and the azimuthal angle cannot be defined. This problem is overcome by using two different reference frames, one with the polar axis pointing to the north pole and the other with the polar axis pointing to the south pole. In order to avoid this computational trouble we choose Cartesian coordinates and integrate the full Eqs. (4a)–(4c). Also we have taken initial conditions over all the sphere and have observed that the condition that $|\mathbf{m}|=1$ is fulfilled with a precision of more than eight orders of magnitude (10^{-8} in 1), for even more than 10^9 integration steps. In Eqs. (4a)–(4c) there are several parameters which are, in principle, free in our calculation: η, h_z, h, and ω. We have fixed η and the bias field h_z and we have chosen as control parameters the amplitude and frequency of the perturbation (which is reasonable from the experimental point of view). Of course the values of η and h_z that we have taken for our calculations are arbitrary and may be changed. Different values for the parameters will give different behaviors (see Sec. III). In what follows we have taken $\eta=0.05$ and $h_z=0.1$. Sweeping in frequency, we have found that the interesting ones are those close to the resonance frequency $\omega_r=1$ in our units, and of the order of several GHz in real units. Thus possible perturbing signals are radio-wave sources. For simplicity in what follows will put the perturbing frequency fixed to the resonance, and sweep in the amplitude, $h=(h_x^2+h_y^2)^{1/2}$.

III. RESULTS

A typical bifurcation diagram is shown in Fig. 1. Here only the components m_θ and m_ϕ are drawn, though in the numerical simulations we followed Eqs. (4a)–(4c), as explained before. The diagram shows the values of the components of $\mathbf{m}$, at time interval multiples of $T=2\pi/\omega$ (Poincaré sections), with the value h shown in the x axis of the figure. There, various kinds of behaviors can be distinguished: periodic, quasiperiodic, and chaotic motion. When there is only one point for a given value of h, it represents a periodic motion with period T; and when there is a continuum of points the behavior is quasiperiodic or chaotic. We will now try to describe the principal features of different critical points shown in Fig. 1. The changes in the diagram found when the dissipation (η) is changed, are mainly quantitative (it changes the value of h at which a given critical behavior is found). Also, the diagram shown has been produced for the magnetic moment pointing in the direction of the external field at $t=0$. Although many initial conditions were considered, this case has been chosen as typical, also and especially because it corresponds to an experimentally reasonable situation. Different initial condition, depending on the basin of attraction, can lead to a slightly different picture. For example, the intermediate odd-period solutions (like period three or period seven in Fig. 1) which normally have a small basin of attraction may appear with a different period. The qualitative picture remains the same.

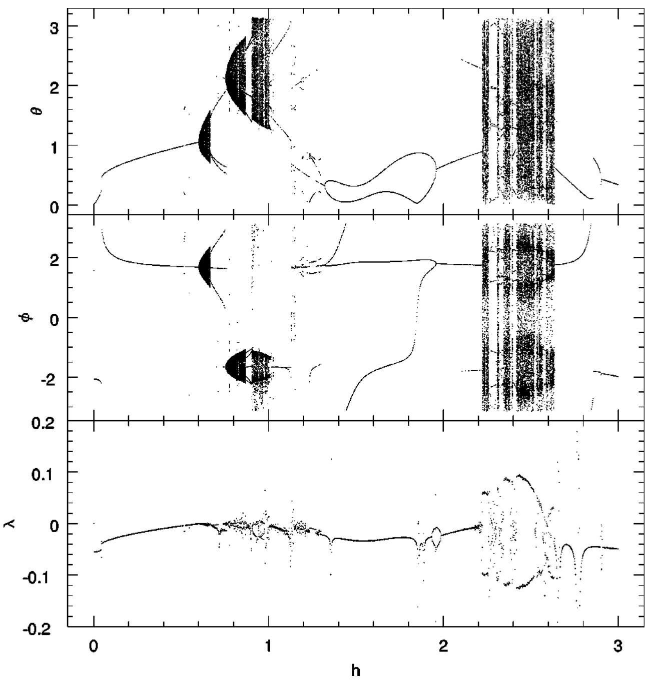

FIG. 1. Bifurcation diagram for the θ and ϕ components of the magnetization and Lyapunov exponents when $h_z=0.1$ and $\eta=0.05$.

For small values of h ($h \sim h_z$) a discontinuity that corresponds to a folding bifurcation is found. This effect, consisting of two independent limit circles, may be the experimental source for the observation of hysteresis when changing the amplitude of the perturbation (h). When increasing h the jump in θ is the one shown in Fig. 1, but if h is decreased, the jump to smaller θ would occur at a lower value of h. Experimentally, this phenomenon could manifest itself in a bistable behavior of a magnetic microwire near the resonance frequency.

Two bifurcations, identified as torus have been observed at $h=0.60$ and $h=0.75$, leading to two regions of complex

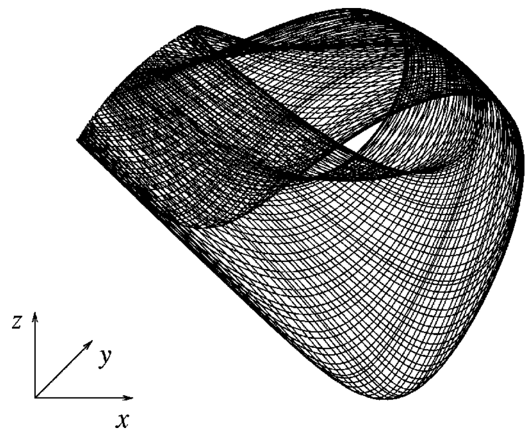

FIG. 2. Quasiperiodic attractor at $h=0.64$ in the diagram of Fig. 1.

(but mainly quasiperiodic) behavior: h from 0.60 to 0.66, and from 0.75 to 0.87. The first one in the upper hemisphere, and the second one in the lower. In the torus bifurcations the stable limit circles become unstable and give rise to quasiperiodic motion on the surface of a torus. In Fig. 2 is shown the phase portrait, in coordinates x, y, and z, of the quasiperiodic attractor at $h=0.64$. There, the Poincaré section changes from just one single point to a closed connected curve.

The region from 0.90 to 1.00 is a mixture of periodic and quasiperiodic behavior, and even chaotic motion. The chaos in this region is characterized by a chaotic attractor at $h \sim 0.9787$, which develops via a global bifurcation of the type of chaotic transients. This means that the system will evolve in the chaotic attractor for some time, and then, feeling the periodic or quasiperiodic stable orbits, will leave it. This is illustrated in Fig. 3, where the time evolution of the Lyapunov exponents is shown. Initially both exponents converge, one to a positive value, and the other to negative, as signature of chaos. But at a given time the positive exponent initiates a decrease towards 0, or even negative value. The Poincaré sections for the initial and final time steps are also shown. Initially the trajectories follow a chaotic map, but after some time they eventually fall in a period-seven orbit. The time spent in the chaotic behavior becomes larger as the chaotic attractor is approached.

Next, there is a wide region of period doubling,[20] with some higher period stripes. Finally, from 2.20 to 2.60, clear chaotic regions (see the Lyapunov exponents in Fig. 1) alternating with quasiperiodicity are observed. In Fig. 4 we show the Poincaré section of the chaotic attractor corresponding to $h=2.50$. In this case the route to chaos is also that of chaotic transients.

If the initial state for the magnetization is in the direction opposite to the external bias field h_z, then, basically, the picture presented above holds. Nevertheless, the folding disappears, and the stable period-one orbit evolves in the lower hemisphere. The same change of hemisphere happens for the period doubling region. The two (upper and lower) torus bifurcations are also preserved.

When the external applied bias h_z is larger than the anisotropy field (larger than 1 in our units) the behavior is slightly different. The bifurcation diagram is shown in Fig. 5. The same kind of structures repeat at $h=3.25\times n$, where n is an integer. A magnification of those structures is shown in the inset, where it is seen that they consist of a torus bifurcation and several periodic windows. The magnetization does not remain at a value close to the saturation, but wanders over the whole sphere.

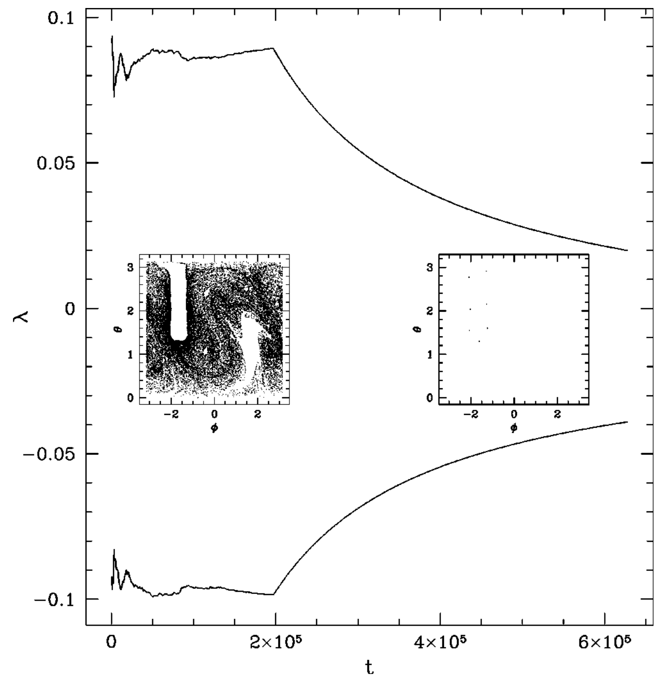

FIG. 3. Time evolution for the Lyapunov exponents when $h=0.9785$ in the diagram of Fig. 1. The left inset is the Poincaré section taken from $t=0$ to $t=1.8\times10^5$. The right one is taking $t=4.5\times10^5$ to $t=6.3\times10^5$.

IV. CONCLUSIONS

In conclusion, we have demonstrated that the dynamics governed by a driven Landau-Lifshitz equation in certain

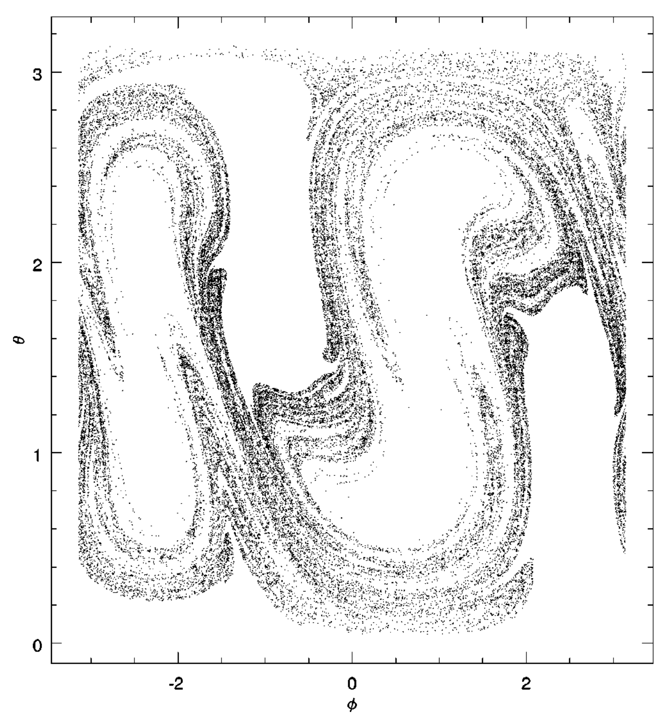

FIG. 4. Chaotic attractor at $h=2.5$ in the diagram of Fig. 1.

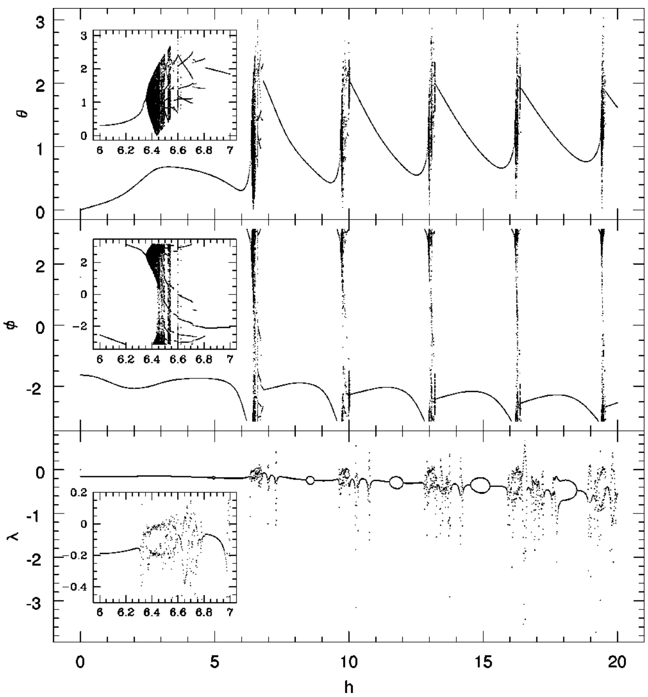

FIG. 5. The same bifurcation as in Fig. 1 for $h_z = 2$. Insets show the magnification of the region of h between 6 and 7.

parameter region can be very complicated. We have observed manifestations of this behavior: the quasiperiodicity alternating with periodic motion, the transient chaotic behavior, and finally a chaotic attractor. The experimental observation of these predictions is limited to the materials which could have a well defined single ferromagnetic resonance frequency, i.e., may be described by a single magnetic moment. As an example, the experimental observation of the phenomena described in this paper in microwires with low anisotropy may be possible. With our election of parameters the interesting phenomena, such as chaotic behavior, occur at values of the microwave field of the order of twice the anisotropy field (of course, the folding bifurcation and the periodic-quasioperiodic alternance occur at even low values). Nevertheless the values of the driving field at which those phenomena occur may be changed by choosing other values for η and the bias field. The damping constant η is fixed by the material but the bias can be modified without difficulty. So we expect that by choosing an appropriate set of parameters the complicated dynamical behavior described above would become accessible by experiments with magnetic microwires of low anisotropy even with the current sources of radiowaves.

On the other hand, the knowledge of the dynamics of a single magnetic moment is relevant for the dynamics of a system of Stoner-Wolfarth particles. Such a description is often made in the approximation of a noninteracting system.[11] Of course, in reality in such a system, the interactions are always present. When dealing with a magnetic material, composed by a large number of magnetic moments, exchange and dipolar interactions play a decisive role in the collective dynamics. The exchange interaction is a local one whereas dipolar interaction is global; the competition between them will determine the complex spatiotemporal behavior of the collective system. At this point the individual behavior of a single magnetic moment is crucial, and we could find a rich variety of complex spatiotemporal dynamics depending on which region of the bifurcation diagram we are.[12] The most interesting feature of this collective systems is the possibility of chaotic synchronization[21] of different parts of the system or even the whole system.

All this complex behavior is relevant in problems such as domain-wall dynamics, the dynamics of ferromagnetic particles, relaxation processes, and the dynamics of magnetostrictive wires and ribbons, and is of great practical importance.

ACKNOWLEDGMENTS

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[18] In a quantum-mechanical x treatment one speaks about the spin $\mathbf{S}_j$ at lattice site j instead of a magnetization per unit volume $\mathbf{M}(\mathbf{r})$ at point $\mathbf{r}$. The equation of motion is then $\dot{\mathbf{S}}_j = -\gamma[\mathbf{S}_j \times \mathbf{H}_{\mathrm{eff}}](j = 1 \ldots N)$; but this equation must be supplemented by a hypothetical (at least a theoretical model for the dissipation processes in the material was available) damping term in order to stabilize normal modes. At this point the Landau-Lifshitz phenomenological treatment turns out to be more convenient.

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