Infinite-randomness quantum Ising critical fixed points

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We examine the ground state of the random quantum Ising model in a transverse field using a generalization of the Ma-Dasgupta-Hu renormalization group (RG) scheme. For spatial dimensionality $d=2$, we find that at strong randomness the RG flow for the quantum critical point is towards an infinite-randomness fixed point, as in one dimension. This is consistent with the results of a recent quantum Monte Carlo study by Pich *et al.* [Phys. Rev. Lett. **81**, 5916 (1998)], including estimates of the critical exponents from our RG that agree well with those from the quantum Monte Carlo. The same qualitative behavior appears to occur for three dimensions; we have not yet been able to determine whether or not it persists to arbitrarily high d. Some consequences of the infinite-randomness fixed point for the quantum critical scaling behavior are discussed. Because frustration is irrelevant in the infinite-randomness limit, the *same* fixed point should govern both ferromagnetic and spin-glass quantum critical points. This RG maps the random quantum Ising model with strong disorder onto a novel type of percolation/aggregation process.

I. INTRODUCTION

Systems with quenched randomness and many degrees of freedom may be divided into three classes based on their coarse-grained behavior in the low-energy, low-frequency, and/or long-distance limit. First, there are many systems where the quenched disorder is irrelevant in the renormalization-group sense. Such systems, even though they are spatially inhomogeneous at the microscopic scale, become asymptotically homogeneous at macroscopic scales; their coarse-grained, low-energy behavior is the same as some ''pure'' system without quenched disorder. In this case, the renormalization-group fixed point governing the coarse-grained system is at zero quenched randomness. The second possibility is systems controlled by fixed points with nonzero, but finite, quenched randomness. In this case the coarse-grained behavior is spatially inhomogeneous, but the relative magnitude of the inhomogeneities remains finite at the fixed point. Examples of this second class include spin glasses and other glassy phases, as well as various critical points with randomness. The third possibility, which is the subject of this paper, occurs when the quenched randomness and thus the relative magnitude of the inhomogeneities grows without limit as the system is coarse grained. So far, we know of a few infinite-randomness fixed points that comprise this third class; all of them are one-dimensional ($d=1$) quantum ground states. These are the random singlet states of certain random antiferromagnetic spin chains,[1,2] the quantum critical point of the random quantum Ising (and Potts) chain,[3–6] and quantum critical points separating random singlet states and the Ising antiferromagnetic phase[2] or the Haldane state in the random spin-1 Heisenberg chain.[7,8]

A natural question is whether such infinite-randomness fixed points can govern the behavior of physical systems with spatial dimensionality $d \geqslant 2$.

Here we study the simplest random quantum system that can exhibit a phase transition, the random ferromagnetic Ising model in a transverse field, which, in all dimensions, has a quantum critical point at zero temperature. We focus on the strong-randomness regime, using a generalization[5] of the Ma-Dasgupta-Hu[1] renormalization group (RG) scheme. This approximate RG is exact in the limit of infinite randomness and thus can, in principle, yield exact results for the scaling behavior of systems governed by infinite randomness fixed points.[5] In one dimension, this RG has been used to analyze the various infinite-randomness fixed points mentioned above and many results can be obtained analytically, in particular for the random quantum Ising model. In higher dimensions, the renormalization group cannot (to our knowledge) be carried out analytically; in this paper we analyze its general structure and consequences, implement it numerically and examine a simple approximation to it. For $d=2$ and 3, we find that the renormalization group flow on the critical manifold for strong randomness is indeed towards even stronger randomness, as in one dimension, indicating that the quantum critical behavior is governed by the infinite-randomness critical fixed point. For $d=2$ we have studied the RG flow thoroughly enough to be fully confident that this is the case, and it appears to remain true for $d=3$.

For $d<4$, the Harris criterion[9] indicates that at the pure Ising quantum critical point weak randomness is relevant, with the RG flow towards stronger randomness; thus the simplest scenario for two and three dimensions is that the infinite-randomness fixed point governs the random quantum critical point with *any* randomness, as in one dimension. A recent quantum Monte Carlo study by Pich *et al.* of the ferromagnetic model with moderate randomness in two dimensions is consistent with this picture.[10]

What do we mean by infinite randomness? This means[5,2] that as the system is coarse grained and the characteristic energy scale decreases, the distributions of the *logarithms* of the magnitudes of the terms in the renormalized Hamiltonian

become arbitrarily broad. As a result, the ratio of the magnitude of any two terms approaches either zero or infinity. In this limit each renormalized coupling completely dominates any weaker renormalized coupling, so even the Ising spin glass becomes unfrustrated.[11] Thus we expect that the same infinite-randomness fixed point governs both the random ferromagnetic and the spin-glass quantum critical points, with simple modifications to account for the antiferromagnetic bonds in the latter case. We should note that recent quantum Monte Carlo studies of the spin-glass case had concluded that for two and three dimensions the scaling near the quantum critical point is conventional,[12] implying a finite-randomness fixed point, in contrast to what we are proposing here. But these finite-size scaling Monte Carlo studies looked at a rather small size range and did not look at distributions of physical properties; thus they were not sensitive to the scaling towards infinite randomness that we now believe occurs for $d=2$ and probably for higher d.

II. CLUSTER RG

We will study the quantum spin-1/2 Ising model with random ferromagnetic interactions, positive transverse fields h_i, and moments μ_i:

$$\hat{H} = -\sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z - \sum_i h_i\sigma_i^x - H\sum_i \mu_i\sigma_i^z, \tag{1}$$

where the σ's are the Pauli spin matrices. In general, we consider an arbitrary random lattice, with all interactions potentially present. More specifically, we are interested in random lattices that may appear in our effective description of some finite-dimensional system. Then the dimensionality of the system is encoded in the interactions J_{ij}, with the strong interactions being between nearby pairs of spins and the interactions between distant spins being extremely weak or absent. The uniform ordering field $H=H_z$ is zero or small; it is included only to probe the system's magnetization and susceptibilities.

The cluster RG finds the system's ground state by successively eliminating the highest energy degrees of freedom. At each step, we find the largest term in the Hamiltonian, which is either a transverse field or an interaction; its strength Ω sets the (maximum remaining) energy scale. If the largest term is the field on spin i, $\Omega=h_i$, that spin is put in the ground state, $\sigma_i^x=1$, of the local field term and virtual excitations to the other state ($\sigma_i^x=-1$) are treated in second-order perturbation theory. For this to be a valid approximation, the field must be much stronger than all this spin's interactions, which is true only in the strong-randomness limit. This step eliminates the spin i, and generates new interactions of the form $J'_{jk}\approx J_{ji}J_{ik}/h_i$. But some of these new interactions may be negligible compared to the interactions that were already present, so the full renormalized interactions are given, for each pair (j,k), by

$$J'_{jk}\approx \max\left(J_{jk}, \frac{J_{ji}J_{ik}}{h_i}\right). \tag{2}$$

We use the maximum here because in the strong-randomness limit the sum of two nonnegative numbers of very different magnitudes is well approximated by the larger number.

If, on the other hand, the largest term in the Hamiltonian is an interaction, $\Omega=J_{ij}$, the two spins involved are combined into one new spin—a cluster—whose two states represent the two ground states ($\sigma_i^z=\sigma_j^z=\pm 1$) of that interaction. Again, the virtual excitations to the states that are eliminated are treated in second-order perturbation theory. With the new spin labeled i (an arbitrary choice) the renormalized field is

$$h'_i \approx \frac{h_i h_j}{J_{ij}}, \tag{3}$$

and the renormalized moment is simply

$$\mu'_i = \mu_i + \mu_j. \tag{4}$$

The renormalized interactions are, for each remaining k,

$$J'_{ik}\approx \max(J_{ik}, J_{jk}). \tag{5}$$

The net result in both cases is the elimination of one spin along with the various renormalizations and reconnections of the lattice. The major complication for $d>1$ is that the RG does not preserve the lattice structure but instead generates a disordered and strongly correlated network of sites—i.e., clusters—and bonds connecting them.

The action of the cluster RG is a novel aggregation/annihilation process. When the strongest term is a field, the corresponding cluster is removed (annihilated), while when it is an interaction the two clusters that it connects are aggregated into one cluster. The clusters thus represent sets of the original spins that are strongly correlated. In the paramagnetic phase, the annihilation process dominates and no large clusters are formed, while in the ordered phase the aggregation dominates producing arbitrarily large clusters. For $d>1$, in the ordered phase an infinite percolating cluster appears during the action of the RG at a *finite* energy scale. When and after this occurs, the infinite cluster is represented by a single renormalized spin that interacts with an infinite number of the other remaining spins (which represent the finite clusters). Thus the topology of the network has by this energy scale completely changed from that of the ''bare'' d-dimensional lattice that had only short-range interactions. This emphasizes that this RG is not simply a ''real-space'' RG; it is more precisely an ''energy-space'' RG that can produce a renormalized lattice with a geometry very different from the bare lattice.

The zero-temperature quantum critical point is a new type of percolation transition at which an infinite cluster first appears (at the quantum critical point it only appears in the zero energy limit), as pointed out by Monthus *et al.*[8] The quantum critical point occurs when the annihilation and aggregation processes balance, so that arbitrarily large clusters are produced but no single cluster dominates at any finite energy scale.

Because of the multiplicative structure of the RG recursion relations, it is convenient to write the renormalized interactions after decimating down to energy scale Ω in logarithmic variables as

$$h_i = \Omega e^{-\beta_i}, \tag{6}$$

$$J_{ij} = \Omega e^{-\zeta_{ij}}. \tag{7}$$

The RG equations involve the full joint distribution of the positive quantities $\{\beta_i\}$ and $\{\zeta_{ij}\}$. The single-field distribution function $R(\beta)$ and the single-bond distribution function $P(\zeta)$ provide partial but important information about the full joint distribution of all the couplings. In general, the coordination number is not fixed, and if there are N spins (clusters) remaining, there may be up to $N(N-1)/2$ interactions. However, only of order N of the strongest interactions matter because the weaker ones will be overruled by the stronger ones in the action of the RG for strong randomness. Thus we normalize the bond distribution function $P(\zeta)$ *per remaining spin*, so that the total number of bonds in the system is $N\int_0^\infty P(\zeta)d\zeta$, and for $d>1$ we define the ''width'' w of this bond distribution by

$$\int_0^w P(\zeta)d\zeta = 1. \tag{8}$$

This width thus includes only the N strongest bonds. [Note that in one dimension, there are always exactly N nearest-neighbor bonds and this normalization of $P(\zeta)$ coincides with that of Ref. 2. If further neighbor bonds exist in one dimension, they will be irrelevant at low energy scales and can be ignored.] The condition for validity of our approximate RG is that the widths of the distributions $R(\beta)$ and $P(\zeta)$ be large; for the RG to be asymptotically exact, these widths should tend to infinity as $\Omega\to 0$; this is the indication that the RG flow goes to an infinite-randomness fixed point. We now discuss the general structure of such a putative infinite-randomness critical fixed point. We will assume that the simplest scaling scenario for an infinite-randomness fixed point occurs in our cluster RG (Ref. 13)—this is the case in one dimension,[6] and is consistent with our numerics for two dimensions.

III. CRITICAL FIXED POINT

We first consider the behavior at the quantum critical point. We expect that at the critical fixed point the distributions $R(\beta;\Gamma)$ and $P(\zeta;\Gamma)$ will asymptotically be given by the simple scaling forms

$$R(\beta;\Gamma)d\beta = B(\beta/\Gamma)d\beta/\Gamma, \tag{9}$$

$$P(\zeta;\Gamma)d\zeta = Z(\zeta/\Gamma)d\zeta/\Gamma \tag{10}$$

for large Γ; where

$$\Gamma \equiv \ln\left(\frac{\Omega_0}{\Omega}\right) > 0 \tag{11}$$

is the logarithm of the energy scale, relative to an energy scale Ω_0 set by the properties of the bare Hamiltonian. As Γ is increased by $d\Gamma$, the fractional decrease in the number of spins or clusters is $(Z_0+B_0)d\Gamma/\Gamma$ with

$$B_0 \equiv B(0), \quad Z_0 \equiv Z(0). \tag{12}$$

The density of remaining clusters per unit volume thus decreases under renormalization as

$$n_\Gamma \sim \Gamma^{-(Z_0+B_0)}. \tag{13}$$

This gives the basic relationship between the length scale L and the energy scale at the quantum critical point,

$$\Gamma = \ln\left(\frac{\Omega_0}{\Omega}\right) \sim L^\psi, \tag{14}$$

with $\psi = d/(Z_0+B_0) < 1$. Note that this is very different from conventional power-law scaling; here it is the *logarithm* of the energy scale that varies as a power of the length scale. Since this is associated with the ''tunneling'' events by which clusters flip, it has been dubbed ''tunneling dynamic scaling.''[13]

The typical moment of a cluster—the number of strongly correlated ''active'' spins in it (i.e., those not yet decimated)—scales as

$$\mu \sim \Gamma^\phi, \tag{15}$$

with some new exponent ϕ, so that the fractal dimension of the set of active spins in a cluster is

$$d_f = \phi\psi. \tag{16}$$

This determines the decay of the average spin-spin correlation function at the critical point.

A. Critical correlations

The correlation function between two spins at distance r,

$$G_{ij} \equiv \langle \sigma_i^z \sigma_j^z \rangle \tag{17}$$

is a random quantity with a very broad probability distribution for large $r \equiv |\mathbf{r}_i - \mathbf{r}_j|$. We first consider the correlation function of a *typical* pair of spins with separation r; typical spin pairs are never active in the same cluster and have only weak correlations that fall off, at criticality, as a stretched exponential function of distance,

$$-\ln G_{\rm typ}(r) \sim r^\psi, \tag{18}$$

with $\psi<1$. These correlations arise from the lowest-order perturbative corrections to the decimation of spin clusters.[5,14] When a cluster is decimated at energy scale Ω, each of the effective spins on its neighboring clusters—in more conventional terms the perturbatively modified wave functions that are labeled by the remaining effective spins—will acquire a component of the decimated cluster's spin whose magnitude is of order J/Ω, with J the effective coupling that links the neighboring cluster to the decimated cluster. Likewise, when these neighboring clusters are decimated, an even smaller component of the original spin will be acquired by the remaining clusters. Correlations between two spins i and j that are never active in the same cluster thus occur when a surviving cluster contains simultaneously a component of *both* the spins i and j. The correlation function G_{ij} is then determined by the maximum over all such mutual clusters that occur, at any energy scale, of the product of the two spins' components contained by the mutual cluster. Typically, the smallest of the multiplied perturbative factors that determine $\ln G$ will dominate; these arise from the stage at which the two spins first have a component on a mutual cluster. Since this occurs when the remaining cluster sizes are of order the separation r between the spins of interest, the dominant per-

turbative factor will be of order $e^{-K_{ij}\Gamma}$ with $\Gamma \sim r^{\psi}$ and K_{ij} random and of order one. This yields the result Eq. (18).[5,14]

We should note that there is another mechanism by which spins become correlated. At any step of the renormalization, when a cluster with field $\tilde{h}$ is decimated, higher-order perturbative effects will give components of the decimated spins on *all* the remaining clusters, with magnitudes that involve products over all the bonds connecting the decimated cluster to the remaining clusters, of factors of the form $\tilde{J}_{jk}/\tilde{h}$. These by themselves would give rise to simple exponential decay of typical correlations as occurs in conventional disordered phases. In contrast to these, the contributions to the correlations that come from the cumulative effects of the successive lowest-order perturbative terms discussed above will have similar form but with each of the $\{\tilde{J}_{jk}\}$ being divided by an effective field from one of the *later* stages of the RG; these are always smaller than $\tilde{h}$. Thus the cumulative lowest-order contributions discussed above will always dominate over the simple exponential decays from the higher-order perturbative effects. We hence conclude that the ψ from the RG must be less than one.

The *average* correlation function $\overline{G(r)}$ behaves quite differently than typical correlations. It is dominated by the rare spin pairs that are active in the *same* cluster at *some* energy scale; such pairs of spins have correlations of order one, reduced from one only by the short-scale high-energy fluctuations that are not included in the approximate RG. As a result, the average correlation function is proportional to the probability of the two spins being active in the same cluster at *some* energy scale. At criticality this occurs—if at all—at scale $\Gamma \sim r^{\psi}$, and the resulting average correlation function hence falls off as a power law:

$$\overline{G(r)} \sim r^{-\eta} \sim r^{-2(d-\phi\psi)}. \tag{19}$$

This is an example of the radically different scaling behavior of the typical and average quantities that is one hallmark of infinite-randomness fixed points.[5,13]

Thermodynamic properties involve averaging over the whole system and will hence be dominated, as are the average correlations, by rare clusters. The low-temperature susceptibility to a small ordering field H ($H \ll T$) can be found easily by stopping the RG at energy scale $\Omega \sim T$. For small T, almost all the decimated spins are frozen and hence nonmagnetic, while almost all the remaining clusters have effective transverse fields and interactions between them that are much less than T. They are hence essentially free and have independent Curie susceptibilities yielding an overall susceptibility at low temperatures near the quantum critical point of

$$\chi \sim \frac{|\ln T|^{2\phi-d/\psi}}{T}. \tag{20}$$

The magnetization in a small ordering field H (at $T \ll H$) can be found similarly: the RG is stopped when the typical magnetic energy $H\mu$ of a cluster is of order Ω. The decimated spins are nonmagnetic while the remaining clusters are almost perfectly polarized by the field. This yields, at the quantum critical point, a magnetization proportional to the fraction of spins that are still active,

$$M \sim |\ln H|^{\phi-d/\psi} = \frac{1}{|\ln H|^{\eta/2\psi}}. \tag{21}$$

B. Simple approximate RG

In order to get a better feeling for the scaling behavior, it is useful to study a simple approximation to the RG flows that is exact in one dimension and in some respects good for higher d; this consists of ignoring correlations among the fields and between the fields and the bonds, but allowing correlations among the bonds. In this approximation, the evolution equation for the field distribution $R(\beta;\Gamma)$ depends on the bond distribution only through $P(0;\Gamma)$, and is identical to its one-dimensional form (see Refs. 5 and 13). Solving for the fixed point gives scaling distributions with $Z_0 = 1$ and $B(\theta) = B_0 e^{-B_0\theta}$ with B_0 undetermined (the exact solution for one dimension has $B_0 = 1$). Within this approximation the exponents are $\psi = d/(1+B_0)$ and $\phi = (1+\sqrt{1+4B_0})/2$. Our numerical studies of the RG flows in two dimensions show that the log-field distribution is very close to the simple exponential form, but the estimated critical exponents for two dimensions differ somewhat from this simple approximation; this must be due to correlations among the fields and between them and the bonds.

IV. NUMERICAL RG STUDY OF THE CRITICAL FIXED POINT IN TWO DIMENSIONS

In order to do better and certainly to test the conjecture of a controlling infinite-randomness fixed point, we must implement the strong-randomness cluster RG numerically. The formulation is the same for a general random network, but we are of course interested in systems that can arise from finite-dimensional lattices with short-range interactions. We have thus studied the RG flows with ''initial conditions'' of finite d-dimensional lattices for two and three dimensions. The program has been tested by verifying that it reproduces (within statistical errors) the analytical results for one dimension.

For $d > 1$, many weak interactions are generated that, for the larger lattices studied, cannot all be stored. Because of this, we keep only interactions above a minimum strength $J_{\min}$, the smallness of $J_{\min}$ being limited by computer memory capacity and speed. Since the discarded bonds (those with $J < J_{\min}$) could not have generated stronger bonds, for the renormalization down to any energy scale with $\Omega > J_{\min}$ the RG decimation sequence is not affected at all, and all the bonds with $J_{\min} \leqslant J \leqslant \Omega$ are retained; thus we know all the fields and all the bonds with $0 \leqslant \zeta \leqslant \zeta_{\mathrm{m}} = \ln(\Omega/J_{\min})$. However, under renormalization ζ_{m} decreases and $\Gamma \equiv \ln(\Omega_0/\Omega)$ increases so that the range of ζ/Γ in the scaled distribution $Z(\zeta/\Gamma)$ that we can study steadily decreases as the system is coarse grained. We have been quite conservative and do not look at all at the ''contaminated'' low-energy part of the bond distribution ($J < J_{\min}$). The limits on memory are most restrictive at the earliest stages of the RG decimation, where the number of clusters is large, so this is when $J_{\min}$ must be set the largest. It might be possible to let $J_{\min}$ decrease later in the decimation and recover more of the renormalized bond distribution with controllable errors.

We have not explored this possibility.

We start with initial conditions of systems of up to 10^5 spins with random short-range interactions and random transverse fields independently chosen from specified initial probability distributions. We run up to 1000 samples for each initial probability distribution to reduce the statistical errors. For each sample we measure properties of the system when the energy scale passes (under renormalization) a predefined set of energies,[14] and then average these properties over different samples.

To reduce transients as much as possible, the shapes of the initial distributions $R(\beta)$ and $P(\zeta)$ are chosen to approximate, as best as we can, the renormalized critical point distributions that we observe. However, these initial conditions are missing any correlations among the fields and interactions that certainly exist in the full joint distribution at the critical fixed point. Thus when we run the RG it does show a fairly strong transient behavior as these correlations are generated and the fixed point is approached. So far, we have only a limited understanding of these transients and the correlations that are generated and we do not have a systematic way of controlling them; we do, however, monitor the simplest types of correlations and they do appear to stabilize after the initial transient in the RG.

In our numerics we primarily concentrate on the individual field and bond distributions $R(\beta;\Gamma)$ and $P(\zeta;\Gamma)$; these are partial but significant indicators of what is happening in the system's full joint probability distribution. For two dimensions we find that at the critical point both distributions do become broader under the action of the RG and the flow towards stronger randomness is clear. This flow is weaker but nevertheless is clearly apparent for three dimensions also.

Numerically we find that under the RG the field distribution maintains fairly accurately a simple exponential form,

$$R(\beta;\Gamma)\cong R_0(\Gamma)e^{-R_0(\Gamma)\beta}, \tag{22}$$

with

$$R_0(\Gamma)\equiv R(\beta=0;\Gamma). \tag{23}$$

The width of the distribution is proportional to $1/R_0(\Gamma)$ and grows steadily as the energy scale is decreased. This is shown in Fig. 1 for a flow near the critical point, but it is also true away from the critical point, and is consistent with the simple approximation to the RG flows discussed above for which the field distribution at low energy scales is always an exponential whose width never decreases.

As initial conditions, we choose for convenience the simple exponential distribution of log-fields with the initial $R_0=1$. [Note that the strong-randomness RG equations, Eqs. (2)–(5), are invariant under a multiplication of all the log-couplings by any constant. Thus, although the initial choice $R_0=1$ would appear to correspond to moderate randomness, we can, without loss of generality, use this in our study of the strong-randomness RG flows.] To search for a fixed point we measure $1/R_0(\Gamma)$ and scale both β and ζ by this width, defining

$$\beta_{sc}=R_0\beta \quad \text{and} \quad \zeta_{sc}=R_0\zeta. \tag{24}$$

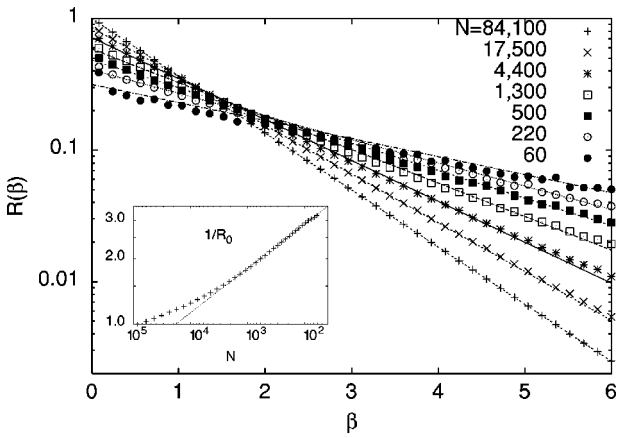

FIG. 1. RG evolution of the field distribution at a putative quantum critical point. Initial conditions are a 290×290 ($N=84{,}100$) triangular lattice with couplings drawn independently from the field distribution $R(\beta)=e^{-\beta}$ and the scaled bond distribution $P_{sc}(\zeta_{sc})=0.1+0.105\zeta_{sc}$, as described in the text. Lines are fits to the simple exponential form $R=R_0e^{-R_0\beta}$ with R_0 depending on the energy scale. Note that the renormalized field distribution fits this form well for all N. Inset: width of the field distribution $1/R_0$ vs the number of remaining spins N; the RG evolution is in the direction of *decreasing* N. The increasing width indicates the RG flow towards infinite randomness. The line here is a power-law fit that gives our estimate of the exponent ψ. Note that this fit works well only after the rather strong initial transient.

(Note that Ω_0 and thus Γ are not defined precisely, so we cannot simply scale the data by Γ.) The scaled field distribution is now $R_{sc}(\beta_{sc})\cong\exp(-\beta_{sc})$ and we can concentrate on the scaled bond distribution $P_{sc}(\zeta_{sc})$. The shape of the bond distribution evolves continuously and its characterization is much less clear. Numerically we observe that for $d\geqslant2$ at and near the quantum critical point the cluster RG always generates positively sloping ($dP/d\zeta>0$) bond distributions. This is in contrast to one dimension, where the exact critical-fixed-point bond distribution $P_{sc}(\zeta_{sc})=\exp(-\zeta_{sc})$ is the same simple exponential as the field distribution due to a duality relation.[5]

For $d=2$ the bond distributions that are generated by the RG near the quantum critical point can be reasonably approximated, in the small ζ_{sc} regime of interest, by a simple linear fit: $P_{sc}(\zeta_{sc})\cong a+b\zeta_{sc}$. We thus choose for initial conditions a $P_{sc}(\zeta_{sc})$ of this form. Our initial lattice for all the data presented here is a triangular lattice with periodic boundary conditions (we also tried others, such as square, to confirm that the results did not depend strongly on this arbitrary choice). Since we expect the stronger bonds to be shorter ranged, we select the nearest-neighbor bonds (there are three such bonds per site) to constitute the strongest-bond part $0<\zeta_{sc}<\zeta_c$ of the bond distribution, with ζ_c chosen so that $\int_0^{\zeta_c}(a+b\zeta)d\zeta=3$, i.e., there are precisely three bonds per site with ζ_{sc} between zero and ζ_c. Then the next batch of the distribution, $\zeta_c<\zeta_{sc}<\zeta_m$, are assigned at random to all the second- and third-neighbor bonds (six more bonds per site), with ζ_m chosen appropriately. (This ζ_m sets our J_{min}, as discussed above.) Thus our initial condition has nine bonds per site, corresponding to a coordination number of 18. Under renormalization, the lattice is quickly randomized, so it no longer resembles the initial triangular lattice, and the

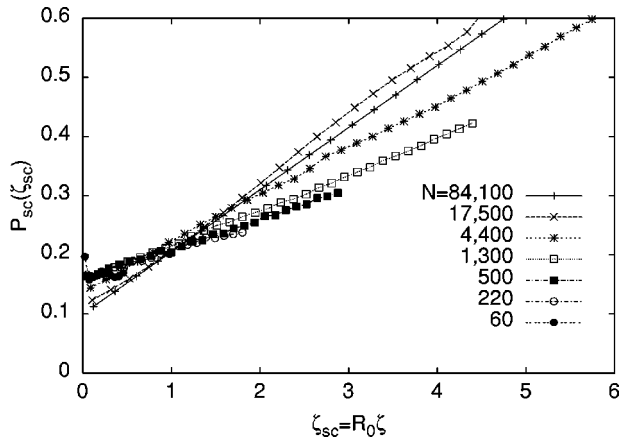

FIG. 2. RG evolution of the scaled bond distribution $P_{sc}(\zeta_{sc})$ at the apparent critical point corresponding to the initial conditions of Fig. 1. Note that although the intercept is fairly stable at $P_{sc}(0) \cong 0.15$ in the later stages of the RG (this is our criterion for locating the critical point—see also, Fig. 3), the shape of the distribution is not as stable (see also, Fig. 4), presumably due to transient effects from our uncorrelated initial conditions. Because we do not keep bonds weaker than J_{min} (see text), the range of the scaled bond distribution that we measure and plot, $0 \leqslant \zeta_{sc} \leqslant \zeta_m = R_0 \log(\Omega/J_{min})$, decreases as we run our RG.

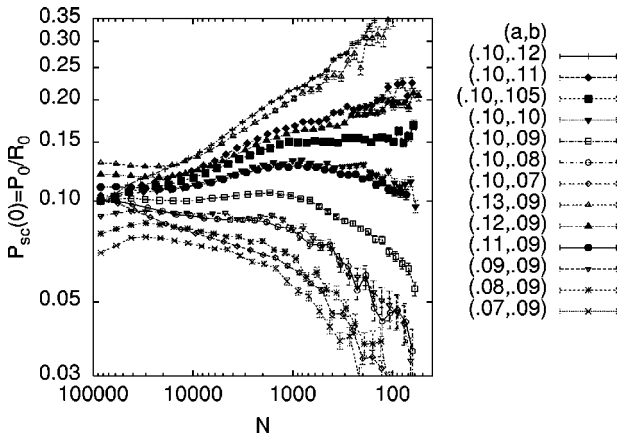

FIG. 3. RG evolution of the intercept $P_{sc}(0)$ of the scaled bond distribution for different initial conditions: all are initially 290×290 triangular lattices with independent couplings and initial field distribution $R(\beta) = e^{-\beta}$; the different curves correspond to various initial scaled bond distributions: $P_{sc}(\zeta_{sc}) = a + b\zeta_{sc}$ with the parameters a and b as indicated (see text). If the intercept saturates to a finite nonzero value as N is decreased, this indicates that the system is critical. Our best estimate of the critical point has $a = 0.10$, $b = 0.105$ (filled squares); this is what is used in all the other figures. Other parameters that we view as possibly critical are also indicated by filled symbols. The error estimates on the various critical exponents include the results from all of these potentially critical systems.

number of bonds kept per remaining cluster initially increases. However, since we do not keep bonds with $J < J_{min}$, the number of bonds kept per cluster decreases in the later stages of the decimation, as Ω decreases towards J_{min}. Note that our choice of the part of the distribution with $\zeta_{sc} > \zeta_c$ (the tail) is only a matter of convenience; its details (and even its very presence) are not important: specifically, we have checked that by the time the RG reaches this energy range, most of the original bonds from the tail are gone, having been dominated by the stronger interactions that arise from the original strong first-neighbor bonds only.

We first searched for a fixed point of the RG by starting from such a linear distribution of initially uncorrelated bonds and monitoring the flow of the two parameters obtained by fitting the renormalized and scaled bond distribution to such a linear form—the intercept $P_{sc}(0)$ and the slope $dP_{sc}/d\zeta_{sc}$. By choosing an initial bond distribution close to the fixed-point distribution we tried to minimize the transients that occur as the full fixed-point *joint* distribution of the fields and bonds is generated by the RG. However, the transients remained too strong for us to accurately locate a fixed point of the RG flow in the plane of these two parameters: we could not fully stabilize this scaled bond distribution. Figure 2 shows an example of the evolution of the scaled bond distribution for initial conditions near what we estimate to be the critical fixed point. The intercept stabilizes at $P_{sc}(0) \cong 0.15$ (see Fig. 3), but the slope is much less stable, although it may be approaching a limit, as shown in Fig. 4.

Since we could not obtain a clear fixed point in the two-parameter space defined by the simple linear fit to the bond distribution, we instead chose as our candidates for critical points those that produce a scaled bond distribution whose intercept $P_{sc}(0)$ appeared to be stabilizing to a fixed-point value under the action of the RG. In running the RG, at each decimation step the maximum-energy term that is ''integrated out'' is either a bond or a field. The ratio of the frequency of these occurrences is simply the intercept $P_{sc}(0)$ [since we have normalized so that the intercept of the scaled *field* distribution is $R_{sc}(0) = 1$], and is easily estimated without fitting any distributions by counting the number of occurrences of the two types of decimations as the RG runs. Stability of the ratio of the frequencies of the two types of

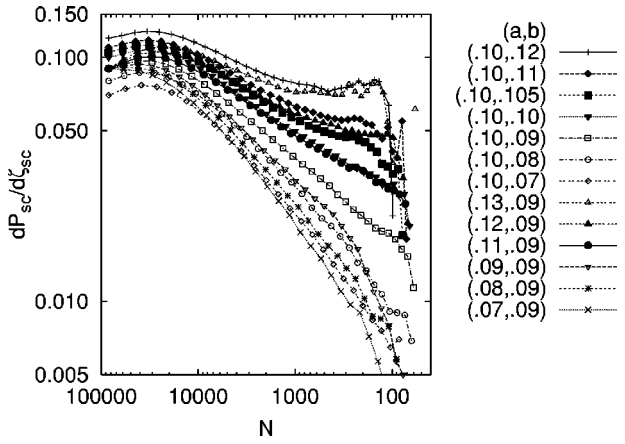

FIG. 4. Evolution of the slope $dP_{sc}/d\zeta_{sc}$ of the scaled bond distribution for the same set of different initial conditions as in Fig. 3. At each energy scale, the slope is calculated by fitting a linear function to the corresponding distribution (as in Fig. 2) in the full available region $0 \leqslant \zeta_{sc} \leqslant \zeta_m$. The observed shape of the bond distribution is only approximately linear, and our data for the bond distributions becomes very limited and noisy for small N. The strongly transient behavior seen here is presumably due to both actual transients in the shapes of the distributions and the reduction with decreasing N of the range $(0, \zeta_m)$, over which the linear fit is made.

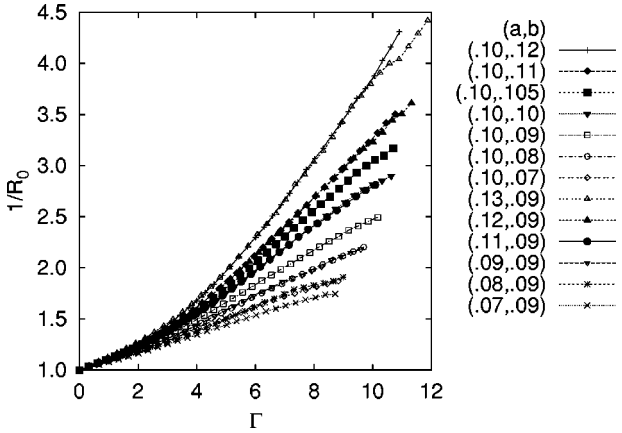

FIG. 5. Width of the field distribution $1/R_0$ vs the log-energy scale Γ for the same set of different initial conditions as in Fig. 3. There is a clear initial transient in all the data for $1/R_0 < 1.5$ (see also plot of $1/R_0$ vs N for our candidate critical point in Fig. 1). After the initial transient the data are consistent with the expected linear behavior at criticality and with the saturation of the width in the disordered phase.

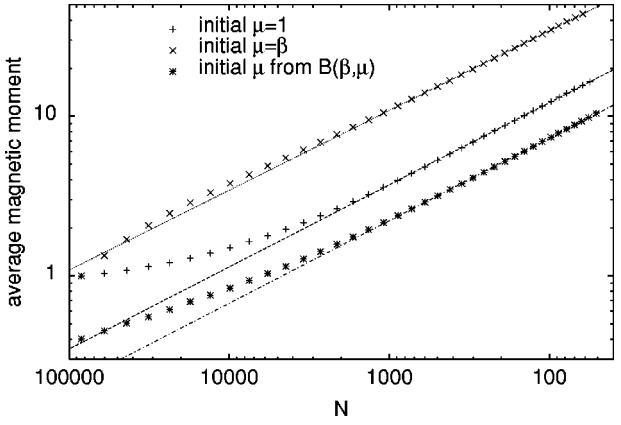

FIG. 6. Scaling of the average magnetic moment per cluster $\overline{\mu}$ with the number of remaining clusters N. Under the RG, the magnetic moment of a cluster μ_i and local log-field magnitude β_i become strongly positively correlated, and we expect significant transients if our initial conditions do not have these correlations. The initial transient is clearly seen if we start with $\mu_i = 1$ for all sites, or with $\mu_i = \beta_i$—an attempt to imitate the positive correlation—but the transient dies off quickly as we run the RG. This transient is slightly suppressed if we generate initial (β_i, μ_i) from the joint distribution function $B(\beta, \mu)$, which is the fixed-point joint distribution in the uncorrelated-field approximation (discussed in the text) with $B(0) = 7$. We consistently find $\mu \sim N^{-0.50 \pm 0.05}$ implying $d_f = 1.0 \pm 0.1$ and $\eta = 2.0 \pm 0.2$.

decimations is a direct indication of a ''balance'' between the field part of the Hamiltonian and the bond part. A detailed study, extended down to a factor of over 1000 in N, shows three different types of behavior of the intercept, which we interpret as follows: If $P_{sc}(0)$ is clearly decreasing towards zero then the system is in the disordered phase. If $P_{sc}(0)$ is steadily increasing then the system is in the ordered phase. Finally, if $P_{sc}(0)$ appears to be saturating at some value then we have a candidate for the critical point. Figure 3 illustrates this. Starting from different initial two-dimensional lattices and different initial distributions, we always find that the apparently ''critical'' (stable) value of the intercept is in the range 0.1–0.2, using what we think are conservatively large uncertainties on when and where the intercept stabilizes. All of the candidate critical points that fall in this range, and that are shown by the filled symbols in Fig. 3, are used in all of our error estimates.

In the simple approximation to the RG that neglects correlations involving fields and has $P_0 \equiv P(0) \approx 1/\Gamma$, the intercept is $P_{sc}(0) \approx 1/B_0$; in general at the critical point $P_0 \approx Z_0/\Gamma$ and the intercept is thus $P_{sc}(0) \approx P_0/R_0 = Z_0/B_0$.

Figure 1 and Fig. 2 show the evolution under the RG of the field distribution and the scaled bond distribution of one candidate for the critical point. Since the bond distribution is not fully stable, our scaling analysis of the critical flow, which we discuss next, is not as certain as our conclusion on the nature of the critical fixed point: i.e., that it is at infinite randomness. As we already mentioned, direct scaling with Γ requires estimating the additional parameter Ω_0. To estimate the ''tunneling scaling'' exponent ψ we therefore consider the evolution of the width $1/R_0$ of the field distribution; this is shown in the inset of Fig. 1. It is expected that $1/R_0 \sim N^{-\psi/d}$ at the critical fixed point since at asymptotically low energy scales, for which the ''bare'' scale Ω_0 is not important, $1/R_0$ should be proportional to Γ. (Figure 5 shows that this is indeed true in the later stages of the renormalization after the initial transient.) It can be seen from the inset in Fig. 1 that during the initial transient the width of the field distribution grows more slowly than later in the renormalization. This accelerating growth of the width occurs for all our candidate critical points for $d = 2$, and emphasizes that the RG flow is certainly towards infinite randomness. Fitting the later stages of the RG for all the initial conditions that appear consistent with being critical—illustrated by the filled symbols in Fig. 3—gives exponent estimates in the range[15]

$$\psi = 0.42 \pm 0.06. \tag{25}$$

For all candidate critical points the estimated exponent ψ is noticeably larger than the $\psi = d/(1+B_0) = 0.2 - 0.3$ from the simple approximation discussed above if one uses the $1/B_0 \cong P_{sc}(0) = 0.1 - 0.2$ obtained from the apparent intercept; this indicates that correlations between the fields and the bonds must be substantial at the critical fixed point. Indeed, after renormalization, correlations between a field and its adjacent bonds are easily detected: the strengths are anticorrelated, so that, for example, a cluster with a weak renormalized field is more likely to have strong renormalized bonds connected to it.

The fractal dimension $d_f = \phi\psi$ of the critical clusters can be obtained directly from the RG flows at the critical fixed point. Figure 6 shows the scaling with N of the average magnetic moment (proportional to the number of bare spins) of surviving clusters. Direct fits to such plots for our candidate critical points give

$$d_f = 1.0 \pm 0.1, \tag{26}$$

in contrast to the prediction from the simple approximation of $d_f = 0.7 - 0.9$. Note, however that if $B_0 \cong 4$ is obtained from our $\psi \cong 0.4$ by using the simple approximation [but ignoring the estimates of $P_{sc}(0)$], the predicted $d_f \cong 1.0$ is close to the value obtained from the full RG. This suggests

that the correlations between the moment and the field on a cluster are reasonably well captured by the simple approximation.

From the scaling relation Eq. (19), the average critical correlations decay with the exponent

$$\eta = 2.0 \pm 0.2. \tag{27}$$

More direct fits for the exponent ϕ alone can be obtained from plots (not shown) of the average magnetic moment vs R_0, giving

$$\phi = 2.5 \pm 0.4. \tag{28}$$

The recent quantum Monte Carlo study by Pich *et al.*[10] of the two-dimensional random Ising ferromagnet has found evidence that the width of the distribution of the logarithms of characteristic energies grows with sample size at the quantum critical point, as for one dimension. They estimate $\psi \cong 0.4$, in good agreement with what we find from the numerical RG. They also measured the spatial correlations $G(r) = \langle \sigma_0^z \sigma_r^z \rangle$ at criticality and found that the median (and hence typical) correlation $G_{\rm typ}(r)$ falls off faster than a power of r, better fit by $-\ln G_{\rm typ} \sim r^{\psi_c}$ with $\psi_c \cong 1/3$, not inconsistent with the scaling prediction $\psi_c = \psi$. In contrast, the average critical correlations exhibit a power-law decay with $\eta \cong 2$, which implies that the fractal dimension of the critical cluster is $d_f = \phi\psi \cong 1$, again in good agreement with the exponents estimated from our RG study.

V. ORDERED AND DISORDERED PHASES

We now turn to a discussion of the ordered and disordered phases. Here and henceforth, we will denote the parameter that controls the difference between the strengths of the typical random fields and those of the typical random bonds in the original Hamiltonian by δ, chosen so that the zero-temperature quantum critical point corresponds to $\delta = 0$, the zero-temperature disordered phase to $\delta > 0$ and the zero-temperature ordered phase to $\delta < 0$.

In our numerical RG studies, we do not know the fixed point accurately enough and do not have sufficient control over initial transients to study the off-critical flows directly. Nevertheless, we can still obtain some information about the near-critical properties indirectly from the critical flows (just as, in conventional systems, the correlation length exponent ν is related to the decay of energy density correlations at the critical point).

The effective field of a cluster is generally a product of some number f of the *original* fields divided by a product of $(f-1)$ *original* interactions (both the original fields and original interactions need not be distinct). At the critical point we expect

$$f \sim \Gamma^{\rho}, \tag{29}$$

with the new exponent ρ satisfying

$$\rho \geqslant \max\left(\phi, \frac{1}{\psi} \right). \tag{30}$$

The first inequality is obtained because any spin that is active in a cluster contributes (at least once) its original field to the effective field of the cluster; the second inequality follows from the observation that the (decimated) bonds that hold the cluster together also contribute to the effective field and must reach across the diameter of the cluster, which is of order $\Gamma^{1/\psi}$.

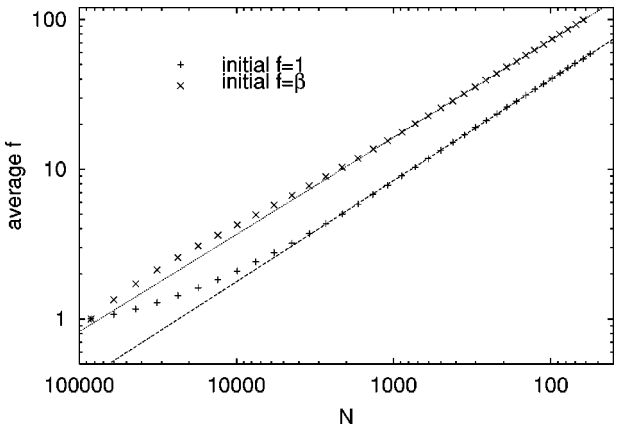

FIG. 7. Scaling of the average cluster ''history'' f—the number of the original fields whose product enters the effective field—with the number of remaining clusters N. Under the RG, f and the log-field magnitude β become positively correlated, but the transient generating this correlation dies off quickly. By starting either from $f_i = 1$ and the corresponding quantity for the bonds $f_{ij} = 1$, or from $f_i = \beta_i$ and $f_{ij} = \zeta_{ij}$, we consistently find $f \sim N^{-0.67 \pm 0.07} \sim L^{1.34 \pm 0.14}$.

An instructive way to understand the effect of deviations from criticality, is to move away from it by simply multiplying all the original fields by an amount $1 + \delta$. Perturbatively, this would change the log-fields at scale Γ by of order $f\delta \sim \Gamma^{\rho}\delta$. If we neglect the effects of δ on changing the order of decimations, we obtain the crossover scale away from criticality

$$\Gamma_{\delta} \sim |\delta|^{-1/(\rho-1)}, \tag{31}$$

as the scale at which the changes in the log-fields become comparable to a typical log-field or log-interaction ($\sim\Gamma$) and thereby substantially alter the distribution. This can be justified as follows. First, note that the cluster RG as given by Eqs. (2)–(5) is essentially *local*: we have considerable freedom in the precise global order of the decimations. The only restriction is that one eliminates, via Eqs. (2)–(5), only the *locally highest* energy degrees of freedom; the eventual result of many such transformations does not depend on the particular order in which they are performed as long as this restriction is respected.[16] Now, before the scale Γ_{δ} is reached, δ only changes the global sequence of the decimations, but not significantly the local sequence. The same terms remain locally strongest; hence, the same local decimation sequence is followed, and the log-fields' changes at scale $\Gamma < \Gamma_{\delta}$ are indeed of order $\Gamma^{\rho}\delta$ and smaller than the typical difference of any two log-energies ($\sim\Gamma$). Thus, the estimate of the crossover scale is consistent. [Note also that ''chaotic'' behavior under the RG flows, as occurs at (and below) the critical point in classical spin glasses,[17] cannot occur here. This is because our RG equations have a form of monotonicity: increasing the original field on one spin in the

quantum model cannot result in a decrease of any renormalized field and cannot result in an increase of any renormalized interaction.]

The scaling argument above yields a correlation length $\xi = \xi_{\rm av} \sim |\delta|^{-\nu}$ with

$$\nu = \frac{1}{(\rho - 1)\psi}. \tag{32}$$

Numerically, we obtain from fitting to $f \sim N^{-\rho\psi/2}$,

$$\rho\psi = 1.34 \pm 0.14 \tag{33}$$

(Fig. 7), and hence, ν. We can also fit more directly for ν using $fR_0 \sim N^{-1/(2\nu)}$, obtaining a similar estimate

$$\nu = 1.07 \pm 0.15, \tag{34}$$

which is consistent with the bounds[18] $\nu \geqslant 2/d = 1$ and

$$\nu \leqslant \frac{1}{\max(1, \phi\psi) - \psi} \cong 1.7. \tag{35}$$

A. Disordered phase: Correlations and Griffiths-McCoy singularities

In the disordered phase, $\delta > 0$, the average spin correlations will be dominated by rare large clusters and decay exponentially,

$$\overline{G(r)} \sim e^{-r/\xi}. \tag{36}$$

The typical correlations, on the other hand, will decay more rapidly, as

$$-\ln G_{\rm typ}(r) \sim r/\xi_{\rm typ}, \tag{37}$$

with

$$\xi_{\rm typ} \sim \delta^{-(1-\psi)\nu} \sim \xi^{1-\psi} < \xi. \tag{38}$$

This can be seen as follows: At the crossover scale away from criticality, Γ_δ, which corresponds to length scale ξ, the ratios between typical remaining bonds and fields are of magnitude $\ln(J/h) \sim -\Gamma_\delta$. Two spins of interest separated by distance r much longer than ξ will each have the maximum—albeit small—component of their spin on a remaining cluster near to them. As these clusters and those between them are decimated until eventually the two spins are contained in the same (mutual) cluster, a multiplicative factor of order the typical J/h ratio at the crossover scale will reduce the components of the spins on the remaining clusters—and thus on the eventual mutual cluster—for each one of the clusters at the crossover scale that is decimated; i.e., for each element of length of order ξ. The result Eq. (38) follows. Note that the typical correlations at and near criticality have the scaling form

$$-\ln G_{\rm typ} \sim r^{\psi} F_{\rm typ}(r/\xi). \tag{39}$$

The behavior of the typical correlations is related to the behavior of the distributions of fields and bonds. The field distribution in the disordered phase has finite width in the limit of low energy (as in one dimension), $R(\beta; \Gamma|\delta) \cong R_0(\delta) e^{-R_0(\delta)\beta}$ for $\Gamma \gg \Gamma_\delta$, while the bonds continue to become weaker. The limiting width is of order

$$1/R_0(\delta) \sim \Gamma_\delta \sim \delta^{-\psi\nu}, \tag{40}$$

which diverges as $\delta \to 0$.[5,19] As in one dimension, this gives rise to a disordered ''Griffiths'' phase with continuously variable dynamical exponent z that relates the scales of energy and length—the typical fields and spacing of surviving clusters—via

$$\Omega \sim L^{-z} \tag{41}$$

with z diverging as

$$z \sim \delta^{-\psi\nu} \tag{42}$$

for $\delta \to 0$, consistent with the numerical data of Ref. 10.

In the disordered phase the distribution of log-fields tends to a simple exponential form at low energies with the limiting width $1/R_0 \approx z/d$.[19,5] Concomitantly, there is a continuously variable power-law singularity in the average ground-state magnetization per spin in an ordering field H:

$$M \sim H^{d/z} |\ln H|^{x}, \tag{43}$$

with the exponent of the logarithmic factor not determined from these simple arguments. This gives the leading low-H behavior near the critical point where $z > d$. For larger δ, where $z < d$, this instead gives a singular correction to an analytic $M(H)$. The low-temperature zero-field susceptibility likewise diverges for $z > d$ as $1/T$ raised to a continuously varying power that is less than unity—a weaker than Curie-law divergence.

For small (positive) H and small $|\delta|$, the magnetization has a scaling form:

$$M(H, \delta) \sim |\ln H|^{\phi - d/\psi} \Xi(C\delta |\ln H|^{1/\psi\nu}), \tag{44}$$

with both C and the cutoff scale implicit in the $\ln H$ representing nonuniversal corrections to scaling. When u is large and positive, the scaling function $\Xi(u) \sim \exp(-u^{\psi\nu})$ times a power of u.

The clusters surviving to low energies in the disordered phase are rare large strongly coupled ferromagnetic clusters that exist even in the disordered phase. The most probable way for such a region to occur is (it appears) for there to be, at the crossover scale Γ_δ, a connected set of n clusters each with linear size $\sim \xi$ with somewhat anomalously weak fields on them and somewhat anomalously strong effective couplings between them; these will make them join together at lower energies into the rare large cluster of interest. Note that because the system has strong randomness, there is no requirement that this large cluster has a compact geometry. Indeed, for large n we expect that the most probable such clusters will have, on scales larger than ξ, a geometry similar to incipient *classical* percolation clusters, as the requirements for a large rare cluster to form out of the scale ξ subclusters is qualitatively like that for a large cluster to form out of the small scale objects in conventional percolation. [See more detailed discussion in next subsection.] Since the probabilities of the occurrence at scale ξ of each of the n such clusters and each of the $(\sim n)$ corresponding couplings are determined primarily by local properties and are hence

roughly independent, the probability of the rare large cluster is of order $1/\alpha^n$ with $\alpha>1$; α is independent of δ for small δ, since the needed clusters and bonds are not very atypical at scale ξ. The field on such a cluster will be of order the energy scale for crossover away from criticality, multiplied by of order n factors, each less than unity with logarithm of order Γ_δ:

$$\tilde{h} \sim \Omega_0 (e^{-a\Gamma_\delta})^n, \tag{45}$$

with a independent of δ near the critical point. The probability of a cluster with an anomalously small field $\tilde{h}$ is thus approximately

$$p(\tilde{h}) \sim e^{-(\ln \alpha)\ln(\Omega_0/\tilde{h})/a\Gamma_\delta}. \tag{46}$$

If one finds the least rare such clusters, i.e., with $(\ln \alpha)/a$ as small as possible, then $R_0(\delta)=(\ln \alpha)/a\Gamma_\delta$ and these rare clusters give rise to the singular scaling in the Griffiths phase quoted above. As stated earlier the dynamic exponent z is given by the asymptotic low-energy value of d/R_0. [Note that we have ignored here differences between $\ln \Omega_0$ and $\ln \Omega_\delta = \ln \Omega_0 - \Gamma_\delta$, since these will not affect the dominant behavior in the limit of interest. Also note that since $|\ln \tilde{h}| \sim n$, a factor $p(\tilde{h})$ controls the asymptotic behavior of the distribution of the $|\ln \tilde{h}|$-variable.]

The RG generates the low-energy tail $\sim e^{-R_0(\delta)\beta}$ of the log-field distribution on scales $L \gg \xi$ precisely because the rare large clusters discussed above are not *too* rare. Note that an exponential tail of the distribution of the cluster effective fields is generated by the RG almost immediately and some rare arbitrarily large clusters are present at any energy scale, as long as there is no infinite cluster that would consume them. In particular, this tail is also generated at the critical point, but it continues to become longer as the energy scale decreases.

B. Ordered phase: Percolation and finite-temperature ordering

The behavior in the ordered phase differs strikingly from that in one dimension. At a *finite* energy scale

$$\Omega_\infty = \Omega_0 e^{-\Gamma_\infty} \sim \Omega_0 \exp[-K|\delta|^{-\psi\nu}], \tag{47}$$

a single infinite cluster (with zero transverse field) develops. Finite clusters, some of which will join the infinite cluster at lower energies, coexist with it. The spontaneous magnetization is proportional to the number of sites in the infinite cluster at $\Omega \to 0$ yielding

$$M_0 \sim |\delta|^\beta, \tag{48}$$

with

$$\beta = \nu\eta/2 = \nu(d-\phi\psi). \tag{49}$$

This implies that the scaling function in Eq. (44) has the asymptotic form in the ordered phase $\Xi(u \to -\infty) \sim (-u)^{d\nu-\phi\psi\nu}$.

As a consequence of the infinite cluster development, in the ferromagnetically ordered phase there is, in contrast to one dimension, an interfacial energy density—albeit exponentially small—for a domain wall for any $\delta<0$. This ''surface tension'' scales as

$$\sigma \sim \xi^{1-d}\Omega_\infty \tag{50}$$

—simply the energy needed to cut the infinite cluster in half. The transition temperature for destruction of the long-range order by thermal fluctuations will similarly be determined by the strength of the bonds that hold together the infinite cluster; thus

$$T_c \sim \Omega_\infty. \tag{51}$$

We note that the arguments given below imply that the width of the classical finite-temperature critical region as the temperature is reduced at fixed negative δ, will be of order T_c.

The behavior near the energy scale $\Omega_\infty(\delta)$ at which the infinite cluster forms has aspects similar to conventional classical percolation (in contrast to the zero-temperature quantum percolation transition at $\delta=0$). At the crossover scale $\Gamma_\delta = \ln(\Omega_0/\Omega_\delta)$ away from criticality, the bonds will start to dominate over the random fields. At this scale the sizes of the clusters and the lengths of the bonds will typically be of order the crossover length scale $\xi \sim \Gamma_\delta^{1/\psi}$; indeed, bonds much longer than this will be exponentially rare. Between the log-energy scales Γ_δ and $\Gamma_\infty \approx \ln(\Omega_0/\Omega_\infty)$, at which the infinite cluster forms, most of the decimations will be of bonds, resulting in the joining together of clusters. The process of decimation of bonds (and occasional clusters) will continue with larger and larger clusters forming until the percolation scale Γ_∞. Note that Γ_∞ will be a fixed (order-one) multiple of the somewhat arbitrarily defined crossover scale Γ_δ, but the corresponding physical energy Ω_∞ is actually exponentially smaller than the energy Ω_δ (for large Γ_δ).

Unlike the case at the quantum critical point, the process by which the large clusters are joined together as the energy scale is decreased near the percolation scale is basically *local*. The key feature of this locality, which occurs when Γ is in the range $\Gamma_\delta<\Gamma<\Gamma_\infty$, is that when Γ is changed by a small amount, whether, say, a large cluster A will become joined to a large cluster B, and whether the same cluster A will become joined to another large cluster C, are roughly independent events, each only depending on the smaller clusters and bonds—which have typical length scale ξ and log-energy scale Γ_δ—in the vicinity of the respective potential connections. As the percolation scale is approached, we expect that this independence will become more and more pronounced as the important connections that make the large clusters grow become further and further apart. On the basis of this argument, we conjecture that the percolation that occurs in the cluster RG at scale Γ_∞ is in the *universality class of classical percolation*, with $(\Gamma_\infty-\Gamma)$ playing the role of (p_c-p) in classical percolation. [Note that the width of the log-bond distribution at scale Γ_δ is of order Γ_δ and will remain so at all scales $\Gamma_\delta<\Gamma<\Gamma_\infty$.]

The nature of the percolation process at the scale Γ_∞ controls the critical behavior associated with the finite-temperature ordering transition at $T_c \sim \Omega_\infty$ over a substantial region of the T-δ plane. On the logarithmic temperature scale, $\Gamma_T = \ln(\Omega_0/T)$, the finite-temperature spin-spin correlation length ξ_T is simply the characteristic length scale of the

distribution of cluster diameters at scale Γ_T: clusters that exist at this scale will each consist of sets of well-correlated (active) spins, while the correlations between these clusters will be destroyed by the thermal fluctuations. On temperature scales above Ω_δ ($\Gamma_T < \Gamma_\delta$), ξ_T will be given by the quantum critical length scale at log-energy scale Γ_T. But on scales between Ω_δ and Ω_∞ ($\Gamma_\delta < \Gamma_T < \Gamma_\infty$), the percolation process will cause the correlation length to diverge as

$$\xi_T \sim \xi \left(\frac{\Gamma_\delta}{\Gamma_\infty - \Gamma_T} \right)^{\nu_p}, \tag{52}$$

where ν_p is the *classical* percolation correlation length exponent—with $\nu_p = 4/3$ in two dimensions—and $\xi \sim (-\delta)^{-\nu}$ is the correlation length associated with the *quantum* critical point (i.e., the characteristic length at the crossover scale Γ_δ).

As the critical temperature is approached, the RG approximation will eventually break down at any nonzero δ. The clean separation of bonds into ''strong'' for those stronger than Γ_T, and ''weak'' for those weaker than Γ_T will not hold for the ''marginal'' bonds whose strength is of order T, which correspond to those whose log-strength is $\Gamma_T \pm O(1)$. This implies an $O(1)$ multiplicative uncertainty in the proportionality between T_c and Ω_∞ and it also implies that the percolation-dominated form of the critical behavior of Eq. (52) breaks down when $\Gamma_\infty - \Gamma_T$ is of order one. Closer to the finite-temperature critical point, the behavior will be dominated by the thermal fluctuations of the marginal bonds that link very large almost-percolating clusters. This will make the critical behavior cross over to that of the conventional *classical d-dimensional random bond Ising* universality class with

$$\xi_T \sim \xi \Gamma_\delta^{\nu_p} \left(\frac{T_c}{T - T_c} \right)^{\nu_I} \sim \xi \Gamma_\delta^{\nu_p} \left(\frac{1}{\Gamma_\infty - \Gamma_T} \right)^{\nu_I}, \tag{53}$$

where ν_I is the classical random Ising correlation length exponent, equal to one in two dimensions (assuming the interactions are not frustrated).

As a function of temperature, this double crossover in the critical behavior will be particularly hard to observe due to the logarithmic temperature scale, which makes the crossover energy scale Ω_δ exponentially larger than T_c for small (negative) δ. But if the temperature is held fixed and extremely small—i.e., $\Gamma_T \gg 1$— then the crossovers can be seen more readily by decreasing the relative strengths of the random fields that are parametrized by δ. The critical value δ_c is of order $-\Gamma_T^{-1/\psi\nu}$. As the random fields are reduced from δ of order one until $\delta \sim +\Gamma_T^{-1/\psi\nu} \sim |\delta_c|$, the thermal effects will be negligible and ξ_T will diverge with the zero-temperature *quantum* exponent that we have denoted simply ν. As δ is further decreased through zero until $\delta - \delta_c \sim \Gamma_T^{-1-1/\psi\nu} \sim |\delta_c|/\Gamma_T$, the intermediate classical-percolation-dominated critical behavior as in Eq. (52) will obtain with the exponent ν_p. Finally, for $\delta - \delta_c \ll \Gamma_T^{-1-1/\psi\nu}$ the classical random Ising critical exponent ν_I will control the divergence of ξ_T. Note that in the limit of asymptotically small $1/\Gamma_T$, all three of these regimes will become very broad on a $\ln[(\delta_c - \delta)/\delta_c]$ plot.

C. Ordered phase: Singularities

We now turn to properties of the ordered phase on energy and temperature scales much lower than the ordering temperature T_c. The low-energy properties of the ordered phase for $d > 1$ will *not* have the strong power-law Griffiths singularities found in the disordered phase and the one-dimensional ordered phase. This can be seen in the RG language that naturally incorporates the role of rare anomalous regions. If we continue the RG much below Ω_∞, i.e., beyond the formation of the infinite cluster, we will find that the remaining finite clusters are almost always connected only to the macroscopic cluster and at lower energy scales almost all of these will either be decimated or will join the infinite cluster; fewer and fewer will join together to make larger finite clusters. Since the finite clusters and the bonds connecting them to the macrocluster are decimated independently, no new clusters or bonds will be generated, and the low-energy tails of the distributions remain essentially the same as they were just below Ω_∞ [when the widths of *both* the log-field and the log-bond distributions are of order $\sim \ln(\Omega_0/\Omega_\infty) \sim \Gamma_\delta \sim |\delta|^{-\psi\nu}$].

As in the disordered phase, the preformed tails of the field and bond distributions represent rare large regions responsible for the low-energy excitations in the system. The role of these rare fluctuations, however, is very different in the ordered phase; although they still make the system gapless, they do not dominate all the low-energy properties (such as, for example, the response to a small ordering magnetic field). In contrast to the disordered phase, the dominant rare regions in the ordered phase are indeed *very rare* and do not produce a power-law singularity in the density of states at zero energy. This can be seen by analyzing the probability that a cluster with a very small effective field $\tilde{h}$ survives down to energy scale $\Omega \sim \tilde{h} \ll \Omega_\infty$. We can consider such a surviving ferromagnetic cluster to be composed of n subclusters (each of diameter $\sim \xi$) with $\tilde{h} \sim \Omega_0 e^{-cn\Gamma_\delta}$, as in the disordered phase. But this cluster must be isolated very effectively from the rest of the system—with effective coupling linking it to the infinite cluster of order $\tilde{h}$ or weaker. In the disordered phase, the typical length of a bond with effective coupling $|\ln \tilde{J}| \sim n\Gamma_\delta$ is $L \sim n\xi$. Thus to achieve sufficient isolation, the disordered region around the droplet must have a *linear* size of order $\sim n\xi$. The probability of such a rare region is very small—of order $1/\alpha^{n^d}$—so that the generic low-energy tail that the RG can generate is

$$R(|\ln \tilde{h}|) \sim e^{-\tilde{c}|\ln \tilde{h}|^d}, \tag{54}$$

very different from the $\sim e^{-\tilde{c}|\ln \tilde{h}|}$ tail in the disordered phase and the similar tail in the one-dimensional ordered phase. It is also strikingly different from an even longer tail, $\sim e^{-\tilde{c}|\ln \tilde{h}|^{1-1/d}}$, that occurs in the ordered phase of the *dilute* quantum Ising system of Ref. 20, which differs from ours by some fraction of the initial J_{ij}'s being *zero*, so perfectly isolated clusters can form even in the ordered phase. The origin of the difference between these cases is easy to understand: in one dimension, the length and the volume of an isolating region are the same, while in the dilute case, with a δ-function weight at zero coupling, a droplet of size n can be

completely isolated by a ''disordered'' region of volume $n^{(d-1)/d}$—just a surrounding surface of missing bonds.

VI. HIGHER DIMENSIONS AND RELATED SYSTEMS

A. Higher-dimensional random ferromagnetic quantum Ising models

So far, we have presented detailed results only for two dimensions, although the general scaling picture, exponent equalities, behavior of correlation functions, etc., should be qualitatively the same for strong randomness in any dimension ($d>1$) for which there is a stable infinite-randomness critical fixed point. Our numerical studies in three dimensions are sufficient to indicate that the infinite-randomness fixed point is stable, although they are not thorough enough to yield reliable estimates of exponents and their uncertainties. Since in both two and three dimensions, weak randomness is a relevant perturbation away from the pure fixed point, we expect the same strong-randomness-dominated critical behavior to occur even for arbitrarily weak randomness.

The situation in higher dimensions—$d \geqslant 4$—is far more uncertain. It is not clear at this point whether or not the direction of the RG flow at strong randomness reverses for d sufficiently large.

For weak disorder, it would appear that the situation is clearer: the Harris criterion would seem to indicate that for $d>4$ weak randomness is irrelevant. But one must be very careful. There are other situations known in which weak randomness formally appears to be irrelevant but for which exponentially rare regions change the behavior for arbitrarily weak randomness.[21] We strongly suspect, as argued below, that this will be the case here. In general, Griffiths singularities and other strong-randomness-like effects will start to appear when the random quantum Ising system is close enough to the quantum transition that the distribution of J's and the distribution of h's overlap in the sense that for some values of (h,J) in the support of these distributions, a *pure* system would be in the ordered phase, while for other values of (h,J), a pure system would be in the disordered phase. This implies that arbitrarily large rare regions will exist that act as if they were in the opposite phase than the full system is. In particular, in the disordered phase sufficiently close to the critical point, strongly correlated clusters will exist with broadly distributed effective fields and effective interactions between them that are broadly distributed and typically decay exponentially with their separation. As the quantum critical point is approached, these rare clusters and their couplings will effectively act like a strongly random system that we expect will dominate the behavior and cause the whole system to be driven to strong (but not necessarily infinite) randomness sufficiently close to the critical point—however weak the original randomness. This intriguing possibility clearly merits further investigation.

B. Other quantum transitions with discrete broken symmetries

As mentioned in the Introduction, the infinite-randomness critical fixed points found here control more than just Ising ferromagnetic quantum transitions. In particular, as pointed out for the one-dimensional case by Senthil and Majumdar,[6] Potts models or any random quantum systems with continuous (second-order) transitions at which a discrete symmetry of a nonconserved order parameter is broken, will have the same critical behavior as the Ising case, with the extra degrees of freedom just ''going along for the ride'' on the basic geometrical transition.

This holds even for systems that are frustrated on small length scales, such as quantum Ising spin glasses. Because of flow to the infinite-randomness fixed point, the frustration will become irrelevant at low energies at the critical point, since in any frustrated loop the weakest interaction will be infinitely weaker than the others, so can be ignored. The primary changes here concern the coupling to a uniform magnetic field in the z direction, and the behavior at nonzero temperature in the ordered phase. Because the uniform field is not an ordering field for the spin glass, the magnetic moment of the large clusters will be random in sign, scaling as the square root of the number of active spins on the cluster. At the critical point this will change the $M(H)$ scaling, yielding

$$M_{SG} \sim |\ln H|^{\phi/2-d/\psi}, \tag{55}$$

in contrast to Eq. (21) for the ferromagnetic case. In the disordered phase, M will scale as the *same* power of H as in the ferromagnetic case with only the logarithmic prefactors modified. In the ordered phase, the behavior of the nonzero-temperature, long-distance correlations will cross over to classical spin-glass behavior at and below a temperature of order Ω_∞. For $d=2$, true long-range spin-glass order will be present only at zero temperature, because the lower critical dimension for the classical spin glass is always more than two.

C. Random quantum XY and Heisenberg antiferromagnets

The simplest example of an infinite randomness quantum fixed point occurs for one-dimensional random Heisenberg (or XY) spin chains. In the corresponding phase, the ''random singlet phase,'' each spin is paired in a singlet with one other spin, usually one close by, but a small fraction of the spins are paired very weakly with spins far away. The RG analysis of this system, first carried out by Ma, Dasgupta, and Hu[1] and then more fully by one of us,[2] is a simpler version of that used in the present paper.

A similar RG analysis was carried out for two- and three-dimensional random antiferromagnets by Bhatt and Lee[22] over a substantial range of energy scales, in particular including those relevant for experiments on the insulating phase of phosphorus-doped silicon. This investigation has been extended by two of us[23] to the strong-randomness limit. We have found that for $d \geqslant 2$, in contrast to one dimension, the infinite-randomness random-singlet fixed point of random Heisenberg or XY quantum antiferromagnets is *unstable* towards a state with finite randomness and, presumably, more conventional scaling; this state includes both antiferromagnetic and ferromagnetic effective interactions, and involves clusters with moments much larger than those of the single spins that dominate the low-energy behavior in the one-dimensional case.

VII. SUMMARY

In summary, we have studied random quantum Ising ferromagnets using an energy space cluster RG that becomes exact for strong randomness. Using the structure of the RG, we presented a scaling picture of the behavior near an infinite-randomness quantum critical fixed point that can occur. Near to this fixed point—corresponding to low energy scales near the zero-temperature quantum phase transition—the RG yields asymptotically exact results. We have implemented the RG numerically, primarily in two dimensions, and found that the critical behavior is indeed controlled by such an infinite-randomness fixed point, as in one dimension. We estimated numerically the corresponding critical exponents in two dimensions, and discussed the properties of the disordered and ordered phases. In the disordered phase we found that rare anomalously strongly coupled ferromagnetic clusters—in the RG language, a low-energy tail of the cluster field distribution generated by the decimation procedure—dominate the low-energy behavior and cause power-law Griffiths-McCoy singularities near the phase transition. In the ordered phase for $d>1$, on the other hand, the Griffiths singularities are much weaker, and do not produce divergences in thermodynamic quantities; the low-energy density of states they produce vanishes faster than any power of the energy.

The universality class controlled by the infinite-randomness quantum Ising critical fixed point is very broad; it includes all continuous quantum transitions in random systems at which a discrete symmetry is broken; since in two dimensions, first-order transitions are not possible in random systems, this class should include *all discrete-symmetry-breaking transitions* providing there are no conservation laws that alter the quantum dynamics in an essential way (for an example of an Ising case with a conserved order parameter, see Ref. 2). The nature of the discrete symmetry breaking quantum transitions we have studied is controlled by a novel type of percolation—rather surprising given the intrinsic quantum nature of the underlying models. As the rules of this percolation process are asymptotically *classical* (although the process is *not* conventional percolation), one might hope that conformal field theory approaches that take advantage of the two-dimensional (rather than ''$2+1$'' dimensional) structure could perhaps be used to obtain analytic results for some of the properties of such two-dimensional random quantum systems.

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