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Shape-preserving two-dimensional solitons in an Abrikosov vortex lattice

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The dynamics of a vortex lattice in a type-II superconductor are investigated under certain simplifying conditions. The Kadomtsev-Petviashvili equation is derived. As this equation admits shape-preserving two-dimensional soliton solutions, it is suggested that these entities be looked for in experiments. Differences with respect to previously found solitons in superconductors are stressed. [S0163-1829(99)01237-0]

In type-II superconductors, sufficiently strong magnetic fields can penetrate as an array of vortices aligned with the applied field.[1] Each vortex consists of one quantum of flux, $\phi_0 = hc/2e$. They appear between two critical values of the applied magnetic induction, B_{c1} and B_{c2}. Below B_{c1}, we observe classical superconductivity with no magnetic field in the body of the material. Above B_{c2}, superconductivity ceases altogether.

Here we consider either an isotropic, type-II superconductor, such as Nb, or else a layered high-T_c superconductor below the decoupling line in B,T parameter space, i.e., before the discreteness of the copper oxide planes becomes significant.[2,3] Vortex drag is assumed small and pinning negligible. These conditions can be satisfied in an ultraclean material.[4]

When the above conditions are satisfied, the basic equations are those of vortex continuity, a vortex equation of motion, and the London equation:

$$\frac{\partial \mathbf{B}_{\mathrm{v}}}{\partial t} = -\nabla \wedge (\mathbf{B}_{\mathrm{v}} \wedge \mathbf{v}), \tag{1}$$

$$\mu \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \alpha_{\mathrm{H}} \mathbf{v} \wedge \mathbf{n}_z + \phi_0 \mathbf{J} \wedge \mathbf{n}_z, \tag{2}$$

$$\mathbf{B}_{\mathrm{v}} = \mathbf{B} - \lambda_{\mathrm{L}}^2 \nabla^2 \mathbf{B}, \tag{3}$$

where

$$\mathbf{J} = \frac{1}{\mu_0} \nabla \wedge \mathbf{B}. \tag{4}$$

A detailed justification of these equations can be found in Ref. 5. Here $\mathbf{B}_{\mathrm{v}}$ is the local vortex generated magnetic field, assumed to be along z, $\mathbf{B}_{\mathrm{v}} = n\phi_0 \mathbf{n}_z$. A static applied induction $B_0 \mathbf{n}_z$ is assumed, and $B_{c1} < B_0 < B_{c2}$. The vortex velocity is $\mathbf{v}$, assumed in the x,y plane, and α_{H} is the Hall coefficient. A mass μ per unit length of the vortex is assumed.[6] The London penetration depth is λ_{L}. In the following analysis, a rectangular cross section of the superconductor is assumed, though cylindrical shapes will be discussed briefly. As $\nabla \cdot \mathbf{B}_{\mathrm{v}} = 0$, z will be cyclic. Under these conditions, Eqs. (1)–(4) simplify to

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \tag{5}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \alpha \mathbf{v} \wedge \mathbf{n}_z - \nabla B, \tag{6}$$

$$n = B - \nabla^2 B. \tag{7}$$

We have scaled the variables such that $t \to \omega_0 t$, $\mathbf{x} \to \mathbf{x}/\lambda_{\mathrm{L}}$, $v \to v/\omega_0 \lambda_{\mathrm{L}}$, $B \to B/B_0$, $n \to n\phi_0/B_0$, and

$$\omega_0^2 = \phi_0 B_0 \lambda_{\mathrm{L}}^2 / \mu_0 \mu, \tag{8}$$

$$\alpha = \alpha_{\mathrm{H}} / \omega_0 \mu. \tag{9}$$

Including $\alpha > 0$ is difficult, and for now we will consider the $\alpha = 0$ limit. This is all the more justified as α is small in most situations (its neglect is, in fact, practiced in many references, e.g., Ref. 5; see also the comment on the plasma theory analogy below). The remainder of this paper will concern possible solitonlike behavior of the vortex lattice in a superconductor so described.

When $\alpha = 0$, and only then, Eqs. (5)–(9) can be derived from a Lagrangian density. Assuming $\mathbf{v}$ to be irrotational, and $\mathbf{v} = \nabla\psi$, this density is

$$L = \frac{1}{2} n (\nabla\psi)^2 + n\psi_t + nB - \frac{1}{2} B^2 - \frac{1}{2} (\nabla B)^2 - n + \frac{1}{2}. \tag{10}$$

Euler-Lagrange variation with respect to ψ yields Eq. (5) whereas the n and B variations yield Eqs. (6) and (7), respectively. [However, to obtain Eq. (6) the gradient of the n Euler-Lagrange equation must be taken.] We will expand this Lagrangian for small amplitudes. The expansion scheme will be partially indicated by linear wave considerations. If we take $n = 1 + \delta n$, $B = 1 + \delta B$; δn, $\delta B \sim \exp[i(\mathbf{k}\cdot\mathbf{v} - \omega t)]$, we obtain, in the linear limit, from Eqs. (5)–(7),

$$\omega^2 = k^2 (1 + k^2)^{-1}. \tag{11}$$

For k small, this can be written as

$$\omega^2 = k_x^2 + k_y^2 - (k_x^2 + k_y^2)^2. \tag{12}$$

If we now extract the positive root and assume $k_y \ll k_x$, we obtain

$$\omega - k_x = -\frac{1}{2} k_x^3 + k_y^2 k_x^{-1}. \tag{13}$$

This suggests the expansion

$$\xi=\epsilon^{1/2}(x-t),\quad \sigma=\epsilon y,\quad \tau=\epsilon^{3/2}t. \tag{14}$$

We also take

$$n=1+\epsilon n^{(1)}+\epsilon^2 n^{(2)}+\cdots,\quad B=1+\epsilon B^{(1)}+\epsilon^2 B^{(2)}+\cdots,\quad \psi=\epsilon^{1/2}\psi^{(1)}+\epsilon^{3/2}\psi^{(2)}+\cdots. \tag{15}$$

We now expand the Lagrangian (10) in ϵ. Our expansion in ϵ limits considerations to a specific class, rather than aborting the physical description. We are now interested in situations such that deviations from a uniform distribution of the vorticies is small. Similarly for the magnetic field; deviations from $\omega_0\lambda_L$ in velocities of propagation are also taken to be small (weak dispersion). Finally, y dependence is slow as compared to that on x. It is often found that models for this limited class of situations are extremely informative, validity extending further than we might expect from the derivation, for examples, see Chap. 8 of Ref. 8, and also the other work referenced there.

$L^{(1)}$ is trivial, as all that survives is $-\psi_\xi^{(1)}$. However, the next order leads to useful information,

$$L^{(2)}=-n^{(1)}\psi_\xi^{(1)}+n^{(1)}B^{(1)}-\frac{1}{2}B^{(1)^2}+\frac{1}{2}\psi_\xi^{(1)^2}+\psi_\tau^{(1)}-\psi_\xi^{(2)}, \tag{16}$$

and the $n^{(2)}$ and $B^{(2)}$ terms cancel. We obtain, as Euler-Lagrange equations,

$$\delta n^{(1)}:\quad \psi_\xi^{(1)}=B^{(1)},$$

$$\delta B^{(1)}:\quad n^{(1)}=B^{(1)},$$

$$\delta\psi^{(1)}:\quad n^{(1)}=\psi_\xi^{(1)}.$$

The third-order Lagrangian in ϵ, when expressed in terms of $\psi^{(1)}$, is

$$L^{(3)}=\psi_\tau^{(1)}\psi_\xi^{(1)}+\frac{1}{2}\psi_\xi^{(1)^3}-\frac{1}{2}\psi_{\xi\xi}^{(1)^2}+\frac{1}{2}\psi_\sigma^{(1)^2} + \text{perfect differentials}. \tag{17}$$

We have not written out the perfect differentials, as they will not contribute to the Euler-Lagrange equations. Our expansion of a Lagrangian is computationally simpler than the standard method, in which each equation, such as our Eqs. (5)–(7) is expanded in ϵ. Although less work is involved, the end result is, of course, the same. The preceding equation yields, as its Euler-Lagrange equation, omitting subscripts and reverting to $(u,v)=(\psi_\xi,\psi_\sigma)$ notation,

$$u_\tau+\frac{3}{2}uu_\xi+\frac{1}{2}u_{\xi\xi\xi}+\frac{1}{2}\partial_\xi^{-1}u_{\sigma\sigma}=0. \tag{18}$$

This is the Kadomtsev-Petviashvili equation. It is integrable[7] and has both one and multiple soliton solutions, as described in Ref. 8. The simplest, one soliton solution, is

$$u=4k_\xi^2\,\mathrm{sech}^2(k_\xi\xi+k_\sigma\sigma-\omega\tau+\delta), \tag{19a}$$

$$D(\mathbf{k},\omega)\equiv k_\xi\omega-2k_\xi^4-\frac{1}{2}k_\sigma^2=0. \tag{19b}$$

Two soliton solutions are X-shaped entities, mobile but shape preserving. Exact formulas and illustrations can be found in Ref. 8, Chap. 7, though in a completely different physical context. Away from the intersection, behavior of each arm of the "X" is similar to Eqs. (19) (with two sets of $\mathbf{k},\omega$, and four phases). Introducing

$$\eta_i=k_{\xi_i}\xi+k_{\sigma_i}\sigma-\omega_i\tau+\delta_i,\quad i=1,2,$$

$$\Delta=-\frac{1}{2}\ln\left|\frac{D(\mathbf{k}_1-\mathbf{k}_2,\omega_1-\omega_2)}{D(\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2)}\right|,$$

we have for large $|\sigma|$,

$$u=\begin{cases}4k_{\xi_1}^2\,\mathrm{sech}^2\,\eta_1+4k_{\xi_2}^2\,\mathrm{sech}^2(\eta_2-\Delta), & \sigma<0\\ 4k_{\xi_1}^2\,\mathrm{sech}^2(\eta_1-\Delta)+4k_{\xi_2}^2\,\mathrm{sech}^2\,\eta_2, & \sigma>0.\end{cases} \tag{20}$$

Thus, a rectangular cross-section, ultraclean type-II superconductor can essentially support shape-preserving solitons. They are stable; see Chap. 8 of Ref. 8.

For *cylindrical* superconductors, the cylindrical Korteweg–de Vries equation[9] has been derived.[5] It is, in variables we prefer,

$$v_\tau+\frac{3}{2}vv_r+\frac{1}{2}v_{rrr}+\frac{1}{2\tau}v=0, \tag{21}$$

where v is in the r direction. The one soliton solution collapses, growing in amplitude as it nears the axis. we would just add to Ref. 5 that interesting two soliton solutions are illustrated in Ref. 8, Chap. 9. N soliton solutions that interpenetrate as they collapse, are known.

Thus our solution, outlined in Eq. (20), differs from that of Ref. 5. First of all, ours is essentially shape preserving, whereas that of Ref. 5 collapses, thus being more difficult to observe. Second, ours cannot possibly be described with one space variable, we must introduce two. In summary, ours is statically more interesting, with Coffey's possibly dynamically more so.

It is tempting to equate the mathematical problem considered here with that of ion acoustic waves in a two component plasma. (Here α is the ratio of the Alfvén velocity to c, usually at most of the order of 10^{-6}.) However, there is a subtlety when so doing. From the plasma problem, in dimensionless units, both Eqs. (5) and (6) can be recovered when a substitution is made: $n\leftrightarrow n_\mathrm{i}$, $\mathbf{v}\leftrightarrow\mathbf{v}_\mathrm{i}$, and $B\leftrightarrow 1+\phi$, where ϕ is the electrostatic potential and the "i" subscript denotes ions. However, the equation corresponding to (7) for a plasma is

$$n=e^\phi-\nabla^2\phi. \tag{22}$$

The approximation $e^\phi\simeq 1+\phi$, which would complete the identity, is, however, not good enough. If we go back to our derivation of the Kadomtsev-Petviashvili equation, we find that new terms quadratic in ϕ would appear (cubic in the

Lagrangian). This leads to different coefficients in the derived Kadomtsev-Petviashvili equation for the two problems.

In conclusion, we have demonstrated the possibility of finding shape-preserving, fully two-dimensional solitons in an ultraclean, type-II superconductor. This shape-preserving quality is in contradistinction to the one space variable entities forseen in the pioneering work of Ref. 5. Many simplifying assumptions were necessary. However, a search for solitons in superconductors should probably begin with the shape-preserving variety. Their detection could lead to a nonintrusive determination of the value of μ.

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