Exact solution of a spin-ladder model

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An integrable spin-ladder model with nearest-neighbor exchanges and biquadratic interactions is proposed. With the Bethe ansatz solutions of the model Hamiltonian, it is found that there are three possible phases in the ground state, i.e., a rung-dimerized phase with a spin gap, and two massless phases. The possible fixed points of the system and the quantum critical behavior at the critical point $J=J_{+}^{c}$ are discussed.
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Recently, there has been growing interest in the spin ladders for their relevance to some quasi-one-dimensional materials, which under hole-doping may show superconductivity.[1] It is well known that the $S=1/2$ isotropic spin ladders with even number of legs have a spin-liquid ground state with an energy gap, while odd-legged ladders have a gapless spin-liquid ground state. On the other hand, generalized ladders including other couplings beyond the nearest-neighbor exchanges, which can interpolate between a variety of systems, can show remarkably rich behavior.[2–4] In a recent paper,[2] Nersesyan and Tsvelik predicted a new gapful phase for the two-leg spin ladders, i.e., the dimerized phase driven by the four-spin interactions, which is essentially different from the well-known Haldane phase.[5] This observation has been demonstrated in a generalized spin-ladder model[3] by constructing the exact ground state. Another interesting phenomenon in the ladder systems is the quantum phase transition from the gaped phase to the gapless phase, which has been studied experimentally in the Heisenberg ladder $Cu_2(C_5H_{12}N_2)_2Cl_4$.[6]

As is well known, the integrable models provided us with very good understanding of the correlated many-body systems in one dimension. However, a satisfactory integrable ladder model, which may play a similar role of the Heisenberg chain,[7] the one-dimensional (1D) Hubbard model,[8] and the supersymmetric t-J model,[9] is still absent. The difficulty to construct an integrable ladder model is almost the same as we encountered in constructing a two-dimensional integrable model due to the strict conditions for the integrability. For example, in a 1D model there is only one path connecting two different sites, while even for two coupled chains, we have a large number of paths connecting two different sites. We note that an integrable ladder model with artificial three-spin interactions has been proposed recently[10] and the integrability of a generalized spin ladder without a free parameter[11] was addressed. The latter is more interesting but still defies the Bethe-ansatz solution. In this paper, we study a spin ladder with biquadratic interactions. By properly choosing the four-spin coupling constants, we show that the model is exactly solvable via algebraic Bethe ansatz. The model Hamiltonian we shall study reads

$$H=\frac{1}{4}J_1\sum_{j=1}^{N}[\vec{\sigma}_j\cdot\vec{\sigma}_{j+1}+\vec{\tau}_j\cdot\vec{\tau}_{j+1}]+\frac{1}{2}J_2\sum_{j=1}^{N}\vec{\sigma}_j\cdot\vec{\tau}_j+\frac{1}{4}U_1\sum_{j=1}^{N}(\vec{\sigma}_j\cdot\vec{\sigma}_{j+1})(\vec{\tau}_j\cdot\vec{\tau}_{j+1})+\frac{1}{4}U_2\sum_{j=1}^{N}(\vec{\sigma}_j\cdot\vec{\tau}_j)\times(\vec{\sigma}_{j+1}\cdot\vec{\tau}_{j+1}), \tag{1}$$

where $\vec{\sigma}_j$ and $\vec{\tau}_j$ are Pauli matrices acting on site j of the upper and lower legs, respectively; J_1 and J_2 are the coupling constants along the legs and the rungs, respectively; $U_{1,2}$ are the biquadratic coupling constants and N denotes the length of the ladder. Without the four-spin terms, Eq. (1) represents the ordinary spin-ladder model. The new terms in Eq. (1) represent an interchain coupling and an interrung coupling, which can be either effectively mediated by spin-phonon interaction or in the doped phase generated by the conventional Coulomb repulsion between the holes moving in the spin correlated background as discussed in Refs. 2 and 12. The importance of biquadratic exchange for some properties of CuO_2 plaquette has been pointed out[13] and recent experiments revealed that such multi-spin-exchange interactions are realized in the two-dimensional (2D) solid ^{3}He,[14] 2D Wigner solid of electrons formed in a Si inversion layer,[15] and the bcc solid ^{3}He.[16] On the other hand, the four spin terms have clear physical meaning in the spin-orbital model,[17] where one leg (say, σ) represents the spins and the other leg (say, τ) represents the orbital. Such interactions are general in the transition-metal oxides and are recent hot interest of research. We note that when $U_2=0$, Eq. (1) is reduced to the model considered in Ref. 2. For general parameters $J_{1,2}$ and $U_{1,2}$, the model (1) is still nonintegrable. However, as we shall show below, when $U_1=J_1$, $U_2=0$ or $U_1=J_1$, $U_2=-J_1/2$, the model is exactly solvable. We shall study these cases through this paper. Not loosing generality, we set $J_1=U_1=1$, $J_2=J$, and $U_2=U$ in the following text.

We study first $U=0$ case. This is the simplest integrable case but shows the main physics of the system. The Hamiltonian (1) for $U=0$ can be rewritten as

$$H=\frac{1}{4}\sum_{j=1}^{N}(1+\vec{\sigma}_j\cdot\vec{\sigma}_{j+1})(1+\vec{\tau}_j\cdot\vec{\tau}_{j+1})+\frac{1}{2}J\sum_{j=1}^{N}(\vec{\sigma}_j\cdot\vec{\tau}_j-1)$$
$$+\tfrac{1}{2}(J-\tfrac{1}{2})N. \quad (2)$$

In this form, the integrability of the model is still somewhat hidden. To show it clearly, we note that the first term in Eq. (2) can be rewritten as $\Sigma_{j=1}^{N}P_{j,j+1}$, where $P_{j,j+1}$ is the permutation operator between two nearest rungs. Therefore, the first term of Eq. (2) is SU(4) invariant as showed for the spin-orbital model.[17] An obvious fact is that $P_{j,j+1}$ can be expressed as $P_{j,j+1}=\Sigma_{\alpha,\beta}X_j^{\alpha\beta}X_{j+1}^{\beta\alpha}$, where $X_j^{\alpha\beta}\equiv|\alpha_j\rangle\langle\beta_j|$ are the Hubbard operators and the Dirac states $|\alpha_j\rangle$ span the Hilbert space of the jth rung and are orthogonal ($\langle\alpha_j|\beta_j\rangle=\delta_{\alpha\beta}$). For convenience, we choose $|0\rangle=1/\sqrt{2}(|\uparrow,\downarrow\rangle-|\downarrow,\uparrow\rangle)$, $|1\rangle=|\uparrow,\uparrow\rangle$, $|2\rangle=1/\sqrt{2}(|\uparrow,\downarrow\rangle+|\downarrow,\uparrow\rangle)$, and $|3\rangle=|\downarrow,\downarrow\rangle$. The first state denotes a singlet rung and the latter three indicate the spin-triplet states of a rung. With these notations, Eq. (2) can be rewritten as

$$H=\sum_{j=1}^{N}\sum_{\alpha,\beta=0}^{3}X_j^{\alpha\beta}X_{j+1}^{\beta\alpha}-2J\sum_{j=1}^{N}X_j^{00}+\frac{1}{2}\left(J-\frac{1}{2}\right)N. \quad (3)$$

Obviously, the operators $N_\alpha\equiv\Sigma_{j=1}^{N}X_j^{\alpha\alpha}$, which denote the numbers of the α rungs in the whole system, are conserved quantities. The constant $2J$ in the last term of Eq. (3) indicates a chemical potential applied on N_0 and reduces the global SU(4) symmetry of the Hamiltonian to U(1) $\times$SU(3). Now we have reduced the Hamiltonian (1) to an SU(4)-invariant spin chain [or equivalently an SU(4) t-J model], which can be solved by following the standard method.[9] There are three branches of flavor waves (generalized spin waves) in this system. If we choose the reference state as $|\Omega\rangle=|0_1\rangle\otimes|0_2\rangle\otimes\cdots\otimes|0_N\rangle$, the Bethe state can be constructed as

$$|\Psi\rangle=\sum_{\{j_m,\alpha_m\}}\Psi_{\alpha_1,\ldots,\alpha_{M_1}}(j_1,\ldots,j_{M_1})$$
$$\times X_{j_1}^{\alpha_1 0}\cdots X_{j_{M_1}}^{\alpha_{M_1}0}|\Omega\rangle, \quad (4)$$

where Ψ is the wave function, the summation for j_m runs from 1 to N and that for α_m runs from 1 to 3. After some manipulation,[9] we obtain the Bethe-ansatz equations (BAE's):

$$\left(\frac{\lambda_j-i/2}{\lambda_j+i/2}\right)^N=\prod_{l\neq j}^{M_1}\frac{\lambda_j-\lambda_l-i}{\lambda_j-\lambda_l+i}\prod_{\alpha=1}^{M_2}\frac{\lambda_j-\mu_\alpha+i/2}{\lambda_j-\mu_\alpha-i/2},$$

$$\prod_{\beta\neq\alpha}^{M_2}\frac{\mu_\alpha-\mu_\beta-i}{\mu_\alpha-\mu_\beta+i}=\prod_{j=1}^{M_1}\frac{\mu_\alpha-\lambda_j-i/2}{\mu_\alpha-\lambda_j+i/2}\prod_{\delta=1}^{M_3}\frac{\mu_\alpha-\nu_\delta-i/2}{\mu_\alpha-\nu_\delta+i/2}, \quad (5)$$

$$\prod_{\gamma\neq\delta}^{M_3}\frac{\nu_\delta-\nu_\gamma-i}{\nu_\delta-\nu_\gamma+i}=\prod_{\alpha=1}^{M_2}\frac{\nu_\delta-\mu_\alpha-i/2}{\nu_\delta-\mu_\alpha+i/2},$$

where $M_1=N_1+N_2+N_3$, $M_2=N_2+N_3$, and $M_3=N_3$; λ_j, μ_α, and ν_δ represent the rapidities of the flavor waves. Note the periodic boundary conditions $X_{N+1}^{\alpha\beta}=X_1^{\alpha\beta}$ has been used in deriving Eq. (5). The eigenenergy of the Hamiltonian (3) is given by

$$E=-\sum_{j=1}^{M_1}\left(\frac{1}{\lambda_j^2+\frac{1}{4}}-2J\right)+\frac{3}{4}(1-2J)N. \quad (6)$$

Obviously, for $0<J<2$, the ground state consists of real λ_j, μ_α, and ν_δ closely packed around the origin. That means we have three ''Fermi seas'' and three branches of gapless excitations. For $J>2$, the reference state becomes the true ground state and any flavor excitation is gapful. The ground state is a product of the singlet rungs, which indicates the dimerization along the rungs. The energy gap can be easily deduced from Eq. (6) as $\Delta=2(J-2)$. $J_+^c=2$ indicates a quantum critical point at which the quantum phase transition from the dimerized phase to the gapless phase occurs. At this critical point, all the three branches of flavor excitations are marginal and the low-temperature thermodynamics of the system shows non-Fermi-liquid behavior as we shall discuss below. For $J<0$, the singlet rungs are unfavorable at low energy scales. For convenience, we choose $|1_1\rangle\otimes|1_2\rangle\otimes\cdots\otimes|1_N\rangle$ as the reference state. The BAE's are still given by Eq. (5) but with $M_1=N_0+N_2+N_3$, $M_2=N_3+N_0$, and $M_3=N_0$. The eigenenergy is given by

$$E=-\sum_{j=1}^{M_1}\frac{1}{\lambda_j^2+\frac{1}{4}}-2JN_0+\frac{1}{4}(3+2J)N. \quad (7)$$

A hidden fact is that there is another critical value J_-^c. For $J<J_-^c$, no singlet rung exists in the ground-state configuration and the excitations consisting of singlet rungs are gapful. In this case, there are only two branches of gapless flavor waves and the effective low-energy Hamiltonian is equivalent to that of the SU(3)-invariant spin chain. The ground state consists of two ''Fermi seas'' (for λ and μ) with $M_1=2N/3$, $M_2=N/3$, and $M_3=0$. We denote the distributions of λ and μ in the ground state as $\rho_1(\lambda)$ and $\rho_2(\mu)$, respectively. A singlet excitation can be constructed by introducing a ν mode and a μ hole μ_h in the BAE's. We denote further the changes of $\rho_1(\lambda)$ and $\rho_2(\mu)$ via the ν mode and the μ hole as $\delta\rho_1(\lambda)$ and $\delta\rho_2(\mu)$, respectively. From the BAE's (5) we can easily obtain

$$\delta\tilde{\rho}_1(\omega)=\frac{1}{4\cosh^2(\omega/2)-1}[e^{-i\nu\omega}-e^{-(1/2)|\omega|}e^{-i\mu_h\omega}], \quad (8)$$

where $\delta\tilde{\rho}_1(\omega)$ is the Fourier transformation of $\delta\rho_1(\lambda)$. Combining Eq. (7) and Eq. (8), we derive the minimum energy (corresponding to $\nu\to0$, $\mu_h\to\infty$) to excite a ν mode from the ground state as

$$\epsilon_{min}=-\frac{1}{2}\int\delta\tilde{\rho}_1(\omega)e^{-(1/2)|\omega|}d\omega-2J$$
$$=2|J|-\pi/2\sqrt{3}+\ln\sqrt{3}. \quad (9)$$

The critical value J_-^c is thus derived from $\epsilon_{min}=0$ as $J_-^c=-\pi/(4\sqrt{3})+(ln3)/4$. For $J_-^c<J<0$, the system behaves as for $0<J<2$. Exactly at the critical point $J=J_-^c$, one

branch of the flavor excitations (the singlet one) is marginal. Therefore, we have three quantum phases in these systems: A rung-dimerized phase when $J>J_+^c$, a gapless phase with three branches of gapless flavor excitations when $J_+^c>J>J_-^c$, and another gapless phase with two branches of gapless flavor excitations when $J<J_-^c$. We note that the dimerized phase shows a long-range order

$$\langle\Omega|X_i^{00}X_j^{00}|\Omega\rangle=1, \tag{10}$$

which indicates the condensation of the singlet rungs. Under hole doping, the system behaves as a t-J ladder and the singlet rungs serve as Cooper pairs. The mobility of the singlet rungs under hole doping may drive the system to show superconductivity. Based on the above observations, we conclude that $J_\pm^c$ represents two unstable fixed points of the system. In addition, the stable fixed points of the system can be conjectured. For $J>J_+^c$, the transverse exchange dominates over the exchange along the legs and the system should flow to a fixed point $J^*=+\infty$. For $J_-^c<J<J_+^c$, the two unstable fixed points $J_\pm^c$ indicate an intermediate stable fixed point $J_-^c<J^*<J_+^c$. For $J<J_-^c$, the singlet excitations are eliminated at low-energy scales and the system should flow to a fixed point $J^*=-\infty$, which is equivalent to an SU(3)-invariant spin chain. The gapless modes in the latter phase is mainly due to the high symmetry. Any small perturbations of J_1 or U_1 breaking this symmetry may drive it to the Haldane phase as in the SU(3)-invariant spin-1 chain.

Based on the BAE's, the thermodynamics of the present model can also be derived by following the standard method.[18,19] In the gapless phases, the system behaves as a Luttinger liquid[20] and nothing is anomalous. However, at the quantum critical points, the system may show non-Fermi-liquid behavior due to the marginal excitations. We consider first the zero-temperature magnetic susceptibility for $J=J_+^c$. Without the external field, the ground state is a condensate of singlet rungs. If we apply a very weak external field on the system, some triplet rungs with $S_z=1$ appear in the ground-state configuration, while N_2 and N_3 still keep to be zero since the levels of these two types of rungs are either lifted ($|3\rangle$) or unchanged ($|2\rangle$). The energy density of the ground state in an external magnetic field ($H>0$) reads

$$E/N=\int_{-\Lambda}^{\Lambda}\left(4-\frac{1}{\lambda^2+\frac{1}{4}}-H\right)\rho_1(\lambda)d\lambda, \tag{11}$$

where $\rho_1(\lambda)$ satisfies

$$\rho_1(\lambda)+\int_{-\Lambda}^{\Lambda}a_2(\lambda-\lambda')\rho_1(\lambda')d\lambda'=a_1(\lambda), \tag{12}$$

with $a_n(\lambda)=n/2\pi[\lambda^2+(n/2)^2]$ and $\Lambda^2=1/(4-H)-1/4$. For a small $H\ll 1$, we have $\Lambda\approx\sqrt{H}/4$ and Eq. (12) can be solved up to $O(H^{3/2},\lambda^2)$ as

$$\rho_1(\lambda)=2/\pi-(1/\pi^2)\sqrt{H}+O(H^{3/2},\lambda^2). \tag{13}$$

Combining Eq. (13) and Eq. (11) we readily obtain the susceptibility as

$$\chi=-\partial^2(E/N)/\partial H^2=(1/2\pi)H^{-1/2}+O(H^{1/2}). \tag{14}$$

The low-temperature susceptibility and the specific heat can also be derived from the so-called thermal Bethe ansatz.[18,19] Via low-temperature expansion of the thermal BAE's (Refs. 21–23) we obtain

$$C\sim T^{1/2},\quad \chi\sim T^{-1/2}, \tag{15}$$

which indicate a typical quantum critical behavior. These results can also be predicted by a simple flavor-wave theory with the dispersion relation $\epsilon(k)\sim k^2$. We note in the gaped phase, the magnetic field can also drive a quantum phase transition. At the quantum critical point $H_c=2(J-2)$, similar quantum critical behavior can be obtained. Another interesting quantity is the order parameter $\langle\vec{\sigma}_i\cdot\vec{\tau}_i\rangle=4(N_1+N_2+N_3)-3$. When $J_-^c<J<2$, it takes the value of -3. Its quantum critical behavior for $J\to 2+0^-$ can be derived by following the same method for the magnetization. The only difference is to put $J-2$ as an effective field (note that the spin triplet is isotropic relative to this field). The result reads $\langle\vec{\sigma}_i\cdot\vec{\tau}_i\rangle+3\sim(J-2)^{1/2}$.

Now we turn to the $U=-1/2$ case. The last term in Eq. (1) can be rewritten as $-\Sigma_j[2X_j^{00}X_{j+1}^{00}-X_j^{00}+1/4]$. Up to an irrelevant constant, we rewrite Eq. (1) as

$$H=\sum_{j=1}^{N}\left[\sum_{\alpha=1}^{3}(X_j^{\alpha 0}X_{j+1}^{0\alpha}+X_j^{0\alpha}X_{j+1}^{\alpha 0})+\sum_{\alpha,\beta=1}^{3}X_j^{\alpha\beta}X_{j+1}^{\beta\alpha}-X_j^{00}X_{j+1}^{00}\right]+(1-2J)\sum_{j=1}^{N}X_j^{00}. \tag{16}$$

The above Hamiltonian has the same algebraic structure as that of an SU(1|3) supersymmetric t-J model, which still allows Bethe-ansatz solution. We choose still $|\Omega\rangle$ as the reference state. The BAE's read

$$\left(\frac{\lambda_j-i/2}{\lambda_j+i/2}\right)^N=\prod_{\alpha=1}^{M_2}\frac{\lambda_j-\mu_\alpha-i/2}{\lambda_j-\mu_\alpha+i/2},$$

$$\prod_{\beta\neq\alpha}^{M_2}\frac{\mu_\alpha-\mu_\beta-i}{\mu_\alpha-\mu_\beta+i}=\prod_{j=1}^{M_1}\frac{\mu_\alpha-\lambda_j-i/2}{\mu_\alpha-\lambda_j+i/2}\prod_{\delta=1}^{M_3}\frac{\mu_\alpha-\nu_\delta-i/2}{\mu_\alpha-\nu_\delta+i/2}, \tag{17}$$

$$\prod_{\gamma\neq\delta}^{M_3}\frac{\nu_\delta-\nu_\gamma-i}{\nu_\delta-\nu_\gamma+i}=\prod_{\alpha=1}^{M_2}\frac{\nu_\delta-\mu_\alpha-i/2}{\nu_\delta-\mu_\alpha+i/2}.$$

The eigenenergy of the Hamiltonian (16) is given by

$$E=\sum_{j=1}^{M_1}\left(\frac{1}{\lambda_j^2+\frac{1}{4}}+2J-1\right)-2JN. \tag{18}$$

The situation is very similar to that of $U_1=1$, $U=0$ case. There are still three phases, i.e., a rung-dimerized phase and two gapless phases. For $J_-^c<J<1/2$, some triplet rungs are allowed in the ground state. The ground-state configuration is described by closely packed real ν modes and the corresponding $\lambda-3$ strings and $\mu-2$ strings:

$$\lambda_\delta^n=\nu_\delta+i(2-n),\quad n=1,2,3,\quad \mu_\delta^\pm=\nu_\delta\pm i/2. \tag{19}$$

For $J>J_+^c=1/2$, we get again a dimerized ground state. Comparing to the SU(4) case, we find J_+^c is remarkably re-

duced by a negative U. Notice that a negative U indicates the attraction between two nearest singlet rungs. This attraction enhances the dimerization along the rung direction.

For small positive J or negative J, the triplet rungs are more stable than the singlet ones. For convenience, we choose $|1_1\rangle\otimes\cdots\otimes|1_N\rangle$ as the reference state. The BAE's (Ref. 24) read

$$\left(\frac{\lambda_j-i/2}{\lambda_j+i/2}\right)^N=\prod_{l\neq j}^{M_1}\frac{\lambda_j-\lambda_l-i}{\lambda_j-\lambda_l+i}\prod_{\alpha=1}^{M_2}\frac{\lambda_j-\mu_\alpha+i/2}{\lambda_j-\mu_\alpha-i/2},$$

$$\prod_{\beta\neq\alpha}^{M_2}\frac{\mu_\alpha-\mu_\beta-i}{\mu_\alpha-\mu_\beta+i}=\prod_{j=1}^{M_1}\frac{\mu_\alpha-\lambda_j-i/2}{\mu_\alpha-\lambda_j+i/2}\prod_{\delta=1}^{M_3}\frac{\mu_\alpha-\nu_\delta-i/2}{\mu_\alpha-\nu_\delta+i/2}, \tag{20}$$

$$\prod_{\alpha=1}^{M_2}\frac{\nu_\delta-\mu_\alpha-i/2}{\nu_\delta-\mu_\alpha+i/2}=1,$$

where $M_1=N_2+N_3+N_0$, $M_2=N_3+N_0$, and $M_3=N_0$. The eigenenergy takes the same form of Eq. (7) but with $J\to J-1/2$. J_-^c can be easily derived from Eq. (20) as $J_-^c=\frac{1}{2}-\pi/4\sqrt{3}+\frac{1}{4}\ln 3$. Interestingly, J_-^c takes a positive value in this case. The space of the phase with three branches of massless excitations is remarkably depressed by the attractive rung-rung interaction. This observation strongly indicates that there is a critical point $U=U_c$. When $U<U_c$, $J_\pm^c$ coincide each other and the intermediate fixed point will be eliminated, implying only two phases can exist in the system. In fact, the model (1) is exactly solvable in the sectors $N_0=0,1$ for arbitrary U. Following the same procedure discussed below Eq. (7) we can easily obtain the energy difference between the ground states of sector $N_0=1$ and sector $N_0=0$ as $\delta\epsilon=-2(J+U)-\pi/(2\sqrt{3})+(\ln 3)/4$. $\delta\epsilon=0$ gives a phase boundary of the J-U phase diagram and when $J=J_+^c=1/2$, we derive $U_c=-1/2-\pi/(4\sqrt{3})+(\ln 3)/8$.

In conclusion, we propose an integrable spin-ladder model that exhibits rich physics. This model may play a similar role in the spin-ladder systems as the supersymmetric t-J model does in the one-dimensional correlated electron systems.

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