Projective construction of non-Abelian quantum Hall liquids

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Using projective construction, a generalized parton construction, we construct many non-Abelian quantum Hall (QH) states, which include the Pfaffian state at filling fraction $\nu=1/2$. The projective construction allows us to calculate bulk and edge effective theories for the constructed QH state. We illustrate how to use the bulk effective theory to calculate the ground-state degeneracy of non-Abelian QH liquids on a torus. We point out that a full description of the effective theory requires both an effective Lagrangian and the definition of electron operators. The latter generates all physical states, and defines the gauge structure of the theory.
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I. INTRODUCTION

Quantum Hall (QH) liquids as a fundamentally new state of matter contain an interesting kind of order—topological order.[1,2] The different topological orders in QH liquids can be divided into two classes. The topological orders in the first class–the Abelian topological orders–are labeled by the K matrix,[3] and were believed to describe most of the observed QH liquids. The second class of topological orders—the non-Abelian topological orders—also exists in QH liquids.[4,5] Quasiparticles in non-Abelian QH states carry non-Abelian statistics, and their edge states cannot be described by ''edge phonons'' (which is a collection of harmonic oscillators).

There are two ways to construct non-Abelian QH states. One[6–10] is through a correlation function in conformal field theory (CFT), and the other[5,6] is through parton construction.[11,12] Both constructions allow us to calculate the structures of edge states.[5,13,14,9,10] However, only the parton construction allows us to calculate the bulk effective theories, which turn out to be Chern-Simons (CS) theories.

In this paper we introduce the projective construction which generalizes parton construction. Using the projective construction, we can construct many old and new non-Abelian (and Abelian) QH states, which include both the $\nu=1$ bosonic Pfaffian state and the $\nu=1/2$ fermionic Pfaffian state,[4,15,7,13,14,8–10] as well as the d-wave paired state introduced in Refs. 7 and 13. The projective construction allows us to calculate both the bulk and edge effective theories for the constructed QH states. The bulk effective theories are complete enough to allow us to calculate the ground-state degeneracies on a torus.

Using the projective construction, we find the effective theories for the $\nu=1$ bosonic Pfaffian state and the $\nu=1/2$ fermionic Pfaffian state to be the $SO(5)_1$ and $U(1)\times SO(5)_1$ CS theories. From these effective theories, we calculated the ground state degeneracies for both states, which are 3 and 6 on the torus. We also calculated the edge effective theories for the two states. The results agree with previous results obtained through the wave functions.[16,9,15,13,14]

The effective theories for the $\nu=1$ bosonic Pfaffian state and the $\nu=1/2$ fermionic Pfaffian state have been obtained before using different approaches. The $SO(5)_1$ CS effective theory for the $\nu=1$ bosonic Pfaffian state obtained here is formally different from the $SU(2)_2$ CS effective theory obtained in Ref. 17. Although both theories give three degenerate ground states on the torus, it is unclear if the two theories are equivalent on high genus Riemann surfaces. At least the meaning of the gauge fields and their coupling to the external electromagnetic field A_μ are quite different. The effective theory for the $\nu=1/2$ fermionic Pfaffian state obtained in Ref. 17 is very unusual (which cannot be regarded as an ordinary CS theory). It is not clear if such an effective theory is equivalent to our $U(1)\times SO(5)_1$ CS effective theory for the $\nu=1/2$ fermionic Pfaffian state. In particular, it is not clear whether the effective theory in Ref. 17 reproduces the six degenerate ground states on the torus. Another form of effective theory—the non-Abelian Ginzburg-Landau CS theory—was obtained in Ref. 18 to describe the Pfaffian states. Since the ground state degeneracies were not calculated, the relation between the effective theories in Ref. 18 and the effective theories obtained in this paper is unclear at the moment.

We would like to point out that two topological theories (CS or Ginzburg-Landau CS theories) with very different forms could be equivalent. This is because the effective theories were written in terms of gauge fields that impose constraint. One can reach the same QH state through different projection schemes which lead to different forms of effective theory. To really know whether the two topological theories are equivalent or not, one needs to study their properties on high genus Riemann surfaces.

In Sec. II we introduce the projective construction using the $U(1)_l\times SU(2)_n$ non-Abelian state as an example. In Sec. III we use the projective construction to construct the $\nu=1$ bosonic Pfaffian state and the $\nu=1/2$ fermionic Pfaffian state. This allows us to obtain the bulk and edge effective theories for the two states. The projective construction also allows us to construct many new non-Abelian states and calculate their bulk and edge effective theories. In Sec. IV we give a general discussion of the projective construction. In particular, we point out the importance of electron operators in defining the effective bulk theory. We illustrate how the discrete gauge structure in the effective theory can affect physical quantities, such as the ground-state degeneracy. The projective construction is a very powerful construction which

can be used to construct many different QH states (both Abelian and non-Abelian). In Sec. V we illustrate how to use the projective construction through some simple examples.

II. PROJECTIVE CONSTRUCTION AND $U(1)_L \times SU(2)_N$ NONABELIAN STATES

In this section we are going to use the projective construction to construct the $U(1)_l \times SU(2)_n$ non-Abelian states.[5] We start with the simplest example, and then generalize to more complicated cases.

We start with a non-Abelian state of spin-1 (bosonic) electrons.[6] The wave function is given by

$$\Phi^b(z_1,m_1;z_2,m_2;\dots) = \sum_{\alpha_1.\beta_1;\dots} \chi_s(z_1,\alpha_1;\dots)\chi_s(z_1,\beta_1;\dots) \times C^{m_1}_{\alpha_1\beta_1}\dots C^{m_N}_{\alpha_1\beta_N}, \quad (1)$$

where $m_i = 0, \pm 1$,

$$C^0 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, C^{+1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, C^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2)$$

and $\chi_s(z_1,\alpha_1;\dots)$ is the wave function of spin-1/2 fermions with the first Landau level filled by the spin-up and spin-down particles. One way to see that the above state is a non-Abelian state is to derive its low-energy effective theory.

To construct the above wave function using a projective construction, we start with the following *free* fermion wave function for two species (labeled by $a=1$ and 2) of spin-1/2 partons $\psi_{a\alpha}$:

$$\Phi_{parton} = \chi_s(z_1^{(1)},\alpha_1;\dots)\chi_s(z_1^{(2)},\beta_1;\dots), \quad (3)$$

where $\alpha_i, \beta_i = \uparrow, \downarrow$ are spin-1/2 indices. Then we combine the two spin-1/2 partons into a spin-1 electron. In terms of the electron operator $\Psi_m(z)$ and the parton operator $\psi_{a\alpha}(z)$, the combination can be expressed as

$$\Psi_m(z) = \psi_{a\alpha}(z)\psi_{b\beta}(z)\epsilon_{ab}C^m_{\alpha\beta} \quad (4)$$

where $a,b = 1,2$. One can easily see that, after setting $z_i^{(1)} = z_i^{(2)} = z_i$ and symmetrizing α_i and β_i using $C^m_{\alpha_i\beta_i}$, Φ_{parton} in Eq. (3) reduces to Φ^b Eq. (1). Or, more precisely,

$$\Phi^b(z_1,m_1;z_2,m_2;\dots) = \langle 0|\prod \Psi_{m_i}(z_i)|\Phi_{parton}\rangle, \quad (5)$$

where $|\Phi_{parton}\rangle$ is the independent parton state described by Φ_{parton}.

To obtain the effective theory for state $\Phi^b(z_i,m_i)$, we start with the effective theory for independent partons,

$$i\psi^\dagger_{a\alpha}\partial_t\psi_{a\alpha} + \frac{1}{2m}\psi^\dagger_{a\alpha}\left(\partial_i - i\frac{e}{2}A_i\right)^2\psi_{a\alpha}, \quad (6)$$

whose ground state is $|\Phi_{parton}\rangle$. The effective theory for state Φ^b is obtained by combining the two kinds of partons in the above effective theory into electrons. Notice that the effective theory for independent partons contains SU(2) excitations created by $\psi^\dagger_{a\alpha}\tau^I_{ab}\psi_{b\alpha}$, where τ^I are Pauli matrices. We will call such a SU(2) the ''color'' SU(2) to distinguish it from the SU(2) spin rotation. From Eq. (4) we see that the electron operator $\Psi_m(z)$ is a color SU(2) singlet. All physical excitations (created by electron operators) are color singlets. Thus, to combine the partons into electrons, we simply need to remove all the ''colored'' excitation from the parton theory [Eq. (6)] and project into the local color singlet sector. The projection can be realized, at the Lagrangian level, by introducing a SU(2) gauge field which couples to the current and the density of the color SU(2):

$$\mathcal{L} = i\psi^\dagger_{a\alpha}[\delta_{ab}\partial_t - i(a_0)_{ab}]\psi_{b\alpha} + \frac{1}{2m}\psi^\dagger_{a\alpha}\left(\delta_{ab}\partial_i - i\frac{e}{2}A_i - i(a_i)_{ab}\right)^2\psi_{b\alpha}. \quad (7)$$

Equation (7) is the effective theory for Φ^b. Only the gauge-invariant operators, such as the electron operator Ψ_m, correspond to physical operators. To see that the effective theory [Eq. (7)] describes a non-Abelian state, we integrate out the parton fields $\psi_{a\alpha}$. We note that the average of the SU(2) gauge field is zero, and the two partons have such a density that they form $\nu = 2$ QH state. Thus, after integrating out the parton fields $\psi_{a\alpha}$, we obtain an SU(2) CS theory at level $k=2$. k is given by the parton filling fraction

$$\frac{k}{4\pi}\mathrm{Tr}\,\epsilon^{\mu\nu\lambda}\left(a_\mu\partial_\nu a_\lambda + \frac{2i}{3}a_\mu a_\nu a_\lambda\right). \quad (8)$$

Although the level $k=1$ SU(2) CS theory contains only Abelian statistics, the level $k>1$ SU(2) CS theory indeed has quasiparticles carrying non-Abelian statistics.[19]

The edge states of the above non-Abelian state can also be obtained from the projective construction. For independent partons described by Eq. (6), the edge theory is simply given by free chiral fermions in one plus one dimension (1+1D):

$$i\psi^\dagger_{a\alpha}(\partial_t - v\partial_x)\psi_{a\alpha}. \quad (9)$$

The above edge theory is also described by the $U(1) \times SU_{spin}(2)_2 \times SU_{color}(2)_2$ Kac-Moody (KM) algebra.[20] The charge associated with U(1) is just the electric charge of the electrons. The combination of the partons into electron is again realized by project into local $SU_{color}(2)$ singlet sector, which can be simply done by removing the sector generated by the $SU_{color}(2)_2$ KM algebra from the edge spectrum. Thus the edge states of the non-Abelian state Φ^b is described by the $U(1)_1 \times SU_{spin}(2)_2$ KM algebra.

Here we have assigned a level 1 to the U(1) KM algebra. The level characterizes how the U(1) charge is quantized. The definition of the level is the following. We know that the edge theory contains electron operators Ψ_m. The operators that create quasiparticles are the operators which are local with respect to the electron operators (i.e., their correlation with the electron operators are single valued). Let $\psi_{U(1)}$ be the $SU_{spin}(2)$ singlet quasiparticle operator which carries the minimum (but nonzero) U(1) charge. The correlation of

$\psi_{U(1)}$ has a form $\langle \psi^\dagger_{U(1)}(x)\psi_{U(1)}(0)\rangle \sim 1/x^h$. Then the level of the U(1) KM algebra is defined as $l=1/h$. [According to this definition, the U(1) KM algebra that describes the edge excitations of the $\nu=1/m$ Laughlin state can be more specifically denoted as $U(1)_m$ KM algebra.] Since its edge state is described by the $U(1)_1\times SU_{spin}(2)_2$ KM algebra, we will call the non-Abelian state Φ^b the $U(1)_1\times SU_{spin}(2)_2$ state. Note that the $U(1)_1\times SU_{spin}(2)_2$ non-Abelian state can have a bulk effective theory which is a purely $SU_{color}(2)_2$ CS theory.

A slightly more complicated non-Abelian state of spin-1 fermionic electrons is given by

$$\begin{aligned}\Phi^f(z_1,m_1;z_2,m_2;\dots)&=\prod (z_i-z_j)\\ &\times \sum_{\alpha_1.\beta_1;\dots} \chi_s(z_1,\alpha_1;\dots)\chi_s(z_1,\beta_1;\dots)\\ &\times C^{m_1}_{\alpha_1\beta_1}\dots C^{m_N}_{\alpha_1\beta_N}.\end{aligned} \tag{10}$$

To obtain the projective construction, we need to split the electron into one charge $e/2$ parton ψ_0 and two charge $e/4$ partons $\psi_a|_{a=1}$ and 2:

$$\Psi_m(z)=\psi_0(z)\psi_{a\alpha}(z)\psi_{b\beta}(z)\epsilon_{ab}C^m_{\alpha\beta}. \tag{11}$$

Following arguments similar to those used above, we obtain the effective theory for Φ^f:

$$\begin{aligned}&i\psi_0^\dagger(\partial_t+2ib_0)\psi_0+\frac{1}{2m}\psi_0^\dagger\left(\partial_i-i\frac{e}{2}A_i+2ib_0\right)^2\psi_0\\ &+i\psi^\dagger_{a\alpha}(\delta_{ab}\partial_t-i(a_0)_{ab}-ib_0\delta_{ab})\psi_{b\alpha}\\ &+\frac{1}{2m}\psi^\dagger_{a\alpha}\left(\partial_i-i\frac{e}{4}A_i-ia_i-ib_0\right)^2_{ab}\psi_{b\alpha}.\end{aligned} \tag{12}$$

Here an extra U(1) gauge field b_μ is introduced to combine ψ_0 and $\psi_{1,2}$. Note that the electron operator Ψ_m carries no b_μ charge. After integrating out the parton fields, the effective theory becomes the $U(1)\times SU(2)_2$ CS theory. The edge states for independent partons are described by the $U(1)\times U(1)\times SU_{spin}(2)_2\times SU_{color}(2)_2$ KM algebra. After the projection, the edge states for the fermion non-Abelian state are described by the

$$\begin{aligned}&\frac{U(1)\times U(1)\times SU_{spin}(2)_2\times SU_{color}(2)_2}{U(1)\times SU_{color}(2)_2}\\ &=U(1)_2\times SU_{spin}(2)_2\end{aligned} \tag{13}$$

KM algebra.

A more general non-Abelian state of spin-$n/2$ electrons is given by

$$\begin{aligned}&\Phi^{(n,k)}(z_1,m_1;z_2,m_2;\dots)\\ &=\prod (z_i-z_j)^k\sum_{\alpha_{ai}} C^{m_1}_{\alpha_{11}..\alpha_{n1}}\dots C^{m_N}_{\alpha_{1N}..\alpha_{nN}}\\ &\times\chi_s(z_1,\alpha_{11};\dots;z_N,\alpha_{1N})\dots\\ &\times\chi_s(z_1,\alpha_{n1};\dots;z_N,\alpha_{nN}),\end{aligned} \tag{14}$$

where $C^m_{\alpha_1,\dots,\alpha_n}$ form a basis of rank-n symmetric tensors, and m is the quantum number of the total spin S_z for each electron. In the bulk, the above state is described by the $[U(1)]^k\times SU_{color}(n)_2$ effective CS theory. The edge excitations are described by the $U(1)_{k+(n/2)}\times SU_{spin}(2)_n$ KM algebra. Such a state will be called a $U(1)_{k+(n/2)}\times SU_{spin}(2)_n$ non-Abelian state. The above result provides an example that the group for the edge KM algebra, $U(1)_{k+(n/2)}\times SU_{spin}(2)_n$, and the group for the bulk CS effective theory, $[U(1)]^k\times SU_{color}(n)_2$, can be quite different.

III. PROJECTIVE CONSTRUCTION AND THE EFFECTIVE THEORY OF PFAFFIAN STATE

Using the $U(1)_{k+(n/2)}\times SU_{spin}(2)_n$ non-Abelian states, we can construct non-Abelian states. Let us start with the simplest $U(1)_1\times SU_{spin}(2)_2$ state.

To construct a non-Abelian state from the $U(1)_1\times SU_{spin}(2)_2$ state, we simply make a further local projection $S_z=0$, in addition to the local color singlet projection. This $S_z=0$ projection can be realized by identifying

$$\Psi_e\equiv\frac{1}{\sqrt{2}}\Psi_{m=0} \tag{15}$$

as the only physical electron operator. $\Psi_{m=\pm1}$ are regarded as unphysical since $S_z\neq0$. The physical Hilbert space is generated by $\Psi_{m=0}$ only. Therefore after the $S_z=0$ projection, the wave function of the non-Abelian state is given by [see Eq. (5)]

$$\Phi^{pf}(z_1,\dots,z_N)=\langle 0|\prod_i \Psi_e(z_i)|\Phi_{parton}\rangle, \tag{16}$$

which can be regarded as a wave function of spinless electrons.

The projective construction allows us to obtain the low-energy effective theory for the above non-Abelian state. At the Lagrangian level, the $S_z=0$ projection can be realized by introducing an extra $U_{S_z}(1)$ gauge field c_μ that couples to S_z. This yields the effective theory for the non-Abelian state:

$$\begin{aligned}&i\psi^\dagger_{a\alpha}[\delta_{ab}\delta_{\alpha\beta}\partial_t-i\delta_{\alpha\beta}(a_0)_{ab}-ic_0\sigma^3_{\alpha\beta}\delta_{ab}]\psi_{b\beta}\\ &+\frac{1}{2m}\psi^\dagger_{a\alpha}\left(\partial_i-i\frac{e}{2}A_i-ia_i-ic_0\sigma^3\right)^2_{a\alpha,b\beta}\psi_{b\beta}.\end{aligned} \tag{17}$$

After integrating out the fermions, we obtain a $U_{S_z}(1)\times SU_{color}(2)_2$ CS theory.

The edge excitations of the Φ^{pf} state can also be obtained through projective construction. Since the Φ^{pf} state is obtained from the $U(1)_1\times SU_{spin}(2)_2$ state by making an additional local $S_z=0$ projection, thus the edge states of the

Φ^{pf} state can also be obtained from that of the $U(1)_1 \times SU_{spin}(2)_2$ state by making a local $S_z=0$ projection. Note that the edge excitations that correspond to the S_z fluctuations is described by the $U_{S_z}(1)$ KM algebra generated by the S_z current. Thus the edge excitations of the Φ^{pf} state is described by the $U(1)_1 \times [SU_{spin}(2)_2/U_{S_z}(1)]$ coset theory.[21,12] Since the $SU_{spin}(2)_2/U_{S_z}(1)$ coset theory is nothing but an Ising model (or a Majorana fermion theory) with a central charge $c=\frac{1}{2}$,[22] the edge theory for the non-Abelian state can also be denoted as the $U(1)_1 \times \mathcal{I}$ theory.

The second way to obtain the edge theory is to note that all edge excitations are generated by Ψ_e in Eq. (15) and $\Psi_e^\dagger$. Thus we can use the algebra of $(\Psi_e, \Psi_e^\dagger)$ to describe the edge excitations.[13] The algebra of $(\Psi_e, \Psi_e^\dagger)$ can be obtained from their operator product expansion (OPE), which can be calculated easily since Ψ_e can be expressed as a product of free chiral fermion operators [Eq. (15)]. We would like to mention that the OPE of electron operators was also used in Ref. 4 to construct a QH wave function. Using the OPE for free chiral fermions $\psi(z)\psi^\dagger(0)=1/z$, we find the following closed OPE generated by $(\Psi_e, \Psi_e^\dagger)$:

$$\Psi_e^\dagger(z)\Psi_e(0)=\frac{1}{z^2}+\frac{J(0)}{z}+O(z^0),$$

$$\Psi_e(z)\Psi_e^\dagger(0)=\frac{1}{z^2}-\frac{J(0)}{z}+O(z^0),$$

$$J(z)\Psi_e(0)=-\frac{\Psi_e(0)}{z}+O(z^0), \tag{18}$$

$$J(z)\Psi_e^\dagger(0)=\frac{\Psi_e(0)}{z}+O(z^0),$$

$$J(z)J(0)=\frac{1}{z^2}+O(z^0).$$

It is not hard to see that

$$J=\tfrac{1}{2}\psi_{a\alpha}^\dagger\psi_{a\alpha}. \tag{19}$$

To obtain the Hilbert space generated by $(\Psi_e, \Psi_e^\dagger)$ [or, equivalently, to find the representation of the above OPE Eq. (18)], we note that in addition to Eq. (15), another representation of Ψ_e,

$$\Psi_e=\psi\eta, \quad \Psi_e^\dagger=\eta\psi^\dagger, \tag{20}$$

also reproduces exactly the same OPE [Eq. (18)]. [Here ψ is a free chiral fermion, $\psi(z)\psi^\dagger(0)=1/z$, and η is a Majorana fermion, $\eta(z)\eta(0)=1/z$.] Thus $\Psi_{m=0}$ and $\psi\eta$ have exactly the same correlations, and we can identify $\Psi_{m=0}=\psi\eta$. Since the electron operator Ψ_e can be expressed as a product of a free chiral fermion ψ in the U(1) theory and a Majorana fermion η in the Ising theory, and since all physical edge excitations are generated by electron operators, the edge theory of the non-Abelian state is described by the $U(1) \times \mathcal{I}$ CFT theory.

To obtain an explicit expression of the wave function for our non-Abelian state, let us first review a relation between the edge theory and the bulk wave function.[13] As we mentioned above, for independent partons the edge state is described by free chiral fermions $\psi_{a\alpha}$ [see Eq. (9)]. This edge theory and the independent-parton wave function [Eq. (3)] are closely related. As pointed out in Ref. 4, in the 1+1D free chiral fermion theory the correlation

$$\left\langle e^{-iN\phi(z_\infty)}\prod_{i=1..N}\psi(z_i)\right\rangle \sim \prod_{ij}(z_i-z_j) \tag{21}$$

is proportional to the analytic part of the spinless electron wave function of the filled first Landau level, $\Pi_{ij}(z_i-z_j)e^{-\Sigma|z_i|^2/4}$. Here ψ is a free chiral fermion field, z is given by $z=x-vt=x+iv\tau$ for complex time $\tau=it$, and $e^{i\phi(z)}=\psi(z)$ is the bosonized form of the free chiral fermion operator. One can show, through bosonization, that $(1/2\pi)\partial_x\phi=\psi^\dagger(x)\psi(x)$.

Generalizing the above relation, we find that

$$\left\langle e^{-(1/2)iN\phi(z_\infty)}\prod_{i=1..N}\psi_{1,\alpha_i}(z_i^{(1)})\psi_{2,\beta_i}(z_i^{(2)})\right\rangle \sim \chi_{an}(z_1^{(1)},\alpha_1;\dots)\chi_{an}(z_1^{(2)},\beta_1;\dots), \tag{22}$$

where $\chi_{an}(z_1,\alpha_1;\dots)$ is the analytic part of $\chi_s(z_1,\alpha_1;\dots)$, and $(1/2\pi)\partial_x\phi=\psi_{a\alpha}^\dagger(x)\psi_{a\alpha}(x)$, which is the total density operator of the fermions $\psi_{a\alpha}$. We see that (the analytic part of) the independent-parton wave function can be expressed as a correlation of the independent-parton operators.

Similarly, after combining the partons into electrons, the wave function Φ^b for the $U(1)_1 \times SU_{spin}(2)_2$ non-Abelian state can be expressed as a correlation of the electron operator. Actually from Eqs. (4) and (22), we see that

$$\left\langle e^{-(1/2)iN\phi(z_\infty)}\prod_{i=1..N}\Psi_{m_i}(z_i)\right\rangle \sim \Phi_{an}^b(z_1,m_1;\dots), \tag{23}$$

where $\Phi_{an}^b(z_1,m_1;\dots)$ is the analytic part of the Φ^b in Eq. (1). Note that both $e^{-2iN\phi}$ and Ψ_{m_i} are $SU_{color}(2)$ singlets. Thus they are operators in the projected $U(1)_1 \times SU_{spin}(2)_2$ theory.

Now it is clear that the wave function of the non-Abelian state Φ^{pf} can also be expressed as a correlation. The analytic part of Φ^{pf} is

$$\left\langle e^{-(1/2)iN\phi(z_\infty)}\prod_{i=1..N}\Psi_e(z_i)\right\rangle \sim \Phi_{an}^{pf}(z_1,\dots,z_N). \tag{24}$$

Since $\Psi_e(z)$ is simply a product of free fermion operators, Eq. (24) can help us calculate the wave function. Note that $\Psi_e(z)$ can also be expressed as $\Psi_e=\psi\eta$. This allows us to calculate the wave function of our non-Abelian state more easily:

$$\Phi^{pf}_{an}(z_1,\ldots,z_N)\propto\left\langle e^{-iN\phi_e(z_\infty)}\prod_{i=1..N}\Psi_e(z_i)\right\rangle = \mathcal{A}\left(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots\right)\prod_{\langle ij\rangle}(z_i-z_j), \tag{25}$$

where $\mathcal{A}$ is the total antisymmetrization operator. The first factor

$$\mathcal{A}\left(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots\right)$$

comes from the η correlation, and the second one $\Pi_{\langle ij\rangle}(z_i-z_j)$ from the ψ correlation. From the explicit form of the wave function, we find that our non-Abelian state is actually not new. It is nothing but the Pfaffian state (for bosonic electrons) first introduced by Moore and Read.[4] However, we did obtain some new results. Let us summarize these here:

(1) Using an algebraic method, we find that the Pfaffian state can be obtained from the projective construction.

(2) The projective construction allows us to derive the bulk low-energy effective theory [Eq. (17)], which is a $\mathrm{U}_{S_z}(1)\times\mathrm{SU}_{color}(2)_2$ CS theory.

(3) The projective construction also allows us to obtain the edge effective theory, the $\mathrm{U}(1)_1\times\mathcal{I}$ CFT theory. This agrees with the old results obtained using different methods.[15,13,14,9,10]

The projective construction can also be applied to other (non-)Abelian states, which allows us to generate many non-Abelian states. In the following, as examples, we will only give the final results of several other non-Abelian states obtained using the projective construction. Making the $S_z=0$ projection for the $\mathrm{U}(1)_2\times\mathrm{SU}_{spin}(2)_2$ non-Abelian state, we obtain the following results.

(1) The Pfaffian state for (fermionic) electrons can be obtained from the projective construction. Let Ψ_e be equal to $\Psi_{m=0}$ in Eq. (11), or

$$\Psi_e(z)=\psi_0(z)[\psi_{a1}(z)\psi_{b2}(z)+\psi_{a2}(z)\psi_{b1}(z)]\epsilon_{ab}; \tag{26}$$

then the wave function of the fermionic Pfaffian state can be expressed as

$$\left\langle e^{-iN\phi_e(z_\infty)}\prod_{i=1..N}\Psi_e(z_i)\right\rangle\propto\mathcal{A}\left(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots\right)\times\prod_{\langle ij\rangle}(z_i-z_j)^2. \tag{27}$$

The above result implies that the fermionic Pfaffian wave function can be obtained from the wave function $\Pi(z_i-z_j)\chi_s^2$ of the $\mathrm{U}(1)_2\times\mathrm{SU}_{spin}(2)_2$ state through the $S_z=0$ projection.

(2) The bulk low-energy effective theory for the fermionic Pfaffian state is given by

$$\begin{aligned} &i\psi_0^\dagger(\partial_t+2ib_0)\psi_0+\frac{1}{2m}\psi_0^\dagger\left(\partial_i-i\frac{e}{2}A_i+2ib_0\right)^2\psi_0\\ &+i\psi^\dagger_{a\alpha}[\delta_{\alpha\beta}\delta_{ab}(\partial_t-ib_0)-i\delta_{\alpha\beta}(a_0)_{ab}-ic_0\delta_{ab}\sigma^3_{\alpha\beta}]\psi_{b\beta}\\ &+\frac{1}{2m}\psi^\dagger_{a\alpha}\left(\partial_i-i\frac{e}{4}A_i-ia_i-ib_i-ic_i\sigma^3\right)^2_{a\alpha,b\beta}\psi_{b\alpha}. \end{aligned} \tag{28}$$

Comparing to Eq. (12), we introduced an additional $\mathrm{U}_{S_z}(1)$ gauge field c_μ to perform the $S_z=0$ projection. After integrating out the fermions, we obtain the $\mathrm{U}(1)\times\mathrm{U}_{S_z}(1)\times\mathrm{SU}_{color}(2)_2$ CS theory. This effective theory, at least formally, is quite different from another effective theory obtained in Ref. 17 for the same fermionic Pfaffian state.

(3) The edge effective theory obtained from the projective construction is the

$$\frac{\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SU}_{spin}(2)_2\times\mathrm{SU}_{color}(2)_2}{\mathrm{U}(1)\times\mathrm{U}_{S_z}(1)\times\mathrm{SU}_{color}(2)_2}=\mathrm{U}(1)_2\times\mathcal{I} \tag{29}$$

KM algebra.

Making the $S_z=0$ projection for the $\mathrm{U}(1)_{k+(n/2)}\times\mathrm{SU}_{spin}(2)_{n=\text{even}}$ state [with wave function $\Phi^{(n,k)}$ in Eq. (14)], we obtain the following results.

(1) The bulk low-energy effective theory of the constructed state is given by $[\mathrm{U}(1)]^k\times\mathrm{U}_{S_z}(1)\times\mathrm{SU}_{color}(n)_2$ CS theory.

(2) The edge effective theory obtained from the projective construction is the $\mathrm{U}(1)_{k+(n/2)}\times[\mathrm{SU}_{spin}(2)_n/\mathrm{U}_{S_z}(1)]=\mathrm{U}(1)_{k+(n/2)}\times\mathcal{P}_n$ CFT theory. Here $\mathcal{P}_n$ is the Z_n parafermion theory.[23] Note that $\mathcal{I}=\mathcal{P}_2$.

(3) The electron operator is given by $\Psi_e=\lambda e^{i\gamma\phi}$ in the $\mathrm{U}(1)_{k+(n/2)}\times\mathcal{P}_n$ theory, where λ is the Z_n parafermion current operator $\psi_{n/2}$(see Refs. 23, 22, and 6). Therefore the wave function for the constructed non-Abelian state is a correlation function of the parafermions times $\Pi_{ij}(z_i-z_j)^{\gamma^2}$. We would like to point out that this state is not the parafermion non-Abelian state studied in Ref. 24 when $n\neq 2$. The latter is constructed using the parafermion current operator ψ_1.

We note that when $n=4$, the ψ_2 has the following OPE:

$$\psi_2(z)\psi_2(0)\sim\frac{1}{z^2}. \tag{30}$$

Thus the above $\mathrm{U}(1)_{k+2}\times\mathcal{P}_4$ state is just the *d*-wave paired state introduced in Ref. 7.

IV. PROJECTIVE CONSTRUCTION—A GENERAL DISCUSSION

The low-energy effective theory of QH liquid, for example the one for the bosonic Pfaffian state [Eq. (17)], has a finite-energy gap for all its excitations (on a space with no boundary). Thus, naively, one might expect the low-energy effective theories for QH liquids are trivial since there are simply no low-energy excitations. Certainly this point of view is incorrect. The effective theories for QH liquids have nontrivial ground-state degeneracies which depend on the topology of the space.[1] Such theories are called topological theories.[19] Different QH liquids (or topological orders) are

described by different topological theories. Thus we can say that the topological orders in QH liquids are characterized by topological theories, just like symmetry broken phases are characterized by Ginzburg-Landau theories. In many cases, topological theories can take many different forms. Thus to know whether two topological theories are equivalent or not, it is important to compare their physical properties, such as the ground-state degeneracies. It is those physical properties that define a topological theory.

One important issue is how to derive the effective theory which describes the topological order in a given QH liquid. From the discussion in the above sections, we see that if a QH liquid can be obtained through projective construction, then there is a way to calculate its effective topological theory. In the following, we will give a general discussion of the projective construction.

One starts with a few parton fields ψ_a (where $a=1,\ldots,n$), each with electric charge Q_a. Thus, for an independent-parton model, the effective theory is

$$\mathcal{L}_{eff}=i\psi_a^\dagger\partial_t\psi_a+\frac{1}{2m}\psi_a^\dagger(\partial_i-iQ_aA_i)^2\psi_a. \tag{31}$$

However, the Hilbert space generated by the parton fields $(\psi_a,\psi_a^\dagger)$ is simply too large. The physical Hilbert space, generated by electron operators $(\Psi_e,\Psi_e^\dagger)$ and the electrical current operators $(J_0,J_i)=(\Sigma_a\psi_a^\dagger Q_a\psi_a,\mathrm{Im}\,\Sigma_a\psi_a^\dagger Q_a\partial_i\psi_a)$, is a subspace of the parton Hilbert space. Thus it is extremely important the give the definition of electron operators, in order to even define the theory. The importance of electron operators in describing the structures of QH states was also pointed out in Ref. 4 for non-Abelain states and in Ref. 25 for Abelian states. In general, there can be several electron operators. Here, for simplicity, we will only consider the case with one electron operator, which takes the form

$$\Psi_e=\sum_m C_m\prod_a \psi_a^{n_a^{(m)}}(z), \tag{32}$$

where $n_a^{(m)}=0,1$. The total charge of the electron operator is e, hence

$$Q_a n_a^{(m)}=e \tag{33}$$

for any m. Since the physical Hilbert space is generated only by the electron operators and the electrical current operators, our model is actually a gauge theory. Let $\mathcal{G}$ be the group of transformations on the parton fields $\psi_a\to W_{ab}\psi_b$ that leave the electron operator and the electrical current operators unchanged. By definition,

$$W^\dagger QW=Q,\quad W\in\mathcal{G}, \tag{34}$$

where Q is a diagonal matrix with diagonal elements $(Q_1,Q_2,\ldots)$. Such a matrix is denoted as $\mathrm{diag}(Q_1,Q_2,\ldots)$. Note that the electron operator is invariant even under a local transformation

$$\psi_a(\boldsymbol{x})\to W_{ab}(\boldsymbol{x})\psi_b(\boldsymbol{x}),\quad W_{ab}(\boldsymbol{x})\in\mathcal{G}. \tag{35}$$

Because all the physical states are generated by the electron operators and the electrical current operators, the transformation $W_{ab}(\boldsymbol{x})$ is actually a gauge transformation. To realize the gauge structure (i.e., to project onto the physical Hilbert space), we need to include gauge fields in Eq. (31) so that it has a proper gauge invariance.

In general the gauge group $\mathcal{G}$ can contain several disconnected pieces. Let $\mathcal{G}_c$ be the connected piece of $\mathcal{G}$ which contain the identity ($\mathcal{G}_c$ itself is a subgroup of $\mathcal{G}$). Then the gauge structure associated with $\mathcal{G}_c$ can be realized through gauge fields $a_\mu(\boldsymbol{x})$ which take value in the Lie algebra of $\mathcal{G}_c$, $L_{\mathcal{G}_c}$: $a_\mu(\boldsymbol{x})\in L_{\mathcal{G}_c}$. [e.g., if $\mathcal{G}_c=\mathrm{SU}(n)$, then $a_\mu(\boldsymbol{x})$ are traceless Hermitian matrices.] After including the gauge fields, the parton theory becomes

$$i\psi_a^\dagger[\delta_{ab}\partial_t-i(a_0)_{ab}]\psi_b+\frac{1}{2m}\psi_a^\dagger(\partial_i-iQA_i-ia_i)^2_{ab}\psi_b. \tag{36}$$

The above Lagrangian is the low-energy effective theory of the QH liquid. We would like to stress that Eq. (36) alone does not provide a complete description of the QH liquid. In particular, Eq. (36) only includes the gauge structure associated with $\mathcal{G}_c$. *Only Eq. (36), together with the definition of electron operators [Eq. (32)], provides a complete description of the QH liquid.* The invariance of the electron operators (and the electrical current operators) gives rise to the full gauge group $\mathcal{G}$ which may contain discrete gauge transformations, in addition to the continuous transformation $\mathcal{G}_c$ described by the gauge fields a_μ. We will see below that discrete gauge transformations are important and can affect physical properties of the theory, such as the ground-state degeneracies.

There is an important issue that we have overlooked in the above discussion. The Lagrangian in general may not describe a state with a finite-energy gap. One way to obtain a state with a finite gap is to assume that each kind of partons form an integral QH state with filling fraction $\nu_a=m_a$. In the following we will examine when this assumption can be self consistent. Under the assumption $\nu_a=m_a$, we can integrate out the parton fields and obtain a CS theory

$$\mathcal{L}=\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}\mathrm{Tr}(Ma_\mu\partial_\nu a_\lambda)+\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\mathrm{Tr}(MQ\partial_\nu a_\lambda)+\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda\mathrm{Tr}(MQ^2), \tag{37}$$

where $M=\mathrm{diag}(m_1,m_2,\ldots)$. Here we have assumed that there is no gauge symmetry breaking (or no Higgs mechanism). Therefore, M and MQ must be invariant under the gauge transformation $\mathcal{G}$, which requires

$$W^\dagger MW=M,\quad W\in\mathcal{G}. \tag{38}$$

Equation (37) is obtained in the following way. First we assume a_μ to be diagonal, which can be regarded as a gauge field for the maximum Abelian subgroup of $\mathcal{G}_c$. In this case one can obtain Eq. (37) easily. Since we assume there is no gauge symmetry breaking, the effective theory has full $\mathcal{G}_c$ gauge invariance. This allows us to show that Eq. (37) is valid for generic a_μ in $\mathcal{G}_c$.

The equation of motion $\partial\mathcal{L}/\partial a_0$ leads to a solution $\overline{a_i}$ which can be chosen to be diagonal: $\overline{a_1}$

$=\mathrm{diag}(\overline{a_i^{(1)}},\overline{a_i^{(2)}},\ldots)$. We note that $\overline{a_i^{(a)}}$ are proportional to A_i: $\overline{a_i^{(a)}}=f_a A_i$. If we shift a_μ to $\tilde{a}_\mu=a_\mu+FA_i$ where $F=\mathrm{diag}(f_1,f_2,\ldots)$, then the equation of motion will give us $\tilde{\overline{a}}_i=0$. The shift changes Q_a. We see that one can redefine Q_a through a shift of a_μ to make $\overline{a_i}=0$. In the following we will assume that Q_a are chosen such that $\overline{a_i}=0$. This requires

$$\mathrm{Tr}(tMQ)=0 \tag{39}$$

for any matrix t in the Lie algebra of $\mathcal{G}_c$. From $\partial\mathcal{L}/\partial(a_0)_{aa}=\rho_a$, we obtain the density of the ath parton:

$$\rho_a=\frac{1}{2\pi}m_a Q_a \partial_i A_j \epsilon^{ij}. \tag{40}$$

Since $Q_a A_i$ happen to be the total gauge field ''seen'' by the ath parton, thus the ath parton always has a filling fraction m_a regardless of how we choose m_a. Here we only require that m_a are chosen such that M satisfies Eq. (38); this leads to $a_i=0$ as a solution of the equation of motion, and ρ_a are all positive.

Equations (36), (32), (34), (38), and (39) form a complete description of the QH liquids. One can calculate all physical properties, such as the ground-state degeneracies, of the QH liquids from those equations.

In the above discussion we only have one-electron operators. In general, there can be several electron operators, and the above discussion can be generalize in a straightforward way to cover those more general cases. For example, the gauge group $\mathcal{G}$ is formed by transformations that leave all the electron operators invariant.

After setting up the bulk effective theory, it is easy to obtain the edge effective theory. For independent partons the edge states contain $n_{edge}=\Sigma_a|m_a|$ branches. Each branch is described by a free chiral fermion theory or a U(1) KM algebra. Thus the edge effective theory for the independent partons is given by

$$\mathcal{L}_{edge}=i\psi_{al}^\dagger(\partial_t-v_a\partial_x)\psi_{al}, \tag{41}$$

where $l=1,\ldots,|m_a|$ and v_a has the same sign as m_a. The above theory is denoted as the $\mathrm{U}^{n_{edge}}(1)$ theory. The true edge effective theory for the physical states is obtained through the coset construction[21] as the $\mathrm{U}^{n_{edge}}(1)/\mathcal{G}$ coset theory. Note that we not only need to remove excitations associated with the $\mathcal{G}$ KM algebra [which give us the $\mathrm{U}^{n_{edge}}(1)/\mathcal{G}_c$ coset theory], we also need to require the physical states to be invariant under all the discrete gauge transformations in $\mathcal{G}$. Another way (which is conceptually better) to obtain the edge theory is to setup the OPE algebra of the electron and the current operators, and generate the edge states through the algebra.

In the following we will outline how to calculate ground-state degeneracy on a torus. From Ref. 26, we see that to obtain the ground state degeneracy on a torus we may reduce the non-Abelian gauge fields to Abelian ones, i.e., to reduce the gauge group $\mathcal{G}_c$ to the maximum Abelian subgroup $\mathcal{G}_{abl}$ which is formed by diagonal matrices. The Abelian version of the effective Lagrangian has a form

$$i\psi_a^\dagger(\partial_t-ia^I p_a^I)\psi_a+\frac{1}{2m}\psi_a^\dagger(\partial_i-iQ_aA_i-ia_i^I q_a^I)^2\psi_a, \tag{42}$$

where $I=1,\ldots,\kappa$. The electron operator is given by Eq. (32). The gauge invariance of the electron operator requires

$$n_a^{(m)}p_a^I=0 \tag{43}$$

for any m and I [see Eq. (32)]. In addition to the Abelian gauge structure described by a_μ^I, there are also discrete gauge transformation generated by $W_i\in\mathcal{G}$ which leave the Abelian subgroup unchanged:

$$W_i^\dagger\mathcal{G}_{abl}W_i=\mathcal{G}_{abl}. \tag{44}$$

The above W_i's form a discrete group. Equation (44) can be reduced to the following matrix equation: W_i is a discrete gauge transformation if and only if $W_i\in\mathcal{G}$, and there exists a $\kappa\times\kappa$ matrix T_i such that

$$\sum_a (W_i^\dagger)_{ba}p_a^I(W_i)_{ac}(T_i)_{IJ}=p_b^J\delta_{bc}. \tag{45}$$

The Lagrangian in Eq. (42) and the electron operator in Eq. (32) are invariant under the discrete gauge transformations

$$\psi_a\to(W_i)_{ab}\psi_b, \tag{46}$$

$$a_\mu^I\to(T_i)_{IJ}a_\mu^J. \tag{47}$$

After integrating out the parton fields from Eq. (42), we obtain a U(1) CS effective theory

$$\frac{\tilde{K}_{IJ}}{4\pi}a_{I\mu}\partial_\nu a_{J\lambda}\epsilon_{\mu\nu\lambda}+\frac{q_I}{2\pi}A_\mu\partial_\nu a_{I\lambda}\epsilon_{\mu\nu\lambda}+\frac{\nu e^2}{4\pi}A_\mu\partial_\nu A_\lambda\epsilon_{\mu\nu\lambda}, \tag{48}$$

where

$$\tilde{K}_{IJ}=\sum_a m_a p_a^I p_a^J,$$

$$q_I=\sum_a m_a Q_a p_a^I,$$

$$\nu=\frac{\sum_a m_a Q_a Q_a}{e^2}. \tag{49}$$

Since we require $a_i=0$ to be a solution to the equation of motion, m_a must be chosen to satisfy

$$\sum_a m_a Q_a p_a^I=q_I=0 \tag{50}$$

for all I. [Note that Eq. (50) is just a special case of Eq. (39).] m_a should also satisfy Eq. (38). For such m_a, ν becomes the total filling fraction of the QH liquid.

Now let us use the Abelian version of the effective theory to calculate the ground-state degeneracy on the torus. Following Refs. 27 and 26, the low-energy degrees of freedom are described by $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$a_1^I(x_1,x_2,t)=2\pi\frac{u_I(t)}{L},\quad a_2^I(x_1,x_2,t)=2\pi\frac{v_I(t)}{L}, \tag{51}$$

where L is the size of the torus. Substituting Eq. (51) into Eq. (48), we obtain

$$L=2\pi\tilde{K}_{IJ}v_J\dot{u}_I, \tag{52}$$

which leads to the following commutator:

$$[u_I,v_J]=i(\tilde{K}^{-1})_{IJ}/2\pi. \tag{53}$$

The large gauge transformation $\psi_a\to e^{i2\pi n_I p_a^I x/L}\psi_a$ generate an equivalence relation

$$\boldsymbol{u}\sim\boldsymbol{u}+\boldsymbol{n}, \tag{54}$$

and $\boldsymbol{n}$ satisfies

$$n_I p_a^I=(\text{integer}) \tag{55}$$

for all a. The vectors $\boldsymbol{n}$ that satisfy the above condition form a lattice whose basis vectors are denoted as $\boldsymbol{e}_I$. This lattice will be called the $\boldsymbol{e}$ lattice. The physically distinct $\boldsymbol{u}$ points are all in the unit cell of the $\boldsymbol{e}$ lattice.

Since the conjugate variables $\boldsymbol{v}$ also have the same equivalence relation Eq. (54), the allowed value of $\boldsymbol{u}$ are quantized. To describe this quantization, let us use the symmetric matrix $\tilde{K}$ to define an inner product $\boldsymbol{u}_1\cdot\boldsymbol{u}_2\equiv u_{1I}\tilde{K}_{IJ}u_{2J}$ (which may not be positive definite). We introduce a dual lattice (which will be called the $\boldsymbol{d}$ lattice) with the basis vectors $\boldsymbol{d}_I$:

$$\boldsymbol{d}_I\cdot\boldsymbol{e}_J=\delta_{IJ}. \tag{56}$$

Then the allowed $\boldsymbol{u}$'s all lie on the $\boldsymbol{d}$ lattice, and the ground states are labeled by the lattice points on the dual lattice .[27,26] However, the points connected by vectors in the $\boldsymbol{e}$ lattice are gauge equivalent:

$$\boldsymbol{u}\sim\boldsymbol{u}+\boldsymbol{e}_I,\quad \boldsymbol{u}\in(\boldsymbol{d}\ \text{lattice}). \tag{57}$$

Thus only the $\boldsymbol{d}$-lattice points which lie inside the unit cell of the $\boldsymbol{e}$ lattice can represent independent ground states.

For ease of calculation, let us redefine the gauge fields a^I to make the basis of the $\boldsymbol{e}$ lattice the standard basis vector [i.e., $(\boldsymbol{e}_I)_J=\delta_{IJ}$]. We will call such a basis the primary basis. For the primary basis, p_a^I have the following two properties.

(1) p_a^I are all integers.

(2) When viewed as n-dimensional vectors, the vectors $\boldsymbol{p}^I$ span a κ-dimensional ''volume'' in n-dimensional space. This κ-dimensional ''volume'' does not contain any n-dimensional integer vectors. (Otherwise, we can choose a new set of $\boldsymbol{p}^I|_{I=1..\kappa}$ that span a smaller cube.)

From Eq. (49), it is clear that $\tilde{K}$ is a symmetric integer matrix. This $\tilde{K}$ matrix is similar to the K matrix in the K-matrix description of Abelian QH states.[3] For the primary basis, the basis vectors of the $\boldsymbol{d}$ lattice are given by the columns of $\tilde{K}^{-1}$. The number of the $\boldsymbol{d}$-lattice points that lie inside the unit cell of the $\boldsymbol{e}$ lattice is given by $|\det(\tilde{K})|$.

For Abelian QH liquids, each point in the unit cell of the $\boldsymbol{e}$ lattice labels a distinct ground state, and the ground-state degeneracy is given by $|\det(\tilde{K})|$.[1,28] However, for non-Abelian states, there are additional equivalent relations:[26]

$$\boldsymbol{u}\sim T_i\boldsymbol{u},\quad \boldsymbol{u}\in(\boldsymbol{d}\ \text{lattice}),\quad i=1,2,\ldots, \tag{58}$$

where T_i are linear maps which map a $\boldsymbol{d}$-lattice point to another $\boldsymbol{d}$-lattice point. Under the equivalence relation [Eq. (58)], different $\boldsymbol{d}$-lattice points in the unit cell of the $\boldsymbol{e}$ lattice can represent the same ground state. Thus only the points in the folded unit cell represent distinct ground states.[26] Therefore, to calculate the ground-state degeneracy on a torus, we need to know $\tilde{K}$ and T_i.

Now let us describe how to obtain the maps T_i. Recall that in addition to the Abelian gauge transformations in $\mathcal{G}_{abl}$, there are additional discrete gauge transformations W_i as defined in Eq. (45). Since W_i is a gauge transformation, the physical states must satisfy $W_i|\text{phys}\rangle=|\text{phys}\rangle$. The gauge transformations W_i induces a gauge transformation on a_μ^I: $a_\mu^I\to(T_i)_{IJ}a_\mu^J$, where T_i is obtained from Eq. (45). Therefore, a^I and $(T_i)_{IJ}a^J$; hence $\boldsymbol{u}$ and $T_i\boldsymbol{u}$, are equivalent points.

We would like to point out that the above result for the ground-state degeneracy on a torus is correct only when $\mathcal{G}$ has no disconnected pieces. When $\mathcal{G}$ has disconnected parts, the above result needs to be modified. Let us assume $\mathcal{G}$ has a form $\mathcal{G}=\mathcal{G}_c\otimes\mathcal{G}_d$ where $\mathcal{G}_d$ is a discrete group. Then the low-energy effective theory is a ($\mathcal{G}_c$ CS theory)$\times$($\mathcal{G}_d$ gauge theory). The above calculation only calculates the ground states from the $\mathcal{G}_c$ CS theory. We know that the discrete $\mathcal{G}_d$ gauge theory has $|\mathcal{G}_d|^{2g}$ degenerate ground states on a genus g surface, where $|\mathcal{G}_d|$ is the number of elements in $\mathcal{G}_d$. Thus the total number of the ground states is given by the number of ground states of the $\mathcal{G}_c$ CS theory times $|\mathcal{G}_d|^{2g}$.

Before ending this section let us summarize the steps of the projective construction as follow:

(1) Introduce a few partons $\psi_a|_{a=1..n}$.

(2) Introduce a few electron operators

$$\Psi_e^{(i)}=\sum_m C_m^{(i)}\prod_a \psi_a^{n_a^{(m)}}(z) \tag{59}$$

[which generalizes Eq. (32)].

(3) Assign charge Q_a to each parton such that the electron operators all have charge e.

(4) Find the gauge group $\mathcal{G}$ [see Eq. (35)] that leaves the electron operators Ψ_e and Q unchanged [see Eq. (34)].

(5) Find the filling fractions m_a of partons which satisfy Eqs. (38) and (39).

This leads to a QH state with wave function

$$\Phi(\{z_1^{(i)},\ldots,z_{N_i}^{(i)}\})$$

$$=\langle 0|\prod_i[\Psi_e^{(i)}(z_1^{(i)})\cdots\Psi_e^{(2)}(z_{N_i}^{(i)})]|\Phi_{parton}\rangle, \tag{60}$$

where Φ_{parton} is the free parton wave function in which the ath kind of partons form a $\nu=m_a$ QH state:

$$\Phi_{parton}=\prod_a \chi_{m_a}(z_1^{(a)},\ldots,z_{N_a}^{(a)}). \tag{61}$$

Here χ_l is the fermion wave function with l filled Landau levels. The bulk effective theory of the above state is given

by Eq. (36) [or Eq. (37)]. The edge effective theory is the $U^{\Sigma|m_a|}(1)/\mathcal{G}$ coset theory. The filling fraction ν is given by $\nu=\Sigma_a m_a Q_a^2/e^2$.

To obtain the Abelian version of the effective theory, we also need to find a set of linearly independent integer vectors $\boldsymbol{p}^I|_{I=1..\kappa}$, such that $P^I=\mathrm{diag}(p_1^I,p_2^I,\ldots,p_n^I)$ is in the Lie algebra of $\mathcal{G}_c$. We also require $\boldsymbol{p}^I$ to span a κ-dimensional ''volume'' in the n-dimensional space that does not contain any integer vectors. Given p^I, Eqs. (44) and (45) determine the T_i matrix. Equation (49) determines the $\tilde{K}$ matrix. T_i, $\tilde{K}$, and $|\mathcal{G}_d|$ allow us to determine the ground-state degeneracy on the torus.

V. APPLICATIONS OF PROJECTIVE CONSTRUCTION

Now let us apply the above general results to some simple cases to gain a better understanding of the projective construction. First let us split an electron into two partons,

$$\Psi_e=\psi_1\psi_2, \tag{62}$$

with charges $Q_1=e[l_1/(l_1+l_2)]$ and $Q_2=e[l_2/(l_1+l_2)]$. If $l_1\neq l_2$ the gauge group is U(1): $\mathcal{G}=\{e^{i\theta\tau_3}\}$. The effective theory is Eq. (42) with $\kappa=1$ and $\boldsymbol{p}^1=(1,-1)$. It is clear that $\boldsymbol{p}^1$ forms a primary basis, since the line from (0,0) to (1, -1) does not contain any integer points. Let us consider a QH state in which the two partons form integral QH states with filling fraction m_1 and m_2. m_a must satisfy Eq. (39) [or Eq. (50)]:

$$m_1l_1-m_2l_2=0. \tag{63}$$

Thus

$$m_1=ml_2, \quad m_2=ml_1 \tag{64}$$

for an integer m. The filling fraction of the QH liquid is

$$\nu=\frac{ml_2l_1^2}{(l_1+l_2)^2}+\frac{ml_1l_2^2}{(l_1+l_2)^2}=m\frac{l_1l_2}{l_1+l_2}. \tag{65}$$

The $\tilde{K}$ matrix is a 1×1 matrix:

$$\tilde{K}=ml_2+ml_1=m(l_1+l_2). \tag{66}$$

Since there is no additional discrete gauge transformations, the ground-state degeneracy on the torus is $m(l_1+l_2)$.

When $m=l_2=1$ and $l_1=l$, we obtain a sequence of hierarchical states with filling fractions 1/2, 2/3,..., $l/(l+1)$, ... [which are similar to the 1/2, 2/5,..., $l/(2l+1)$, ... states for the fermionic electrons]. The ground-state degeneracies for those states are given by $N_D=2,3,\ldots,l+1,\ldots$. Since the first parton has a filling fraction $\nu=1$ and the second parton has $\nu=l$, the electron wave function has a form $\chi_1\chi_l$.

According to the K-matrix description,[3] the effective theory of the $\chi_1\chi_l$ state is given by

$$\mathcal{L}=\frac{K_{IJ}}{4\pi}a_{I\mu}\partial_\nu a_{J\lambda}\epsilon_{\mu\nu\lambda}, \tag{67}$$

with $K=I_l+C_l$, where I_l is the $l\times l$ identity matrix and C_l is the $l\times l$ matrix with all its elements equal to 1. However, according to the projective construction, the $\chi_1\chi_l$ state is described by

$$\tilde{\mathcal{L}}=\frac{\tilde{K}_{IJ}}{4\pi}a_{I\mu}\partial_\nu a_{J\lambda}\epsilon_{\mu\nu\lambda}, \tag{68}$$

with a 1×1 matrix $\tilde{K}=(l+1)$. Actually, there is no contradiction here. $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are equivalent topological theories (for example, they have the same number of degenerate ground states). $\tilde{\mathcal{L}}$ can be regarded as a dual form of $\mathcal{L}$.

We would like to remark here that showing that the $\chi_1\chi_l$ state can be described by a bulk effective CS theory with only one U(1) gauge field does not imply that the $\chi_1\chi_l$ state has only one branch of edge excitations. Actually the $\chi_1\chi_l$ state has l branches of edge excitations.

When $m=-l_2=1$ and $l_1=l$, we obtain a sequence of hierarchical states with filling fractions 2, 3/2,..., $l/(l-1)$, ... [which are similar to the 2/3, 3/5,..., $l/(2l-1)$, ... states for the fermionic electrons]. The ground-state degeneracies for those states are given by $N_D=1,2,\ldots,l-1,\ldots$.

If $l_1=l_2=1$, the gauge group is SU(2): $\mathcal{G}=\{e^{i\boldsymbol{\theta}\cdot\boldsymbol{\tau}}\}$. Now m_1 and m_2 must be equal [see Eq. (38)]: $m_1=m_2=m$. The resulting state is nothing but the $\mathrm{SU}(2)_m$ non-Abelian state discussed in Ref. 5 (the $m=2$ case was discussed in detail in Sec. II). Its wave function is given by χ_m^2. The filling fraction is $\nu=m/2$.

In the Abelian version of effective theory [Eq. (42)], we have $\kappa=1$ and $\boldsymbol{q}^1=(1,-1)$ as a primary basis. In addition to the Abelian gauge transformation, we also have a discrete gauge transformation $W=i\tau_2$. Such a discrete gauge transformation induces a ''gauge'' transformation in the Abelian gauge field $a_\mu^1\rightarrow-a_\mu^1$ (i.e., $T=-1$ in Eq. (45)]. The $\tilde{K}$ matrix is $\tilde{K}=2m$. The $2m$ $\boldsymbol{d}$-lattice points in the unit cell of the $\boldsymbol{e}$ lattice are $0,1/2m,\ldots,l/2m,\ldots,(2m-1)/2m$. The $T=-1$ transformation leads to an equivalence relation $l/2m\sim-l/2m\sim(2m-l)/2m$. Thus the $\mathrm{SU}(2)_m$ non-Abelian state has $m+1$ degenerate ground states on the torus.

Next we start with four different partons, all with the same charge $Q_a=1/2$. The electron operator is chosen to be

$$\Psi_e=\frac{1}{\sqrt{2}}[\psi_1(z)\psi_4(z)-\psi_3(z)\psi_2(z)] \tag{69}$$

[which is Eq. (15) if we identify $(\psi_1,\ldots,\psi_4)=(\psi_{1\uparrow},\psi_{1\downarrow},\psi_{2\uparrow},\psi_{2\downarrow})$]. The gauge group $\mathcal{G}_c$ is generated by ten generators: $\tau_i\otimes\sigma_0$, $\tau_i\otimes\sigma_1$, $\tau_i\otimes\sigma_2$, and $\tau_0\otimes\sigma_3$, where $\tau_0=\sigma_0$ are the 2×2 identity matrix. It turns out that $\mathcal{G}_c$ is the SO(5) (or Sp_4) group in its four-dimensional representation. It appears that $\mathcal{G}$ has no disconnected pieces, and $\mathcal{G}=\mathcal{G}_c$. To be consistent with the gauge invariance [Eq. (38)], the partons must all have the same integer filling fraction $\nu_a=m$. The effective theory is given by Eq. (36) with a_μ in the Lie algebra of the SO(5) gauge group $\mathcal{G}_c$. After integrating out the fermions, we obtain a $\mathrm{SO}(5)_m$ CS theory.

The Abelian version of the effective theory [Eq. (42)] has $\kappa=2$ and

$$(p_1^1,\ldots,p_4^1)=(1,0,0,-1),\quad (p_1^2,\ldots,p_4^2)=(0,-1,1,0). \tag{70}$$

The parallelogram spanned by $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$ does not contain any integer points. Thus $\boldsymbol{p}^{1,2}$ is a primary basis. The $\tilde{K}$ matrix is [see Eq. (49)] $\tilde{K}=\left(\begin{smallmatrix}2m & 0\\ 0 & 2m\end{smallmatrix}\right)$. The $\boldsymbol{d}$ lattice is generated by the bases $\boldsymbol{d}^1=(1/2m,0)$ and $\boldsymbol{d}^2=(0,1/2m)$. The $4m^2$ $\boldsymbol{d}$-lattice points in the unit cell of the $\boldsymbol{e}$ lattice are $(k_1/2m,k_2/2m)$, with $k_1,k_2=0,\ldots,2m-1$. The electron operator is invariant under the following three transformations

$$W_1=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},$$

$$W_2=\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\-1&0&0&0\end{pmatrix}, \tag{71}$$

$$W_3=\begin{pmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

This induces three mappings on $\boldsymbol{u}$ [see Eq. (58)]:

$$T_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

$$T_2=\begin{pmatrix}-1&0\\0&1\end{pmatrix}, \tag{72}$$

$$T_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

$T_{2,3}$ leads to the equivalence relations $(k_1/2m,k_2/2m)\sim[(2m-k_1)/2m,k_2/2m]\sim[k_1/2m,(2m-k_2)/2m]$. Thus $k_1,k_2=0,\ldots,m$ label all the independent states. T_1 gives rise to an equivalence relation $(k_1/2m,k_2/2m)\sim(k_2/2m,k_1/2m)$. Therefore, the QH state obtained through the above projective construction has $[(m+1)^2/2]+[(m+1)/2]=[(m+1)(m+2)/2]$ degenerate ground states on a torus [represented by points $(k_1/2m,k_2/2m)$ with $k_1,k_2=0,\ldots,m$ and $k_1\leqslant k_2$].

Note that when $m=1$ the above projective construction is just the construction used in Sec. III to construct the filling fraction $\nu=1$ bosonic Pfaffian state with wave function Φ^{pf} in Eq. (25). We see that the $\nu=1$ bosonic Pfaffian state has three degenerate ground state on the torus. This result agrees with a previous result obtained from the wave function.[16,9] When $m>1$, the above construction produces other non-Abelian states.

In Sec. III, the bulk effective theory for the $\nu=1$ bosonic Pfaffian state Φ^{pf} was found to be the $\mathrm{U}_{S_z}(1)\times\mathrm{SU}_{color}(2)_2$ CS theory [see Eq. (17)]. From the above discussion, we see that the correct bulk effective theory should be the $\mathrm{SO}(5)_1$ CS theory. If the two effective theories are inequivalent, the $\mathrm{SO}(5)_1$ CS theory is the correct effective theory for the $\nu=1$ bosonic Pfaffian state, since the $\mathrm{SO}(5)_1$ CS theory has full gauge symmetry. However, it is also possible that the $\mathrm{U}_{S_z}(1)\times\mathrm{SU}_{color}(2)_2$ effective CS theory is equivalent to the $\mathrm{SO}(5)_1$ CS theory. On the torus both theories give three degenerate ground states. Also, the edge excitation for the $\nu=1$ bosonic Pfaffian state should be described by the $\mathrm{U}^4(1)/\mathrm{SO}(5)_1$ coset theory. Note that $\mathrm{U}^4(1)=\mathrm{U}(1)\times\mathrm{SU}_{spin}(2)_2\times\mathrm{SU}_{color}(2)_2$ theory can be described by eight free Majorana fermions. The U(1) KM algebra can be described by two Majorana fermions $(\mathrm{Re}\,\psi,\mathrm{Im}\,\psi)$, the $\mathrm{SU}_{spin}(2)_2$ KM algebra by three Majorana fermions $(\eta_s^m|_{m=1,2,3})$, and the $\mathrm{SU}_{color}(2)_2$ KM algebra also by three Majorana $(\eta_c^a|_{a=1,2,3})$. The SO(5) gauge field couples to $(\eta_c^{1,2,3},\eta_s^{1,2})$ and the projection to the SO(5) singlet sector gives us $\mathrm{U}^4(1)/\mathrm{SO}(5)_1=\mathrm{U}(1)\times\mathcal{I}$ theory described by (ψ,η_s^3). Thus effective edge theory—the $\mathrm{U}(1)\times\mathcal{I}$ theory—obtained in Sec. III is still valid.

We can also start with five different partons with four partons $\psi_{1,2,3,4}$ carrying charge $el_1/2(l_2+l_2)$, and the fifth parton carrying charge $el_2/(l_1+l_2)$. The electron operator can be chosen to be

$$\Psi_e=\frac{1}{\sqrt{2}}[\psi_1(z)\psi_4(z)+\psi_3(z)\psi_2(z)]\psi_5 \tag{73}$$

[which is Eq. (26) if we identify $(\psi_1,\ldots,\psi_5)=(\psi_{1\uparrow},\psi_{1\downarrow},\psi_{2\uparrow},\psi_{2\downarrow},\psi_0)$]. The gauge group $\mathcal{G}_c$ is generated by 11 generators. The first ten generators $\tau_i\otimes\sigma_0$, $\tau_i\otimes\sigma_1$, $\tau_i\otimes\sigma_2$, and $\tau_0\otimes\sigma_3$ act only on $\psi_{1,2,3,4}$. The last generator is given by $\mathrm{diag}(1,1,1,1,-2)$. $\mathcal{G}_c$ is the $\mathrm{SO}(5)\times\mathrm{U}(1)$ group. Again $\mathcal{G}$ has no disconnected pieces and $\mathcal{G}=\mathcal{G}_c$. To be consistent with the gauge invariance [Eq. (38)], the first four partons all have the same integer filling fraction $\nu_a=m_1$, and the last parton ψ_5 has filling fraction m_5. Equation (39) [or Eq. (50)] requires $m_1l_1-l_2m_5=0$. Therefore,

$$m_1=ml_2,\quad m_5=ml_1. \tag{74}$$

The effective theory is given by Eq. (36) with a_μ in the Lie algebra of the $\mathrm{SO}(5)\times\mathrm{U}(1)$ gauge group $\mathcal{G}_c$. After integrating out the fermions, we obtain a $\mathrm{SO}(5)_{m_1}\times\mathrm{U}(1)$ CS theory.

The Abelian version of the effective theory is Eq. (42) with

$$(p_1^1,\ldots,p_5^1)=(1,0,0,-1,0),$$

$$(p_1^2,\ldots,p_5^2)=(0,-1,1,0,0), \tag{75}$$

$$(p_1^3,\ldots,p_5^3)=(0,1,0,1,-1).$$

The ''volume'' spanned by $\boldsymbol{p}^{1,2,3}$ does not contain any integer points, and $\boldsymbol{p}^{1,2,3}$ is a primary basis. The $\tilde{K}$ matrix becomes

$$\tilde{K}=\begin{pmatrix} 2m_1 & 0 & -m_1 \\ 0 & 2m_1 & -m_1 \\ -m_1 & -m_1 & 2m_1+m_5 \end{pmatrix}. \tag{76}$$

The electron operator is invariant under the following three transformations which also leave the Abelian gauge structure unchanged:

$$W_1=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$W_2=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{77}$$

$$W_3=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This induces three mappings on $\boldsymbol{u}$ [see Eq. (58)]:

$$T_1=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$T_2=\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{78}$$

$$T_3=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

To calculate the ground-state degeneracy on the torus, let us consider only a simple case $l_1=l_2=m=1$. The corresponding QH state is just the fermionic Pfaffian state at a filling fraction $\frac{1}{2}$ with wave function equation (27). From

$$\tilde{K}^{-1}=\tfrac{1}{8}\begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & 2 \\ 2 & 2 & 4 \end{pmatrix},$$

we find that the $\boldsymbol{d}$ lattice is generated by the bases $\boldsymbol{d}^1=(5/8,1/8,1/4)$, $\boldsymbol{d}^2=(1/8,5/8,1/4)$, and $\boldsymbol{d}^3=(1/4,1/4,1/2)$. The eight $\boldsymbol{d}$-lattice points in the unit cell of the $\boldsymbol{e}$ lattice are (5/8,1/8,1/4), (1/8,5/8,1/4), (7/8,3/8,3/4), (3/8,7/8,3/4), (0,0,0), (1/4,1/4,1/2), (1/2,1/2,0), and (3/4,3/4,1/2). $T_{2,3}$ do not lead to any new equivalence relations. However, T_1 gives rise to two equivalence relations (5/8,1/8,1/4) $\sim$(1/8,5/8,1/4) and (7/8,3/8,3/4)$\sim$(3/8,7/8,3/4). Thus the fermionic $\nu=1/2$ Pfaffian state has six degenerate states on a torus. This result again agrees with a previous result obtained from wave function.[16,9]

VI. SUMMARY

In this paper we introduced a powerful method—the projective construction—to construct many non-Abelian (and Abelian) states, which include the fermionic $\nu=1/2$ and the bosonic $\nu=1$ Pfaffian states, and the d-wave-paired non-Abelian state. What is more significant is that the projective construction allows us to calculate bulk and edge effective theories. We find that the bluk effective theory is a $SO(5)_1$ CS theory for the bosonic $\nu=1$ Pfaffian state, and a $U(1)\times SO(5)_1$ CS theory for the fermionic $\nu=1/2$ Pfaffian state. Using the bulk effective theory, the ground-state degeneracy on a torus was calculated.

However, it is unclear if the projective construction can produce all the QH states or not. We still do not known how to use the projective construction to construct the Haldane-Rezayi state.[29] Although we understand a lot of physical properties of the Haldane-Rezayi state,[13,14,30] we still do not know its bulk effective theory.

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