Microscopic theory of weakly coupled superconducting multilayers in an external magnetic field

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We present a fully microscopic, self-consistent, and self-contained theory of superconducting weakly coupled periodic multilayers with tunnel barriers in the presence of externally applied parallel magnetic fields, in the local Ginzburg-Landau regime. We solve a nontrivial mathematical problem of a microscopic derivation and exact minimization of the free-energy functional. In the thin-layer limit that corresponds to the domain of validity of the phenomenological Lawrence-Doniach model, our physical results strikingly contrast with those of our predecessors. In particular, we completely revise previous calculations of the lower critical field and refute the concept of a triangular Josephson vortex lattice. We show that Josephson vortices penetrate into all the barriers simultaneously and form peculiar structures that we term ''vortex planes.'' We calculate the superheating field of the Meissner state and predict hysteresis in the magnetization. In the vortex state, the magnetization exhibits distinctive oscillatory behavior and jumps due to successive penetration of the vortex planes. We prove that the vortex-plane penetration and pinning by the edges of the sample cause the Fraunhofer pattern of the critical Josephson current. We calculate the critical temperature and the upper critical field of infinite (along the layers) multilayers. For finite multilayers, we predict a series of first-order phase transitions to the normal state and oscillations of the critical temperature versus the applied field. Finally, we discuss some theoretical and experimental implications of the obtained results. [S0163-1829(99)08633-6]

I. INTRODUCTION

In this paper, we present a fully microscopic, self-consistent, and self-contained theory of superconducting weakly coupled periodic multilayers (superlattices) of the S/I-type (S for a superconductor, I for an insulator or a semiconductor) in the presence of externally applied parallel magnetic fields, in the local Ginzburg-Landau[1] (GL) regime [i.e., temperatures are close to the bulk critical temperature T_{c0}, all characteristic dimensions of the S layers are much larger than the BCS (Ref. 2) coherence length ξ_0].

In Sec. II, we derive a microscopic free-energy functional that describes a smooth transition from the single-junction case to the thin-layer limit, when the S-layer thickness a is small compared to all other relevant length scales. [By contrast, previous treatment was based predominantly on the phenomenological Lawrence-Doniach[3] (LD) model, applicable only in the thin-layer limit.] Our analysis of implications of gauge invariance reveals important facets of the theory. In particular, we establish the existence of fundamental constraint relations coupling the phases of the order parameter to the vector potential and equalizing the phase differences at neighboring barriers. Mathematically, these constraints complement the usual Euler-Lagrange equations and make the free energy a minimum. Physically, they state that the average intralayer currents are always equal to zero and the local magnetic field has, in general, the periodicity of the multilayer. (The latter has recently received strong experimental support in polarized neutron reflectivity measurements on artificial Nb/Si multilayers.[4]) As a result of this investigation, we obtain a closed, self-consistent set of mean-field equations. These equations have a particularly simple form in the thin-layer limit: Remarkably, the phase differences (the same at all the barriers) obey a Ferrell-Prange-type[5] equation with a single length scale, the Josephson penetration depth $\lambda_J=(8\pi e p j_0)^{-1/2}$ (p is the period, j_0 is the critical Josephson current). (This should be contrasted with a mathematically ill-defined infinite set of differential equations containing two length scales, proposed without appropriate justification in the literature.[6,7]) Furthermore, due to the absence of screening by the intralayer currents in the thin-layer limit, the local magnetic field proves to be independent of the coordinate normal to the layers.

In Sec. III, we obtain exact analytical solutions to the equations of the thin-layer limit. This limit corresponds to the domain of validity of the phenomenological LD model, which allows us to draw a comparison with the results of our predecessors. Thus we show that previously suggested single Josephson vortex penetration[8,9] as well as the occurrence[6] of a triangular Josephson vortex lattice are incompatible with the above-mentioned fundamental constraints of the theory.[10] We also disprove the claim of Ref. 7 that the Fraunhofer pattern of the total critical Josephson current occurs in the absence of Josephson vortices. However, our consideration contains as a limiting case a particular exact solution for the vortex state obtained in the framework of the LD model by Theodorakis[11] and fully supports phenomenological calculations[12,13] of the upper critical field $H_{c2\infty}$ in infinite (along the layers) multilayers.

Among interesting physical results of Sec. III are the following. We provide comprehensive description of the Meissner state in semi-infinite (along the layers) multilayers and show that at the field $H_s=(ep\lambda_J)^{-1}$ (the superheating field) the Meissner phase becomes unstable with regard to Josephson vortex penetration. We predict simultaneous and coherent penetration into all the barriers. (This prediction has been confirmed experimentally.[4]) We show that in the absence of screening by the intralayer currents (see above) the ''tails'' of Josephson vortices overlap in the layering direction forming peculiar structures, ''vortex planes.'' The lower critical field at which the formation of a single vortex plane becomes energetically favorable in an infinite multilayer is found to be $H_{c1\infty}=2(\pi e p \lambda_J)^{-1}$. For the lower critical field in a finite multilayer with $W \ll \lambda_J$ (W is the S-layer length) we obtain $H_{c1W}=\pi/epW$, which corresponds to the first minimum of the Fraunhofer pattern. We prove that the Fraunhofer oscillations occur due to successive penetration of the vortex planes and their pinning by the edges of the sample. We show that vortex-plane penetration leads also to jumps of the magnetization. (Such features have been already observed.[4]) For a certain field range, we predict a small paramagnetic effect. We calculate the critical temperature and the upper critical field of an infinite multilayer. The obtained implicit dependence $H_{c2\infty}(T)$ exhibits the well-known three-dimensional–two-dimensional ''(3D-2D) crossover'' and is free from the unphysical ''low-temperature'' divergence[14] of the LD model. In addition, we predict interesting size effects in finite multilayers: a series of first-order phase transitions to the normal state and oscillations of the critical temperature versus the applied field.

In Sec. IV, we discuss some theoretical and experimental implications of the obtained results. In the Appendix, we write down a few mathematical formulas related to the application of Mathieu functions in Sec. III.

II. BASIC EQUATIONS OF THE THEORY

A. Derivation and exact minimization of the microscopic free-energy functional

Our starting point is a microscopic second-quantized BCS-type Hamiltonian of the form[15,16]

$$H=\int_R d^3\mathbf{r}\psi_\alpha^+(\mathbf{r})\left\{-\frac{1}{2m}[\boldsymbol{\nabla}-ie\tilde{\mathbf{A}}(r)-ie\mathbf{A}_{\rm ext}(\mathbf{r})]^2-E_F\right\}$$

$$\times\psi_\alpha(\mathbf{r})-\frac{|g|}{2}\int_{R_s}d^3\mathbf{r}g(\mathbf{r})\psi_\alpha^+(\mathbf{r})\psi_{-\alpha}^+(\mathbf{r})\psi_{-\alpha}(\mathbf{r})\psi_\alpha(\mathbf{r})$$

$$+\int_{R_s}d^3\mathbf{r}\psi_\alpha^+(\mathbf{r})V_{\rm imp}(\mathbf{r})\psi_\alpha(\mathbf{r})+U_0\int_{R_b}d^3\mathbf{r}\psi_\alpha^+(\mathbf{r})\psi_\alpha(\mathbf{r})$$

$$+\frac{1}{8\pi}\int_R d^3\mathbf{r}\tilde{\mathbf{h}}^2(\mathbf{r}),$$

$$R_s=\bigcup_{n=-\infty}^{+\infty}R_{s_n},\quad R_b=\bigcup_{n=-\infty}^{+\infty}R_{b_n},\quad R=R_s\cup R_b, \tag{1}$$

$$R_{s_n}=[-a/2+np\le x\le a/2+np]\times[L_{y1}\le y\le L_{y2}]$$

$$\times(-\infty<z<+\infty),$$

$$R_{b_n}=[a/2+(n-1)p\le x\le -a/2+np]\times[L_{y1}\le y\le L_{y2}]$$

$$\times(-\infty<z<+\infty),$$

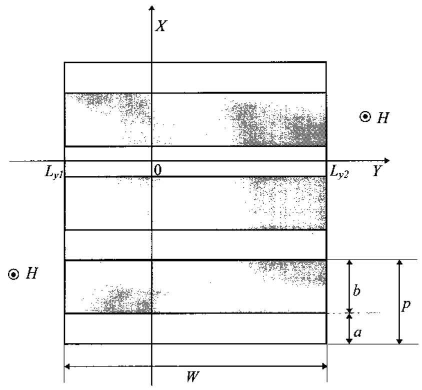

FIG. 1. Geometry of the problem. Alternating superconducting layers and nonsuperconducting barriers are shown by white and gray rectangles, respectively. The system is supposed to be infinite in the x and z directions. An external magnetic field H is applied along the z axis.

$$\tilde{\mathbf{h}}(\mathbf{r})=\boldsymbol{\nabla}\times\tilde{A}(r),\quad \mathbf{H}=\boldsymbol{\nabla}\times\mathbf{A}_{\rm ext}(\mathbf{r})\equiv(0,0,H),$$

$$\boldsymbol{\nabla}=(\nabla_x,\nabla_y,\nabla_z)\equiv\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right). \tag{2}$$

Here $\hbar=c=1$, $E_F=k_F^2/2m$ is the Fermi energy (with k_F being the Fermi momentum), R_s and R_b correspond respectively to the superconducting and barrier regions (with a being the S-layer thickness, b the barrier thickness and $p=a+b$ the period, the x axis being normal to the barrier interfaces), $\psi_\alpha(\mathbf{r})$ is the electron field operator for spin α (a summation over repeated spin indices is implied), $g<0$ is the BCS coupling constant, $V_{\rm imp}(\mathbf{r})$ is the nonmagnetic impurity potential, $U_0>0$ is the repulsive barrier potential, $\mathbf{A}_{\rm ext}$ and $\tilde{\mathbf{A}}$ are the external (classical) and induced (operator) vector potentials.[17] The system is taken to be infinite in the direction of the x and z axes, while no restrictions on the linear dimensions along the y axis is so far imposed. The external magnetic field $\mathbf{H}$ is directed along the z axis (see Fig. 1).

Using field-theoretical methods of Ref. 15, we can derive from Eq. (1) a microscopic free-energy functional of the system $\Omega[\Delta_n,\Delta_n^*,\mathbf{A};H]$, where Δ_n and $\mathbf{A}$ are classical variables: Δ_n is the pair potential (order parameter) of the nth S layer, and $\mathbf{A}=\tilde{\mathbf{A}}+\mathbf{A}_{\rm ext}$ is the total vector potential, $\mathbf{h}(\mathbf{r})=\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r})$ being the corresponding local magnetic field. For external fields satisfying the quasiclassical condition $H\ll k_F/e\xi_0$, in the GL regime

$$\tau\equiv\frac{T_{c0}-T}{T_{c0}}\ll 1, \tag{3}$$

$$\xi_0\ll a,\quad W\equiv L_{y2}-L_{y1}, \tag{4}$$

where T_{c0} is the bulk critical temperature, $\xi_0=v_0/2\pi T_{c0}$ is the BCS coherence length ($v_0=k_F/m$), this functional takes on the form

$$\Omega[f_n,\phi_n,A_x,A_y;H]=\frac{H_c^2(T)}{4\pi}W_z\int_{L_{y1}}^{L_{y2}}dy\Bigg[\sum_{n=-\infty}^{+\infty}\int_{-a/2+np}^{a/2+np}dx\Bigg[-f_n^2(x,y)+\frac{1}{2}f_n^4(x,y)+\zeta^2(T)\times\sum_{i=x,y}\{[\nabla_i f_n(x,y)]^2+[\nabla_i\phi_n(x,y)-2eA_i(x,y)]^2f_n^2(x,y)\}+\frac{\alpha\zeta^2(T)}{2a\xi_0}\times\{f_{n-1}^2[a/2+(n-1)p,y]+f_n^2(-a/2+np,y)-2f_n(-a/2+np,y)f_{n-1}[a/2+(n-1)p,y]\cos\Phi_{n,n-1}(y)\}+4e^2\zeta^2(T)\lambda^2(T)\int_{L_{x1}}^{L_{x2}}dx[h(x,y)-H]^2\Bigg],\tag{5}$$

$$\Phi_{n,n-1}(y)=\phi_n(-a/2+np,y)-\phi_{n-1}[a/2+(n-1)p,y]-2e\int_{a/2+(n-1)p}^{-a/2+np}dxA_x(x,y),$$

$$\alpha=\frac{3\pi^2}{7\zeta(3)\chi(\xi_0/l)}\int_0^1 dt\,tD(t),$$

$$D(t)=\frac{16E_Ft^2(U_0-E_Ft^2)}{U_0^2}\exp[-2b\sqrt{2m(U_0-E_Ft^2)}],$$

$$\chi(\xi_0/l)=\frac{8}{7\zeta(3)}\sum_{n=0}^{+\infty}(2n+1)^{-2}(2n+1+\zeta_0/l)^{-1},$$

$$H_c^2(T)=4\pi N(0)\Delta_\infty^2(T)\tau,$$

$$h(x,y)=\frac{\partial A_y(x,y)}{\partial x}-\frac{\partial A_x(x,y)}{\partial y}.\tag{6}$$

In Eq. (5), we have introduced the reduced modulus $0\leqslant f_n\leqslant 1$ and the phase ϕ_n of the pair potential in the nth S layer via the relation $\Delta_n=\Delta_\infty f_n\exp(i\phi_n)$, where $\Delta_\infty(T)=\sqrt{8\pi^2T_{c0}^2\tau/7\zeta(3)}$ is the bulk gap, $\zeta(m)$ is the Riemann zeta function.[18] The rest of the notations are as follows: $W_z=L_{z2}-L_{z1}$ is the length of the system in the z direction, $D(t)$ is the tunneling probability of an insulating barrier between two successive S layers $[D(1)\ll 1]$, $\chi(\xi_0/l)$ is the impurity factor[19] (l is the electron mean free path), $\zeta(T)=\xi_0\sqrt{7\zeta(3)\chi(\xi_0/l)/12\tau}$ is the GL coherence length, $\lambda(T)=\sqrt{3}[\pi\chi(\xi_0/l)\xi_0N(0)\tau]^{-1/2}/8\pi eT_{c0}$ is the GL penetration depth, $N(0)=mk_F/2\pi^2$ is the one-spin density of states at the Fermi level, and $H_c(T)$ is the bulk thermodynamic critical field near T_{c0}.[19] The term proportional to $\alpha\ll 1$ determines the interlayer Josephson coupling. Equation (6) is merely the Maxwell equation for the local magnetic field $\mathbf{h}=(0,0,h)$.

The microscopic free-energy functional (5) covers all well-known limiting cases. In the limit $\alpha=0$ (no Josephson interlayer coupling), Eq. (5) reduces to a sum of free-energy functionals of independent S layers. Making a shift of the coordinate system $x\rightarrow x-a/2-b/2$ and taking the limit $a\rightarrow\infty$, one gets the case of a single SIS junction. Shifting $x\rightarrow x-a/2$ and taking $a\rightarrow\infty$, $b\rightarrow\infty$, we recover the limit of a semi-infinite superconductor in contact with vacuum.

Our task now is to establish mean-field equations of the theory, which is mathematically equivalent to the problem of minimization of Eq. (5) with respect to f_n, ϕ_n, and $\mathbf{A}=(A_x,A_y,0)$. This problem should be approached with certain caution, because Euler-Lagrange equations for ϕ_n, and A_x,A_y are not independent.

Indeed, the functional (5) is invariant under the general gauge transformation

$$\phi_n(x,y)\rightarrow\phi_n(x,y)+\eta(x,y),$$

$$A_i(x,y)\rightarrow A_i(x,y)+\frac{1}{2e}\nabla_i\eta(x,y),\tag{7}$$

where $\eta(x,y)$ is an arbitrary gauge function, defined in the whole region R. As a result, the variational derivatives with respect to ϕ_n, and A_x,A_y are related through fundamental functional identities

$$\frac{\delta\Omega}{\delta\phi_n(x,y)}\equiv\frac{1}{2e}\sum_{i=x,y}\nabla_i\frac{\delta\Omega}{\delta A_i(x,y)}.\tag{8}$$

The occurrence of such identities is typical of gauge theories.[20] Moreover, identities relating variational derivatives appear already in some problems of classical variational calculus with degenerate (i.e., invariant under symmetry transformations) functionals.[21] As in degenerate theories the number of variables exceeds the number of independent Euler-Lagrange equations, complementary relations should be normally imposed to eliminate irrelevant degrees of freedom and close the system mathematically. Whereas in bulk superconductors and single junctions the elimination of unphysical degrees of freedom amounts merely to an appropriate choice of gauge, in periodic weakly coupled structures this problem has additional implications. Namely, in the presence of the Josephson interlayer coupling phase differences $\Phi_{n,n-1}$ and $\Phi_{n+1,n}$ at two successive barriers are in themselves not independent, which means, mathematically, that we are dealing with a variational problem with constraints. Unfortunately, this fundamental feature was not noticed in previous literature.

The variations with respect to f_n are independent and can be taken first. Varying under the assumption of arbitrary δf_n at the boundaries, we obtain

$$\left[1+\zeta^2(T)\sum_{i=x,y}\{\nabla_i^2-[\nabla_i\phi_n(x,y)-2eA_i(x,y)]^2\}\right]\times f_n(x,y)-f_n^3(x,y)=0.\tag{9}$$

$$(x,y)\in R_{s_n};$$

$$\frac{\partial f_n}{\partial y}(x,L_{y1})=\frac{\partial f_n}{\partial y}(x,L_{y2})=0; \tag{10}$$

$$\frac{\partial f_n}{\partial x}(-a/2+np,y)=\frac{\alpha}{2\xi_0}\{f_n(-a/2+np,y)-f_{n-1}\times[a/2+(n-1)p,y]\cos\Phi_{n,n-1}(y)\}, \tag{11}$$

$$\frac{\partial f_n}{\partial x}(a/2+np,y)=-\frac{\alpha}{2\xi_0}\{f_n(a/2+np,y)-f_{n+1}\times[-a/2+(n+1)p,y]\cos\Phi_{n+1,n}(y)\}. \tag{12}$$

Here, Eq. (9) is the usual GL equation for the bulk order parameter. Relations (10) are the usual GL boundary conditions at the superconductor/vacuum interfaces. Boundary conditions (11) and (12), describing the suppression of the order parameter due to the Josephson currents at the superconductor/insulator interfaces, are of the type derived by de Gennes[22] for a single junction.

By contrast to f_n, the variables A_x, A_y are defined in the whole region R. For these variables, continuity up to the second-order partial derivatives at the superconductor/insulator interfaces should be assumed. The corresponding Euler-Lagrange equations are

$$-\frac{\partial h(x,y)}{\partial x}=\frac{1}{2e}\frac{f_n^2(x,y)}{\lambda^2(T)}\left[\frac{\partial\phi_n(x,y)}{\partial y}-2eA_y(x,y)\right]\equiv 4\pi j_{ny}(x,y), \tag{13}$$

$$\frac{\partial h(x,y)}{\partial y}=\frac{1}{2e}\frac{f_n^2(x,y)}{\lambda^2(T)}\left[\frac{\partial\phi_n(x,y)}{\partial x}-2eA_x(x,y)\right]\equiv 4\pi j_{nx}(x,y),\quad (x,y)\in R_{s_n}; \tag{14}$$

$$\frac{\partial h(x,y)}{\partial y}=4\pi j_0 f_n(-a/2+np,y)\times f_{n-1}[a/2+(n-1)p,y]\sin\Phi_{n,n-1}(y)\equiv 4\pi j_{n,n-1}(y), \tag{15}$$

$$j_0=\frac{7\zeta(3)\alpha\chi(\xi_0/l)}{6}eN(0)\xi_0\Delta_\infty^2(T), \tag{16}$$

$$\frac{\partial h(x,y)}{\partial x}=0,\quad (x,y)\in R_{b_n}. \tag{17}$$

Here, Eqs. (13) and (14) are the Maxwell equations in the S layers, with $\mathbf{j}_n$ being the intralayer supercurrent densities. Equations (15) and (17) are the Maxwell equations in the barrier regions, with $j_{n,n-1}$ being the Josephson current density between the nth and the $(n-1)$th layers. Relation (16) is the definition of the Josephson critical current density in a single SIS junction.[15]

Equations (13)–(17) should be complemented by boundary conditions at the outer interfaces $y=L_{y1},L_{y2}$. [When deriving these equations, we have only assumed $\delta A_x(x,L_{y1})=\delta A_x(x,L_{y2})=0$.] As we do not consider here externally applied currents in the y direction, the first set of boundary conditions follows from the requirement $[j_{ny}]_{y=L_{y1},L_{y2}}=0$:

$$\left[\frac{\partial\phi_n(x,y)}{\partial y}-2eA_y(x,y)\right]_{y=L_{y1},L_{y2}}=0. \tag{18}$$

Applied to Eq. (13), these boundary conditions show that the local magnetic field at the outer interfaces is independent of the coordinate x: $h(x,L_{y1})=h(L_{y1})$, $h(x,L_{y2})=h(L_{y2})$. The boundary conditions imposed on h should be compatible with Ampere's law $h(L_{y2})-h(L_{y1})=4\pi I$ obtained by integration of Eqs. (13) and (15) over y, where

$$I\equiv\int_{L_{y1}}^{L_{y2}}dy\,j_{nx}(x,y)=\int_{L_{y1}}^{L_{y2}}dy\,j_{n,n-1}(y) \tag{19}$$

is the total current in the x direction. Throughout this paper, depending on a physical situation under consideration, we will employ three types of boundary conditions on h:

$$\begin{cases} h(L_{y1})=h(L_{y2})=H, & \text{(i)};\\ h(L_{y1})=H-2\pi I,\quad h(L_{y2})=H+2\pi I, & \text{(ii)};\\ h(L_{y1})=H,\quad h(L_{y2})=H+4\pi I, & \text{(iii)}.\end{cases} \tag{20}$$

As usual, the Maxwell Eqs. (13) and (14) yield the current-continuity equations inside the S layers:

$$\sum_{i=x,y}\nabla_i\{f_n^2(x,y)[\nabla_i\phi_n(x,y)-2eA_i(x,y)]\}=0. \tag{21}$$

The conservation of Josephson interlayer current is readily verified from Eq. (15). Using Eqs. (14) and (15) and assumed continuity of $\partial h/\partial y$, we arrive at the boundary conditions

$$\left[\left(\frac{\partial\phi_n(x,y)}{\partial x}-2eA_x(x,y)\right)f_n(x,y)\right]_{x=-a/2+np}=\frac{\alpha}{2\xi_0}f_{n-1}[a/2+(n-1)p,y]\sin\Phi_{n,n-1}(y), \tag{22}$$

$$\left[\left(\frac{\partial\phi_n(x,y)}{\partial x}-2eA_x(x,y)\right)f_n(x,y)\right]_{x=a/2+np}=\frac{\alpha}{2\xi_0}f_{n+1}[-a/2+(n+1)p,y]\sin\Phi_{n+1,n}(y), \tag{23}$$

reflecting the continuity of the x component of the supercurrent at the internal interfaces $x=\pm a/2+np$.

Integrating Eqs. (9) and (21) over x and applying boundary conditions (11), (12), (22) and (23), respectively, we obtain very useful integrodifferential representations

$$\overline{f_n(x,y)}-\overline{f_n^3(x,y)}-\zeta^2(T)\times\sum_{i=x,y}\overline{[\nabla_i\phi_n(x,y)-2eA_i(x,y)]^2f_n(x,y)}+\zeta^2(T)\frac{\partial^2\overline{f_n(x,y)}}{\partial y^2}=\frac{\alpha\zeta^2(T)}{2a\xi_0}\{f_n(-a/2+np,y)+f_n(a/2+np,y)-f_{n+1}[-a/2+(n+1)p,y]\cos\Phi_{n+1,n}(y)-f_{n-1}[a/2+(n-1)p,y]\cos\Phi_{n,n-1}(y)\},\quad(24)$$

$$\frac{\partial\overline{\{f_n^2(x,y)[\partial\phi_n(x,y)/\partial y-2eA_y(x,y)]\}}}{\partial y}=\frac{\alpha}{2a\xi_0}\{f_n(-a/2+np,y)f_{n-1}[a/2+(n-1)p,y]\times\sin\Phi_{n,n-1}(y)-f_n(a/2+np,y)\times f_{n+1}[-a/2+(n+1)p,y]\sin\Phi_{n+1,n}(y)\},\quad(25)$$

where $\overline{(x,\ldots)}\equiv(1/a)\int_{-a/2+np}^{a/2+np}(x,\ldots)dx$ denotes averaging over the interval $-a/2+np\leqslant x\leqslant a/2+np$.

By summing Eqs. (25) over the layer index n, integrating and applying boundary conditions (18), we obtain the integral

$$\sum_{n=-\infty}^{+\infty}\overline{f_n^2(x,y)\left[\frac{\partial\phi_n(x,y)}{\partial y}-2eA_y(x,y)\right]}=0,\quad(26)$$

which is, physically, the conservation law for the total supercurrent in the y direction. Mathematically, Eq. (26) has the form of a constraint relation between variables $\partial\phi_n/\partial y$ and A_y.[20,23] To find the rest of constraints of the theory, closing the system of equations, we must minimize the functional (5) with respect to ϕ_n and $\partial\phi_n/\partial y$.

By virtue of fundamental identities (8), a naive variation of Eq. (5) with respect to ϕ_n (with arbitrary $\delta\phi_n$ at the boundaries) does not yield new equations. Indeed, the corresponding Euler-Lagrange equation reduces to the conservation law (21), while surface variations merely reproduce boundary conditions (18), (22), and (23). Considering variations of the type $\phi_n(x,y)\to\phi_n(x,y)+\epsilon\psi_n(y)$, $\partial\phi_n(x,y)/\partial y\to\partial\phi_n(x,y)/\partial y+\epsilon\partial\psi_n(y)/\partial y$, where ϵ is a small parameter and $\psi_n(y)$ are arbitrary functions of y, we arrive at Eqs. (25). To obtain genuinely new equations, minimizing Eq. (5) with respect to ϕ_n and $\partial\phi_n/\partial y$, we must enlarge the class of allowed variations.

A mathematically rigorous approach to this problem is as follows. While varying $\phi_n(x,y)\to\phi_n(x,y)+\delta\phi_n(y)$, $\partial\phi_n(x,y)/\partial y\to\partial\phi_n(x,y)/\partial y+\partial\delta\phi_n(y)/\partial y$, with $\delta\phi_n(y)$ being small arbitrary functions of y, instead of integrating by parts, we impose additional constraints

$$\overline{f_n^2(x,y)\left[\frac{\partial\phi_n(x,y)}{\partial y}-2eA_y(x,y)\right]}=0,\quad(27)$$

compatible with boundary conditions (18) and constraint relation (26). The requirement of compatibility with the current-conservation law (25) automatically yields another set of constraints

$$f_n(-a/2+np,y)f_{n-1}[a/2+(n-1)p,y]\sin\Phi_{n,n-1}(y)=f_n(a/2+np,y)f_{n+1}[-a/2+(n+1)p,y]\sin\Phi_{n+1,n}(y).\quad(28)$$

The above procedure is formally equivalent to minimization of Eq. (5) with respect to independent variations of ϕ_n and $\partial\phi_n/\partial y$. As this class of variations of ϕ_n and $\partial\phi_n/\partial y$ is larger than that employed in deriving Eq. (25), we can argue that Eqs. (27) and (28) provide the sought necessary conditions for the true minimum of the free-energy functional (5).

The physical meaning of Eqs. (27) and (28) is quite transparent. Constraints (27) minimize the kinetic-energy term in Eq. (5) with respect to variations $\partial\phi_n(x,y)/\partial y\to\partial\phi_n(x,y)/\partial y+\delta\psi_n(y)$, where $\delta\psi_n(y)$ are small arbitrary functions of y. They show that the average intralayer currents in the y direction are always equal to zero, and, as a result.

$$h(-a/2+np,y)=h(a/2+np,y)\quad(29)$$

[see Eq. (13)]. These constraints appear already in the case of decoupled S layers. By contrast, constraints (28) are uniquely imposed by the Josephson interlayer coupling. Their function is to make the Josephson energy stationary with respect to variations $\phi_n(x,y)\to\phi_n(x,y)+\delta\phi_n(y)$ and to assure the conservation of the total Josephson current I in neighboring barriers [see Eqs. (15) and (19)].

As no other conditions are imposed on the variables, we can satisfy Eq. (28) by choosing

$$f_n(x,y)=f_{n-1}(x-p,y)\equiv f(x,y),f(x+np,y)=f(x,y),\quad(30)$$

$$\Phi_{n+1,n}(y)=\Phi_{n,n-1}(y)\equiv\Phi(y).\quad(31)$$

These relations finalize the determination of a closed, complete, self-consistent system of mean-field equations for a S/I superlattice in the GL regime.

Constraints (27), (28), and their corollaries (29)–(31) belong to key results of this paper. Derived by means of a rigorous mathematical analysis of the impact of gauge invariance, they are not restricted to the functional (5), but should hold for any superconducting weakly coupled periodic structure. To illustrate their importance, we point out that Eqs. (28) and (29), for example, completely rule out any possibility of single Josephson vortex penetration[8,9] and triangular Josephson vortex lattice,[6] proposed without appropriate physical and mathematical justification. On the contrary, they imply that the distribution of the local magnetic field due to the Josephson vortices has, in general, the periodicity of the multilayer, as recently verified experimentally.[4] It should be noted, however, that although the role of constraints (28) and (29) in minimizing the free energy and closing the system of Euler-Lagrange equations for f_n, ϕ_n, and $\mathbf{A}$ has not been realized until now, relations (30) and (31) were implicitly employed in phenomenological calculations of $H_{c2\infty}$.[3,12–14,24] Moreover, relations of the type

(27), (30), and (31) were used by Theodorakis[11] in his particular exact solution of the LD model in a parallel field.

The equations of this subsection admit exact solutions in two limiting situations: the single-junction case, when $a \gg \max\{\zeta(T),\lambda(T)\}$; the thin-layer limit, when $\xi_0 \ll a \ll \min\{\zeta(T),\lambda(T),\alpha^{-1}\xi_0,W\}$ ($W \equiv W_y = L_{y2} - L_{y1}$ is the length of the S layer in the y direction). The single-junction case is well known. The thin-layer limit will be extensively discussed in the next subsection and in Sec. III.

B. The thin-layer limit

The mean-field equations of the previous subsection allow remarkable simplification in the thin-layer limit, when $\xi_0 \ll a \ll \min\{\zeta(T),\lambda(T),\alpha^{-1}\xi_0,W\}$.

First, we can neglect the x dependence of f, defined by Eq. (30): $f(x,y) \equiv f(y)$. Second, fixing the gauge by the condition

$$A_x(x,y) \equiv 0, \quad A_y(x,y) \equiv A(x,y), \tag{32}$$

we can neglect the x dependence of ϕ_n as well: $\phi_n(x,y) \equiv \phi_n(y)$. In the gauge (32), $\Phi_{n,n-1}(y) = \phi_n(y) - \phi_{n-1}(y)$, and Eqs. (31) become

$$\phi_{n+1}(y) + \phi_{n-1}(y) = 2\phi_n(y), \phi_n(y) - \phi_{n-1}(y) = \phi(y),$$

with the solution

$$\phi_n(y) = n\phi(y) + \chi(y), \tag{33}$$

where $\phi(y)$ is the coherent phase difference (the same for all the barriers), and $\chi(y)$ is an arbitrary gauge function, allowed by particular gauge transformations $\phi_n(y) \to \phi_n(y) + \chi(y)$, $A(x,y) \to A(x,y) + (1/2e)[\partial\chi(y)/\partial y]$. Without any loss of generality, we can set $\chi \equiv 0$.

In view of independence of f and ϕ_n from x in the thin-layer limit, the physical meaning of constraints (27) and (28) becomes even more obvious. Thus Eqs. (28) are now the conditions of stationarity of the Josephson energy with respect to all allowed variations of ϕ_n. Due to Eqs. (27), the term in Eq. (24) responsible for the kinetic energy of the intralayer currents becomes

$$\begin{aligned}
&\zeta^2(T)\sum_{i=x,y}\overline{[\nabla_i\phi_n(x,y) - 2eA_i(x,y)]^2 f(x,y)} \\
&\to \zeta^2(T)\overline{\left[\frac{d\phi_n(y)}{dy} - 2eA(x,y)\right]^2} f(y) \\
&= \zeta^2(T)\left[\frac{d\phi_n(y)}{dy} - 2e\overline{A(x,y)}\right]^2 f(y) \\
&\quad + 4e^2\zeta^2(T)[\overline{A^2(x,y)} - \overline{A(x,y)}^2] f(y) \\
&= 4e^2\zeta^2(T)[\overline{A^2(x,y)} - \overline{A(x,y)}^2] f(y),
\end{aligned}$$

which shows that conditions (27) minimize the kinetic energy for a given configuration of the vector potential A.

Concerning the Maxwell equations, the right-hand side of Eq. (13) is of order $a^2/\lambda^2(T)$ and can be discarded. Equation (14) can be altogether dropped. Thus we arrive at a closed set of equations

$$\frac{\partial^2 A(x,y)}{\partial x^2} = 0, \quad (x,y) \in R; \tag{34}$$

$$\frac{1}{4\pi}\frac{\partial^2 A(x,y)}{\partial y \partial x} = j_0 f^2(y)\sin\phi(y), \quad (x,y) \in R_{b_n}; \tag{35}$$

$$n\frac{d\phi(y)}{dy} - 2e\overline{A(x,y)} = 0, \quad (x,y) \in R_{s_n}; \tag{36}$$

$$h(x,y) = \frac{\partial A(x,y)}{\partial x}, \quad (x,y) \in R, \tag{37}$$

$$\begin{aligned}
&f(y) - f^3(y) - 4e^2\zeta^2(T)[\overline{A^2(x,y)} - \overline{A(x,y)}^2] f(y) \\
&\quad + \zeta^2(T)\frac{d^2 f(y)}{dy^2} \\
&= \frac{\alpha\zeta^2(T)}{a\xi_0}[1 - \cos\phi(y)] f(y), \\
&\qquad (x,y) \in R_{s_n}.
\end{aligned} \tag{38}$$

These equations, of course, should be complemented by continuity conditions on A, $\partial A/\partial x$ and boundary conditions (10) and (20).

It is worth noting that an immediate consequence of Eqs. (34) and (37) is independence of the local field h from the coordinate x in the whole region R: $h(x,y) = h(y)$, $-\infty < x < +\infty$. This result is fully compatible with the requirement (29) and demonstrates that the intralayer supercurrents in the thin-layer limit are unable to screen out the magnetic field: The situation is very familiar from the physics of isolated superconducting films with $a \ll \lambda(T)$.[14,22,25]

Our next objective is to eliminate the vector potential and obtain a closed set of equations involving only f and ϕ. Equations (25) and (35) can be easily solved for A in the nth "elementary cell" $R_n = R_{s_n} \cup R_{b_n}$ (the S layer plus the adjacent barrier). Applying the continuity conditions on A, $\partial A/\partial x$, boundary conditions (20), and the constraint relation (36), we get

$$\begin{aligned}
A(x,y) &= \left[4\pi j_0\int_{L_{y1}}^{y} du f^2(u)\sin\phi(u) + H_1\right](x - np) \\
&\quad + \frac{n}{2e}\frac{d\phi(y)}{dy},
\end{aligned} \tag{39}$$

where $H_1 \equiv H$ for Eq. (20) (i) and (iii), and $H_1 \equiv H - 2\pi I$ for Eq. (20) (ii). Matching Eq. (39) to an analogous solution in the adjacent cell R_{n-1} leads to the solvability condition

$$\frac{d\phi(y)}{dy} = 8\pi e j_0 p\int_{L_{y1}}^{y} du f^2(u)\sin\phi(u) + 2epH_1. \tag{40}$$

Equation (40) is nothing but an analog of the Ferrell-Prange[5] relation for a single Josephson junction, which can be readily verified by differentiation. From this point of view, the quantity $(8\pi e j_0 p)^{-1/2}$ should be identified with the Josephson penetration depth λ_J. Note that instead of the factor 2λ entering the definition of λ_J in the single-junction case,[26,27] in our case we get the period p.

With the help of Eq. (39), we arrive at the expression for the vector potential in the whole region $R=\cup_{n=-\infty}^{+\infty}R_n=R_s\cup R_b$:

$$A(x,y)=\frac{1}{2ep}\frac{d\phi(y)}{dy}x,\quad (x,y)\in R. \tag{41}$$

This equation should be substituted into Eqs. (37) and (38).

In this manner, we obtain a closed, complete set of equations describing a thin-layer S/I superlattice in an external parallel magnetic field:

$$\Delta(x,y)=\Delta_\infty f(y)\sum_n \delta_{R_{s_n}}(x,y)\exp[in\phi(y)], \tag{42}$$

$$\delta_{R_{s_n}}(x,y)=\begin{cases}1, & \text{for } (x,y)\in R_{s_n},\\ 0, & \text{for } (x,y)\notin R_{s_n};\end{cases}$$

$$\left[1-\frac{1}{12}\zeta^2(T)\left(\frac{a}{p}\right)^2\left(\frac{d\phi(y)}{dy}\right)^2\right]f(y)+\zeta^2(T)\frac{d^2f(y)}{dy^2}-f^3(y)$$
$$-\frac{\alpha\zeta^2(T)}{a\xi_0}[1-\cos\phi(y)]f(y)=0, \tag{43}$$

$$\frac{df}{dy}(L_{y1})=\frac{df}{dy}(L_{y2})=0, \tag{44}$$

$$\frac{d^2\phi(y)}{dy^2}=\frac{f^2(y)}{\lambda_J^2}\sin\phi(y), \tag{45}$$

$$\lambda_J=(8\pi e j_0 p)^{-1/2}, \tag{46}$$

$$h(y)=\frac{1}{2ep}\frac{d\phi(y)}{dy}, \tag{47}$$

$$j(y)\equiv j_x(y)\equiv j_0 f^2(y)\sin\phi(y)=\frac{1}{4\pi}\frac{dh(y)}{dy}, \tag{48}$$

with boundary conditions (20) and $I\equiv\int_{L_{y_1}}^{L_{y^2}}dy\,j(y)$, where $j(y)$ is the x component of the supercurrent density (both in the S layers and the barriers). The y component of the intralayer supercurrent, whose average over the layer thickness is equal to zero, within the accepted accuracy enters the theory only implicitly, via the average kinetic-energy term in Eq. (43).

Significantly, the coherent phase difference ϕ (the same for all the barriers) obeys only one nonlinear second-order differential Eq. (45) with only one length scale, the Josephson penetration depth λ_J, as in the case of a single junction.[26,27] Due to the factor f^2, Eq. (45) is coupled to nonlinear second-order differential Eq. (43), describing the spatial dependence of the superconducting order parameter f (the same for all the S layers). In the latter equation, the term proportional to a^2/p^2 accounts for the average kinetic energy of the intralayer currents, while the term proportional to α accounts for the kinetic energy of the interlayer Josephson currents. The Maxwell Eq. (47) and (48), combined together, yield Eq. (45), as they should by virtue of self-consistency.

It is instructive to compare the above equations with those now circulating in literature concerned with the phenomenological LD model. As already mentioned, neither mutual dependence of the Euler-Lagrange equations for ϕ_n and $\mathbf{A}$, nor fundamental complementary relations of the type (27) and (28), minimizing the free energy, have been established in previous publications. Left with an incomplete set of equations, some authors make a non-self-consistent approximation $f_n=1$ and, regarding the phase differences, propose a mathematically ill-defined infinite set of differential equations with two different length scales (see, e.g., Refs. 6 and 7). In view of the conditions (31), these equations reduce to our Eq. (45) with $f=1$.

Finally, the free-energy functional (5) in the thin-layer limit after a transition to the mean-field approximation with respect to $\mathbf{A}$ takes the form

$$\Omega[f,\phi;H]=\frac{H_c^2(T)}{4\pi}W_xW_z\int_{L_{y1}}^{L_{y2}}dy\left\{\frac{a}{p}\left[-f^2(y)+\frac{1}{2}f^4(y)\right.\right.$$
$$+\zeta^2(T)\left(\frac{df(y)}{dy}\right)^2$$
$$+\frac{\zeta^2(T)}{12}\left(\frac{a}{p}\right)^2\left(\frac{d\phi(y)}{dy}\right)^2f^2(y)+\frac{\alpha\zeta^2(T)}{\alpha\xi_0}$$
$$\left.\times[1-\cos\phi(y)]f^2(y)\right]+4e^2\zeta^2(T)\lambda^2(T)$$
$$\left.\times\left[\frac{1}{2ep}\frac{d\phi(y)}{dy}-H\right]^2\right\}, \tag{49}$$

where $W_x=L_{x2}-L_{x1}$. As expected, minimizing Eq. (49) with respect to f and the phase difference ϕ, and neglecting terms of order a^2/λ^2, we arrive at Eqs. (43)–(45).

The functional (49) and complementing Maxwell Eqs. (48) and (49) contain much more physical information than the phenomenological LD model in a parallel field: the domain of validity is exactly determined, all the coefficients are microscopically defined, and a finite S layer thickness is explicitly taken into account. (As we show in Sec. III, this factor removes unphysical divergence of $H_{c2\infty}$, typical[14] of the LD model.) Another important difference is the proportionality of the condensation energy in Eq. (49) to the layer thickness a, instead of the period p in the LD functional.

The equations of the thin-layer limit admit exact solutions for all physical situations of interest. These solutions are the subject of the next section.

III. MAJOR PHYSICAL EFFECTS IN THE THIN-LAYER LIMIT

A. The Meissner state in a semi-infinite multilayer: The superheating field $H_s=(ep\lambda_J)^{-1}$

Consider a semi-infinite (in the y direction) multilayer with $L_{y1}=0$, $L_{y2}=+\infty$ in the external fields

$$0\leqslant H\leqslant H_s=(ep\lambda_J)^{-1}, \tag{50}$$

with boundary conditions of the type (20) (iii):

$$h(0)=\frac{1}{2ep}\frac{d\phi}{dy}(0)=H,\ \ h(+\infty)=\frac{1}{2ep}\frac{d\phi}{dy}(+\infty)$$
$$=H+4\pi I=0,\ \ \phi(+\infty)=0. \tag{51}$$

For

$$\frac{\alpha\zeta^2(T)}{a\xi_0}\ll 1, \tag{52}$$

the Meissner solutions of Eqs. (43)–(48) are

$$\phi(y)=-4\ \arctan\frac{H\exp[-y/\lambda_J]}{H_s+\sqrt{H_s^2-H^2}}, \tag{53}$$

$$h(y)=\frac{2HH_s[H_2+\sqrt{H_s^2-H^2}]\exp[-y/\lambda_J]}{[H_s+\sqrt{H_s^2-H^2}]^2+H^2\exp[-2y/\lambda_J]}, \tag{54}$$

$$j(y)=-\frac{HH_s}{2\pi\lambda_J}[H_s+\sqrt{H_s^2-H^2}]\frac{[[H_s+\sqrt{H_s^2-H^2}]^2-H^2\exp[-2y/\lambda_J]]\exp[-y/\lambda_J]}{[[H_s+\sqrt{H_s^2-H^2}]^2+H^2\exp[-2y/\lambda_J]]^2}, \tag{55}$$

$$f(y)=1-\frac{4\alpha\zeta^2(T)}{a\xi_0}\frac{H^2[H_s+\sqrt{H_s^2-H^2}]^2\exp[-2y/\lambda_J]}{[[H_s+\sqrt{H_s^2-H^2}]^2+H^2\exp[-2y/\lambda_J]]^2}. \tag{56}$$

The Meissner solutions persist up to the field $H_s=(ep\lambda_J)^{-1}$ that should be regarded as the superheating field of the Meissner state.

Indeed, as we will show below, the presence of Josephson vortices inside an infinite multilayer becomes energetically favorable at a field $H=H_{c1\infty}<H_s$. As in the case of the well-known Bean-Livingston barrier[28,22,25] in semi-infinite type-II superconductors, the penetration of Josephson vortices at fields $H_{c1\infty}\leqslant H<H_s$ is prevented by a surface barrier due to surface currents $j(0)$. [Compare the discussion of the superheating field in the case of a single junction,[26] where it is given by the expression $H_s=(2e\lambda\lambda_J)^{-1}$.] Equation (55) shows that $|j(0)|$ increases in the interval $0\leqslant H<H_s/\sqrt{2}$, reaches its maximum value at $H=H_s/\sqrt{2}$, decreases in the interval $H_s/\sqrt{2}<H<H_s$ and vanishes at $H=H_s$. Moreover, the phase difference at the surface $\phi(0)$, being a nonpositive, monotonously decreasing function of H in the whole interval $0\leqslant H\leqslant H_s$, also reaches its minimum value $\phi(0)=-\pi$ at $H=H_s$. The appearance of the phase difference $-\pi$ can be attributed to the formation of a line singularity of the amplitude of condensation $\langle\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rangle$ at the outer interface of the barrier (''the Josephson vortex core''). In addition, the magnetic flux per ''elementary cell'' at $H=H_s$ is $\Phi=\Phi_0/2$, where $\Phi_0=\pi/e$ is the flux quantum.

Finally, from the second of Eqs. (51) and the condition $H\leqslant H_s$ for the Meissner solutions, the maximal value of the total Josephson current $|I|=I_{\max}$ in a semi-infinite multilayer is

$$I_{\max}=\frac{H_s}{4\pi}=2\lambda_J j_0. \tag{57}$$

For $|I|>I_{\max}$, the field at the boundary is $h(0)>H_s$, and the stationary flow of the Josephson current is disrupted by the penetration of Josephson vortices that move under the influence of the Lorentz force.

Thus in fields $H>H_s$, only vortex solutions are possible. Owing to the specific feature of the thin-layer limit, i.e., the absence of screening by the intralayer currents, the ''tails'' of magnetic field distribution of individual Josephson vortices overlap in the layering direction, causing the formation of unique vortex structures that we term here ''Josephson vortex planes.'' We begin the discussion of these structures form a single ''vortex plane,'' forming in an infinite layered superconductor at the lower critical field $H=H_{c1\infty}$.

B. The lower critical field $H_{c1\infty}=2(\pi e\lambda_J p)^{-1}$ in infinite multilayers: Vortex planes

Consider now an infinite (in the y direction) layered superconductor with $L_{y1}=-\infty$, $L_{y2}=+\infty$, subject to boundary conditions of the type (20) (i), with $H=0$. The condition (52) is supposed to be fulfilled. We are looking for a vortex solution with one flux quantum Φ_0 per ''elementary cell'', i.e., with

$$\phi(+\infty)-\phi(-\infty)=2\pi,\ \ \frac{1}{2ep}\frac{d\phi}{dy}(\pm\infty)=0,\ \ \phi(0)=\pi. \tag{58}$$

The sought solution has the form of a kink:

$$\phi(y)=4\ \arctan\exp\left[\frac{y}{\lambda_J}\right]. \tag{59}$$

This solution describes a single vortex plane positioned at $y=0$. [Compare the phase difference $\phi(0)=\pi$ of Eq. (59) with the phase difference $\phi(0)=-\pi$ of Eq. (53) at the surface of a semi-infinite superconductor in the field $H=H_s$, when a vortex plane only starts to penetrate. After the actual penetration, the phase difference changes by 2π, as expected from general considerations.[29]]

Corresponding distribution of the local magnetic field is given by

$$h(y)=(ep\lambda_J)^{-1}\cosh^{-1}\left[\frac{y}{\lambda_J}\right]. \tag{60}$$

Notice that at the vortex plane $h(0)=H_s$. The density of the Josephson currents is

$$j(y)=-2j_0\cosh^{-2}\left[\frac{y}{\lambda_J}\right]\sinh\left[\frac{y}{\lambda_J}\right]. \tag{61}$$

At the vortex plane, $j(0)=0$. The Josephson currents vanish exponentially at $y\to\pm\infty$ and reach their peak values at $y=\pm\ln(1+\sqrt{2})\lambda_J\approx\pm0.88\lambda_J$. As regards the order parameter, we get

$$f(y)=1-\frac{4\alpha\zeta^2(T)}{a\xi_0}\frac{\exp[-2y/\lambda_J]}{(1+\exp[-2y/\lambda_J])^2}. \tag{62}$$

Notice that Eqs. (60)–(62), considered in the half space $0\leqslant y<+\infty$, have exactly the same form as the solutions (54)–(56) for a semi-infinite multilayer in the external field $H=H_s$, in full agreement with our interpretation of H_s as the penetration field for a single vortex plane.

To find the lower critical field $H_{c1\infty}$ at which the solution (59) becomes energetically favorable, we must consider the free-energy functional (49), which in this case takes the form

$$\Omega[\phi(y);H]-\Omega[H]_{N_v=0}$$
$$=N_{\text{cell}}W_z\left\{\frac{j_0}{2e}\int_{-\infty}^{+\infty}dy\left[1-\cos\phi(y)+\frac{\lambda_J^2}{2}\left(\frac{d\phi(y)}{dy}\right)^2\right]\right.$$
$$\left.-\frac{1}{4\pi}\frac{[\phi(+\infty)-\phi(-\infty)]H}{2e}\right\}, \tag{63}$$

where $\Omega[H]_{N_v=0}\equiv\Omega[\phi=0;H]$ is the free energy in the absence of vortices (N_v is the number of vortex planes), and $N_{\text{cell}}=W_x/p$ is the number of elementary cells. Inserting Eq. (59) into Eq. (63), we obtain the free-energy contribution of a single vortex plane:

$$\Omega[H]_{N_v=1}-\Omega[H]_{N_v=0}=N_{\text{cell}}W_z\left[\frac{4\lambda_J j_0}{e}-\frac{\Phi_0 H}{4\pi}\right], \tag{64}$$

with $\Phi_0=\pi/e$ the flux quantum. From the condition $\Omega[H_{c1\infty}]_{N_v=1}=\Omega[H_{c1\infty}]_{N_v=0}$, the lower critical field is

$$H_{c1\infty}=2(\pi ep\lambda_J)^{-1}=\frac{2}{\pi}\frac{\Phi_0}{\pi p\lambda_J}, \tag{65}$$

as in the case of a single junction, apart from the factor p in the denominator instead of $2\lambda(T)$.[26,27] As expected, $H_{c1\infty}=2H_s/\pi<H_s=h(0)$. On the contrary, Eq. (65) is completely different from previously proposed ones for layered superconductors,[8] based on an invalid assumption of single-vortex penetration.

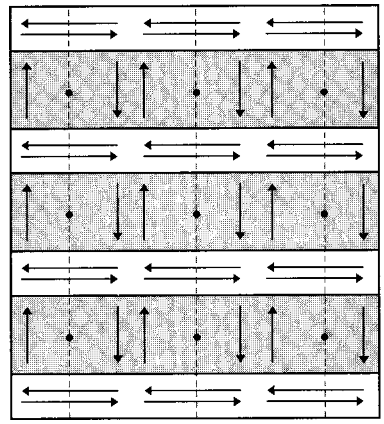

FIG. 2. Vortex state in a finite LD multilayer (in cross section). Josephson vortices (i.e., singularities of the amplitude of condensation) are conventionally denoted by black dots. The vortex planes [i.e., maxima of the microscopic magnetic field $h(y)$] are shown by dashed lines. Arrows show the direction of supercurrents.

From the proportionality of the right-hand side of Eq. (64) to N_{cell}, we infer that the total number of Josephson vortices (i.e., 1D singularities of the amplitude of condensation) in one vortex plane is equal to the total number of elementary cells. This means that Josephson vortices penetrate all the cells simultaneously and coherently. As in the case of a single junction,[26] the quantity

$$E_0=\frac{4\lambda_J j_0}{e} \tag{66}$$

can be identified with the self-energy of a single Josephson vortex per unit length (in the z direction). In higher external fields ($H\gg H_{c1\infty}$), we expect to get a ''stack'' of N_v vortex planes with the total number of Josephson vortices $N_{v\,\text{tot}}=N_v N_{\text{cell}}$ (see Fig. 2).

C. The vortex state in intermediate fields $H_{c1\infty}\ll H\ll[ea\zeta(T)]^{-1}$. The lower critical field $H_{c1W}=\pi/epW$ in finite-size samples ($W\ll\lambda_J$): A paramagnetic effect

Consider a finite-size (in the y direction) multilayer with $-L_{y1}=L_{y2}=L$, $W\equiv2L$, in the field range $H_{c1\infty}\ll H\ll[ea\zeta(T)]^{-1}$ and in the absence of externally applied current ($I=0$), i.e., subject to the boundary conditions (20) (i). The validity of the condition (52) is again assumed. [The upper bound $(ea\zeta(T))^{-1}$ for the field range means that we rare concerned with $H\ll H_{c2\infty}(T)$.]

Under these assumptions, the phase difference up to first order in the small parameter $(ep\lambda_J H)^{-2}$ is

$$\phi(y)=2epHy+\pi N_v(H)-\frac{(-1)^{N_v}}{4(ep\lambda_J H)^2}[\sin(2epHy)$$
$$-2epHy\cos(epWH)]. \tag{67}$$

The constant of integration $\pi N_v(H)$ accounts here for the phase shift due to N_v vortex planes [π per each vortex plane, see the last of Eqs. (58)]. The number of vortex planes N_v is itself a singular function of the applied field H:

$$N_v(H)=\left[\frac{epWH}{\pi}\right]=\left[\frac{\Phi}{\Phi_0}\right], \tag{68}$$

where $[u]$ means the integer part of u, and $\Phi=pWH$ is the flux through an elementary cell. This choice of the constant of integration guarantees that the energy of the Josephson coupling E_J in Eq. (49) takes the minimal value for a given H:

$$E_J[H]=\frac{H_c^2(T)}{4\pi}W_xWW_z\frac{\alpha\zeta^2(T)}{a\xi_0}\left[1-\frac{\Phi_0}{\pi\Phi}\left|\sin\frac{\pi\Phi}{\Phi_0}\right|\right] \tag{69}$$

(this expression should be compared with its analog for a single Josephson junction[26]).

The physical quantities corresponding to Eq. (67) are

$$h(y)=H\left[1-\frac{(-1)^{N_v}}{4(ep\lambda_JH)^2}[\cos(2epHy)-\cos(epWH)]\right], \tag{70}$$

$$j(y)=(-1)^{N_v}j_0\sin(2epHy), \tag{71}$$

$$f(y)=1-\frac{\alpha\zeta^2(T)}{2a\xi_0}\left[1-\frac{(-1)^{N_v}\cos(2epHy)}{1+2[epH\zeta(T)]^2}-\frac{\sqrt{2}epH\zeta(T)|\sin(epHW)|}{1+2[epH\zeta(T)]^2}\frac{\cosh[\sqrt{2}y/\zeta(T)]}{\sinh[W/\sqrt{2}\zeta(T)]}\right]. \tag{72}$$

[The term $2(epH\zeta(T))^2$ in the denominators of Eq. (72) can only be retained if $p\gg a$.]

In the limit $W\gg\zeta(T)$, $|y|\ll W/2$, Eq. (72) becomes

$$f(y)=1-\frac{\alpha\zeta^2(T)}{2a\xi_0}\left[1-\frac{(-1)^{N_v}\cos(2epHy)}{1+2[epH\zeta(T)]^2}\right]. \tag{73}$$

Equations of the type (67), (70), and (73) for $N_v=2m$ (m is an integer) were first obtained by Theodorakis[11] in the framework of the LD model. Our Eq. (67) for $N_v=2m$ should also be compared with an analogous solution for an infinite single junction given, for instance, in Ref. 25.

The singular function $N_v(H)$ introduces discontinuities in Eqs. (67) and (70)–(73). These discontinuities witness that the system undergoes a first-order phase transition when a vortex plane penetrates or leaves the sample (compare with the discussion of a single junction in Ref. 26).

The positions of vortex planes y_v correspond to local maxima of the field $h(y)$ in Eq. (70). [In the case (73), y_v exactly coincide with local minima of $f(y)$.] In the vortex planes $y=y_v$, the microscopic magnetic field is higher than the applied one:

$$h(y_v)=H\left[1+\frac{1}{4(ep\lambda_JH)^2}[1+(-1)^{N_v}\cos(epWH)]\right]>H, \tag{74}$$

which is expected for any vortex solution. The Josephson current density $j(y)=(1/4\pi)[dh(y)/dy]$ vanishes both in the vortex planes $y=y_v$ and in the planes of local minima of $h(y)$, $y=y_v\pm\pi/2epH$. When passing through zero in these planes, $j(y)$ changes the sign, as depicted in Fig. 2.

From Eq. (68) with $N_v(H)=1$, we obtain the lower critical field H_{c1W} in a finite multilayer with $W\ll\lambda_J$:

$$H_{c1W}=\frac{\pi}{epW}=\frac{\pi^2}{2}H_{c1\infty}\frac{\lambda_J}{W}\gg_{c1\infty}. \tag{75}$$

The definition of the magnetization M,[25]

$$4\pi M=\frac{1}{W}\int_{-L}^{+L}dy\,h(y)-H, \tag{76}$$

and Eq. (70) yield

$$M(H)=-\frac{1}{16\pi H(ep\lambda_J)^2}\left[\frac{|\sin(epWH)|}{epWH}-(-1)^{N_v}\cos(epWH)\right]. \tag{77}$$

The magnetization (77) shows distinctive oscillatory behavior and discontinuities at $epWH\to\pi N$ (N is an integer), when a vortex plane penetrates or leaves the sample.

Interestingly enough, the right-hand side of Eq. (77) passes through zero and may have both signs. Thus, for $\Phi=pWH\gg\Phi_0$, the sample exhibits a small paramagnetic effect, if $N_v\Phi_0<\Phi<(N_v+\frac{1}{2}-\Phi_0/\pi^2\Phi)\Phi_0$:

$$M(H)=\frac{1}{16\pi H(ep\lambda_J)^2}\left[\left|\cos(epWH)\right|-\frac{|\sin(epWH)|}{epWH}\right]>0. \tag{78}$$

D. Fraunhofer oscillations of the Josephson current in multilayers with $W\ll\lambda_J$, in the field range $0\leq H\ll[ea\zeta(T)]^{-1}$: "Edge pinning" of the vortex planes

Now we proceed to the case of a finite-size (along the layers) multilayer with $-L_{y1}=L_{y2}\equiv L$, $W\equiv 2L$ in the presence of an externally applied current I, i.e., subject to the boundary conditions (20) (ii). (Compare the discussion by Owen and Scalapino[30] of the single-junction case.) The relation (52) is supposed to hold. The applied magnetic fields are within the range $0\leq H\ll[ea\zeta(T)]^{-1}$.

Assuming $W\ll\lambda_J$, we can consider W^2/λ_J^2 as a small expansion parameter in Eq. (45). In this way, we obtain

$$\phi(y)=2epHy+\pi N_v(H)+\varphi-\frac{(-1)^{N_v}}{4}\frac{W^2}{\lambda_J^2}\left[\frac{\Phi_0}{\pi\Phi}\right]^2\left[\sin(2epHy+\varphi)-2epHy\cos\frac{\pi\Phi}{\Phi_0}\cos\varphi-\sin\varphi\right], \tag{79}$$

$$I(\varphi,\Phi)\equiv\int_{-L}^{+L}dy\,j(y)=j_0W\frac{\Phi_0}{\pi\Phi}\left|\sin\frac{\pi\Phi}{\Phi_0}\right|\sin\varphi, \tag{80}$$

$$h(y)=H\left\{1-\frac{(-1)^{N_v}}{4}\frac{W^2}{\lambda_J^2}\left[\frac{\Phi_0}{\pi\Phi}\right]^2\left[\cos(2epHy+\varphi)-\cos\frac{\pi\Phi}{\Phi_0}\cos\varphi\right]\right\}, \quad (81)$$

$$f(y)=1-\frac{\alpha\zeta^2(T)}{a\xi_0}\left[1-\frac{(-1)^{N_v}\cos(2epHy+\varphi)}{1+2[epH\zeta(T)]^2}-\frac{\sqrt{2}epH\zeta(T)|\sin(epHW)|\cos\varphi}{1+2[epH\zeta(T)]^2}\frac{\cosh[\sqrt{2}y/\zeta(T)]}{\sinh[W/\sqrt{2}\zeta(T)]}-\frac{(-1)^{N_v}\sqrt{2}epH\zeta(T)\cos(epHW)\sin\varphi}{1+2[epH\zeta(T)]^2}\times\frac{\sinh[\sqrt{2}y/\zeta(T)]}{\cosh[W/\sqrt{2}\zeta(T)]}\right], \quad (82)$$

where $N_v(H)=[epWH/\pi]$ is the number of vortex planes, $\Phi=pWH$ is the flux through an elementary cell, $\Phi_0=\pi/e$, as usual, and the constant φ ($|\varphi|\leq\pi/2$) parameterizes the total Josephson current I given by Eq. (80). Equation (80) yields the well-known Fraunhofer pattern, the only difference from the single-junction case being the occurrence of the period p in place of $2\lambda(T)$.[26,27] Note that the first zero of the Fraunhofer pattern, by virtue of Eq. (75), corresponds to the lower critical field H_{c1W}. In the absence of the transport current, i.e., for $\varphi=0$, Eqs. (79), (81), and (82) reduce, respectively, to Eqs. (67), (70), and (72), as they should.

The self-consistency of our calculations can be easily verified by means of Ampere's law $h(+L)-h(-L)=4\pi I$. It is assured by terms proportional to W^2/λ_J^2 in Eqs. (79) and (81) that explicitly take into account the effect of self-induced fields. Although Eq. (80) was first derived in the framework of the LD model in Ref. 7, the authors of that publication did not calculate the phase differences self-consistently and did not evaluate the local magnetic field in first order in W^2/λ_J^2. As a result, they arrived at an incorrect conclusion that Fraunhofer oscillations of I could be observed in the absence of Josephson vortices. Unfortunately, this misunderstanding is shared in some other recent publication.[31] Therefore we provide below a detailed and rigorous clarification.

As we see from Eq. (81), in the presence of the transport current I, the vortex planes are shifted by the Lorentz force to new equilibrium positions [local maxima of $h(y)$]:

$$\overline{y_v}=y_v-\frac{\varphi}{2epH}, \quad (83)$$

where y_v correspond to local maxima of the right-hand side of Eq. (81) for $\varphi=0$. The local magnetic field in the vortex planes now is

$$h(\overline{y_v})=H\left(1+\frac{1}{4(ep\lambda_J H)^2}\times[1+(-1)^{N_v}\cos(epWH)\cos\varphi]\right)>H. \quad (84)$$

In equilibrium, the Lorentz force f_L per elementary cell acting on the vortex planes is counterbalanced by the pinning force f_{pin} that can be defined as[32]

$$f_{\mathrm{pin}}(Y)=-\frac{1}{N_{\mathrm{cell}}W_z}\frac{dU_{\mathrm{pin}}(Y)}{dY}, \quad (85)$$

where $U_{\mathrm{pin}}(Y)$ is the pinning potential arising owing to the shift by Y of the vortex planes from their equilibrium positions in the absence of the transport current I. To evaluate the pinning potential, we consider the increase of the free energy in first order in $\alpha\zeta^2(T)/a\zeta_0$, caused by such a shift. Noting that first-order corrections to $f(y)\approx 1$ and $h(y)\approx H$ do not contribute to the free energy, taking $\phi(y)\approx 2epHy+\pi N_v(H)$, making the transformation $y\to y-Y$ and substituting into Eq. (49), we obtain

$$U_{\mathrm{pin}}(Y)\equiv\Omega[H;Y]-\Omega[H;Y=0]=N_{\mathrm{cell}}WW_z\frac{j_0}{2e}\frac{\Phi_0}{\pi\Phi}\left|\sin\frac{\pi\Phi}{\Phi_0}\right|\left[1-\cos\left(\frac{2\pi\Phi}{\Phi_0}\frac{Y}{W}\right)\right]. \quad (86)$$

It is very instructive to rewrite Eq. (86) as

$$U_{\mathrm{pin}}(Y)=N_{\mathrm{cell}}WW_z\frac{1}{2e}\frac{\Phi_0}{\pi\Phi}[2j(+L;Y=0)+j(-L;Y)-j(+L;Y)], \quad (87)$$

where

$$j(\pm L;Y)=(-1)^{N_v}j_0\sin(\pm 2epHL+2epHY)$$

are the surface currents in the presence of the shift Y. We see that the pinning potential for $|Y|<\pi/4epH$ arises owing to the emergence of additional surface currents on the opposite side of the superconductor. At $2epHL=\pi N+\frac{1}{2}$ (N is an integer), $j(-L;Y)=-j(+L;Y)$, i.e., these currents flow in the opposite directions, and the pinning potential reaches its maximum. On the contrary, at $2epHL=\pi N$ (N is an integer), $j(-L;Y)=j(+L;Y)$, i.e., the surface currents flow in the same direction and mutually compensate each other in Eq. (87), the pinning potential vanishes, and vortex planes freely penetrate or leave the sample (compare with the discussion at the beginning of this section of the case of a semi-infinite multilayer for $H=H_s$). The surface currents also flow in the same direction and mutually compensate each other when the magnitude of the shift $|Y|$ reaches the value $|Y|=Y_{\max}\equiv\pi/4epH$. Moreover, the pinning potential vanishes for $\Phi\gg\Phi_0$.

From Eqs. (85) and (86), we obtain the pinning force for the shift Y:

$$f_{\mathrm{pin}}(Y)=-I\left(\frac{2\pi\Phi}{\Phi_0}\frac{Y}{W};\Phi\right)\Phi,$$

$$I\left(\frac{2\pi\Phi}{\Phi_0}\frac{Y}{W};\Phi\right)\equiv j_0W\frac{\Phi_0}{\pi\Phi}\left|\sin\frac{\pi\Phi}{\Phi_0}\right|\sin\left(\frac{2\pi\Phi}{\Phi_0}\frac{Y}{W}\right). \quad (88)$$

From these expressions we infer that the maximal value of the pinning force $|f_{\rm pin}|$ for given flux Φ is $f_{\rm pin}^{\rm max} = |I(|\pi/2|;\Phi)|\Phi$.

In the presence of the transport current $I(\varphi;\Phi)$ [Eq. (80)], the shift of the positions of the vortex planes, according to Eq. (83), is $Y = -\varphi/2epH$, with the maximal equilibrium value $|Y| = Y_{\rm max} \equiv \pi/4epH$. Taking into account the fact that in equilibrium $f_L = -f_{\rm pin}$, we arrive at the expression for the corresponding Lorentz force:

$$f_L = -I(\varphi;\Phi)\Phi. \tag{89}$$

This expression was to be expected from general considerations,[32] which prescribe for the magnitude of the Lorentz force the relation $|f_L| = |I|\Phi$, where I is the transport current. It is therefore absolutely clear that the stationary Josephson effect becomes impossible if the magnitude of the transport current $|I|$ exceeds the value $I_{\rm max} = |I(|\pi/2|;\Phi)|$, because in this situation $|f_L| > |f_{\rm pin}|$, and the vortex planes are completely depinned.

Notice that the physics of the Fraunhofer pattern in single junctions was discussed in terms of a series of first-order phase transitions due to successive penetration of Josephson vortices long ago.[26] A qualitative explanation by means of the edge pinning was proposed in the book by Tinkham.[14] In general, the pinning of Josephson vortices in weakly coupled superconducting structures with $W \ll \lambda_J$ is completely analogous to the pinning of Abrikosov vortices by the edges of a thin [compared to $\lambda(T)$] type-II superconducting film.[33]

Finally, we observe that the magnetization in the presence of the transport current $I(\varphi;\Phi)$, according to Eqs. (76) and (81), is given by

$$M(H) = -\frac{1}{16\pi H(ep\lambda_J)^2}\left[\frac{|\sin(epWH)|}{epWH} - (-1)^{N_v}\cos(epWH)\right]\cos\varphi. \tag{90}$$

For $\varphi = 0$, Eq. (90) reduces to Eq. (77). For $\Phi = pWH \gg \Phi_0$, $N_v\Phi_0 < \Phi < (N_v + \frac{1}{2} - \Phi_0/\pi^2\Phi)\Phi_0$, we again obtain the paramagnetic effect [compare with Eq. (78)].

E. Critical parameters of an infinite multilayer: $T_{c\infty}$, $H_{c2\infty}$

At the point of the second-order phase transition to the normal state, f^2 can be considered as a small parameter. Thus the term f^3 in Eq. (68) and the right-hand side of Eq. (70) can be dropped. Applying boundary conditions (20) (i) yields $\phi(y) = 2epHy + \pi N_v(H)$. With this phase difference, the linearized version of Eq. (19) can be transformed into

$$\frac{d^2 f(t)}{dt^2} + [A(T,H) - (-1)^{N_v+1} q(H)\cos 2t] f(t) = 0,$$

$$A(T,H) \equiv \frac{1 - (1/3)e^2H^2a^2\zeta^2(T) - \alpha\zeta^2(T)/a\xi_0}{[ep\zeta(T)H]^2},$$

$$q(H) \equiv \frac{\alpha}{2a\xi_0(epH)^2}, \tag{91}$$

where we have introduced a dimensionless variable $t \equiv epHy$: $f(t) \equiv f(t/epH)$. Hence one gets two independent equations: for the odd $N_v = 2m+1$ ($m = 0,1,2,\ldots$) and the even $N_v = 2m$ number of vortex planes. Both of them have the usual form of Mathieu equations (see the Appendix). As to the boundary conditions, it is convenient to take $-L_{y1} = L_{y2} \equiv L \equiv W/2$ and, by symmetry, consider Eq. (91) in the interval $0 \leqslant y \leqslant L$, with

$$\frac{df}{dt}(0) = \frac{df}{dt}(epHL) = 0. \tag{92}$$

The critical parameters T_c and H_{c2} are now determined by the smallest eigenvalue of the boundary problem (91) and (92).

In an infinite in the y direction multilayer ($L \to \infty$), the only bounded at the infinity solutions of Eq. (91) are periodic Mathieu functions, with $f_{N_v = 2m+1}(t) \propto {\rm ce}_0(t,q)$ and $f_{N_v = 2m}(t) \propto {\rm ce}_0(\pi/2 - t, q)$ corresponding to the smallest eigenvalues $a_0(q)$ and $a_0(-q) = a_0(q)$, respectively. Thus the critical parameters are given by the equation

$$[A(T,H)]_{c\infty} = [a_0(q)]_{c\infty}, \tag{93}$$

where one should fix H to obtain $T_{c\infty}$ or, alternatively, fix T to obtain $H_{c2\infty}$. As in the case of Eq. (73), local minima of the reduced order parameter $f(t)$ in Eq. (91) correspond to the positions of the vortex planes: in conventional units the distance between two successive minima is $\Delta y_v = \pi/epH$, which gives the flux $\Phi = \Delta y_v pH = \Phi_0$ per single vortex.

1. The critical temperature $T_{c\infty}$

For magnetic fields H in the range

$$H \ll \frac{\Phi_0}{a\xi_0\chi^{1/2}(\xi_0/l)}, \tag{94}$$

the general expression for $T_{c\infty}$ resulting from Eq. (93) is

$$T_{c\infty} = T_{c0}\left\{1 - \frac{7\zeta(3)}{12}\xi_0^2\chi(\xi_0/l)\left[\frac{1}{3}e^2H^2a^2 + \frac{\alpha}{a\xi_0} + (epH)^2 a_0\left(\frac{\alpha}{2a\xi_0(epH)^2}\right)\right]\right\}. \tag{95}$$

In weak fields

$$H \ll \frac{1}{ep}\sqrt{\frac{\alpha}{2a\xi_0}} \equiv \frac{\Phi_0}{\pi p}\sqrt{\frac{\alpha}{2a\xi_0}}, \tag{96}$$

pair breaking effect of intralayer supercurrents is unimportant, and we get

$$T_{c\infty} = T_{c0}\left[1 - \frac{7\sqrt{2}\zeta(3)}{12}\xi_0^2\chi(\xi_0/l)\sqrt{\frac{\alpha}{a\xi_0}}\,epH\right]. \tag{97}$$

In strong fields

$$\frac{\Phi_0}{\pi p}\sqrt{\frac{\alpha}{2a\xi_0}} \ll H \ll \frac{\Phi_0}{a\xi_0\chi^{1/2}(\xi_0/l)}, \tag{98}$$

Eq. (95) becomes

$$T_{c\infty}=T_{c0}\left[1-\frac{7\zeta(3)}{12}\xi_0^2\chi(\xi_0/l)\left(\frac{1}{3}e^2H^2a^2+\frac{\alpha}{a\xi_0}\right)\right]. \tag{99}$$

The term proportional to α takes into account pair breaking by the Josephson currents, locally equal to the critical ones.[15] In the absence of weak coupling (α=0), Eq. (99) reduces to the well-known expression for an isolated thin superconducting film.[22,14]

2. The upper critical field $H_{c2\infty}$

For a fixed T, Eq. (93) yields an implicit expression for $H_{c2\infty}$ as a function of T:

$$[ep\zeta(T)H_{c2\infty}]^2\left[\frac{1}{3}\left(\frac{a}{p}\right)^2+a_0\left(\frac{\alpha\zeta^2(T)}{2a\xi_0[ep\zeta(T)H_{c2\infty}]^2}\right)\right]=1-\frac{\alpha\zeta^2(T)}{a\xi_0}. \tag{100}$$

This expression exhibits the so-called[14] 3D-2D crossover, experimentally verified, for example, on Nb/Ge multilayers.[34] The crossover temperature T^* can be conventionally defined by the relation $\alpha\zeta^2(T^*)/a\xi_0=1$.

For temperatures close to T_{c0}, when

$$\frac{\alpha\zeta^2(T)}{a\xi_0}\gg 1, \tag{101}$$

$$H_{c2\infty}(T)=\frac{\Phi_0}{\sqrt{2}\pi p\zeta(T)}\frac{\sqrt{a\xi_0}}{\sqrt{\alpha}\zeta(T)}=\frac{12}{7\sqrt{2}\zeta(3)\pi\sqrt{\alpha}}\frac{\Phi_0\sqrt{a\xi_0}}{p\xi_0^2\chi(\xi_0/l)}\left(1-\frac{T}{T_{c0}}\right). \tag{102}$$

In this 3D regime, the positive kinetic energy of small interlayer Josephson currents in Eq. (49) competes with the negative intralayer condensation energy. The superconductivity of the S layers is strongly depressed by the vortex planes, as a comparison between local maxima $f_{\max}$ and local minima $f_{\min}$ of the order parameter shows

$$\frac{f_{\min}}{f_{\max}}\equiv\frac{f(y_v)}{f(y_v\pm\pi/2epH_{c2\infty})}=2\sqrt{2}\exp\left[-\frac{2\alpha\zeta^2(T)}{a\xi_0}\right]\ll 1. \tag{103}$$

At lower temperatures, when

$$\frac{\alpha\zeta^2(T)}{a\xi_0}\ll 1, \tag{104}$$

$$H_{c2\infty}(T)=\frac{\sqrt{3}\Phi_0}{\pi a\zeta(T)}\left[1-\frac{1}{2}\frac{\alpha\zeta^2(T)}{a\xi_0}\right]. \tag{105}$$

In this 2D regime, the energy of the Josephson interlayer coupling is small relative to the intralayer condensation energy.[15] The transition to the normal phase occurs owing mainly to pair breaking by the intralayer supercurrents, and the order parameter is almost unperturbed by the vortex planes:

$$f(y)\propto 1-\frac{(-1)^{N_v}}{12}\frac{a^2}{p^2}\frac{\alpha\zeta^2(T)}{a\xi_0}\cos(2epHy). \tag{106}$$

This expression should be compared with Eq. (73) for intermediate fields in the same temperature range (104). In the limit α=0 (no Josephson coupling), Eq. (105) goes over into the well familiar one for an isolated thin superconducting film.[22,25,14] Equation (105) explains the origin of the well-known[14] unphysical ''low-temperature'' divergence of the LD model: taking a formal limit $a\to 0$ while keeping $\alpha\zeta^2(T)/a\xi_0=\text{const}$, we get $H_{c2\infty}(T)\to\infty$.

Aside from microscopically determined parameters, dependence (102) for layered superconductors was first obtained within the framework of the LD model.[3,24,14] Expressions of the type (105) were derived phenomenologically in Refs. 12 and 13. In all these publications relations (30) and (31) were implicitly adopted as physically plausible assumptions. The very fact that these results are contained as limiting cases in our Eqs. (42)–(48) once again demonstrates the generality and self-consistency of the approach of this paper.

Finally, we emphasize that the concept of Josephson vortex planes applies both in limits (101) and (104). Contrary to previous suggestions,[24,7] there is no transition from the ''Abrikosov-core regime'' to the ''Josephson-core regime'' at T^*: The existence of Abrikosov vortices with normal cores in the thin-layer limit is not allowed by the solutions of Eq. (91) [mathematically, the function $\mathrm{ce}_0(t,q)$ is strictly positive].

F. Size effects: Oscillations of T_{cW}

Aside from a special case $epHL=\pi k/2$ ($\Phi=k\Phi_0$, $k=0,1,2,\ldots$), for multilayers with finite S-layer length $W=2L$ only approximate solutions of the boundary problem (91) and (92) can be obtained. Using Galerkin's method,[35] we have found two groups of solutions corresponding to the smallest eigenvalues $[A]_c$:

$$f_{\mu,N_v=2n+1}(t)\propto\cosh(\mu t)\mathrm{ce}_0(t,q),$$

$$f_{\mu,N_v=2n}(t)\propto\cosh(\mu t)\mathrm{ce}_0\left(\frac{\pi}{2}-t,q\right), \tag{107}$$

$$\mu=\coth(\mu epHL)\frac{|(d\mathrm{ce}_0/dt)(epHL,q)|}{\mathrm{ce}_0(epHL,q)}, \tag{108}$$

$$[A(T,H)]_c=[a_0(q)-\mu^2]_c, \tag{109}$$

and

$$f_{\nu,N_v=2n+1}(t)\propto\cos(\nu t)\mathrm{ce}_0(t,q),$$

$$f_{\nu,N_v=2n}(t)\propto\cos(\nu t)\mathrm{ce}_0\left(\frac{\pi}{2}-t,q\right), \tag{110}$$

$$\nu=-\cot(\nu epHL)\frac{|(d\mathrm{ce}_0/dt)(epHL,q)|}{\mathrm{ce}_0(epHL,q)}, \tag{111}$$

$$[A(T,H)]_c=[a_0(q)+\nu^2]_c, \tag{112}$$

where Eqs. (108) and (111) implicitly define parameters μ and ν, and Eqs. (109) and (112) determine the critical point.

From Eqs. (109) and (112) we see, that the eigenvalue corresponding to the eigenfunctions f_μ is smaller than $a_0(q)$, while that corresponding to the eigenfunctions f_ν is larger. Physically, this means that with f_μ in a finite multilayer we can achieve higher values of the critical parameters T_c and H_{c2} than in an infinite one [compare with Eq. (93)]. At $epHL=\pi k/2$ ($\Phi=k\Phi_0$, $k=0,1,2,\ldots$), these equations yield $\mu=\nu=0$.

For $epHL\to\infty,\mu$ is a bounded, oscillating function of *epHL* and does not tend to any limit. On the contrary, $\nu\to\pi/2epHL$, when $epHL\to\infty$. This signifies that at certain values of *epHL* the solution f_μ becomes unstable and gives way to the solution f_ν with lower values of the critical parameters, presumably by means of a first-order phase transition. As the parameters μ and ν can enter the free energy only via the combinations $\mu epHL$ and $\nu epHL$, we expect the transitions $f_\mu\leftrightarrow f_\nu$ to occur when $[\mu epHL]_*=[\nu epHL]_*$, hence the relation

$$\cot[\mu epHL]_*=-\coth[\mu epHL]_*$$

with the numerical solution $[\mu epHL]_*\approx 2.37$. Thus the solution f_μ with Eq. (109) is realized when

$$\frac{epHL|(d\mathrm{ce}_0/dt)(epHL,q)|}{ce_0(epHL,q)}<[\mu epHL]_*\tanh[\mu epHL]_*\approx 2.33, \tag{113}$$

while for

$$\frac{epHL|(d\mathrm{ce}_0/dt)(epHL,q)|}{\mathrm{ce}_0(epHL,q)}>2.33, \tag{114}$$

the system ''chooses'' f_ν with Eq. (112). The condition (113) is met, for instance, when $\Phi\equiv pWH\approx k\Phi_0$ ($W=2L$, $k=0,1,2,\ldots$). Because of the oscillating character of the left-hand sides of Eqs. (113) and (114), the system oscillates between the states with f_μ and f_ν with increasing *epHL*. For larger *epHL*, the domain of existence of f_μ becomes narrower, while that of f_ν widens. For $epHL\to\infty$, the solution f_ν goes over smoothly into that of an infinite multilayer.

We want to point out here that the exact character of the transitions $f_\mu\leftrightarrow f_\nu$ can only be established by solving the nonlinear boundary problem and comparing the corresponding free energies, which is beyond the scope of the present paper.

As an important application of the above results, we consider the critical temperature of a finite multilayer T_{cW} in the field range given by Eq. (98), and with $W\ll p\lambda_J/\lambda(T)$. Under such circumstances, the condition (113) is satisfied, and the solution of Eq. (108) is

$$\mu^2=\frac{\alpha}{a\xi_0(epH)^2}\,\frac{\Phi_0}{\pi\Phi}\left|\sin\frac{\pi\Phi}{\Phi_0}\right|,$$

which on substituting into Eq. (109) yields

$$T_{cW}=T_{c0}\left\{1-\frac{7\zeta(3)}{12}\,\xi_0^2\chi(\xi_0/l)\left[\frac{1}{3}e^2H^2a^2+\frac{\alpha}{a\xi_0}\times\left(1-\frac{\Phi_0}{\pi\Phi}\left|\sin\frac{\pi\Phi}{\Phi_0}\right|\right)\right]\right\}. \tag{115}$$

Thus in a finite-size multilayer the critical temperature can exhibit the same oscillations with changing the flux through the elementary cell as the total Josephson current I does [see Eq. (80)]. However, a significant difference lies in the fact that while the oscillations of I are observable in any types of Josephson systems, the oscillations of T_c is an interesting feature, because in a single Josephson junction with thick superconducting electrodes any shifts of T_c are negligible. In the limit $\Phi\gg\Phi_0$, Eq. (115) reduces to Eq. (99), as anticipated.

IV. DISCUSSION

Based solely on the microscopic Hamiltonian (1), we have constructed a self-consistent theory that provides a comprehensive, unified picture of physical effects in S/I multilayers in parallel magnetic fields in the GL regime.

Employing rigorous technique of variational calculus, we have derived in Sec. II fundamental constraint relations (27) and (28) and solved a nontrivial problem of exact minimization of the microscopic functional (5). Up until the present study, such a problem has not been solved even for the much simpler phenomenological LD functional. Surprisingly, even mutual dependence and incompleteness of the Euler-Lagrange equations for ϕ_n and $\mathbf{A}$ were not noticed in previous publications. This incompleteness fully manifests itself in unphysical degrees of freedom and an irrelevant length scale of equations for the phase differences proposed, e.g., in Refs. 6 and 7: Making use of constraints of the type (28), one can reduce these equations to our Eq. (45) with $f(y)=1$. Emerging as a direct mathematical consequence of such general physical properties as gauge invariance and Josephson interlayer coupling, constraints (27) and (28) should apply to any superconducting weakly coupled periodic structure. The discovery of their role in minimizing the free energy makes further progress in the development of the theory possible.

In the thin-layer limit which corresponds to the domain of validity of the phenomenological LD model, we have derived a remarkably simple, closed set of self-consistent microscopic mean-field Eqs. (42)–(48) and the generating functional (49). The fact that the solution (67), (70), and (73) of these equations describing the vortex state in an infinite multilayer reproduces the result obtained by Theodorakis[11] in the framework of the LD model is not an occasional coincidence. The application of the mathematical methods of this paper allows us to obtain the complete exact solution of the LD model in parallel fields as well (this solution will be published elsewhere). The resulting mean-field equations are merely a limiting case of our Eqs. (42)–(48) for $a/p\to 0$. As our equations contain more physical information, we propose that they should replace the LD model in parallel fields.

Concerning some major physical results of Sec. III in the thin-layer limit, we have completely revised previous calculations[8,9] of H_{c1} based on an invalid assumption of single Josephson vortex penetration and refuted the concept[6] of a triangular Josephson vortex lattice. Our consideration envisages simultaneous and coherent penetration in the form of the vortex planes. Our prediction of the superheating field H_s for semi-infinite multilayers implies hysteretic behavior of the magnetization. In the vortex state, the magnetization should exhibit jumps due to the vortex-plane penetration. We have fully clarified the widespread[7,31] misunderstanding of the physics of the Fraunhofer oscillations: our self-consistent treatment of the Josephson effect unambiguously proves that the Fraunhofer pattern occurs due to successive penetration of the vortex planes and their pinning by the edges of the sample. Our prediction of novel size-effects in finite multilayers, a series of first-order phase transitions to the normal state and oscillations of the critical temperature versus the applied field, should stimulate further experimental investigation.

The results of our investigation directly apply to artificial superconductor/insulator[36] and superconductor/semiconductor[34,37,4] multilayers. As regards the high-T_c superconductors BSCCO and TBCCO, believed to be atomic-scale weakly coupled superlattices,[38] the application is restricted by the limitation (4). However, we expect that such basic features of the thin-layer limit as simultaneous and coherent penetration in the form of the vortex planes will hold. For high-T_c samples exhibiting a clear Fraunhofer pattern,[39] we anticipate the presence of the related effect of oscillations of the critical temperature (115) as well.

As to direct experimental verification of basic concepts of our theory, the best evidence is provided by the recent magnetization and polarized neutron reflectivity measurements on Nb/Si multilayers in parallel fields.[4] These measurements clearly revealed simultaneous penetration of Josephson vortices into all Si layers and a companion effect of jumps of the magnetization, exactly as predicted in our paper. The distribution of the magnetic field attributed to Josephson vortices was found to have the periodicity of the Nb/Si layering, in agreement with the general consideration of Sec. II A. (The experimental conditions[4] did not fully match the requirements of the thin-layer limit for which the screening by the intralayer currents could be neglected.) Finally, it is quite natural that our general implicit expression (100) for $H_{c2\infty}(T)$ exhibits the so-called 3D-2D crossover, well-known from the experiment,[34] and is free from the unphysical low-temperature divergence, typical[14] of the phenomenological LD model.

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APPENDIX: THE APPLICATION OF MATHIEU FUNCTIONS

The canonical form of the Mathieu equations is[18,40]

$$\frac{d^2 f}{dt^2} + (a - 2q \cos 2t) f = 0. \tag{A1}$$

If $f(t)$ is a solution to Eq. (A1), then $f((\pi/2) - t)$ stisfies

$$\frac{d^2 f}{dt^2} + (a + 2q \cos 2t) f = 0. \tag{A2}$$

In the class of periodic solutions of Eq. (A1), the smallest eigenvalue $a_0(q)$ is a nonpositive, continuous, even, monotonously decreasing function of q. The corresponding eigenfunction $\mathrm{ce}_0(l,q)$ has a period π, is even and strictly positive.

For $0 \leqslant q \ll 1$, we have the asymptotics

$$\mathrm{ce}_0(t,q) \approx \frac{1}{\sqrt{2}} \left[1 - \frac{q}{2} \cos 2t + \cdots \right], \tag{A3}$$

$$a_0(q) \approx -\frac{q^2}{2} + \dots . \tag{A4}$$

For $q \gg 1$,

$$a_0(q) \sim 2\sqrt{q} - 2q, \tag{A5}$$

but there is no uniform asymptotics for $\mathrm{ce}_0(t,q)$. In this case, the behavior of $\mathrm{ce}_0(t,q)$ may be characterized by the formulas

$$\mathrm{ce}_0(t,q) \sim \left(\frac{\pi}{2}\right)^{1/4} q^{1/8} e^{-\sqrt{q}\cos^2 t/4}, \quad |\cos t| < \frac{2^{1/4}}{\sqrt{q}}; \tag{A6}$$

$$\mathrm{ce}_0(0,q) \sim 2\sqrt{2} e^{-2\sqrt{q}} \mathrm{ce}_0(\pi/2,q) \sim 2(2\pi)^{1/4} q^{1/8} e^{-2\sqrt{q}}. \tag{A7}$$

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