Microscopic theory of vortex dynamics in homogeneous superconductors

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Vortex dynamics in a type-II superconductor is systematically investigated by the influence functional method. The irrelevant fermionic degrees of freedom are integrated out and their effects on the dynamics are treated in terms of the vortex coordinate. When an isolated vortex is moving against its background, forces proportional to the first order of vortex velocity on the vortex are calculated within the present formulation. The total transverse force on the moving vortex is explicitly shown to be proportional to the superfluid number density and insensitive to impurities. Its equivalent expressions in terms of the Berry phase and the various summations of transitions between quasiparticle (hole) states are discussed. At finite temperatures, due to the finite population of quasiparticle (hole) excitations above (below) the energy gap, there is a friction against vortex motion which diverges logarithmically in the low-frequency limit. Nonmagnetic impurities give rise to an additional friction from the core states which saturates to a value independent of the normal-state resistivity in the dirty limit. In this limit, the coupling to the electromagnetic field does not change the conclusions if charge neutrality in the superconductor is maintained. Macroscopic constraints on vortex dynamics by the second law of thermodynamics and by the fluctuation-dissipation theorems are also discussed. [S0163-1829(99)10129-2]

I. INTRODUCTION

In a type-II superconductor, vortex motion is responsible for a variety of low-frequency transport phenomena. It is the only topological singularity whose dynamical properties are widely accessible to experimental studies in both classical and quantum regimes, and its importance has long been realized.[1–4] Despite decades of research, the theoretical agreement reached so far is very limited: At zero temperature, in the absence of any impurity potentials, a vortex follows the local superfluid velocity. In the absence of a local superfluid velocity, when a vortex follows the motion of an external trapping potential, there is a momentum change in the superfluid transverse to the direction of vortex motion. In order to provide this momentum change, a force must be applied by the external trapping potential to the superfluid through the vortex. The vortex experiences a transverse force proportional to the superfluid number density, balanced by the external force from the trapping potential. Beyond this simplest and idealized situation many aspects of vortex dynamics have remained unsettled and even controversial. In the present paper we attempt to provide an influence functional formulation of vortex dynamics from the microscopic Bardeen-Cooper-Schrieffer (BCS) theory, and a few detailed microscopic calculations under realistic conditions.

The current microscopic understandings of vortex dynamics in the presence of impurities and at finite temperatures may be classified into two different physical pictures, which are based on different theoretical approaches and give contradictory results. In one picture the magnitude of the total force experienced by the vortex in the transverse direction is proportional to the superfluid number density.[5,6] The superfluid momentum change caused by the vortex motion is provided by an externally controlled trapping potential in the absence of a local superfluid velocity, regardless of the existence of the normal fluid.[6] It is an exact consequence of the global topological constraint on the vortex. The normal fluid at finite temperatures gives rise to friction for the vortex motion in the longitudinal direction. Furthermore, the global methods used in Refs. 5 and 6 indicate that the total transverse force is insensitive to random impurities, though there are additional frictional effects. In this picture, in the absence of the externally controlled trapping potential, the pinning and friction should be used to obtain vortex motion perpendicular to the direction of an externally applied current. For the other picture, the essence of the results is that there are additional forces in the transverse direction of vortex velocity, provided by unbounded quasiparticle excitations or the normal fluid, by bounded vortex core states, by the substrate, or by a certain combination of them.[7–11] The total transverse force is reduced, which is most clearly represented by the alleged gradual turning on the cancellation between two topological effects by a relaxation time:[7] the spectral flow of vortex core state transitions and the Berry phase counting far away from the core. To discuss this controversy from a detailed and straightforward approach is one of the main purposes of the present paper.

We now state precisely the physical quantities which we are going to address. In the classical limit, we are looking for an effective equation of motion for a vortex. In two dimensions (2D), or for a straight vortex line in 3D, the equation for a vortex specified by $\mathbf{r}_v$ takes the form of a Langevin equation:

$$m_v\ddot{\mathbf{r}}_v=\mathbf{F}(\mathbf{r}_v,t)-B\dot{\mathbf{r}}_v\times\hat{z}-\eta\dot{\mathbf{r}}_v+\mathbf{f}. \quad (1)$$

Here m_v is the vortex mass, η is the friction coefficient, B is the coefficient for the transverse force with $\hat{z}$ the direction perpendicular to the plane of vortex motion, and the fluctuating force $\mathbf{f}$ related to the friction force by the fluctuation-dissipation theorem. The force $\mathbf{F}$ contains all other forces which are not functions of vortex velocity: the force from the trapping potential, pinnings, the force due to an externally applied supercurrent,[12] the force due to other vortices, etc. We may classify the terms in the above equation into three types, according to the order of time derivatives of $\mathbf{r}_v$:

(i) Forces contain no explicit dependence on any time derivative of vortex coordinate $\mathbf{r}_v$, represented by $\mathbf{F}$. These types of forces may be regarded as conceptually well understood, corresponding to the Born-Oppenheimer potentials, and are not controversial, though practically they can be difficult to evaluate. They are contained in our formulation, but will not be discussed in the present paper.

(ii) Forces have a first-order time derivative of the vortex coordinate, the vortex velocity, represented by the transverse and longitudinal coefficients B and η. Calculating those forces explicitly is the focus of the present paper. It is our purpose to clarify the physical origins behind those forces, starting from a well-defined microscopic theory, the BCS theory, using a well-defined and rigorous procedure, the influence functional method.[13] We will show in detail that the total transverse force is insensitive to details, and is proportional to the superfluid number density, and present calculations leading to finite vortex friction contributions.

(iii) The term contains the second-order time derivative of the vortex coordinate, the vortex mass m_v. This is also an unclear quantity, and the subject of the recent active study.[14,15] Though we believe our present formulation also provides the framework to address the dynamical effects on the vortex mass, we will not explore them here.[16] This term will be ignored by assigning the vortex acceleration $\ddot{\mathbf{r}}_v=0$.

We organize the rest of the paper as follows. In Sec. II the total transverse force is studied from a macroscopic point of view. We first demonstrate from a thermodynamic consideration that the magnitude of the total transverse force should be proportional to the superfluid number density. A reduction from this value will lead to a violation of the second law of thermodynamics. Then we put the transverse force and friction in the context of fluctuation-dissipation theorems, and illustrate that the relaxation time approximation in the microscopic derivations of vortex dynamics should be avoided. In Sec. III we first present a general formulation based on the BCS theory. Then we relate this formulation to that of the influence functional approach which has been proved to be rigorous and effective to calculate friction in quantum dissipative dynamics of a subsystem, where the total system is described by a Hamiltonian. A few general properties of our formulation will be discussed. In Sec. IV we give detailed evaluations of both longitudinal and transverse correlations in the clean limit for arbitrary temperatures, which lead to both the friction and the total transverse force. In particular, detailed evaluations of the total transverse force from either extended state counting or core state transitions are given there, and are explicitly shown to be equivalent. In Sec. V the effects of impurity potentials are considered. We will show that the total transverse force will not be affected. However, impurity potentials strongly affect the core state spectrum, which leads to a contribution to the vortex friction in addition to that of extended states. In Sec. VI we show that the electromagnetic field does not affect the total transverse force and friction, under the condition that the charge neutrality in the superconductor is maintained. In Sec. VII some experimental tests are briefly discussed, and we summarize in Sec. VIII.

II. THERMODYNAMICS AND STATISTICAL MECHANICS

A. Force balance and thermodynamics

The microscopic calculations which we will present later are unavoidably lengthy and technical. It may be helpful to obtain an overall picture and useful information (as much as we can) under general but elementary considerations. In this subsection we give a thermodynamic consideration of vortex dynamics to show that there is a constraint on the total transverse force, and in the next subsection the derivation of vortex dynamics will be put in the context of fluctuation-dissipation theorems.

We may write down a possible equation of motion for a quantized vortex in the absence of impurities, taking into account the possible role of the normal fluid in the motion of the vortex, in the limit of the vortex acceleration equal to zero:

$$h\rho_s\hat{z}\times(\mathbf{u}_v-\mathbf{v}_s)-D(\mathbf{u}_v-\mathbf{v}_n)-D'\hat{z}\times(\mathbf{u}_v-\mathbf{v}_n)+\mathbf{F}^{\rm ext}=0. \quad (2)$$

Here ρ_s is the number density of the superfluid, $\mathbf{u}_v$, $\mathbf{v}_s$, and $\mathbf{v}_n$ are the velocities of vortex, the superfluid, and the normal fluid with respect to the substrate or the wall of the container. Those velocities are independent variables. The velocity dependences are only in first order in Eq. (2). The first term in the left-hand side of Eq. (2) is the Magnus force,[2] whose magnitude is proportional to the superfluid number density. The last term represents a possible external force on the vortex. The other two terms are possible contributions coming from the interaction of the vortex with the normal fluid. Initially, both the normal fluid and superfluid velocities are set to zero.

We will demonstrate that the conditions of force balance and thermodynamics put a constraint on the value of D'. For this purpose let us imagine a torus-shaped tank filled with a superfluid, or a torus-shaped superconductor film. The tank can be considered as a thermal reservoir to the superfluid. This implies that at finite temperatures there is also a normal fluid. After creating a vortex-antivortex pair, we keep the antivortex at rest and move the vortex to wind once with a small velocity u_v around one of the two circumferences of the torus, say L_y, in time $t_{\rm total}$ before the annihilation with the antivortex. We take $t_{\rm total}$ much longer than the relaxation time of the normal fluid such that the normal fluid velocity always stays close to zero, by transferring a possible momentum gained from the vortex motion to the substrate, via the relaxation process represented by the normal fluid viscosity. A physical realization may be the electron-phonon interaction. Hence, the normal fluid velocity is always negligible

comparing with the vortex velocity $\mathbf{u}_v$ which is an order of $u_v \approx L_y/t_{\text{total}}$. As a result of the vortex motion, the momentum of circulating superfluid particles along the torus has been changed from zero to $p_{s,f}=h/L_x$, because of the change of the winding number of the superfluid. This occurs regardless how slow the vortex motion is. The kinetic energy of the superfluid has been changed from zero to $E=\rho_s L_x L_y p_{s,f}^2/2m^*$ for a neutral superfluid, or when the effective magnetic screening length is larger than L_y for the superconductor film. Here m^* is the effective mass of superfluid particles and L_x and L_y are the circumferences of the torus.

The total momentum change of the superfluid requires a force in the transverse direction of the vortex motion

$$F_\perp = \frac{dP}{dt} = \frac{\rho_s L_x L_y p_{s,f}}{t_{\text{total}}} = h\rho_s u_v ,$$

here P is the total momentum of the superfluid. Since the normal fluid velocity stays zero, no kinetic energy can be transferred from the normal fluid to the superfluid. However, if the normal fluid would contribute to this force to superfluid by changing its internal energy, an additional transverse force on the vortex, $-D'\hat{z}\times\mathbf{u}_v$, arises. The magnitude of the external force in the transverse direction of vortex motion should be equal to the total transverse force according to Eq. (2),

$$F_\perp^{\text{ext}} = (h\rho_s - D')u_v .$$

Now we are ready to consider the thermodynamic relations. The process of creating a vortex-antivortex pair and its annihilation after the vortex crossing one circumference L_y of the torus leaves only a finite increase of superfluid circulation in the tank, corresponding to the change of winding number. The initial and final normal fluid velocities are zero. The increase of kinetic energy of the superfluid needs to be provided from somewhere. There are only two possible sources: the external trapping potential and the normal fluid. Here we need to be reminded of a significant difference between the superfluid and the normal fluid: The superfluid carries no entropy, while the normal fluid does. Therefore, according to the second law of thermodynamics,[17] the superfluid cannot gain kinetic energy by lowering the internal energy or entropy of the normal fluid.

We need to consider the work performed on the system by the external force to move this vortex. In the longitudinal direction of vortex motion, the interaction between the normal fluid and the vortex gives rise to a vortex friction $-Du_v$. Thus the external force on the normal fluid in the longitudinal direction is Du_v. This friction does not dissipate energy. Rephrased alternatively, the energy dissipated can be arbitrarily small by taking the time to complete the process arbitrarily long, $t_{\text{total}}\to\infty$. The process is then quasistatic. The normal fluid velocity is always negligible in the process because of its finite viscosity. Thus the external force on the normal fluid in the longitudinal direction of vortex motion does not provide any work to the system. The external force acting on the superfluid will be able to provide enough work for the kinetic energy increase only if $F_\perp^{\text{ext}} \geqslant F_\perp$, i.e., $D' \leqslant 0$, which leads to the conclusion that the magnitude of total transverse force cannot be reduced from that determined by the superfluid number density. The work done by the external force is exactly equal to the kinetic energy increase if the magnitude of the external force in the transverse direction is the product of the superfluid number density, the Planck constant h, and the vortex velocity.

The next question is whether or not the total transverse force on a moving vortex can be larger than that determined by the superfluid number density. If the normal fluid would carry a vortex with a vorticity in the same direction as that of the superfluid, the answer to this question is positive. However, since we have assumed that the normal fluid is viscous, the vortex of the normal fluid will eventually disappear. This is true for a slow process whose time scale is much larger than the relaxation time of the normal fluid assumed here. This consideration leads to that the total transverse force cannot be larger than the value determined by the superfluid number density. Combining with the thermodynamic argument we conclude $D'=0$.

The above discussion has explicitly made use of the assumption of a finite normal fluid viscosity. In case that the normal fluid relaxation time would be infinite, that is, the normal fluid viscosity would be zero, a process which generates a vortex circulation in the superfluid would also generate a vortex circulation in the normal fluid. This would be the limiting situation of a dynamical process in which the internal relaxation time of the normal fluid is much shorter than its relaxation time to the substrate and the time scale for the process is between them. An example would be the creation of vortices by a magnetic flux in an ultraclean superconductor. In such a case, the normal fluid velocity will not relax to zero. Under this ideal condition $-D'=h\rho_n$, corresponding to that the normal fluid has a vortex, which is what has been discussed in Ref. 18.

If impurities are present, a phenomenological equation of motion for the vortex may be written down if the impurities are homogeneously distributed and vary only at a scale much smaller compare with the size of the vortex core. We have two more possible parameters from the vortex-impurity interaction:

$$h\rho_s\hat{z}\times(\mathbf{u}_v-\mathbf{v}_s) - D(\mathbf{u}_v-\mathbf{v}_n) - D'\hat{z}\times(\mathbf{u}_v-\mathbf{v}_n) - d\mathbf{u}_v - d'\hat{z}\times\mathbf{u}_v + \mathbf{F}^{\text{ext}} = 0. \quad (3)$$

Parallel to what we have discussed for the normal fluid case, d' must be zero in order for the external force to provide the energy gain needed by the superfluid. The impurities cannot provide energy to the superfluid either by lowering their internal energy or entropy because of the second law of thermodynamics. We note that impurities introduce another contribution to the normal fluid viscosity.

The microscopic global considerations[5,6] have already suggested $D'=0$ and $d'=0$. The conclusion here will be borne out by detailed and independent microscopic calculations in the following sections.

B. Friction and fluctuation-dissipation theorems

The derivation of the equation of motion for the vortex is different from the usual linear-response theory. In the linear-

response theory, a driving force is given, that is, the Hamiltonian is known, and we look for the average responding velocity. It is a calculation of conductivity or mobility. In the present case the force on a moving vortex is the unknown quantity which we need to find out. The vortex velocity is, however, readily defined through the vortex coordinate. It is a calculation of resistivity or friction. To appreciate this difference, we will examine the different correlation functions involved in these different types of calculations and their relationships. The focus point in this subsection in on the condition for using the relaxation time approximation.

When the normal fluid is at rest, the vortex motion is governed by a classical Langevin equation with parameters to be determined microscopically. This equation has the same form as a classical electron moving in a magnetic field. We adopt the language in transport theory to make it easier to relate to the early work in that field.[19–22]

We start by considering a classical charged particle in a magnetic field obeying a generalized Langevin equation:

$$m\dot{u}_i(t) = -\int_{t_0}^{t} dt'\,\eta_{ik}(t-t')u_k(t') + F_i^{\rm ext}(t) - B\epsilon_{ik}u_k(t) + f_i(t). \tag{4}$$

Here $i=x$ or y, $\mathbf{u}(t)=[u_x(t),u_y(t)]$ is the velocity of the particle, m is the mass, $\mathbf{F}^{\rm ext}(t)=[F_x^{\rm ext}(t),F_y^{\rm ext}(t)]$ is an external force, $\mathbf{f}(t)=[f_x(t),f_y(t)]$ is a random force which simulates the effect of the thermal reservoir. The Einstein convention of the repeated indices as summation has been used. $B\epsilon_{ik}u_k(t)$ represents the transverse force, the Lorentz force $-\mathbf{u}(t)\times\mathbf{B}$ in the Langevin equation with the magnetic field taken along the z direction. The matrix $\eta(t-t')=\{\eta_{ij}\}$ represents friction in both longitudinal and transverse directions of the particle motion. Its possible finite off-diagonal elements will change the effect of the original Lorentz force on the particle. In addition, we have

$$\langle f_i(t)\rangle = 0,$$

$$\langle u_i(t_0)f_j(t_0+t)\rangle = 0, \quad t>0, \tag{5}$$

$$\langle u_i(t_0)u_j(t_0)\rangle = \frac{k_BT}{m}\delta_{ij}.$$

The first equation is obvious: no average fluctuating force. The second one is due to the causality and the last one is due to the equipartition theorem.

If the Hamiltonian of the particle is known, the problem of particle responding to a perturbation can be formulated in two different but equivalent ways. We can calculate the velocity of the particle while the applied force is given. In such a case, it is to obtain a conductivity or mobility formula. The conductivity or mobility may be obtained by the Nakano-Kubo's formula, a calculation of velocity-velocity correlation function. It may also be obtained by solving the Boltzmann equation in the presence of an electric field.[21] Otherwise, we can consider a given velocity for the particle and calculate the applied force needed to maintain this motion. It is to obtain a resistivity or friction formula, i.e., calculating an electric field needed to maintain the given current. The derivation of vortex dynamics belongs to the second kind, where we consider a steady motion of the vortex and calculate the external force acted on the vortex. Unfortunately, we do not have the choice to formulate vortex dynamics in superconductors in terms of conductivity or mobility formula because the effective vortex Hamiltonian is unknown *a priori*.

Introducing a Laplace transform

$$\eta[\omega] = \int_0^\infty dt\, e^{-i\omega t}\eta(t),$$

the mobility is given from Eq. (4) in the limit $t_0\to-\infty$ by

$$\mu[\omega] = (im\omega + \eta[\omega] + i\sigma_y B)^{-1}. \tag{6}$$

Here the mobility $\mu[\omega]$ is defined through

$$\langle \mathbf{u}_i[\omega]\rangle = \mu_{ij}[\omega]\overline{\mathbf{F}}_j^{\rm ext}[\omega],$$

with an applied external force $\overline{\mathbf{F}}^{\rm ext}(t)=\overline{\mathbf{F}}^{\rm ext}[\omega]e^{i\omega t}$.

Defining the velocity-velocity correlation function matrix

$$\mathcal{U}_{ij}(t) = \langle u_i(t_0+t)u_j(t_0)\rangle,$$

with $\mathcal{U}_{ij}(t=0)=\delta_{ij}k_BT/m$ according to Eq. (5), the mobility is related to the velocity-velocity correlation function

$$\mu[\omega] = \frac{\mathcal{U}[\omega]}{k_BT}. \tag{7}$$

This is the ''first'' fluctuation-dissipation theorem described by Kubo,[19] equivalent to the Nakano-Kubo's formula for the electrical conductivity.

It is easy to demonstrate that the relaxation time approximation can be valid in the Nakano-Kubo's formula. Without the thermal reservoir, the velocity-velocity correlation is given by

$$\mathcal{U}[\omega] = (im\omega + i\sigma_y B)^{-1}m\mathcal{U}(0). \tag{8}$$

When using a relaxation time approximation by the standard rule, $i\omega\to i\omega+\eta[\omega]/m$ and substituting it into Eq. (8), we find the velocity-velocity correlation under the relaxation time approximation is given by

$$\mathcal{U}[\omega] = (im\omega + \eta[\omega] + i\sigma_y B)^{-1}k_BT,$$

which is exactly the same as the one obtained by the rigorous calculation, Eqs. (6) and (7). Therefore, the relaxation time approximation can be a valid one for velocity-velocity correlations when used in a conductivity or mobility formula.

The resistivity or friction formula is known to be difficult and it is worthwhile to examine it closely.[19–22] First, we calculate the total force-force correlation function matrix

$$\mathcal{F}_{ij}(t)=m^2\langle \dot{u}_i(t_0+t)\dot{u}_j(t_0)\rangle.$$

Taking the Laplace transform, using the translational invariance in time

$$\langle u_i(t_0+t)\dot{u}_j(t_0)\rangle=-\langle \dot{u}_i(t_0+t)u_j(t_0)\rangle,$$

and the total force-velocity correlation function

$$m\langle \dot{u}_i(t_0+t)u_j(t_0)\rangle[\omega]=-m\mathcal{U}_{ij}(0)+im\omega\mathcal{U}_{ij}[\omega],$$

we have

$$\mathcal{F}[\omega]=\left(-iB\sigma_y+im\omega+\frac{(m\omega)^2}{im\omega+\eta[\omega]+i\sigma_yB}\right)k_BT. \tag{9}$$

In the limit $\omega\to 0$, it is reduced to

$$\mathcal{F}[0]=-iB\sigma_y\ \ k_BT, \tag{10}$$

and is independent of the frictional coefficient η.

The random force-force correlation matrix is defined as

$$\mathcal{R}_{ij}(t)=\langle f_i(t_0+t)f_j(t_0)\rangle. \tag{11}$$

From the Langevin equation, Eq. (4), we can express $\mathcal{R}(t)$ in terms of total force-force, total force-velocity, and velocity-velocity correlation functions. Taking the Laplace transform and integrating by part, we obtain

$$\mathcal{R}[\omega]=\eta[\omega]m\mathcal{U}(0)=\eta[\omega]k_BT, \tag{12}$$

or $\eta(t)=\mathcal{R}(t)/(k_BT)$. This is the ''second'' fluctuation-dissipation theorem described by Kubo.[19] We emphasize that the generalized frictional coefficient $\eta(t)$ is directly given by the random force-force correlation. The frictional coefficient matrix $\eta(t)$ has no off-diagonal part if the random force-force correlation matrix has not. This fluctuation-dissipation theorem allows us to obtain some general properties of the generalized friction. For example, for a charged particle described by a single relaxation time in the Boltzmann equation moving in a magnetic field, there will be no frictional effect on the force in the transverse direction.

Next we consider that the particle is moving at a given velocity $\overline{\mathbf{u}}(t)$ and find out what is the external force needed to sustain such a motion. This is exactly the situation which we encounter in vortex dynamics and it is equivalent to the calculation of resistivity or friction. From Eq. (4), the average force is given by

$$\langle F_i^{\rm ext}[\omega]\rangle=(im\omega+\eta[\omega]+i\sigma_yB)_{ij}\overline{u_j}[\omega], \tag{13}$$

which is trivially identical to the reciprocal of conductivity formula. Obviously, this process does not provide us an independent way of calculating resistivity.

However, if we are only interested in the average force $\langle\mathbf{F}^{\rm ext}\rangle$ in the dc limit, we do have an alternative resistivity formula. After taking $\omega\to 0$ and using Eqs. (10) and (12), Eq. (13) gives

$$\langle F_i^{\rm ext}\rangle[0]=\frac{1}{k_BT}(\mathcal{R}_{ij}[0]-\mathcal{F}_{ij}[0])\overline{u_j}[0]. \tag{14}$$

Taking η to be a scalar, the external force can be written into a more suggestive form,

$$\langle\mathbf{F}^{\rm ext}[0]\rangle=\eta[0]\overline{\mathbf{u}}[0]+\overline{\mathbf{u}}[0]\times\mathbf{B}, \tag{15}$$

where the longitudinal component depends on $\mathcal{R}[0]$, the random force-force correlation function, and the transverse component only on $\mathcal{F}[0]$, the total force-force correlation function. Equation (14) is the dc resistivity formula. It provides a direct way to obtain dc resistivity from force correlation functions. The straightforward interpretation of Eq. (15) is the force balance: The externally applied force to keep the constant velocity is equal in magnitude but opposite in sign to the sum of the frictional and the Lorentz forces.

We will show that the relaxation time approximation is invalid when used in the force calculation. We start with the force-force correlations. Without thermal reservoir, the random force correlation is zero, that is, $\mathcal{R}(t)=0$. If we switch on the effect of a thermal reservoir by using a relaxation time approximation $i\omega\to i\omega+\eta[\omega]/m$, the random force correlation is still incorrectly set to zero. This shows that the relaxation time approximation cannot be used to calculate the random force correlation.

The total force correlation without thermal reservoir is

$$\mathcal{F}_{ij}[\omega]=\left(-iB\sigma_y+im\omega+\frac{(m\omega)^2}{im\omega+i\sigma_yB}\right)_{ik}m\mathcal{U}_{kj}(0). \tag{16}$$

When we switch on the thermal reservoir by using a relaxation time approximation $i\omega\to i\omega+\eta[\omega]/m$ in Eq. (16), we have

$$\mathcal{F}[\omega]=\left(-iB\sigma_y+im\omega+\eta[\omega]-\frac{(im\omega+\eta[\omega])^2}{im\omega+\eta[\omega]+i\sigma_yB}\right)k_BT.$$

This is a rather complicated expression. In the limit $\omega\ll\eta[\omega]$, or $\omega\tau\ll 1$, we can simplify it to

$$\mathcal{F}[0]=\frac{B}{1+(\omega_0\tau)^2}[\omega_0\tau-i\sigma_y(\omega_0\tau)^2]k_BT \tag{17}$$

with $\omega_0=B/m$. Here $\tau=m/\eta[0]$ is a relaxation time. Let us use the resistivity formula Eq. (14) to calculate the external force needed to keep the particle moving with a given velocity. With $\mathcal{R}[\omega]=0$ and $\mathcal{F}[0]$ given by Eq. (17), the external force is

$$\langle \mathbf{F}^{\rm ext}[0]\rangle = \frac{\omega_0\tau}{1+(\omega_0\tau)^2}(-B\overline{\mathbf{u}}[0]+\omega_0\tau\overline{\mathbf{u}}[0]\times\mathbf{B}). \quad (18)$$

These results have no connection at all to the rigorous results shown in Eq. (15). Even the sign for the longitudinal force is wrong. Evidently, the relaxation time approximation cannot be valid in such a calculation, because such an approximation in the force balance equation cannot consider the random force properly, and leads to results violating the fluctuation-dissipation theorems.

By the simple and exactly solvable model, we have demonstrated the essential conditions for the validity of the relaxation time approximation in velocity-velocity correlation function calculations, and for its invalidity in force-force correlation function calculations. We refer readers to Refs. 19–22 for more sophisticated discussions in the context of the Green's function or Boltzmann equation.

With a redefinition of constants $B=h\rho_s$ with h being the Planck constant, ρ_s being the superfluid particle number density, and $\omega_0=\epsilon_0$ as the core-level spacing, this force becomes the same in magnitude as the one which appeared in the derivation of vortex dynamics using the relaxation time approximation,[7–9,11] including the same sign error.[10] The inappropriate use of the relaxation time approximation in vortex dynamics in d-wave superconductors has also been pointed out recently.[23] In the following we show how to obtain the vortex friction without the relaxation time approximation, and demonstrate at the same time that the total transverse force remains unchanged as dictated by the topology.

III. VORTEX DYNAMICS IN HOMOGENEOUS BCS SUPERFLUID

A. Formulation of the problem

We present now our microscopic derivation, from the standard BCS Lagrangian for s-wave pairing in the imaginary time path-integral formulation of the influence functional method. The connection of the total transverse force to the Berry phase is straightforward in this formulation. We believe the present formulation has some advantages: a transparent crossover from the quantum to the classical description via the semiclassical approximation, and a flexible treatment of the general dissipative effect arising from the integration out of irrelevant degrees of freedom, fermionic quasiparticles, and holes. The relevant degree of freedom is the vortex coordinate.[24]

We consider a neutral fermionic superfluid first. The coupling to electromagnetic fields will be discussed later. The Lagrangian is given by

$$L_{\rm BCS}=\sum_\sigma \psi_\sigma^\dagger(x,\tau)\left(\hbar\partial_\tau-\mu_F-\frac{\hbar^2}{2m}\nabla^2+V(x)+U_0(x-x_\nu)\right)\psi_\sigma(x,\tau) - g\psi_\uparrow^\dagger(x,\tau)\psi_\downarrow^\dagger(x,\tau)\psi_\downarrow(x,\tau)\psi_\uparrow(x,\tau), \quad (19)$$

where ψ_σ describes electrons with spin $\sigma=(\uparrow,\downarrow)$, μ_F is the chemical potential determined by the electron number density, $V(x)$ is the impurity potential, U_0 is the trapping potential, and $x=(x,y,z)$. A vortex at x_ν has been assumed through the trapping potential. A more explicit implementation of the vortex coordinate will be discussed after Eq. (39). The partition function is

$$Z=\int \mathcal{D}\{x_\nu,\psi^\dagger,\psi\}\exp\left\{-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\int d^3x L_{\rm BCS}\right\}, \quad (20)$$

with $\beta=1/k_BT$, and $d^3x=dxdydz$. Inserting the identity in the functional space,

$$1=\int \mathcal{D}\{\Delta^*,\Delta\} \times\exp\left\{-\frac{g}{\hbar}\int_0^{\hbar\beta}d\tau\int d^3x\left|\psi_\downarrow\psi_\uparrow+\frac{1}{g}\Delta(x,\tau)\right|^2\right\},$$

into Eq. (20) we have

$$Z=\int \mathcal{D}\{x_\nu,\psi^\dagger,\psi,\Delta^*,\Delta\}\exp\left\{-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\int d^3x(\psi_\uparrow^\dagger,\psi_\downarrow) \times(\hbar\partial\tau+\mathcal{H})\begin{pmatrix}\psi_\uparrow\\ \psi_\downarrow^\dagger\end{pmatrix}-\frac{1}{\hbar g}\int_0^{\hbar\beta}d\tau\int d^3x|\Delta|^2\right\}. \quad (21)$$

Here the Hamiltonian is defined as

$$\mathcal{H}(\Delta,\Delta^*)=\begin{pmatrix}H & \Delta\\ \Delta^* & -H^*\end{pmatrix} \quad (22)$$

with $H=-(\hbar^2/2m)\nabla^2-\mu_F+V(x)+U_0(x-x_\nu)$.

Exactly integrating out the electron fields $\psi_\sigma^\dagger$ and ψ_σ first, then integrating out the auxiliary (pair) fields Δ under the mean-field approximation, one obtains the partition function for the vortex

$$Z=\int \mathcal{D}\{x_\nu\}\exp\left\{-\frac{S_{\rm eff}}{\hbar}\right\} \quad (23)$$

with the effective action

$$\frac{S_{\rm eff}}{\hbar}=-\,{\rm Tr}\ln G^{-1}+\frac{1}{\hbar g}\int_0^{\hbar\beta}d\tau\int d^3x|\Delta|^2, \quad (24)$$

where Tr includes internal and space-time indices, and the Nambu-Gor'kov Green's function G is defined by

$$(\hbar\partial_\tau+\mathcal{H})G(x,\tau;x',\tau')=\delta(\tau-\tau')\delta^3(x-x'), \quad (25)$$

together with the BCS gap equation, or the self-consistent equation,

$$\Delta(x,\tau)=\hbar g G_{12}(x,\tau;x,\tau). \tag{26}$$

In the presence of impurity potentials, the averaging over them is implied in Eq. (26), unless explicitly specified.

Since the effective action is a smooth function of vortex coordinate in the functional space of $\{x_\nu(\tau)\}$, we consider that the vortex has made a small move from its original place x_0, which allows a small parameter expansion in terms of the difference between the vortex position x_ν and x_0. We look for the long-time behavior of vortex dynamics under this small parameter expansion. In the final step, the forces on vortex are to be calculated by varying Lagrangian to this small motion. The influence of the eliminated degrees of freedom on the vortex dynamics will then be obtained. As an example, for the mean-field value of the order parameter, this small parameter expansion to the second order is

$$\Delta(x,\tau,x_\nu)=\left(1+\delta x_\nu(\tau)\cdot\nabla_{x_0}+\frac{1}{2}[\delta x_\nu(\tau)\cdot\nabla_{x_0}]^2\right)\Delta_0(x,x_0). \tag{27}$$

Here $\delta x_\nu=x_\nu-x_0$. In Eq. (27) we have used the fact that when $x_\nu=x_0$ the vortex is static. The effective action for the vortex to the same order is, after dropping a constant term,

$$\frac{S_{\rm eff}}{\hbar}=\frac{1}{2}\mathrm{Tr}(G_0\Sigma')^2+\frac{1}{\hbar g}\int_0^{\hbar\beta}d\tau\int d^3x \times\delta x_\nu\cdot\nabla_{x_0}\Delta_0^*\,\delta x_\nu\cdot\nabla_{x_0}\Delta_0, \tag{28}$$

with

$$\Sigma'=\delta x_\nu\cdot\nabla_{x_0}\begin{pmatrix}U_0 & \Delta_0\\ \Delta_0^* & -U_0\end{pmatrix}=\delta x_\nu\cdot\nabla_{x_0}\mathcal{H}_0. \tag{29}$$

Here the Hamiltonian $\mathcal{H}_0=\mathcal{H}|_{x_\nu=x_0}$ for the static vortex at x_0, G_0 is the Nambu-Gor'kov Green's function with $\mathcal{H}$ replaced by $\mathcal{H}_0$, the gradient ∇_{x_0} is with respect to x_0, and $G^{-1}=G_0^{-1}+\Sigma'$.

Now we construct the Nambu-Gor'kov Green's function G_0 following the usual procedure.[25] First, we consider the eigenfunctions of $\mathcal{H}_0$. The stationary equation, the Bogoliubov–de Gennes equation, is

$$\mathcal{H}_0\Psi_\alpha(x)=E_\alpha\Psi_\alpha(x), \tag{30}$$

with

$$\Psi_\alpha(x)=\begin{pmatrix}u_\alpha(x)\\ v_\alpha(x)\end{pmatrix}.$$

No confusion with the vortex velocity in Sec. II should arise here.

Given the eigenfunctions of Eq. (30), G_0 can be expressed as

$$G_0(x,\tau;x',\tau')=\sum_{n,\alpha}\frac{-1}{\hbar\beta}\frac{e^{-i\omega_n(\tau-\tau')}}{i\hbar\omega_n-E_\alpha}\Psi_\alpha(x)\Psi_\alpha^\dagger(x'). \tag{31}$$

Here $\omega_n=n\pi/\hbar\beta$, with n odd integers.

Direct substituting of Eq. (31) into Eq. (28) leads to

$$\begin{aligned}\frac{S_{\rm eff}}{\hbar}=&\frac{1}{2(\hbar\beta)^2}\int d^3x d^3x'd\tau d\tau'\\ &\times\sum_{n\alpha,n'\alpha'}\frac{e^{-i\omega_n(\tau-\tau')}}{i\hbar\omega_n-E_\alpha}\frac{e^{-i\omega_{n'}(\tau'-\tau)}}{i\hbar\omega_{n'}-E_{\alpha'}}\\ &\times\Psi_\alpha^\dagger(x')\,\delta x_\nu(\tau')\cdot\nabla_{x_0}\mathcal{H}_0(x')\Psi_{\alpha'}(x')\Psi_{\alpha'}^\dagger(x)\\ &\times\delta x_\nu(\tau)\cdot\nabla_{x_0}\mathcal{H}_0(x)\Psi_\alpha(x)\\ &+\frac{1}{\hbar g}\int_0^{\hbar\beta}d\tau\int d^3x\,\delta x_\nu(\tau)\cdot\nabla_{x_0}\Delta_0^*\,\delta x_\nu(\tau)\cdot\nabla_{x_0}\Delta_0.\end{aligned} \tag{32}$$

Keeping only terms relevant to vortex dynamics and assuming global rotational symmetry after summing over all the states, we have

$$\begin{aligned}\frac{S_{\rm eff}}{\hbar}=&\frac{1}{2(\hbar\beta)^2}\int d^3x d^3x'd\tau d\tau'\\ &\times\sum_{n\alpha,n'\alpha'}\frac{e^{-i\omega_n(\tau-\tau')}}{i\hbar\omega_n-E_\alpha}\frac{e^{-i\omega_{n'}(\tau'-\tau)}}{i\hbar\omega_{n'}-E_{\alpha'}}\\ &\times[\Psi_\alpha^\dagger(x')\nabla_0\mathcal{H}_0\Psi_{\alpha'}(x')\cdot\Psi_{\alpha'}^\dagger(x)\nabla_{x_0}\mathcal{H}_0\Psi_\alpha(x)\\ &\times\delta x_\nu(\tau')\cdot\delta x_\nu(\tau)+(\Psi_\alpha^\dagger(x')\nabla_{x_0}\mathcal{H}_0\Psi_{\alpha'}(x')\Psi_{\alpha'}^\dagger(x)\\ &\times\nabla_{x_0}\mathcal{H}_0\Psi_\alpha(x))\cdot\hat z(\delta x_\nu(\tau')\times\delta x_\nu(\tau))\cdot\hat z].\end{aligned} \tag{33}$$

With a rearrangement, finally we arrive at

$$\begin{aligned}S_{\rm eff}=&\frac{1}{2}\int_0^{\hbar\beta}d\tau\left\{\int_0^{\hbar\beta}d\tau' F_\parallel(\tau-\tau')|\delta x_\nu(\tau)-\delta x_\nu(\tau')|^2\right.\\ &\left.-\int_0^{\hbar\beta}d\tau' F_\perp(\tau-\tau')(\delta x_\nu(\tau)\times\delta x_\nu(\tau'))\cdot\hat z\right\}\end{aligned} \tag{34}$$

with

$$\begin{aligned}F_\parallel(\tau-\tau')=&-\frac{1}{2\hbar\beta^2}\int d^3x d^3x'\\ &\times\sum_{n\alpha,n'\alpha'}\frac{e^{-i\omega_n(\tau-\tau')}}{i\hbar\omega_n-E_\alpha}\frac{e^{-i\omega_{n'}(\tau'-\tau)}}{i\hbar\omega_{n'}-E_{\alpha'}}\\ &\times\Psi_\alpha^\dagger(x')\nabla_{x_0}\mathcal{H}_0\Psi_{\alpha'}(x')\cdot\Psi_{\alpha'}^\dagger(x)\nabla_{x_0}\mathcal{H}_0\Psi_\alpha(x),\end{aligned} \tag{35}$$

and

$$F_{\perp}(\tau-\tau')=\frac{1}{\hbar\beta^2}\int d^3x d^3x' \times\sum_{n\alpha,n'\alpha'}\frac{e^{-i\omega_n(\tau-\tau')}}{i\hbar\omega_n-E_\alpha}\frac{e^{-i\omega_{n'}(\tau'-\tau)}}{i\hbar\omega_{n'}-E_{\alpha'}} \times[\Psi^\dagger_\alpha(x')\nabla_{x_0}\mathcal{H}_0\Psi_{\alpha'}(x') \times\Psi^\dagger_{\alpha'}(x)\nabla_{x_0}\mathcal{H}_0\Psi_\alpha(x)]\cdot\hat{z}. \quad (36)$$

Equation (34) has the form of influence functional in quantum dissipative dynamics.[13] Please note that $\int_0^{\hbar\beta}F_\parallel(\tau-\tau')d\tau=0$. Therefore, there is no so-called ''counterterm'' in Eq. (33) as discussed in Ref. 13. Here we have generalized the influence functional to include the transverse force as a response from the environment.

Before proceeding to evaluate these correlations, we discuss some properties of the wave functions of the Bogoliubov–de Gennes equation, which will be used later. First, because $\mathcal{H}_0$ is Hermitian, all its eigenstates form a complete and orthonormal set, that is,

$$\int d^3x\Psi^\dagger_\alpha(x)\Psi_{\alpha'}(x)=\delta_{\alpha,\alpha'}$$

and

$$\sum_\alpha \Psi_\alpha(x)\Psi^\dagger_\alpha(x')=\mathbf{1}.$$

Here $\Psi^\dagger(x)=[u^*(x),v^*(x)]$ and the wave function $\Psi(x)$ is normalized to 1 over a cylinder of radius R and length L, the box normalization. In the thermodynamic limit, $R=\infty$, one may consider the scattering states. In this case the Dirac δ function normalization for extended states should be the better choice. Furthermore, Eq. (30) has the property that if

$$\mathcal{H}_0\Psi(x)=E\Psi(x),\quad \overline{\Psi}(x)=\begin{pmatrix} v^*(x)\\ -u^*(x)\end{pmatrix},$$

then

$$\mathcal{H}_0\overline{\Psi}(x)=-E\overline{\Psi}(x). \quad (37)$$

There is no specific assumption about the Hamiltonian $H(H^*)$ in Eq. (22) for this identity. There is another important property implied by Eq. (30). Since both the Hamiltonian $\mathcal{H}_0$ and its eigenfunctions are the function of the vortex coordinate at x_0, taking the derivative with respect to x_0 at both sides of Eq. (30), we have

$$(\nabla_{x_0}\mathcal{H})|\Psi_{\alpha'}\rangle+\mathcal{H}|\nabla_{x_0}\Psi_{\alpha'}\rangle=E_{\alpha'}|\nabla_{x_0}\Psi_{\alpha'}\rangle.$$

Multiplying both sides of this equation by $\langle\Psi_\alpha|$, and using the relation that $\langle\Psi_\alpha|\mathcal{H}=E_\alpha\langle\Psi_\alpha|$, the Hermitian conjugation of Eq. (30), for $\alpha\neq\alpha'$ we have

$$\int d^3x\Psi^\dagger_\alpha(x)(\nabla_{x_0}\mathcal{H}_0)\Psi_{\alpha'}(x) =(E_{\alpha'}-E_\alpha)\int d^3x\Psi^\dagger_\alpha(x)\nabla_{x_0}\Psi_{\alpha'}(x) \quad (38)$$

with

$$(\nabla_{x_0}\mathcal{H}_0)\equiv\nabla_{x_0}\begin{pmatrix}U_0 & \Delta\\ \Delta^* & -U_0\end{pmatrix}.$$

Here we have used $\nabla_{x_0}E_{\alpha'}=0$ to get Eq. (38), under the assumption that the system is homogeneous. Hence, there is no vortex velocity-independent potential for the vortex arising from Eq. (38), that is, no Born-Oppenheimer-type potential, in accordance with the present purpose of looking for the effects which are first order in vortex velocity. Starting from the Hermitian conjugate of Eq. (30), taking the derivative with respect to the vortex coordinate we have

$$\langle\nabla_{x_0}\Psi_\alpha|\mathcal{H}+\langle\Psi_\alpha|(\nabla_{x_0}\mathcal{H})=E_\alpha\langle\nabla_{x_0}\Psi_\alpha|.$$

Then multiplying this equation by $|\Psi_{\alpha'}\rangle$ we have

$$\int d^3x\Psi^\dagger_\alpha(x)(\nabla_{x_0}\mathcal{H}_0)\Psi_{\alpha'}(x) =-(E_{\alpha'}-E_\alpha)\int d^3x\nabla_{x_0}\Psi^\dagger_\alpha(x)\Psi_{\alpha'}(x). \quad (39)$$

We note that both Eqs. (38) and (39) are exact, following from the general property of Eq. (30). They relate the transition elements of the Hamiltonian after the differentiation with respect to a parameter to the connections between wave functions. Though the wave functions have to be determined as an eigenvalue problem, the usefulness of Eqs. (38) and (39) is that it allows one to concentrate on wave functions instead of the original Hamiltonian, which is particularly convenient in the discussion of certain topological properties described better by wave functions, such as a vortex in a BCS superfluid here. In the rest of the paper, we will take the trapping potential to be zero, $U_0\to 0$ unless specified, and determine the vortex position self-consistently through the gap equation, Eq. (26).

For the convenience of calculation, sometimes we wish to use $\nabla\Psi_\alpha(x)$ instead of $\nabla_{x_0}\Psi_\alpha(x)$ in the expression. It can be done in the following way. We split the gap function, or the order parameter Δ into

$$\Delta=\overline{\Delta}(x-x_0)+\Delta'(x,x_0),$$

where $\overline{\Delta}$ is a smooth part of the self-consistent potential, Δ' is the fluctuating part for a given impurity configuration. The

impurity average gives $\langle \Delta' \rangle = 0$. In the presence of impurity potentials the gap function $\overline{\Delta}$ may differ from the one in the clean limit. The Hamiltonian becomes $\mathcal{H}_0 = \overline{\mathcal{H}}_0 + \delta\mathcal{H}$, with

$$\overline{\mathcal{H}}_0 = \begin{pmatrix} H_0 & \overline{\Delta} \\ \overline{\Delta}^* & -H_0^* \end{pmatrix}, \tag{40}$$

where $H_0 = -(\hbar^2/2m)\nabla^2 - \mu_F$, and

$$\delta\mathcal{H} = \begin{pmatrix} V(x) & \Delta' \\ \Delta'^* & -V(x) \end{pmatrix}. \tag{41}$$

Using

$$\int d^3x \Psi_\alpha^\dagger(x) \nabla (\mathcal{H}_0 \Psi_{\alpha'}(x)) = \int d^3x [\Psi_\alpha^\dagger(x)(\nabla \mathcal{H}_0)\Psi_{\alpha'}(x) + \Psi_\alpha^\dagger(x)\mathcal{H}_0 \nabla \Psi_{\alpha'}(x)],$$

and $(\nabla + \nabla_{x_0})\overline{\mathcal{H}}_0 = 0$, and defining

$$\nabla \mathcal{H}' \equiv (\nabla + \nabla_{x_0})\delta\mathcal{H},$$

we have the desired relations

$$\begin{aligned}(E_{\alpha'} - E_\alpha) \int d^3x \Psi_\alpha^\dagger(x) \nabla_{x_0} \Psi_{\alpha'}(x) &= -(E_{\alpha'} - E_\alpha) \int d^3x \Psi_\alpha^\dagger(x) \nabla \Psi_{\alpha'}(x) \\ &\quad + \int d^3x \Psi_\alpha^\dagger(x) \nabla \mathcal{H}' \Psi_{\alpha'}(x)\end{aligned} \tag{42}$$

and

$$\begin{aligned}(E_{\alpha'} - E_\alpha) \int d^3x \nabla_{x_0} \Psi_\alpha^\dagger(x) \Psi_{\alpha'}(x) &= -(E_{\alpha'} - E_\alpha) \int d^3x \nabla \Psi_\alpha^\dagger(x) \Psi_{\alpha'}(x) \\ &\quad - \int d^3x \Psi_\alpha^\dagger(x) \nabla \mathcal{H}' \Psi_{\alpha'}(x).\end{aligned} \tag{43}$$

The last part is obtained from Eq. (42) by a partial integration, which can be carried through because the wave function is normalizable, either by the box normalization or by the Dirac delta function.

B. Longitudinal correlation

We now discuss the general properties of the longitudinal correlation function, Eq. (35). We find

$$\begin{aligned}\sum_{n,n'} \frac{e^{-i\omega_n(\tau-\tau')}}{i\hbar\omega_n - E_\alpha} \frac{e^{-i\omega_{n'}(\tau'-\tau)}}{i\hbar\omega_{n'} - E_{\alpha'}} &= \sum_{n,n'} \frac{e^{-i(\omega_n-\omega_{n'})(\tau-\tau')}}{i(\hbar\omega_n - \hbar\omega_{n'}) - (E_\alpha - E_{\alpha'})} \times \left(\frac{1}{i\hbar\omega_{n'} - E_{\alpha'}} - \frac{1}{i\hbar\omega_n - E_\alpha} \right) \\ &= \sum_{n-n'} \beta(f_{\alpha'} - f_\alpha) \frac{e^{-i(\omega_n-\omega_{n'})(\tau-\tau')}}{i(\hbar\omega_n - \hbar\omega_{n'}) - (E_\alpha - E_{\alpha'})},\end{aligned} \tag{44}$$

after using

$$\sum_n \frac{e^{-i\omega_n\delta}}{i\hbar\omega_n - E_\alpha} = \begin{cases} \beta f_\alpha, & \delta = 0^- \\ -\beta(1-f_\alpha), & \delta = 0^+ \end{cases}$$

with the Fermi distribution function $f_\alpha = 1/(1+e^{\beta E_\alpha})$. To complete the calculation, we also need

$$\sum_{n-n'} \frac{\cos[(\omega_n - \omega_{n'})(\tau-\tau')]}{i(\hbar\omega_n - \hbar\omega_{n'}) - (E_\alpha - E_{\alpha'})} = -\frac{\beta}{2} \frac{\cosh[(E_\alpha - E_{\alpha'})/\hbar(\hbar\beta/2 - |\tau-\tau'|)]}{\sinh[(E_\alpha - E_{\alpha'})\beta/2]} + :\delta(\tau-\tau):. \tag{45}$$

Here $:\delta(\tau):$ is a periodic delta function with period $\hbar\beta$. The term with $\Sigma_{n-n'} \sin[(\omega_n - \omega_{n'})(\tau-\tau')]/[i(\hbar\omega_n - \hbar\omega_{n'}) - (E_\alpha - E_{\alpha'})]$ is zero inside the double imaginary time integration in Eq. (34), because the integrand is an odd function of $\tau - \tau'$. Dropping the periodic δ function, whose contribution is zero in Eq. (34), we are ready then to write down the longitudinal correlation function as

$$F_\parallel(\tau) = \frac{1}{\pi} \int_0^\infty d\omega J(\omega) \frac{\cosh[\omega(\hbar\beta/2 - |\tau|)]}{\sinh[\omega(\hbar\beta/2)]} \tag{46}$$

with the spectral function

$$J(\omega) = \frac{\pi}{4} \sum_{\alpha,\alpha'} \delta(\hbar\omega - |E_\alpha - E_{\alpha'}|) |f_{\alpha'} - f_\alpha| \times \left| \int d^3x \Psi_\alpha^\dagger(x) \nabla_{x_0} \mathcal{H}_0 \Psi_{\alpha'}(x) \right|^2. \tag{47}$$

It is interesting to point out that in terms of the spectral function the longitudinal correlation function, Eq. (46), is in exactly the same form of the influence functional in quantum dissipative dynamics.[13] The apparent difference is that the spectral function in Ref. 13 has been obtained by integrating out a set of independent harmonic oscillators (bosons), while

here it has come from the elimination of independent fermionic modes determined by the Bogoliubov–de Gennes equation.

It is important to remember that in order to have a smooth spectral function $J(\omega)$, the thermodynamic limit must be taken first before the implementation of the δ function in Eq. (47).[26] Otherwise the spectral function consists of sum of discrete δ functions, and there would be no dissipation. This limit procedure is in accordance with the requirement in nonequilibrium statistical mechanics: The thermodynamic limit must be taken first to have well-defined low-lying modes in the zero-frequency limit. After this consideration of the thermodynamic limit, in the low-frequency limit the spectral function may have the following generic form:

$$J(\omega)=\eta\omega^{s}, \quad \omega\rightarrow 0^{+}, \tag{48}$$

with $s>1$ being the super-Ohmic case, $1>s>0$ being the sub-Ohmic case, and $s=1$ being the Ohmic case, following from the influence functional formulation of quantum dissipative dynamics.[13] For the physically important Ohmic case, the longitudinal force, friction, is given by $-\eta\mathbf{v}_V$, and from Eq. (47) we have the frictional coefficient

$$\eta=\frac{\pi}{4}\sum_{\alpha'\neq\alpha}\hbar\frac{f_\alpha-f_{\alpha'}}{E_{\alpha'}-E_\alpha}\times\delta(0^{+}-|E_\alpha-E_{\alpha'}|)|\langle\Psi_\alpha|\nabla_{x_0}\mathcal{H}_0|\Psi_{\alpha'}\rangle|^2. \tag{49}$$

This equation is the familiar Fermi Golden rule for dissipation. The matrix elements of $\nabla_{x_0}\mathcal{H}_0$ are well behaved. If we use Eqs. (38) and (39) to re-express the frictional coefficient η in terms of the overlap integral between the wave functions $|\nabla_{x_0}\Psi_\alpha\rangle$ and $|\Psi_{\alpha'}\rangle$, we turn it into the form of ratio 0/0 when $E_\alpha-E_{\alpha'}\rightarrow 0$. Then attention should be paid to the divergence of the overlap integral when $|E_\alpha-E_{\alpha'}|\rightarrow 0^{+}$. This limiting behavior has been discussed in Ref. 27, and we refer the reader to the Appendix for a detailed discussion. Equation (49) clearly shows that the coefficient of friction η is determined by low-energy excitations such as phonons, extended quasiparticles, and bounded core quasiparticles when their energy spectrum is smeared out by impurities. The equivalence of Eq. (49) in the context of vortex dynamics to a more conventional partial wave phase-shift analysis has been discussed in Ref. 28 for a few well-defined situations. A more formal discussion can be found in Ref. 27. It may also be instructive to mention here that the friction experienced by a moving object in a normal Fermi liquid has been analyzed in the influence functional approach.[29] Those considerations suggests that nonzero extended states friction contributions exist, as will be borne out in detail in the next section. Finally, it should also be pointed out that Eq. (49) is a special case of Eq. (47). It will not pick up any super-Ohmic contributions, and will give infinity for any sub-Ohmic contributions. If such cases occur, we need to return to the general expressions, Eqs. (46) and (47).

To close this subsection, there are two general remarks in order. First, the present result of expressing the frictional coefficient in terms of low-lying excitations is in accordance with Landau's quasiparticle picture: In the zero-frequency limit the lifetime of those excitations approaches infinity. Those excitations give an exact description of the dynamics of the whole system in this limit. Secondly, to relate to the discussion in Sec. II B, the spectral function given by Eq. (47) completely determines the spectral representation of the second kind of fluctuation-dissipation theorems. Equation (47) is indeed a quantitative description of dissipation.

C. Transverse correlation

To obtain physically more transparent expressions for the transverse correlation function in Eq. (36), we use Eqs. (38) and (39) to rewrite it as

$$F_\perp(\tau-\tau')=\frac{1}{\hbar\beta^2}\int d^3x d^3x'\times\sum_{n\alpha,n'\alpha'}\frac{e^{-i\omega_n(\tau-\tau')}}{i\hbar\omega_n-E_\alpha}\frac{e^{-i\omega_{n'}(\tau'-\tau)}}{i\hbar\omega_{n'}-E_{\alpha'}}\times(E_\alpha-E_{\alpha'})^2[(\Psi^\dagger_\alpha(x')\nabla_{x_0}\Psi_{\alpha'}(x'))\times(\nabla_{x_0}\Psi^\dagger_{\alpha'}(x)\Psi_\alpha(x))]\cdot\hat{z}. \tag{50}$$

According to Eqs. (44) and (45),

$$\begin{aligned}\sum_{n,n'}\frac{e^{-i(\omega_n-\omega_{n'})(\tau-\tau')}}{(i\hbar\omega_n-E_\alpha)(i\hbar\omega_{n'}-E_{\alpha'})}&=\beta\sum_n\frac{e^{-i\tilde{\omega}_n(\tau-\tau')}}{i\hbar\tilde{\omega}_n-(E_\alpha-E_{\alpha'})}(f_{\alpha'}-f_\alpha)\\&=\frac{\beta^2}{2}\left[-1+\frac{\hbar}{E_\alpha-E_{\alpha'}}\partial_{\tau-\tau'}\right]\times\frac{\cosh[(E_\alpha-E_{\alpha'})/\hbar(\hbar\beta/2-|\tau-\tau'|)]}{\sinh[(E_\alpha-E_{\alpha'})\beta/2]}(f_{\alpha'}-f_\alpha).\end{aligned}$$

Because of the symmetry with the interchange of α and α', the -1 term in the above square brackets does not contribute to the transverse correlation function, and we have

$$F_\perp(\tau-\tau')=\frac{1}{2}\int d^3x d^3x'\sum_{\alpha,\alpha'}\partial_{\tau-\tau'}\times\frac{\cosh[(E_\alpha-E_{\alpha'})/\hbar(\hbar\beta/2-|\tau-\tau'|)]}{\sinh[(E_\alpha-E_{\alpha'})\beta/2]}\times(E_\alpha-E_{\alpha'})(f_{\alpha'}-f_\alpha)[(\Psi^\dagger_\alpha(x')\nabla_{x_0}\Psi_{\alpha'}(x'))\times((\nabla_{x_0}\Psi^\dagger_{\alpha'}(x)\Psi_\alpha(x))]\cdot\hat{z}.$$

The corresponding term in the effective action, Eq. (34), is then

$$-\frac{1}{2}\int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' F_\perp(\tau-\tau')(\delta x_\nu(\tau)\times\delta x_\nu(\tau'))\cdot\hat{z}$$

$$=-\frac{1}{4}\int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' \int d^3x d^3x' \sum_{\alpha,\alpha'} \partial_{\tau-\tau'} \frac{\cosh[(E_\alpha-E_{\alpha'})/\hbar(\hbar\beta/2-|\tau-\tau'|)]}{\sinh[(E_\alpha-E_{\alpha'})\beta/2]}(E_\alpha-E_{\alpha'})(f_{\alpha'}-f_\alpha)$$
$$\times[(\Psi^\dagger_\alpha(x')\nabla_{x_0}\Psi_{\alpha'}(x'))\times(\nabla_{x_0}\Psi^\dagger_{\alpha'}(x)\Psi_\alpha(x))]\cdot\hat{z}(\delta x_\nu(\tau)\times\delta x_\nu(\tau'))\cdot\hat{z}$$

$$=-\frac{1}{4}\int_0^{\hbar\beta} d\tau \int_{-\infty}^{\infty} d\tau' \int d^3x d^3x' \sum_{\alpha,\alpha'} \partial_{\tau-\tau'} \exp\left\{-\frac{|E_\alpha-E_{\alpha'}|}{\hbar}|\tau-\tau'|\right\}\mathrm{sgn}(E_\alpha-E_{\alpha'})(E_\alpha-E_{\alpha'})(f_{\alpha'}-f_\alpha)$$
$$\times[(\Psi^\dagger_\alpha(x')\nabla_{x_0}\Psi_{\alpha'}(x'))\times(\nabla_{x_0}\Psi^\dagger_{\alpha'}(x)\Psi_\alpha(x))]\cdot\hat{z}(\delta x_\nu(\tau)\times\delta x_\nu(\tau'))\cdot\hat{z}. \quad (51)$$

In the last equality we have used the periodicity of the function $\delta x_\nu(\tau)=\delta x_\nu(\hbar\beta+\tau)$ to turn the hyperbolic function into the exponential function. Now we look for the slow motion expansion to the leading order in velocity: $\delta x_\nu(\tau')=\delta x_\nu(\tau)+\delta\dot{x}_\nu(\tau)(\tau'-\tau)$. Substituting this expansion into Eq. (51), after the integration over $(\tau'-\tau)$ we have

$$-\frac{1}{2}\int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' F_\perp(\tau-\tau')(\delta x_\nu(\tau)\times\delta x_\nu(\tau'))\cdot\hat{z}$$
$$=i\int_0^{\hbar\beta} d\tau B[\delta x_\nu(\tau)\times\delta\dot{x}_\nu(\tau)]\cdot\hat{z},$$

with the quantity B which determines the transverse force defined as

$$B=i\frac{\hbar}{2}\sum_{\alpha,\alpha'}(f_{\alpha'}-f_\alpha)\int d^3x\int d^3x'\hat{z}\cdot(\Psi^\dagger_\alpha(x')\nabla_{x_0}\Psi_{\alpha'}(x')$$
$$\times\nabla_{x_0}\Psi^\dagger_{\alpha'}(x)\Psi_\alpha(x)). \quad (52)$$

We demonstrate next that the contribution to the transverse correlation can be evaluated by counting extended states contributions. First, we regroup terms in Eq. (52):

$$B=i\frac{\hbar}{2}\hat{z}\cdot\mathrm{tr}\sum_{\alpha,\alpha'}(-)2\sum_n\frac{1}{\beta}\frac{e^{-i\omega_n\delta}}{i\hbar\omega_n-E_\alpha}$$
$$\times\int d^3x\int d^3x'\Psi_\alpha(x)\Psi^\dagger_\alpha(x')\nabla_{x_0}\Psi_{\alpha'}(x')$$
$$\times\nabla_{x_0}\Psi^\dagger_{\alpha'}(x)$$
$$=i\frac{\hbar}{2}\hat{z}\cdot\mathrm{tr}\sum_\alpha(-)2\sum_n\frac{1}{\beta}\frac{e^{-i\omega_n\delta}}{i\hbar\omega_n-E_\alpha}$$
$$\times\int d^3x(\nabla_{x_0}\Psi_\alpha(x)\times\nabla_{x_0}\Psi^\dagger_\alpha(x)), \quad (53)$$

because that $\{\Psi_{\alpha'}\}$ form a completed set. Here tr stands for summing over spinor indices. The replacement of f_α by the summation is to take care of the delicate equal-time limit in the trace: $\delta=0^-$ for spin up and $\delta=0^+$ for spin down in Nambu spin space. We encounter such a choice of time limit only in the case of taking the trace of the Nambu-Gor'kov Green's function directly. This choice will not be there if we only need to take trace of higher powers of the Nambu-Gor'kov Green's function, e.g., $\mathrm{tr}\,G_0^2$. This implies that the functions containing occupation numbers $\{f_\alpha\}$ in their differences are well defined. Therefore, Eq. (52) can be safely used if we directly put in the eigenstates of the Bogoliubov–de Gennes equation. An alternative natural way of deriving Eq. (53) is to leave the summation over ω_n in place throughout Eq. (50) to Eq. (53).

After substituting Eq. (53) into Eq. (52), we will write it explicitly in terms of the eigenstates of Bogoliubov–de Gennes equation. Since

$$\mathrm{tr}\sum_\alpha\sum_n\frac{1}{\beta}\frac{e^{-i\omega_n\delta}}{i\hbar\omega_n-E_\alpha}\int d^3x\nabla_{x_0}\Psi_\alpha(x)\times\nabla_{x_0}\Psi^\dagger_\alpha(x)$$
$$=\sum_\alpha\int d^3x\{f_\alpha\nabla_{x_0}u_\alpha(x)\times\nabla_{x_0}u^*_\alpha(x)$$
$$-(1-f_\alpha)\nabla_{x_0}v_\alpha(x)\times\nabla_{x_0}v^*_\alpha(x)\},$$

we obtain

$$B=-i\hbar\hat{z}\cdot\sum_\alpha\int d^3x[f_\alpha\nabla_{x_0}u^*_\alpha(x)\times\nabla_{x_0}u_\alpha(x)$$
$$-(1-f_\alpha)\nabla_{x_0}v^*_\alpha(x)\times\nabla_{x_0}v_\alpha(x)]. \quad (54)$$

After using $\nabla_{x_0}\to-\nabla$, we evaluate Eq. (54) with the help of the current definition[30]

$$\mathbf{j}=-\frac{i\hbar}{2}\sum_\alpha\{f_\alpha u^*_\alpha\nabla u_\alpha+(1-f_\alpha)v_\alpha\nabla v^*_\alpha\}+\mathrm{c.c.}$$

Equation (54) becomes

$$B=\int dx\ \hat{z}\cdot(\nabla\times\mathbf{j})=\oint_{|x-x_0|\to\infty}d\mathbf{l}\cdot\mathbf{j}=2\pi\hbar\rho_s(T). \quad (55)$$

Here we have used the fact that the current is zero at the vortex position, as the explicit calculation in Ref. 30 has shown. It can be understood as the requirement of quantum mechanics: At the phase singular point, the amplitude of any wave function carrying this singular phase must be zero. In reaching Eq. (55) we have also made the assumption that the vortex does not have a normal fluid circulation. The normal fluid is in equilibrium with the substrate or the walls of the container.

From Eq. (55) one may conclude that when counting the contribution from individual states, only extended states give rise to the contribution to the transverse response, because the loop of the line integral can be chosen arbitrarily large to make the core state contributions arbitrarily small. It corresponds to the fact that only extended states can contribute to the Berry phase of the vortex.[5] This result is valid even when the trapping potential U_0 is finite, which we demonstrate here. Since Eqs. (38) and (39) are valid in the presence of a finite trapping potential, the transverse correlation function can be expressed by wave functions in exactly the same form as that of vanishing trapping potential, up to Eq. (54). The wave functions, particularly those for core states, may be strongly affected by the trapping potential, and may even become ill defined. An example may be the trapping of a vortex by a physical wire. Now, one may perform the same calculation of turning the area integration into line integrations, as done in Eq. (55). Since the trapping potential will not affect the superfluid number density far away from the vortex, and since the circulation current is still zero at the vortex position, one then gets the same result as Eq. (55) in the presence of a trapping potential.

The validity of Eq. (55) in the presence of a finite trapping potential implies that the transverse force is independent of the trapping potential U_0 at the vortex center. In Eq. (29), the main function of the trapping potential is to specify the vortex position, a symmetry breaking in an otherwise homogeneous system. This is similar to the symmetry breaking by an infinitesimal field near a continuous phase transition in statistical mechanics. Hence, it can be effectively taken to be zero, as we have explicitly done in the present paper.

Next, we turn to the calculation of the superfluid number density ρ_s. At zero temperature, it is straightforward. It is equal to the total fluid number density $\rho_0=\Sigma_{\alpha,E_\alpha>0}|v_\alpha(|x-x_0|\to\infty)|^2$, the number of Cooper pairs per unit area. At finite temperatures, there is a reduction of superfluid number density due to the backflow carried by quasiparticle excitations. In principle, one may directly calculate the current density together with the gap equation, or self-consistent equation, to find out ρ_s. This would be prohibitively difficult. Instead, one may proceed in the following manner: Far away from the vortex core, the current varies slowly. One may take the current to be locally uniform. Following the same way as that in superfluid He3 using the backflow contribution,[31] the superfluid number density can be found as

$$\rho_s(T)=\rho_0[1-Y_0(T)] \tag{56}$$

with the Yosida function Y_0 defined as

$$Y_0(T)=\int_{-\infty}^{\infty} d\epsilon \frac{e^{\sqrt{\epsilon^2+\Delta_\infty^2}/k_BT}}{k_BT(e^{\sqrt{\epsilon^2+\Delta_\infty^2}/k_BT}+1)^2},$$

which accounts for the quasiparticle excitations contributions. At the superconducting transition temperature T_c, $\Delta_\infty=0$ and $Y_0(T_c)=1$, the superfluid number density is zero as expected. This expression is the same as that obtained from the London penetration depth for a clean type-II superconductor.[3]

Using Eq. (55), the transverse term in the effective action, Eq. (34), is

$$\begin{aligned}
&-\frac{1}{2}\int d\tau d\tau' F_\perp(\tau-\tau')[\delta x_v(\tau)\times\delta x_v(\tau')]\cdot\hat{z}\\
&=i\int d\tau d\tau' B[\delta x_v(\tau)\times\delta\dot{x}_v(\tau)]\cdot\hat{z}\\
&=-i2\pi\hbar\rho_s\int d\tau\delta\dot{x}_v(\tau)\cdot\mathbf{A}_t
\end{aligned} \tag{57}$$

with

$$\mathbf{A}_t=\frac{1}{2}(\delta x_v\times\hat{z}),$$

which has the same form of the action for a charged particle in a uniform magnetic field. The geometric phase or the Berry phase for the vortex moving along a closed trajectory Γ is

$$\begin{aligned}
\Theta=2\pi\hbar\rho_s\int d\tau\delta\dot{x}_v(\tau)\cdot\mathbf{A}_t&=2\pi\hbar\rho_s\oint_\gamma d(\delta x_v)\cdot\mathbf{A}_t\\
&=-2\pi\hbar\rho_s S(\Gamma),
\end{aligned}$$

with $S(\Gamma)$ being the area enclosed by Γ. The total transverse force on a vortex is then

$$\mathbf{F}=-2\pi\hbar\rho_s\delta\dot{x}_v\times\hat{z}.$$

In view of the foregoing discussions, we may rewrite our general formulation, Eq. (34), in a more suggestive form. The effective action for the vortex is

$$\begin{aligned}
S_{\text{eff}}=\int_0^{\hbar\beta}d\tau\Big\{&-i2\pi\hbar\rho_s\delta\dot{x}_v(\tau)\cdot\mathbf{A}_t\\
&+\frac{1}{2}\int_0^{\hbar\beta}d\tau' F_\parallel(\tau-\tau')|\delta x_v(\tau)-\delta x_v(\tau')|^2\Big\}
\end{aligned} \tag{58}$$

with $\mathbf{A}_t=\frac{1}{2}(\delta x_v\times\hat{z})$. The rewriting of Eq. (46) for the correlation function is

$$F_\parallel(\tau)=\frac{1}{\pi}\int_0^\infty d\omega J(\omega)\frac{\cosh[\omega(\hbar\beta/2-|\tau|)]}{\sinh[\omega(\hbar\beta/2)]},$$

and the rewriting of Eq. (47) for the spectral function is

$$J(\omega)=\frac{\pi}{4}\sum_{\alpha,\alpha'}\delta(\hbar\omega-|E_\alpha-E_{\alpha'}|)|f_{\alpha'}-f_\alpha| \times\left|\int d^3x\Psi^\dagger_\alpha(x)\nabla_{x_0}\mathcal{H}_0\Psi_{\alpha'}(x)\right|^2.$$

The thermodynamic limit must be taken first to have a smooth spectral function, which is crucial for obtaining a finite vortex friction.

IV. VORTEX DYNAMICS IN CLEAN LIMIT

The example of an extremely clean limit of fermionic superfluids is the superfluid He 3: The impurity concentration can be made to be smaller than 1 in 10^{12}. For superconductors, the impurity effect can, in principle, be made arbitrarily small, but no clear experimental realization has been reported yet. In view of this experimental situation, the discussions in this section are more relevant to He 3. However, from the methodological point of view, it is instructive to see how the formulation developed in Sec. III works for such a clean situation.

A. Extended states contribution to vortex friction

In this subsection we first calculate the extended state, quasiparticle and hole excitations, contributions to the vortex friction to illustrate the usefulness of the present longitudinal response formula. The formula, Eq. (47) or (49), is formally exact. However, for a given problem it is difficult to obtain an exact detailed expression for friction, except in some rare cases.[32,29,28] Hence a WKB-type approximation will be used below. The responses of fermions, or electrons, governing by Hamiltonian dynamics generates a finite friction for the vortex.

At finite temperatures the extended states above (below) the Fermi level [the quasiparticles (holes)] are partially occupied. The vortex motion causes transitions between these states, which gives rise to vortex friction. The transitions between different single quasiparticle levels $\langle\Psi_\alpha|\nabla_{x_0}\mathcal{H}_0|\Psi_{\alpha'}\rangle$ are considered here since they dominate the low-energy process. The quasiparticles are described by the eigenstates, u_α and v_α, of the Bogoliubov–de Gennes equation. Their behavior in the presence of a vortex has been well studied in Ref. 30. We may take

$$\Psi_\alpha=\begin{pmatrix}u_\alpha(x)\\ v_\alpha(x)\end{pmatrix}=\frac{e^{ik_zz}}{\sqrt{L}}\frac{e^{i\mu\theta+i\sigma_z\theta/2}}{\sqrt{2\pi}}\hat{f}(r) \tag{59}$$

with $\mathbf{r}$ measured from the vortex position, θ is the azimuthal angle around the vortex, L is the thickness of the superconductor film (the length of the vortex line), and ξ_0 is the coherence length. In order to obtain a concrete form for the transition elements, we use a WKB-type solution for $\hat{f}(r)$

$$\hat{f}(r)=\frac{1}{\sqrt{2}}\begin{pmatrix}\left[1\pm\frac{\sqrt{E^2-|\Delta(r)|^2}}{E}\right]^{1/2}\\ \left[1\mp\frac{\sqrt{E^2-|\Delta(r)|^2}}{E}\right]^{1/2}\end{pmatrix}J_{\mu\pm1/2}(k_\pm(E)r). \tag{60}$$

Here $k_\pm(E)=\sqrt{k_\rho^2\pm2m\sqrt{E^2-|\Delta(r)|^2}/\hbar^2}$ with $k_\rho^2=k_f^2-k_z^2$. The negative energy wave functions determined by the Bogoliubov–de Gennes equation may be constructed according to Eq. (37) from the positive energy ones. We will use the approximation that $k_\rho\approx k_F$ for the prefactor by assuming that the significant contributions come from the region near the Fermi surface. This WKB-type solution is valid when r is outside the classical turning point $r_t=|\mu|/k_\rho$. Here r_t is the impact parameter. A WKB-type solution also exists inside the turning point. However, because it approaches zero as $(rk_\rho)^{|\mu|}/|\mu|!$, the contribution to the transition elements from this region is small, and will be set to zero. The transition elements are then given by

$$|\langle\Psi_\alpha|\nabla_{x_0}\mathcal{H}_0|\Psi_{\alpha'}\rangle|^2=\left|\int d^3x[u^*_{\alpha'}(x)(\nabla_{x_0}\Delta)v_\alpha(x)+v^*_{\alpha'}(x)\times(\nabla_{x_0}\Delta^*)u_\alpha(x)]\right|^2 =\begin{cases}\frac{\Delta_\infty^2}{2\pi^2k_F^2}\delta_{k_z,k_z'}\delta_{\mu',\mu\pm1}, & |\mu|\leqslant\xi_0k_\rho\\ 0, & |\mu|>\xi_0k_\rho.\end{cases} \tag{61}$$

Here Δ_∞ is the value of $|\Delta(r)|$ far away from the vortex core. Physically, it means that if the classical quasiparticle trajectory is far away from the vortex core, it will not contribute to the vortex friction. The summation over states in Eq. (47) or Eq. (49) is replaced by

$$\sum_{\alpha'\neq\alpha}=\sum_{\mu,\mu',k_z,k_z'}\int dEdE'\frac{E}{\sqrt{E^2-\Delta_\infty^2}}\frac{E'}{\sqrt{E'^2-\Delta_\infty^2}}\left(\frac{2m}{\hbar^2}\right)^2, \tag{62}$$

after considering the density of states.

Substituting Eqs. (61) and (62) into Eq. (49) and using the quasiparticle distribution function $f_\alpha=1/(e^{\beta E_\alpha}+1)$, the coefficient of friction is given by

$$\eta=\frac{Lm^2\xi_0\Delta_\infty^2\beta}{4\pi^2\hbar^3}\int_{\Delta_\infty}^\infty dE\frac{E^2}{E^2-\Delta_\infty^2}\frac{1}{\cosh^2(\beta E/2)}. \tag{63}$$

The integral in Eq. (63) diverges logarithmically. It implies that the spectral function corresponding to the vortex-quasiparticle coupling is not strictly Ohmic but has an extra frequency factor proportional to $\ln(\Delta_\infty/\hbar\omega)$. When $\hbar\omega$ is not very small comparing to Δ_∞, which may be realized when close to T_c, we can ignore the logarithmic divergence in Eq. (63) by using the density of states for normal electrons to

obtain a finite friction, i.e., replacing $E^2/(E^2-\Delta_\infty^2)$ with 1 in Eq. (63). Close to T_c, the vortex friction approaches zero the same way as Δ_∞^2, which is proportional to the superfluid number density ρ_s. When $-\ln(\hbar\omega/\Delta_\infty)$ is large, we need to use Eq. (47) instead. Straightforward evaluation shows that in such a case

$$\eta=\frac{Lm^2\xi_0\Delta_\infty^3\beta}{8\pi^2\hbar^3}\frac{1}{\cosh^2(\beta\Delta_\infty/2)}\ln(\Delta_\infty/\hbar\omega_c). \tag{64}$$

Here ω_c is the low-frequency cutoff. It is determined by the size of the system for a single vortex, and by the intervortex distance for a vortex array.

We discuss briefly here the connection of our results to previous ones. The partial wave analysis has been performed for quasiparticle scattering off a vortex in a superconductor.[33] Though the phase shifts were obtained approximately, it is clear from the analysis that they are not all zero. Using the formal relationship between the phase shift and the friction,[27–29,32] the extended states have a contribution to the vortex friction, in accordance with our results.

It should be emphasized that the logarithmic divergence comes from the interplay between the divergence in the density of states and the off-diagonal potential scattering. We can consider a situation where we physically create a pinning center to trap the vortex and guide its motion. In such a case the vortex has a diagonal potential. If the scattering is dominated by the diagonal potential, e.g., by the trapping potential U_0, an additional factor coming from $|u_\alpha|^2-|v_\alpha|^2$ will remove this logarithmic divergence. The friction on the physical trapping potential will be finite even above T_c without the vortex, as indicated in Refs. 27–29, and 32, though the total transverse force disappears because $\rho_s=0$. This again shows the sensitivity of the vortex friction to details.

The vortex friction from extended states exists for both clean and dirty superconductors at finite temperatures. Close to the transition temperature, it scales linearly with the superfluid density, and is exponentially small when $T\ll\Delta_\infty$. For intermediate temperatures $T\sim\Delta_\infty$, using $\xi_0\sim\hbar^2k_F/m\Delta_\infty$ and $N(0)=mk_F/\pi^2\hbar^2$, $\eta\sim L\hbar N(0)\Delta_\infty^2/k_BT$. When the impurity potential is nonzero, there is an additional contribution to the friction, to be discussed in the next section.

We mention here that there is another type of low-lying excitation, phonons, which may lead to an additional contribution to vortex friction. This type of excitation can be described by the phase dynamics of the gap function Δ, and has been ignored here by the assumption of an adiabatic following up of the gap function to the vortex coordinate. Based on general considerations we expect that the phonon contribution is super Ohmic.[14] Hence, it is asymptotically weaker than the (sub)Ohmic damping contribution from quasiparticle excitations discussed above and the core state contribution to be discussed below.

The nonzero friction contribution from extended states found above is in accordance with the linear-response theory in nonequilibrium statistical mechanics, where transport coefficients are related to the fluctuations near the equilibrium by the fluctuation-dissipation theorem. The fluctuations are completely determined by the underlying Hamiltonian. Here they are quasiparticles determined by the Bogoliubov–de Gennes equation.

It should be pointed that the expression leading to vortex friction, Eq. (47) or (49), are absent in Refs. 7, 9, and 10. To obtain a finite friction in those work, a finite relaxation time needs to be inserted into the denominator of the force-force correlation function,[10] or into the denominator of the Nambu-Gor'kov Green's function at a convenient point,[9] which at the same time leads to the reduction of the total transverse force. As discussed in Sec. II B, such a procedure should be avoided.

B. Core vs extended states transitions for total transverse force

There are various ways to express the transverse correlation function in Eq. (36) or (52), with emphasis on different aspects of the transition elements. In Sec. III C we have shown how to obtain the total transverse force from the consideration of extended states. In this subsection we show that it can also be obtained from the consideration of core states, even some combination of both types of states.

Explicitly, we may evaluate the transverse response directly from Eq. (36) or (52). We will show that the total transverse force can be expressed as contributions from only core to core transitions, or from core to extended states transitions. For a clean superconductor we replace $\nabla_{x_0}\to-\nabla$. We define following symbols for the transition elements:

$$\alpha(l)\alpha'(l')=-i\hbar(f_\alpha-f_{\alpha'})\frac{1}{2}\hat{z}\cdot\int d^3x d^3x' \times\Psi_\alpha^\dagger(x')\nabla\Psi_{\alpha'}(x')\times\nabla\Psi_{\alpha'}^\dagger(x)\Psi_\alpha(x), \tag{65}$$

which groups the transition elements into core state to core state, core state to extended states, extended state to core states, and extended state to extended state transitions. Here $l=c,e$ represents the core c or extended e states, and α represents other indices: k_z, μ. For example, $\alpha(c)\alpha'(c)$ represents the elements in Eq. (52) when Ψ_α and $\Psi_{\alpha'}$ are both core states. More explicit examples will be given below. From Eq. (52) and (65), the summing over these transition elements as well as over α and α' gives B,

$$B=\sum_{\alpha,\alpha'}[\alpha(c)\alpha'(c)+\alpha(c)\alpha'(e)+\alpha(e)\alpha'(c)+\alpha(e)\alpha'(e)]. \tag{66}$$

First, we note that for a core state, the sum of its transition elements to all other states is zero:

$$\sum_{\alpha'}[\alpha(c)\alpha'(c)+\alpha(c)\alpha'(e)]=0. \tag{67}$$

In fact, we have already obtained Eq. (67) and used this identity earlier in Sec. III C from Eqs. (52)–(55) to exclude

the core state contribution to the circulating current far away from the vortex core in Eq. (55). In Eq. (52) both extended and core state contributions are there. Then we summed over all the $\Psi_{\alpha'}$ to reach Eq. (53) or (54), because they form a complete set. Thus in the last expression of Eq. (52) inside the sum over α, if Ψ_α is an extended state, its contribution to the transverse response is actually $\Sigma_{\alpha'}[\alpha(e)\alpha'(c)+\alpha(e)\alpha'(e)]$, because all the α's have been summed over. The same procedure applies if Ψ_α is a core state. In Eq. (55), we have shown that the area integral can be converted into a line integral and we can choose the loop large enough compared to the core size. If Ψ_α is a core state, its contribution to Eq. (55) is zero. Thus we conclude the validity of Eq. (67).

With the aid of Eq. (67), the transverse correlation can be expressed in the following two additional forms:

$$B=\sum_{\alpha,\alpha'}[\alpha(e)\alpha'(c)+\alpha(e)\alpha'(e)] \tag{68}$$

$$=\sum_{\alpha,\alpha'}[\alpha(e)\alpha'(e)-\alpha(c)\alpha'(c)]. \tag{69}$$

In reaching Eq. (69), we have used the identity $\alpha(c)\alpha'(e)=\alpha(e)\alpha'(c)$. Here Eq. (68) can be reduced to Eq. (55) after using the completeness of the eigenfunctions, as discussed above. Next, we present more detailed discussions on the core-core and extended-extended transition element contributions.

To be specific, we consider the zero-temperature case. For the core states, because of the topology the energy is uniquely determined by μ. The only transition elements contributed to the transverse correlation function $F_\perp$ are between states $\mu=\pm\frac{1}{2}$:

$$\sum_{\alpha,\alpha'}\alpha(c)\alpha'(c)=-i2\hbar(f_{\mu=-1/2}-f_{\mu'=1/2})\times\frac{1}{2}\hat{z}\cdot\int d^3x\int d^3x'\times[\Psi^\dagger_{-1/2}(x')\nabla_{x_0}\Psi_{1/2}(x')\times\nabla_{x_0}\Psi^\dagger_{1/2}(x)\Psi_{-1/2}(x)]. \tag{70}$$

The additional factor 2 accounts for the transition from the $\mu=1/2$ state to the $\mu'=-1/2$ state, which gives the identical contribution. The transition element in Eq. (70) may be expressed as

$$\int d^3x\int d^3x'[\Psi^\dagger_{-1/2}(x')\nabla_{x_0}\Psi_{1/2}(x')\times\nabla_{x_0}\Psi^\dagger_{1/2}(x)\Psi_{-1/2}(x)]=-i\hat{z}\,\delta_{k_z,k_z'}\frac{(a_{-1/2,1/2}+b_{-1/2,1/2})(a^*_{-1/2,1/2}+b^*_{-1/2,1/2})}{2(E_{-1/2}-E_{1/2})^2},$$

where

$$a_{-1/2,1/2}=-\int_0^\infty rdr|\Delta|'_r[\hat{f}^*_{+,-1/2}(r)\hat{f}_{-,1/2}(r)+\hat{f}^*_{-,-1/2}(r)\hat{f}_{+,1/2}(r)],$$

and

$$b_{-1/2,1/2}=-\int_0^\infty dr|\Delta|[\hat{f}^*_{+,-1/2}(r)\hat{f}_{-,1/2}(r)-\hat{f}^*_{-,-1/2}(r)\hat{f}_{+,1/2}(r)],$$

which follow the definitions in Eqs. (A10)–(A15) in the Appendix.

Now we evaluate Eq. (70) explicitly. For the deep core states in clean superconductors $E=-\mu\epsilon_0$ with ϵ_0 the core level spacing, $f_{\mu=-1/2}=0$ and $f_{\mu=1/2}=1$. The relation between the energy of core states and the quantum number μ in our case is different from the one in Ref. 30 in sign because we are considering a vortex with positive vorticity. The wave functions for deep core states take the form

$$\Psi^\dagger_\mu(x)\approx\frac{1}{2}\sqrt{\frac{k_F}{\xi_0}}\begin{pmatrix}e^{i(\mu+1/2)\theta}J_{\mu+1/2}(k_Fr)\\ e^{i(\mu-1/2)\theta}J_{\mu-1/2}(k_Fr)\end{pmatrix}e^{-r/\xi_0}. \tag{71}$$

This leads to $a_{-1/2,1/2}(E,E')\approx 0$ and $b_{-1/2,1/2}(E,E')\approx\Delta_\infty/\xi_0$.

Using Eq. (71), it is straightforward to show

$$\hat{z}\cdot\sum_{\mu,\mu'}(f_\mu-f_{\mu'})\int d^3x\int d^3x'\times[\Psi^\dagger_\mu(x')\nabla\Psi_{\mu'}(x')\times\nabla\Psi^\dagger_{\mu'}(x)\Psi_\mu(x)]=ik_F^2.$$

Therefore,

$$\sum_{\alpha,\alpha'}\alpha(c)\alpha'(c)=\frac{\hbar k_F^2}{2}=B. \tag{72}$$

The last equality is due to the fact that in $2D$, the electron density $n_e=k_F^2/2\pi$. The additional factor 1/2 accounts for the pairing. The conclusion is that at zero temperature the sum of the core-core state transitions alone gives rise to the total transverse force. It corresponds to the fact that the core-core state transitions are a local and differential form of the geometric phase, and the Berry phase is the global and integral form. In the next section we will show that Eq. (72) is unchanged in the presence of impurities.

Here we wish to point out an interesting feature explicitly manifested in Eq. (72): the transverse response is insensitive to the size of the system, because the core states are exponentially localized. This implies that the thermodynamic limit is not important for the total transverse force. We attribute this feature to the topological constraint on the transverse response, corresponding to the well-known fact that the Berry phase exists for a discrete energy spectrum.

There are two more interesting results which follow Eq. (72). At zero temperature, using Eqs. (69) and (72), we have

$$\sum_{\alpha,\alpha'} \alpha(e)\alpha'(e)=2B. \tag{73}$$

This implies that for a fermionic superfluid, the sum of all extended state transitions lead to twice that of the total transverse force. Combining Eq. (67) with Eq. (72), we have

$$B=-\sum_{\alpha,\alpha'} \alpha(e)\alpha'(c), \tag{74}$$

which shows that the core-extended state transitions can also be used to calculate the total transverse force. We believe that this property has been explored before in the case considering the contributions from states whose energies are around Δ_∞, the interface between the core and extended states.[34]

In the literature, after a transverse response equivalent to Eq. (36) or (52) was reached, it had always been assumed that only one core-to-core state transition contributes.[9,10] However, as we have found out, core to extended state transitions are of the same order. The above discussions show that there are many equivalent ways to compute the total transverse force. Because of the topological constraint, the total transverse force can even be evaluated by partial summations of the transition elements, expressed by Eqs. (68), (69), (72), (73), and (74). This is completely different from the computation of the longitudinal force (the friction), where a partial summation contributes only a part of the total friction. The demonstration in this subsection suggests that the alleged cancellation between the core spectrum flow contribution and the Berry phase counting is a consequence of the combination of double counting, treating core and extended contributions to the transverse force as different quantities, and the misuse of the relaxation time approximation in the force-force correlation functions.

To briefly summarize this subsection, we have formulated the transverse response in terms of transitions between core state or between core and extended states. Equivalently, we have also formulated it in terms of a summation over the extended state contributions. In a clean and neutral superfluid at finite temperatures, the total transverse force is given by the product of the superfluid number density, the Planck constant h, and the vortex velocity, though the vortex friction exists.

V. EFFECTS OF IMPURITIES

A. No effect on total transverse force

The presence of impurities is unavoidable in superconductors. In this subsection we consider this realistic situation of the influence of the impurity potential to the transverse force on the moving vortex. In Sec. IV B we have shown that the transverse correlation function can be evaluated by either considering the extended states or by considering only the core states in a clean superconductor. The same also holds in the case with impurities, as we will demonstrate below. We first give a formal demonstration from the counting of individual state contributions, then explicitly consider the core state transitions, to pave the way for the core state contribution to vortex friction. The robust conclusion is that random impurities do not affect the total transverse force.

It is more convenient to change the gradient from ∇_{x_0} to ∇ when an impurity potential $V(x)$ is involved. Applying Eqs. (42) and (43) to Eq. (52), the transverse correlation becomes

$$B=-i\sum_{\alpha,\alpha'}\int d^3x\int d^3x'(f_\alpha-f_{\alpha'})\frac{\hbar}{2}\hat{z}\cdot[\Psi_\alpha^\dagger(x')\nabla\Psi_{\alpha'}(x')\times\nabla\Psi_{\alpha'}^\dagger(x)\Psi_\alpha(x)]$$
$$+\sum_{\alpha,\alpha'}(f_\alpha-f_{\alpha'})(E_\alpha-E_{\alpha'})^{-2}\frac{\hbar}{2}\hat{z}\cdot\int d^3x\int d^3x'[\Psi_\alpha^\dagger(x')\nabla\mathcal{H}'(x')\Psi_{\alpha'}(x')\times\Psi_{\alpha'}^\dagger(x)\nabla\mathcal{H}'(x)\Psi_\alpha(x)]. \tag{75}$$

Other terms are identically equal to zero after summing over α,α'.

We first show that the second term in Eq. (75) is zero after the impurity average. To be concrete, we expand the wave function Ψ_α in terms of eigenfunctions of $\bar{\mathcal{H}}_0$, $\{\phi_\gamma\}$,

$$\Psi_\alpha=\sum_\gamma \chi_{\alpha\gamma}\phi_\gamma. \tag{76}$$

Here $\chi_{\alpha\gamma}=a_{\alpha\gamma}e^{i\varphi_{\alpha\gamma}}$, and $a_{\alpha\gamma}$ and $\varphi_{\alpha\gamma}$ are the modulus and phase of the expansion coefficients. We remind the reader that $\bar{\mathcal{H}}_0$ has dependence on the impurity potential because it includes the smooth part of the self-consistent potential Δ. Because of the normalization requirement, the coefficients $\{a_{\alpha\gamma}e^{i\varphi_{\alpha\gamma}}\}$ form a unitary matrix,

$$\sum_\gamma a_{\alpha\gamma}^2=1,\quad \sum_\alpha a_{\alpha\gamma}^2=1.$$

The second term of Eq. (75) is now

$$\int d^3x \int d^3x' \Psi^\dagger_\alpha(x')\nabla\mathcal{H}'(x')\Psi_{\alpha'}(x')\times\Psi^\dagger_{\alpha'}(x)\nabla\mathcal{H}'(x)\Psi_\alpha(x)$$
$$= \sum_{\gamma,\gamma',\gamma_1,\gamma_1'} a_{\alpha\gamma}a_{\alpha'\gamma'}a_{\alpha'\gamma_1'}a_{\alpha\gamma_1}e^{-i\varphi_{\alpha\gamma}+i\varphi_{\alpha'\gamma'}-i\varphi_{\alpha'\gamma_1'}+i\varphi_{\alpha\gamma}}\int d^3x \int d^3x' \phi^\dagger_\gamma(x')\nabla\mathcal{H}'(x')\phi_{\gamma'}(x')\times\phi^\dagger_{\gamma_1'}(x)\nabla\mathcal{H}'(x)\phi_{\gamma_1}(x). \tag{77}$$

From the random-matrix theory,[35] the phase $\{\varphi_{\alpha\gamma}\}$ are random numbers. The impurity average makes $\gamma=\gamma_1$ and $\gamma'=\gamma_1'$. Under the assumption, i.e., the core size much larger than the average distance between impurities, the averaging over impurities restores the homogeneity and isotropy of the spatial space. This implies that all the odd power of $\nabla\mathcal{H}'$ will be averaged to zero. Since each term in Eq. (77),

$$\int d^3x \int d^3x' \phi^\dagger_\gamma(x')\nabla\mathcal{H}'(x')\phi_{\gamma'}(x')\times\phi^\dagger_{\gamma'}(x)\nabla\mathcal{H}'(x)\phi_\gamma(x)$$
$$= \hat{z}\int d^3x \int d^3x' \{(\phi^\dagger_\gamma(x')\nabla\mathcal{H}'(x')\phi_{\gamma'}(x'))_x(\phi^\dagger_{\gamma'}(x)\nabla\mathcal{H}'(x)\phi_\gamma(x))_y$$
$$-(\phi^\dagger_\gamma(x')\nabla\mathcal{H}'(x')\phi_{\gamma'}(x'))_y(\phi^\dagger_{\gamma'}(x)\nabla\mathcal{H}'(x)\phi_\gamma(x))_x\},$$

consists of elements of the odd power of x and y components of $\nabla\mathcal{H}'$, the second term of Eq. (75) is zero after the impurity average.

The average transverse correlation is then, following the same procedure from Eqs. (52)–(55),

$$B = \oint \oint_{|x-x_0|\to\infty} d\mathbf{l}\cdot\mathbf{j}. \tag{78}$$

Here the average current

$$\mathbf{j} = -\left\langle \frac{i\hbar}{2}\sum_\alpha [f_\alpha u^*_\alpha\nabla u_\alpha + (1-f_\alpha)v_\alpha\nabla v^*_\alpha]\right\rangle + \text{c.c.}$$
$$= -\frac{i\hbar}{2}\sum_{\alpha\gamma}\langle a^2_{\alpha,\gamma}\rangle\{[f_\alpha u^*_\gamma\nabla u_\gamma + (1-f_\alpha)v_\gamma\nabla v^*_\gamma] + \text{c.c.}\},$$

and $\langle\cdots\rangle$ stands for the impurity average over the expansion coefficients. In the limit $r\to\infty$, we have

$$\mathbf{j} = -\frac{i\hbar}{2}\sum_{\alpha,\gamma}\langle a^2_{\alpha,\gamma}\rangle\{[f_\alpha|u_\gamma|^2 + (1-f_\alpha)|v_\gamma|^2] + \text{c.c.}\}\nabla\theta.$$

At zero temperature, we have

$$\sum_{\alpha,E_\alpha>0}\sum_\gamma \langle a^2_{\alpha\gamma}\rangle|\overline{v}_\gamma(|x-x_0|\to\infty)|^2$$
$$= \sum_{\alpha,E_\alpha>0}\langle|v_\alpha(|x-x_0|\to\infty)|^2\rangle = \rho_0.$$

The above second equality is the Anderson theorem, in that nonmagnetic impurities do not affect the density of states near the Fermi surface, hence there is no effect on the superconducting transition temperature. We will come back to this point after the discussion of the impurity effect on vortex friction. This result may also be reached from the envelope wave function argumentation.[36] Therefore Eq. (55) also holds in the presence of impurities. Thus we have shown in general that the transverse correlation is not influenced by impurity potentials. Physical understanding of this result is straightforward: There is no average circulation current associated with impurity potentials.

Next, the insensitivity of the total transverse force to random impurities will be illustrated by a different demonstration. We evaluate core state transitions with impurity potentials, a part of the first term in the right-hand side of Eq. (75). First, we will explicitly consider the total transverse force from core state contributions with a weak impurity potential. Because of the factor $(f_\mu - f_{\mu'})$ and the selection rule in the transition elements, deep core states are the ones making the main contributions to the total transverse force for temperatures well below T_c. When the impurities are weak enough, for deep core states $\Psi_\mu = \Sigma_\nu a_{\mu\nu}e^{i\varphi_{\mu\nu}}\phi_\nu$ the expansion coefficient $a_{\mu\nu}$ is large only for the neighboring states around μ, which are also deep core states. In a clean superconductor, core states are uniquely specified by the azimuthal number ν. Therefore, with weak impurity potentials, we may only consider deep core states close to the Fermi surface, and ignore the mixing of deep core-level states with the extended states in Eq. (75).

Substituting Eq. (76) into Eq. (75) and using Eq. (71) to calculate transition elements among core states $\{\phi_\nu\}$, we have

$$\int d^3x \int d^3x' \Psi_\mu^\dagger(x') \nabla \Psi_{\mu'}(x') \times \nabla \Psi_{\mu'}^\dagger(x) \Psi_\mu(x)$$

$$= \sum_{\nu,\nu',\nu_1,\nu_1'} a_{\mu\nu} a_{\mu'\nu'} a_{\mu'\nu_1'} a_{\mu\nu_1} \times e^{-i\varphi_{\mu\nu}+i\varphi_{\mu'\nu'}-i\varphi_{\mu'\nu_1'}+i\varphi_{\mu\nu_1}} (\mp) i t_c^2 \delta_{\nu',\nu\pm1} \delta_{\nu_1',\nu_1\pm1} \hat{z}. \tag{79}$$

Here we have used

$$\int d^3x \phi_\nu^\dagger(x) \nabla \phi_{\nu'}(x) \approx i t_c (\pm \hat{x} + i\hat{y}) \delta_{\nu',\nu\pm1}$$

with $t_c = k_F/2$. If μ and μ' are interchanged, there is a sign change in the right-hand side of Eq. (77). Including the factor $(f_\mu - f_{\mu'})$ and summing over μ and μ', core-core state transitions become

$$\sum_{\mu,\mu'} (f_\mu - f_{\mu'}) \int d^3x \int d^3x' \hat{z} \cdot [\Psi_\mu^\dagger(x') \nabla \Psi_{\mu'}(x') \times \nabla \Psi_{\mu'}^\dagger(x) \Psi_\mu(x)]$$

$$= \sum_{\mu,\mu'} (f_\mu - f_{\mu'}) \times \sum_{\nu,\nu',\nu_1,\nu_1'} a_{\mu\nu} a_{\mu'\nu'} a_{\mu'\nu_1'} a_{\mu\nu_1} \times e^{-i\varphi_{\mu\nu}+i\varphi_{\mu'\nu'}-i\varphi_{\mu'\nu_1'}+i\varphi_{\mu\nu_1}} 2(\mp) i t_c^2 \delta_{\nu',\nu\pm1} \delta_{\nu_1',\nu_1\pm1}. \tag{80}$$

Because of the randomness in the phase factor of $e^{-i\varphi_{\mu\nu}+i\varphi_{\mu'\nu'}-i\varphi_{\mu'\nu_1'}+i\varphi_{\mu\nu_1}}$, the dominant contribution comes from states $\nu = \nu_1, \nu' = \nu_1'$. Equation (80) becomes

$$\sum_{\mu,\mu'} (f_\mu - f_{\mu'}) \sum_{\nu,\nu'} a_{\mu\nu}^2 a_{\mu'\nu'}^2 2(\mp) i t_c^2 \delta_{\nu',\nu\pm1}.$$

For a given distribution of $a_{\alpha\mu}$ this summation can be evaluated. For the purpose of demonstration, let us assume a simple distribution centered at α: $a_{\mu\nu}^2 = 1/(2l+1)$ when $|\nu - \mu| < l$, $a_{\mu\nu} = 0$ otherwise, and consider only zero temperature. Here $1 \ll l \ll N_c$, with N_c being the total number of core states. With this assumption, each of the original states is spread into a band of $2l+1$ states around it when impurities are present. Equation (79) takes the value

$$\sum_{\mu,\mu'} (f_\mu - f_{\mu'}) \sum_{|\nu-\mu|\le l, |\nu'-\mu'|\le l} \frac{1}{(2l+1)^2} 2(\mp) i t_c^2 \delta_{\nu',\nu\pm1}.$$

We note that for the pair of states $\mu = \mp 1/2$, $\mu' = \pm 1/2$ closest to the Fermi surface their contribution is reduced by a factor of $1/(2l+1)^2$. However, all the states within the energy shell $|E_\mu| \le l\epsilon_0$ near the Fermi surface contribute now. In order to have a nonzero contribution at zero temperature, we have $E_\mu E_{\mu'} < 0$, one above and one below the Fermi surface. Because of the restriction of the band distribution and the selection for ν' and ν, we have an additional constraint on the E_μ and $E_{\mu'}$: $|E_\mu|, |E_{\mu'}| < l$. The net contribution is

$$\sum_{\mu,\mu'} (f_\mu - f_{\mu'}) \sum_{|\nu-\mu|\le l, |\nu'-\mu'|\le l} \frac{1}{(2l+1)^2} 2(\mp) i t_c^2 \delta_{\nu',\nu\pm1} = i k_F^2/2, \tag{81}$$

which is approximately the same value for the clean superconductor. The factor of 1/2 is due to the approximation for t_c using same value for all transitions between μ and $\mu \pm 1$. One can check that the above result also holds for Gaussian distribution of $a_{\alpha\mu}$. This completes the discussion of the first term in Eq. (75) in the weak impurity potential limit.

Although at zero temperature in a clean superconductor only the core states closest to the Fermi level contribute to Eq. (52) or (66), the transition elements of other states are not small. Their contributions cancel each other completely. With impurities, more states than those closest to the Fermi level give contributions to the transverse response. These contributions from other core states restore the transverse response to its original value of a clean superconductor. In the calculations with the relaxation time approximation, the reduction of the contribution from the two states closest to the Fermi level has been taken into account.[9,10] The contribution due to other core states, which arises after introducing impurities, has not been included in those calculations.

Next we consider the dirty limit, and we will again make use of arguments in the random matrix theory.[35] We assume that there is no mixing between the core and extended states. In the weak impurity potential limit, it is not difficult to justify this assumption: The band width in Eq. (81) caused by impurities is much smaller than Δ. In the dirty limit, the number of core states remains the same, since the energy gap away from the vortex core remains the same. Hence, there is a conservation of the number of core states, because of the topological nature of the vortex. We also note that there is no degeneracy for the core states, in contrast to the extended states. In additional, impurities do not cause an additional violation of time-reversal symmetry. For those reasons we do not expect that they would mix two topologically distinct types of solutions, the core and extended states, of the Bogoliubov–de Gennes equation.

With increasing impurity potential strength, eventually any core states Ψ_μ in Eq. (79) consist of all the core states of $\bar{\mathcal{H}}_0$, $\Psi_\mu = \Sigma_\nu \chi_{\mu\nu} \phi_\nu$. Here the summation Σ_ν runs over core states only. The total number of core states does not change after introducing impurity potentials, because the core-level spacing for ϕ_ν only depends on the values of E_F and Δ_∞, the value of Δ far from the core. A specific approximate realization may still be in the form of the band distribution as given in Eq. (81), with $l \sim N_c$, the total number of core states:

$$|\chi_{\mu\nu}|^2=\begin{cases}\dfrac{1}{N_c-2|\mu|}, & |\mu-\nu|<N_c/2-|\mu|\\ 0, & |\mu-\nu|>N_c/2-|\mu|.\end{cases}\tag{82}$$

It is widely spread for deep core states $|\mu|\ll N_c/2$. One may check that $\Sigma_\nu|\chi_{\mu\nu}|^2=1$ and $\mu=\Sigma_\nu|\chi_{\mu\nu}|^2\nu$. The latter corresponds to the requirement for the energy spectrum $\langle E_\mu\rangle=\langle\Psi_\mu|\mathcal{H}_0|\Psi_\mu\rangle=\Sigma_\nu|\chi_{\mu\nu}|^2E_\nu$. The condition of $\Sigma_\mu|\chi_{\mu\nu}|^2=1$ is only approximately satisfied: we have found that $\Sigma_\mu|\chi_{\mu\nu}|^2\approx\frac{1}{2}\ln(N_c^2)/[(N_c/2)^2-\nu^2]$ which gives $\ln 2$ when $\nu\ll N_c/2$ and $\ln N_c$ when $\nu\sim N_c/2$.

In this limit, Eq. (79) becomes

$$\begin{aligned}&\sum_{\mu,\mu'}(f_\mu-f_{\mu'})\hat{z}\cdot\int d^3x\int d^3x'\\&\quad\times[\Psi^\dagger_\mu(x')\nabla\Psi_{\mu'}(x')\times\nabla\Psi^\dagger_{\mu'}(x)\Psi_\mu(x)]\\&=\sum_{\mu,\mu'}(f_\mu-f_{\mu'})\sum_{\nu,\nu',\nu_1,\nu_1'}\chi^*_{\mu\nu}\chi_{\mu'\nu'}\chi^*_{\mu'\nu_1'}\chi_{\mu\nu_1}\\&\quad\times2(\mp)it_c^2\delta_{\nu',\nu\pm1}\delta_{\nu_1',\nu_1\pm1}.\end{aligned}\tag{83}$$

Its average value is $ik_F^2/2$ at zero temperature, approximately the same as in clean superconductor, by using the distribution function given in Eq. (82).

We have gone into great detail to calculate the total transverse force from core state transitions in the presence of impurities. Indeed, there is a reduction in transition amplitude between any pair of neighboring states. Nevertheless, the summation over all possible core state transitions restores the total transverse force to its value in the clean limit. Hence, the impurities have a negligibly small effect on the total transverse force from both the core state transitions consideration and the extended state counting, though the friction contributions are strongly affected by impurities, to be discussed in the next subsection.

In addition, we check the self-consistent condition with respect to Δ here and show that they are satisfied for our choice of $\chi_{\mu\nu}$. Because in $\overline{\mathcal{H}_0}$ we have already assumed that the profile of $\overline{\Delta}$ is the smooth part of the self-consistent potential with impurity potentials included, we need to make sure that the decomposition of the eigenfunction does not introduce an extra term to the self-consistent potential. We have

$$\begin{aligned}\Delta&=-g\sum_\alpha u_\alpha v^*_\alpha(1-2f_\alpha)\\&=-g\sum_{\mu,\nu,\nu'}\chi_{\mu\nu}\chi^*_{\mu\nu'}u^0_\nu v^{0*}_{\nu'}(1-2f_\alpha)\\&\quad-g\sum_{\alpha(e)}u_\alpha v^*_\alpha(1-2f_\alpha),\end{aligned}\tag{84}$$

where u^0,v^0 are the components of core eigenfunctions $\{\phi_\nu\}$, and (e) denotes the extended states. In the last equation, the summation is split into those of core states and extended states. Using a distribution function p for χ such as defined by Eq. (82), the average core state contribution to Δ is the same as the one calculated by using $\{\phi_\nu\}$,

$$\begin{aligned}&-g\left\langle\sum_{\mu,\nu,\nu'}\chi_{\mu\nu}\chi^*_{\mu\nu'}u^0_\nu v^{0*}_{\nu'}(1-2f_\mu)\right\rangle\\&=-g\sum_\mu u^0_\mu v^{0*}_\mu(1-2f_\mu).\end{aligned}$$

The extended states also need to be self-consistent. We assume the impurity strength is strong enough to mix the core states on the scale of $1/\xi$ but too weak to cause extended state distortion on the scale of $1/k_F$. Then for extended states the distribution of $\chi_{\alpha\gamma}$ is a function $p(E_\alpha-E_\nu)$. It is straightforward to check that the extended state contribution to Δ is the same as that of those calculated by using $\{\phi_\nu\}$.

B. Impurity contribution to vortex friction

In the present of impurity potentials, there are two kinds of contributions to the friction. The extended state contribution remains basically the same as what we have discussed before. The main difference is that the coherence length in Eq. (61) will change when impurities are present. We will give a brief discussion here. Since the density of states remains unchanged, we only need to evaluate Eq. (61) again. The transition elements are given by

$$\begin{aligned}|\langle\Psi_\alpha|\nabla_{x_0}\mathcal{H}_0|\Psi_{\alpha'}\rangle|^2&=\sum_{\nu,\nu',\gamma,\gamma'}|\chi_{\alpha;\nu\gamma}|^2|\chi_{\alpha';\nu\gamma'}|^2\\&\times\begin{cases}\dfrac{\Delta_\infty^2}{2\pi^2k_F^2}\delta_{k_z,k_z'}\delta_{\nu,\nu'\pm1}, & |\nu|<\xi k_\rho\\ 0, & |\nu|>\xi k_\rho.\end{cases}\end{aligned}\tag{85}$$

Here χ is the expansion coefficient in Eq. (76), ξ is the coherence length in the presence of impurities, l is the angular index of the state, and E is the energy of the state. With a given distribution of χ such that the expansion coefficient is confined to the neighborhood of its original energy, it can be shown that the extended state contributions to the friction remain unchanged, except the change of the coherence length of $\xi_0\to\xi$.

In the presence of impurities, the core state energy levels are no longer monotonically arranged according to azimuthal number or the angular momentum. In addition, it may become quasicontinuous under the impurity average. The mixing caused by impurity potentials makes it possible to have transitions into energetically nonadjacent core states, as discussed in the previous subsection, as well as into energetically nearly degenerate core states. Thus the core states can give another contribution to the vortex friction, similar to the residual resistance in a metal.

First, we consider the weak impurity potential limit. We assume the effect of impurities is not so strong such that we can treat their influence on core states perturbatively. Using Eqs. (42) and (43) we obtain

$$\left| \int d^3x \Psi_\mu^\dagger(x) \nabla_{x_0} \mathcal{H}_0 \Psi_{\mu'}(x) \right|^2$$
$$= \left| -(E_{\mu'} - E_\mu) \int d^3x \Psi_\mu^\dagger(x) \nabla \Psi_{\mu'}(x) - \int d^3x \Psi_\mu^\dagger(x) \nabla \mathcal{H}' \Psi_{\mu'}(x) \right|^2. \quad (86)$$

For the leading-order contribution, we only need to use the unperturbed Ψ_μ and Δ. The first term in Eq. (86) will not give any contribution to the dissipation because of the discreteness of $(E_{\mu'} - E_\mu)$ and the factor $\delta(\hbar\omega - |E_\mu - E_{\mu'}|)$ in $J(\omega)$. After summing over μ and μ', this term will become terms of $\delta(\hbar\omega \pm \epsilon_0)$, which will not give any dissipation. The contribution to the dissipation comes from the second term.

We may assume the impurity potential has a length scale small compared with the coherence length so that we can describe it by a delta potential $V(x) = \Sigma_i V_0 \delta(x - x_i)$. We have

$$\left\langle \left| \int d^3x \Psi_\mu^\dagger(x) \nabla \mathcal{H}' \Psi_{\mu'}(x) \right|^2 \right\rangle = n_i (\pi \xi^2 L) V_0^2 \left(\frac{k_F}{\pi \xi^2 L} \right)^2,$$

with the impurity concentration n_i. Under this assumption we will make connections to the normal-state transport parameters. Note that for normal states the electronic transport relaxation time and the electron scattering cross section have the following relations:[37]

$$\tau_{\rm tr}^{-1} = n_i v_F \sigma_{\rm tr}$$

with

$$\sigma_{\rm tr} = \int d\Omega (1 - \cos\theta) |V(\theta)|^2.$$

Here

$$V(\theta) = -\frac{m}{2\pi\hbar^2} \int d^3x V(x) e^{-i\mathbf{q}\cdot\mathbf{r}}$$

with $\mathbf{q} = \mathbf{k} - \mathbf{k}'$, θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$, and $v_F = \hbar k_F / m$. For our choice of impurity potential, $\tau_{\rm tr}^{-1}$ can be calculated,

$$\tau_{\rm tr}^{-1} = n_i v_F \left(\frac{m}{2\pi\hbar^2} \right)^2 V_0^2 \int d\Omega (1 - \cos\theta).$$

We emphasize here that the electronic transport relaxation time $\tau_{\rm tr}$ is directly determined by the Hamiltonian of Eq. (22).

Expressed in $\tau_{\rm tr}$, the spectral function now becomes

$$J(\omega) = 2\hbar\omega \left(\frac{k_F L}{2\pi\epsilon_0} \right)^2 n_i (\pi \xi^2 L) V_0^2 \left(\frac{k_F}{\pi \xi^2 L} \right)^2 = \omega \frac{3}{2} \frac{m n_e (\pi \xi^2 L)}{\tau_{\rm tr}}. \quad (87)$$

Here $\epsilon_0 = \Delta_\infty^2 / E_F$ is the core-level spacing, and $E_F = m v_F^2 / 2$. In Eq. (87) $k_F L / 2\pi\epsilon_0$ is the approximate density of core states near the Fermi surface. It appears with the factor $2\hbar\omega$ because

$$\sum_{\mu,\mu'} \delta(\hbar\omega - |E_\mu - E_{\mu'}|) |f_\mu - f_{\mu'}| = \int dE_\mu dE_{\mu'} \delta(\hbar\omega - |E_\mu - E_{\mu'}|) |f_\mu - f_{\mu'}| n_c^2(E) = 2\hbar\omega n_c^2(E)$$

with $n_c(E) \approx k_F L / (2\pi\epsilon_0)$. The scattering time $\tau_{\rm tr}$ is linked to the residual resistivity by

$$\rho = \frac{m}{n_e e^2 \tau_{\rm tr}}$$

and can be measured independently.

The above spectral function $J(\omega)$ gives the vortex friction in the weak impurity limit

$$\eta = \frac{3}{2} \frac{m n_e (\pi \xi^2 L)}{\tau_{\rm tr}}. \quad (88)$$

It has a simple interpretation. For a normal electron moving in the metal, the friction is simply $m / \tau_{\rm tr}$. Equation (88) can be interpreted as that in the weak impurity limit, the friction for a vortex is the friction for each electron times the total number of electrons inside the core, $n_e(\pi \xi^2 L)$.

This vortex friction increases with impurity concentration and strength. We will show that this increase eventually saturates in the dirty limit. Using Eq. (76), we expand localized states Ψ in terms of $\{\phi_\nu\}$, the of eigenfunctions of $\mathcal{H}_0$.

$$\left\langle \left| \int d^3x \Psi_\mu^\dagger(x) \nabla_{x_0} \mathcal{H}_0 \Psi_{\mu'}(x) \right|^2 \right\rangle = \sum_{\nu,\nu',\nu_1,\nu_1'} \langle \chi_{\mu\nu} \chi_{\mu'\nu'} \chi_{\mu\nu_1} \chi_{\mu'\nu_1'} \rangle (E_\nu - E_{\nu'})(E_{\nu_1} - E_{\nu_1'}) \times \int d^3x \phi_\nu^\dagger(x) \nabla \phi_{\nu'}(x) \cdot \int d^3x \phi_{\nu_1}^\dagger(x) \nabla \phi_{\nu_1'}(x)$$
$$= \sum_{\nu,\nu'} |\chi_{k\nu}|^2 |\chi_{k'\nu'}|^2 2\epsilon_0^2 |t_c|^2 \delta_{\nu',\nu\pm1}. \quad (89)$$

With the distribution function given in Eq. (82), the average value

$$\left\langle \left| \int d^3x \Psi^\dagger_\mu(x) \nabla_{x_0} \mathcal{H}_0 \Psi_{\mu'}(x) \right|^2 \right\rangle = \frac{4\epsilon_0^2 |t_c|^2}{N_c}.$$

Here the total number of core levels is

$$N_c = 2\Delta_\infty \frac{k_F L}{2\pi\epsilon_0} = \frac{E_F}{\Delta_\infty} \frac{k_F L}{\pi}.$$

Finally, the spectral function is

$$J(\omega) = \hbar\omega \left(\frac{k_F L}{2\pi\epsilon_0} \right)^2 \frac{4\epsilon_0^2 |t_c|^2}{N_c} = \omega \frac{3\pi^2}{8} \hbar n_e \frac{\Delta_\infty}{E_F} L, \quad (90)$$

which gives the friction per unit length

$$\eta = \frac{3\pi^2}{8} \hbar n_e \frac{\Delta_\infty}{E_F}.$$

This result is similar to what is obtained in Ref. 1. Hence its microscopic base has been provided. In the low-temperature limit, the magnitude of the vortex friction is smaller than the total transverse force by a factor of Δ_∞ / E_F.

In the above derivation, we have ignored the localization effect which suppresses the density of state, or the superfluid number density. We justify our assumption here. There are three energy scales involved in the derivation of vortex dynamics, the Fermi energy E_F, the energy gap Δ_∞, and the core-level spacing Δ_∞^2 / E_F. The effect of impurities on vortex dynamics is believed appreciable at $\tau_{tr} \Delta_\infty^2 / \hbar E_F \leqslant 1$,[9] and the equality of Eqs. (90) and (87) suggests that the impurity starts to be effective at $\tau_{tr} \Delta_\infty / \hbar (\Delta_\infty / E_F)^2 \sim 1$. They indicate that the impurity effect on vortex friction occurs at a rather weak level, determined by the smallest energy scale in the problem. The dirty limit is given by $\Delta_\infty / E_F < \tau_{tr} \Delta_\infty / \hbar < 1$. The localization effect is only pronounced in the extremely dirty limit, the localization regime, when $\tau_{tr} E_F / \hbar \leqslant 1$.[38] Because $\Delta_\infty / E_F \ll 1$, away from the localization regime the suppression of density is indeed negligible. The unsuppressed electronic density applies, and the present results are valid well into the dirty limit of the superconductors.

To summarize this section, we have shown that the total transverse force is insensitive to impurities by two different methods, but the additional core contribution to the vortex friction arises. For a weak enough impurity potential, a perturbative calculation leads to the core friction proportional to the normal-state resistivity. In the dirty limit the core friction contribution saturates to a value determined only by the energy gap and the Fermi energy.

VI. COUPLING TO ELECTROMAGNETIC FIELD

Now let us discuss a superconductor when the penetration depth is finite but still much larger than the coherence length. The Lagrangian is given by

$$L_{\rm BCS} = \sum_\sigma \psi^\dagger_\sigma(x,\tau) \Big[\hbar\partial_\tau - \mu_F - eA_0 + \frac{1}{2m} \left(\frac{\hbar}{i} \nabla - \frac{e}{c} \mathbf{A} \right)^2 + V(x) \Big] \psi_\sigma(x,\tau) - g \psi^\dagger_\uparrow(x,\tau) \times \psi^\dagger_\downarrow(x,\tau) \psi_\downarrow(x,\tau) \psi_\uparrow(x,\tau) + \frac{1}{8\pi}(E^2 + B^2) + eA_0 n_0, \quad (91)$$

here $-en_0$ is the charge density of the ionic background. The coupling to the electromagnetic field is in the usual minimum coupling form. The fermionic degrees of freedom can be integrated out to give

$$\frac{S_{\rm eff}}{\hbar} = -\,{\rm Tr} \ln G^{-1} + \frac{1}{\hbar g} \int_0^{\hbar\beta} d\tau \int d^3x |\Delta|^2 + \int_0^{\hbar\beta} d\tau \int d^3x \frac{1}{8\pi}(E^2 + B^2) + eA_0 n_0 \quad (92)$$

with

$$(\hbar\partial_\tau + \mathcal{H}) G(x,\tau;x',\tau') = \delta(\tau - \tau') \delta^3(x - x') \quad (93)$$

and

$$\mathcal{H} = \begin{pmatrix} H & \Delta \\ \Delta^* & -H^* \end{pmatrix}. \quad (94)$$

Here

$$H = -eA_0 + \frac{1}{2m} \left(\frac{\hbar}{i} \nabla - \frac{e}{c} \mathbf{A} \right)^2 - \mu_F + V(x),$$

$$H^* = -eA_0 + \frac{1}{2m} \left(\frac{\hbar}{i} \nabla + \frac{e}{c} \mathbf{A} \right)^2 - \mu_F + V(x),$$

and c is the speed of light in a vacuum. Variation with respect to A_0 and $\mathbf{A}$ gives

$$\nabla \cdot \mathbf{E} = 4\pi e (n - n_0)$$

and

$$\nabla \times \mathbf{B} - \frac{1}{c} \partial_\tau \mathbf{E} = \frac{4\pi e}{c} \mathbf{j}$$

with $\mathbf{E} = -\nabla A_0 + (1/c) \partial \mathbf{A} / \partial\tau$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Here en and $e\mathbf{j}$ are the electric charge and current densities. They should be obtained through the electronic Green's function. In the Lorentz gauge, the equations for A_0 and $\mathbf{A}$ from the above equations are[39] adapted to the imaginary time here,

$$\left[\nabla^2+\frac{1}{c^2}\partial_\tau^2\right]A_0=-4\pi e(n-n_0) \tag{95}$$

and

$$\left[\nabla^2+\frac{1}{c^2}\partial_\tau^2\right]\mathbf{A}=-\frac{4\pi}{c}e\mathbf{j}. \tag{96}$$

Assuming that for a static vortex at x_v the vector potential is $\overline{\mathbf{A}}(x-x_v)$, then for a slow moving vortex, the correction $\delta\mathbf{A}$ to the vector potential $\mathbf{A}$ from $\overline{\mathbf{A}}=\overline{\mathbf{A}}[x-x_v(\tau)]$ starts from second order in $\dot{x}_v(\tau)$ and can be ignored. This can be directly demonstrated from Eq. (96) to the leading order in $\delta\mathbf{A}$:

$$\nabla^2\delta\mathbf{A}+\frac{1}{c^2}\partial_\tau^2\overline{\mathbf{A}}=0.$$

The same is true for the scalar potential A_0. For our purpose of keeping to linear order in $\delta\dot{x}_v$, we may use $\mathbf{A}=\overline{\mathbf{A}}[x-x_v(\tau)]$.

Now we expand

$$A_l[x-x_v(\tau)]=(1+\delta x_v(\tau)\cdot\nabla_{x_0})\overline{A_l}[x-x_v(\tau)],$$

with $l=(0,x,y,z)$. The effective action for the vortex is

$$\frac{S_{\rm eff}}{\hbar}=\frac{1}{2}\mathrm{Tr}(G_0\Sigma')^2+\frac{1}{\hbar g}\int_0^{\hbar\beta}\int d^3x\,\delta x_v\cdot\nabla_{x_0}\Delta_0^*\,\delta x_v\cdot\nabla_{x_0}\Delta_0$$
$$+\int_0^{\hbar\beta}d\tau\int d^3x\frac{1}{8\pi}(E^2+B^2)+eA_0n_0 \tag{97}$$

with

$$\Sigma'=\delta x_v\cdot\nabla_{x_0}\begin{pmatrix}-\frac{e}{2mc}\frac{\hbar}{i}(\overline{\mathbf{A}}\cdot\nabla) & \Delta_0\\ \Delta_0 & -\frac{e}{2mc}\frac{\hbar}{i}(\overline{\mathbf{A}}\cdot\nabla)\end{pmatrix}$$
$$=\delta x_v\cdot\nabla_{x_0}\mathcal{H}, \tag{98}$$

where both $\mathbf{E}$ and $\mathbf{B}$ are substituted by the stationary values calculated from $A_0=0$ and $\mathbf{A}=\overline{\mathbf{A}}[x-x_v(\tau)]$. Equation (97) is in the same form as Eq. (36). There is then no change of the transverse response from the superfluid, reflecting the fact that the canonical momentum of the superfluid is not changed by the coupling to the electromagnetic field. A similar conclusion has also been reached by others phenomenologically.[40]

However, there are relativistic corrections to the solutions [Eqs. (95) and (96)] of the Maxwell equations due to the motion of the current and charge sources associated with the vortex. They are determined according to the Lorentz transformation of the four vector formed by the scalar and vector potentials, or equivalently, the four vector by the charge density and current. Those relativistic corrections give rise to additional terms in the action and can, in principle, contribute to the total transverse force in vortex dynamics. The relevant term in the effective action takes the form

$$\int_0^{\hbar\beta}d\tau\int d^3x\frac{e}{c}\delta\dot{x}_v\cdot\overline{\mathbf{A}}(x-x_v)[n(x-x_v)-n_0],$$

arising from the relativistic correction to the scalar potential, the Aharonov-Casher phase.[41] This contribution is due to the interaction between the moving magnetic flux carried by a vortex to the electric charges of both conducting electrons and the background charges. Since the charge neutrality condition is maintained in a superconductor, this contribution is zero, as also been noticed in Ref. 42. Hence, we do not need to consider it here. Other relativistic corrections do not affect the total transverse force, and the rest of the terms have the same structure as the uncharged superconductor with the replacement of Eq. (98) by Eq. (29). Therefore, all the steps from Eqs. (24)–(36) remain unchanged. We arrive at the same expressions [Eqs. (35) and (36)]. There are two differences, however. First, the Bogoliubov–de Gennes equation includes a vector potential, which can be served to generate a vortex, not by a rotation of superfluid. Second, we have now a compelling physical reason to neglect the phonon (density fluctuation) mode compared to the neutral case, because it is the plasma mode with a big energy gap.

Let us discuss the effect of including the vector potential in the Bogoliubov–de Gennes equation to the final results. For the extreme type-II superconductor the vector potential near the core is $A_\theta=\frac{1}{2}rh_0$, here h_0 is the magnetic field along the vortex line. When the penetration depth is large, h_0 is small. For small r, when solving the vortex core structure, $\hbar c/2er>A_\theta$ and we can safely ignore $A_\theta=\frac{1}{2}rh$. The core structure is insensitive to the vector potential in the extreme type-II case. Because the total transverse force can be expressed in core state transitions, it is insensitive to coupling to the electromagnetic field. Equivalently, when expressing the transverse response in terms of the summation over extended states, Eq. (55), we need the large-r behavior of the Bogoliubov–de Gennes equation. When $r\gg\lambda$, $A_\theta\to0$, so that the the coupling to the electromagnetic field will not influence the results in Eq. (55). In the presence of impurities, the vortex friction is also insensitive to the coupling to the electromagnetic field, as noticed long ago.[1]

To summarize this section, in a charge neutral extreme type-II superconductor, the vortex dynamics is the same as that in an uncharged BCS superfluid. This is a known result, but we have sketched how to obtain it within the present formulation.

VII. EXPERIMENTS

A. Transport measurement

It had been assumed that the forces on a vortex could be extracted from transport measurements. Let us first review this apparently plausible proposal and discuss ideas which are crucial to the understanding of transport measurements. Considering our previous derivations, the equation of motion

of the ith vortex takes the form of the Langevin equation similar to that of a charged particle in the presence of a magnetic field:

$$0 = q_v \rho_s h(\mathbf{v}_s - \dot{\mathbf{r}}_i) \times \hat{z} - \eta \dot{\mathbf{r}}_i + \mathbf{F}_{\text{pin}} + \mathbf{f} + \sum_j \mathbf{F}_{ij}. \quad (99)$$

Here $q_v = \pm 1$ represent different vorticity. The total transverse force $-q_v \rho_s h \dot{\mathbf{r}}$ and viscosity η are the ones we have calculated in the previous sections. In addition, there are a fluctuating force $\mathbf{f}$ related to the viscosity by the fluctuation-dissipation theorem, a pinning force $\mathbf{F}_{\text{pin}}$, and we should also include the forces due to other vortices $\mathbf{F}_{ij}$ because of vortex interaction. Here we have explicitly written out the external current term in Eq. (99), though in a real situation its effect is always through the rearrangement of vortices in the superconductor. The motion of vortices is a genuine many-body problem. A general exact solution does not exist.

Equation (99) may be solved after a drastic simplification by ignoring the pinnings. This is equivalent to the situation that a perfect vortex lattice is sliding through the sample. Together with with the Josephson relation, we can determine longitudinal and transverse resistivity for superconductors. The Hall angle, defined as $\theta_{\text{Hall}} = \tan^{-1}(\rho_{xy}/\rho_{xx})$, is nearly 90° for almost all situations. However, in transport measurement, most of the samples show a small Hall angle and some show a sign change in the Hall angle upon entering the superconducting state. This simplified model certainly disagrees with experiments.

Now let us consider whether or not this simplification can be made by considering the magnitude of pinning. The equation of motion of a vortex in a superconductor, Eq. (99), has the form of a particle with zero mass in a strong magnetic field. Because the kinetic energy is zero, it is always confined to a local energy minimum in space, formed by vortex interactions and pinning potentials. An applied current tilts the potential and the vortex moves by thermal activation. If the applied current is so large that no minima due to pinning and vortex interaction exist, then indeed we expect a large Hall angle. However, in order to have a truly free vortex flow, the current should be large enough to overcome the largest pinning potential, the edge pinning. The current needed is on the order of 10^8 A/cm^2, which is too large to be relevant to experiments. Therefore, in the real experiments, the vortices must be helped by their many-body interactions to overcome this energy barrier. We need to consider transport measurement by solving the lattice structure formed by vortices and by what mechanism their transport is made possible. It has been quantitatively suggested that vortex many-body effects can be responsible for the Hall effect, in agreement with recent experimental indications.[43]

B. Direct measurement of total transverse force

The total transverse force on a moving vortex in thin Y-Ba-Cu-O films has been directly measured via a mechanical device.[44] This experiment used a small vibrating magnet mounted above the center of a superconducting film to generate moving vortices in the film. The vortices follow the motion of the magnet for samples with less twin boundaries. The experiments were performed on those samples. The force was measured by measuring the motion of the superconducting film in response to the vibration of the magnet. The experimental results have provided a qualitative confirmation of the insensitivity of the total transverse force to impurities.

C. Measurement of friction

In a rf resistance measurement, the vortices are moving around their local minima, rather than over potential barriers in a dc measurement. In such a case, it is possible to observe intrinsic friction after using a potential to describe the periodic vortex interaction and making some assumptions about the pinning. The rf resistance was analyzed early with a vortex dynamics model without transverse force.[45] In order to compare with our theory, the total transverse force needs to be included. We will not go into any detail other than to suggest this possibility.

VIII. CONCLUSIONS

We summarize here what we have achieved in the present paper. With respect to the microscopic derivation, we have developed an influence functional formulation started from the BCS theory. This formulation has allowed us to discuss several difficult questions regarding vortex dynamics. One question has been whether the total transverse force originates from core states, extended states, or from both. This question is unique to a fermionic superfluid because of the vortex core structure. We have shown that the total transverse force can be calculated equivalently by considering exclusively transitions between core states, by transitions between core and extended states, or by counting contributions from extended states. The total core-state transition contribution to the total transverse force is shown not to be affected by impurities when calculated by using random matrices instead of the relaxation time approximation.

On the thermodynamics and statistical mechanics level, we need to consider the increase of the superfluid kinetic energy associated with the increase of superfluid momentum due to the vortex motion. This kinetic energy needs to be provided from somewhere. If there are no normal fluid and no impurities, this kinetic energy is provided from the work done by the external trapping potential on the vortex. When either the normal fluid or impurities, or both, are present, there is a question whether or not a vortex can extract the internal energy from the normal fluid or substrate, which carries entropy, and can transfer it into kinetic energy of the superfluid, which carries no entropy. If not allowed, the increase of superfluid kinetic energy due to vortex motion can only be provided by an external force and the transverse force on a vortex cannot be reduced by the normal fluid or random impurities. We have discussed this question and demonstrated that thermodynamics gives a powerful constraint on phenomenological models of vortex dynamics: The total transverse force cannot be reduced.

We have located the source for contradicting theoretical results in the two pictures: the use of the relaxation time approximation in the force calculation. This problem is rather subtle. It is well known that the relaxation time approximation has been used successfully in some applications,

particularly in calculations of conductivity or mobility, where the average velocity is computed under a given driving force, through velocity-velocity correlations. However, the special feature of vortex dynamics is that it belongs to the same category as resistivity or friction formulas in transport theory, where the average force is computed with a given velocity. The direct calculation of resistivity is known to be difficult. To derive vortex dynamics microscopically, the vortex velocity-velocity correlation is not calculable directly, because the effective vortex Hamiltonian is unknown and is precisely what we are looking for. We are forced to abandon the usual approaches of the Nakano-Kubo type, and to tackle the problem from the difficult side. Nevertheless, in the dc limit, the transverse force on a moving vortex can be calculated from the force-force correlation function, in analogy to a dc resistivity formula. This limit makes the relaxation time approximation invalid, because a significant part of the frequency dependence is lost, and the common way of introducing the relaxation time approximation by substituting $i\omega \rightarrow i\omega + 1/\tau$ requires the correct frequency dependence. For example, in transport theory, the relaxation time approximation is used in an ac conductivity formula, then taking the dc limit subsequently. In addition, the relaxation time approximation in a force-force correlation function is always erroneous. With an exactly solvable model, we have shown that when the relaxation time approximation is used in a dc resistivity formula, it leads to results violating fluctuation-dissipation theorems.

Introducing the relaxation time approximation, even done correctly, is not a necessary step in obtaining dissipation. One of the goals of nonequilibrium statistical mechanics is to compute various transport coefficients, including the relaxation time, for a given Hamiltonian system. In a Hamiltonian system, dissipation appears after irrelevant degrees of freedom are integrated out. What determines dissipation are quantities like temperature and strength of impurity potentials, as well as the density of state of low-frequency modes of irrelevant degrees of freedom. In the present paper, irrelevant degrees of freedom are the fermionic quasiparticles. When those quasiparticle degrees of freedom are eliminated, one obtains the vortex friction. The friction formula obtained here follows the one used in dissipative quantum dynamics,[13] where it has been explicitly shown that the friction obtained by eliminating irrelevant degrees of freedom is equivalent to an evaluation of the random force-force correlation function. It also corresponds to the familiar Fermi Golden rule for dissipation. We believe that a rather detailed study of vortex dynamics based on the BCS theory have been presented here, with the key issue of the sources for the vortex friction. We have shown that the vortex friction can come from two contributions: At finite temperatures, the finite population of quasiparticles above and quasiholes below the energy gap give rise to a friction which diverges logarithmically at low frequency; The nonmagnetic impurities give rise to an extra friction which saturates to a value independent of the normal-state resistivity in the dirty limit. This core state contribution corresponds to the phenomenological value obtained in Ref. 1. We have also considered the effect of coupling to the electromagnetic field and have found that it does not change the neutral superfluid conclusions when the superconductor is charge neutral, consistent with earlier phenomenological treatments.

Finally, we expect that the method developed here to formulate the vortex dynamics in an *s*-wave superconductor will find applications in other systems represented by dynamics of collective variables, such as vortex dynamics in *d*-wave superconductors, fission and fusion in atomic nuclei, and even the quasiparticle dynamics in quantum Hall systems.

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APPENDIX: DIVERGENT OVERLAP INTEGRALS AND VORTEX FRICTION

There have been some questions above the implications of Eqs. (38) and (39), repeatedly raised by referees as well as by others during private discussions, in particular on the diverging nature of the overlap integrals on the right-hand side of Eqs. (38) and (39), when the energy difference between two eigenfunctions vanishes. This question may have already been addressed in the literature. Nevertheless, we believe it is helpful to give it an explicit discussion in the present context.

We note that Eqs. (38) and (39) are exact consequences of the fact that the Hamiltonian $\mathcal{H}_0$ is the function of the parameter x_0. To make the connection to the scattering problem of quasiparticles scattered off a vortex, the thermodynamic limit must be taken first to allow the existence of the continuous spectrum. This implies that it is appropriate to use the Dirac delta function normalization for extended states,

$$\Psi_\alpha = \begin{pmatrix} u_\alpha(x) \\ v_\alpha(x) \end{pmatrix} = \frac{e^{ik_z z}}{\sqrt{L}} \frac{e^{i\mu\theta}}{\sqrt{2\pi}} \begin{pmatrix} e^{i(\theta/2)} \hat{f}_{+,\mu,E}(x) \\ e^{-i(\theta/2)} \hat{f}_{-,\mu,E}(x) \end{pmatrix}, \quad \text{(A1)}$$

the same normalization condition as in Ref. 30. The function satisfies the Bogoliubov–de Gennes equation, Eq. (30),

$$\frac{\hbar^2}{2m}\left[-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{(\mu+\frac{1}{2})^2}{r^2} - k_\rho^2 \right] \hat{f}_{+,\mu,E}(x) + |\Delta(r)| \hat{f}_{-,\mu,E}(x) = E \hat{f}_{+,\mu,E}(x) \quad \text{(A2)}$$

and

$$-\frac{\hbar^2}{2m}\left[-\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+\frac{(\mu-\frac{1}{2})^2}{r^2}-k_\rho^2\right]\hat{f}_{-,\mu,E}(x)$$
$$+|\Delta(r)|\hat{f}_{+,\mu,E}(x)=E\hat{f}_{-,\mu,E}(x). \quad \text{(A3)}$$

Here $r=|x-x_0|$ and $k_z^2+k_\rho^2=k_F^2$. Inside the vortex core, we may set the energy gap to zero, $|\Delta(r)|=0$. There are two independent solutions in this region, which we may choose to be the following forms:

$$\hat{f}_{1,\mu,E}(x)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}J_{\mu+1/2}(\sqrt{k_\rho^2+2m|E|/\hbar^2}\,r) \quad \text{(A4)}$$

and

$$\hat{f}_{2,\mu,E}(x)=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}J_{\mu-1/2}(\sqrt{k_\rho^2-2m|E|/\hbar^2}\,r). \quad \text{(A5)}$$

Here $J_{\mu\pm1/2}$ are Bessel functions. Away from the zero-energy gap region, the corresponding solutions may take the forms

$$\hat{f}_{1,\mu,E}(x)=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1+\sqrt{E^2-|\Delta|^2}/E}\\ \sqrt{1-\sqrt{E^2-|\Delta|^2}/E}\end{pmatrix}J_{\mu+1/2}(k_+(E)r) \quad \text{(A6)}$$

and

$$\hat{f}_{2,\mu,E}(x)=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1-\sqrt{E^2-|\Delta|^2}/E}\\ \sqrt{1+\sqrt{E^2-|\Delta|^2}/E}\end{pmatrix}J_{\mu-1/2}(k_-(E)r), \quad \text{(A7)}$$

where $k_\pm(E)=\sqrt{k_\rho^2\pm2m\sqrt{E^2-|\Delta|^2}/\hbar^2}$. One may check that Eqs. (A6) and (A7) give the asymptotically exact solutions when $r=\infty$. They are WKB-type solutions connected to the solutions at $r=0$ and $r=\infty$, valid under the condition that the energy gap $|\Delta|$ is smooth on the scale of $1/k_F$. Exact solutions may be difficult to find. However, for the present purpose of demonstration of the diverging overlap integrals they are good enough. The solutions for a negative energy $-E$ can be constructed by using Eq. (37):

$$\hat{f}_{1,-\mu,-E}(x)=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1-\sqrt{E^2-|\Delta|^2}/E}\\ -\sqrt{1+\sqrt{E^2-|\Delta|^2}/E}\end{pmatrix}J_{\mu+1/2}(k_+(E)r) \quad \text{(A8)}$$

and

$$\hat{f}_{2,-\mu,-E}(x)=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1+\sqrt{E^2-|\Delta|^2}/E}\\ -\sqrt{1-\sqrt{E^2-|\Delta|^2}/E}\end{pmatrix}J_{\mu-1/2}(k_-(E)r). \quad \text{(A9)}$$

The immediate conclusion of the thermodynamic limit is that there is an infinite degeneracy for a given energy characterized by μ, corresponding to the angular momenta of quasiparticles. Those states form the base functions for the partial wave analysis of the quasiparticle scattering, and make the transitions between states with the same energy meaningful.

We now consider the left-hand side of Eq. (38) with an arbitrary small energy difference,

$$\mathbf{I}\equiv\int d^3x\Psi_\alpha^\dagger(x)(\nabla_{x_0}\mathcal{H}_0)\Psi_{\alpha'}(x). \quad \text{(A10)}$$

We will show that it can be a finite value (nonzero). For the vanishing small trapping potential U_0,

$$(\nabla_{x_0}\mathcal{H}_0)=\begin{pmatrix}0 & \nabla_{x_0}\Delta\\ \nabla_{x_0}\Delta^* & 0\end{pmatrix}. \quad \text{(A11)}$$

Since

$$\nabla_{x_0}\Delta(x)=-e^{i\theta}|\Delta(r)|_r'(\hat{x}\cos\theta+\hat{y}\sin\theta)$$
$$-i\,e^{i\theta}|\Delta(r)|\frac{-\hat{x}\sin\theta+\hat{y}\cos\theta}{r}, \quad \text{(A12)}$$

the integral $\mathbf{I}$ may be expressed as

$$\mathbf{I}=\delta_{k_z,k_z'}\int_0^\infty rdr\int_0^{2\pi}\frac{d\theta}{2\pi}e^{-i(\mu-\mu')\theta}$$
$$\times[\hat{f}_{+,\mu,E}^*(r)\nabla_{x_0}\Delta\hat{f}_{-,\mu',E'}(r)$$
$$+\hat{f}_{-,\mu,E}^*(r)\nabla_{x_0}\Delta^*\hat{f}_{+,\mu',E'}(r)]$$
$$=\frac{1}{2}\delta_{k_z,k_z'}\delta_{\mu',\mu\pm1}[\hat{x}(a_{\mu,\mu'}(E,E')\pm b_{\mu,\mu'}(E,E'))$$
$$+i\hat{y}(\pm a_{\mu,\mu'}(E,E')+b_{\mu,\mu'}(E,E'))], \quad \text{(A13)}$$

where $\hat{x}(\hat{y})$ is the unit vector in the $x(y)$ direction,

$$a_{\mu,\mu'}(E,E')=-\int_0^\infty rdr|\Delta|_r'[\hat{f}_{+,\mu,E}^*(r)\hat{f}_{-,\mu',E'}(r)$$
$$+\hat{f}_{-,\mu,E}^*(r)\hat{f}_{+,\mu',E'}(r)] \quad \text{(A14)}$$

and

$$b_{\mu,\mu'}(E,E')=-\int_0^\infty dr|\Delta|[\hat{f}_{+,\mu,E}^*(r)\hat{f}_{-,\mu',E'}(r)$$
$$-\hat{f}_{-,\mu,E}^*(r)\hat{f}_{+,\mu',E'}(r)]. \quad \text{(A15)}$$

The $\cos\theta$ and $\sin\theta$ inside the integral give rise to the selection rule for the transition elements: $\int_0^{2\pi}d\theta e^{-i(\mu-\mu')\theta}\cos\theta$

$=\pi\delta_{\mu',\mu\pm1}$ and $\int_0^{2\pi}d\theta e^{-i(\mu-\mu')\theta}\sin\theta=\pm i\pi\delta_{\mu',\mu\pm1}$. Only the transition between neighboring $\mu's$, that is, $\mu'=\mu\pm1$, can be nonzero.

For those nonzero $a_{\mu,\mu\pm1}(E,E')$, the only possible place which may give rise to an infinite value for the integral is the region far away from the vortex core. In this region, the eigenfunctions are given in the form of Bessel functions, Eqs. (A6)–(A9) [cf. Eq. (4.10) of Ref. 30], which are well behaved. Away from the core $|\Delta(r)|_r'$ goes to zero rapidly, we conclude that the integral $a_{\mu,\mu'}(E,E')$ containing $|\Delta(r)|_r'$ is finite. It may be instructive to give an estimation of $a_{\mu,\mu\pm1}(E,E')$. For this purpose we consider $\mu>0$ and $\mu'=\mu+1$ for positive energy states. The negative μ and negative energy cases give similar results. We will also restrict to the case that E and E' are close to each other, i.e., $|E-E'|<\Delta_\infty$, and that both are sufficiently close to the energy gap, i.e., $E\sim\Delta_\infty$. In this case the integral has the largest value, and it incorporates the equal energy limit implied in Eqs. (47) and (49). First, we note that since $k_\pm(E)\approx k_F$ and $J_\mu(k\pm(E)r)=(k_\pm(E)r)^\mu/\mu!$ for small $k(E)r$, the Bessel function is negligible small if $r<r_t=\mu/k_F$ and μ is large. The integral for region $r<r_t$ is negligible. Second, we also neglect the integral in the region $r>\xi_0$, because $|\Delta(r)|_r'$ is small. The integral now becomes

$$a_{l,\mu}\approx\begin{cases}-\int_{r_t}^{\xi_0}rdr|\Delta|_r'[\hat f^*_{+,\mu,E}(r)\hat f_{-,\mu+1,E'}(r)\\ +\hat f^*_{-,\mu,E}(r)\hat f_{+,\mu+1,E'}(r)], & r_t<\xi_0\\ 0, & r_t>\xi_0.\end{cases}\tag{A16}$$

Here we have used l to denote the various combinations from the solutions, Eqs. (A6)–(A9), specified below. In the region $r<\xi_0$, $|\Delta|_r'\approx\Delta_\infty/\xi_0$ in the above integral. For $k_+(E),k_+(E')$ [$k_-(E),k_-(E')$ can be considered in the same manner], following Eqs. (A16) and (A6) we have

$$\begin{aligned}a_{1,\mu}=&-\int_{r_t}^{\xi_0}rdr\frac{\Delta_\infty}{\xi_0}\frac{1}{2}\\&\times\Bigg[\sqrt{\left(1+\frac{\sqrt{E^2-|\Delta|^2}}{E}\right)\left(1-\frac{\sqrt{E'^2-|\Delta|^2}}{E'}\right)}\\&+\sqrt{\left(1-\frac{\sqrt{E^2-|\Delta|^2}}{E}\right)\left(1+\frac{\sqrt{E'^2-|\Delta|^2}}{E'}\right)}\Bigg]\\&\times J_{\mu+1/2}(k_+(E)r)J_{\mu+3/2}(k_+(E')r).\end{aligned}\tag{A17}$$

Using the asymptotic form of Bessel function, $J_\nu(z)=\sqrt{2/\pi z}\cos(z-\nu\pi/2-\pi/4)$, and approximating the factor in the square bracket by 2, we find

$$a_{1,\mu}\approx-\frac{\Delta_\infty}{\xi_0}\frac{1}{2\pi k_F}\int_{r_t}^{\xi_0}dr\sin[\Delta k\,r]=-\frac{\Delta_\infty}{2\pi k_F}\Delta k\xi_0/2.\tag{A18}$$

Here $\Delta k\equiv k_+(E)-k_+(E')$. In reaching Eq. (A18) we have dropped a smaller contribution from $\int_{r_t}^{\xi_0}dr\cos[(k_+(E)+k_+(E'))r-(2\mu+3)\pi/2]$, because $k_+(E)r_t=\mu>1$, and have also used the fact that $|k_+(E)-k_+(E')|\xi_0<1$. Equation (18) gives $|a_{\mu,\mu+1}(E,E')|<\Delta_\infty/k_F$.

For $k_+(E),k_-(E')$, following Eqs. (A16), (A6), and (A7) we have

$$\begin{aligned}a_{2,\mu}=&-\int_{r_t}^{\xi_0}rdr\frac{\Delta_\infty}{\xi_0}\frac{1}{2}\\&\times\Bigg[\sqrt{\left(1+\frac{\sqrt{E^2-|\Delta|^2}}{E}\right)\left(1+\frac{\sqrt{E'^2-|\Delta|^2}}{E'}\right)}\\&+\sqrt{\left(1-\frac{\sqrt{E^2-|\Delta|^2}}{E}\right)\left(1-\frac{\sqrt{E'^2-|\Delta|^2}}{E'}\right)}\Bigg]\\&\times J_{\mu+1/2}(k_+(E)r)J_{\mu+1/2}(k_-(E')r).\end{aligned}\tag{A19}$$

Please note the difference between Eqs. (A17) and (A19) in the indices of the Bessel functions. Since $k_+(E)-k_-(E')\approx1/\xi_0$, and again approximating the factor in the square bracket by 2, we find that

$$a_{2,\mu}=\frac{\Delta_\infty}{\xi_0}\frac{1}{2\pi k_F}\int_{r_t}^{\xi_0}dr\cos[(k_+(E)-k_-(E'))r]\approx-\frac{\Delta_\infty}{2\pi k_F}.\tag{A20}$$

Now we consider the phase integral part of **I**, the integral $b_{\mu,\mu'}(E,E')$. For $r=|x-x_0|\to\infty$, $|\Delta|\to\Delta_\infty$. We may ignore the integral in the region $r<r_t$, but not when $r_t>\xi_0$. Keeping the leading contribution, $b_{\mu,\mu'}(E,E')$ may be expressed as

$$\begin{aligned}b_{l,\mu}\approx&-\Delta_\infty\int_{r_t}^{\infty}dr[\hat f^*_{+,\mu,E}(r)\hat f_{-,\mu',E'}(r)\\&-\hat f^*_{-,\mu,E}(r)\hat f_{+,\mu',E'}(r)].\end{aligned}\tag{A21}$$

The Bessel functions will be replaced by their asymptotic forms inside Eq. (A21). In the following we consider four cases as done for $a_{\mu,\mu'}(E,E')$. For $k_+(E),k_+(E')$, following Eqs. (A21) and (A6) we have

$$\begin{aligned}b_{1,\mu}=&-\Delta_\infty\int_{r_t}^{\infty}dr\frac{1}{2}\\&\times\Bigg[\sqrt{\left(1+\frac{\sqrt{E^2-|\Delta|^2}}{E}\right)\left(1-\frac{\sqrt{E'^2-|\Delta|^2}}{E'}\right)}\\&-\sqrt{\left(1-\frac{\sqrt{E^2-|\Delta|^2}}{E}\right)\left(1+\frac{\sqrt{E'^2-|\Delta|^2}}{E'}\right)}\Bigg]\\&\times J_{\mu+1/2}(k_+(E)r)J_{\mu+3/2}(k_+(E')r).\end{aligned}\tag{A22}$$

Since $E'\to E$ and both are close to the energy gap, the factor inside the square bracket is always an order of unity, and we approximate by 2. However, we note that when $E=E'$, the term in the square bracket approaches to zero when $r\gg\xi_0$. Using the asymptotic form of the Bessel functions,

$$b_{1,\mu} \approx -\Delta_\infty \int_{r_t}^{\infty} dr \frac{1}{2\pi k_F} \frac{\sin[\Delta k_{++} r]}{r} = -\frac{\Delta_\infty}{2\pi k_F} \begin{cases} \mathrm{sgn}(\Delta k)\pi/2, & |\Delta k| r_t < 1 \\ 1/\Delta k r_t, & |\Delta k| r_t > 1. \end{cases} \tag{A23}$$

Here $\Delta k \equiv k_+(E) - k_+(E')$. Again, the contribution from $\int_{r_t}^{\infty} dr \cos[(k_+(E)+k_+(E'))r - (2\mu+3)\pi/2]/r$ has been ignored, because $k_+(E) r_t > 1$. Since $|k_+(E) - k_+(E')|\xi_0 < 1$, the condition $|k_+(E) - k_+(E')| r_t < 1$ will be satisfied if $r_t < \xi_0$.

For $k_+(E), k_-(E')$, following Eqs. (A21), (A6), and (A7) we have

$$b_{2,\mu} = -\Delta_\infty \int_{r_t}^{\infty} dr \frac{1}{2} \times \left[\sqrt{\left(1 + \frac{\sqrt{E^2 - |\Delta|^2}}{E}\right)\left(1 + \frac{\sqrt{E'^2 - |\Delta|^2}}{E'}\right)} - \sqrt{\left(1 - \frac{\sqrt{E^2 - |\Delta|^2}}{E}\right)\left(1 - \frac{\sqrt{E'^2 - |\Delta|^2}}{E'}\right)} \right] \times J_{\mu+1/2}(k_+(E) r) J_{\mu+1/2}(k_-(E') r). \tag{A24}$$

Using a similar procedure for $b_{1,\mu}$ we find

$$b_{2,\mu} \approx -\Delta_\infty \int_{r_t}^{\infty} dr \frac{1}{2\pi k_F} \frac{\cos[(k_+(E) - k_-(E'))r]}{r} = -\frac{\Delta_\infty}{2\pi k_F} \begin{cases} O(1), & r_t/\xi_0 < 1 \\ \xi_0/r_t, & r_t/\xi_0 > 1. \end{cases} \tag{A25}$$

We have used $k_+(E) - k_-(E') \approx 1/\xi_0$ in Eq. (A29). If one is concerned about the logarithmic divergence of the cosine integral when $k_+(E) - k_-(E') \to 0$, we point out that it only occurs when both $|E|$ and $|E'|$ are approaching the energy gap Δ_∞. In this limit the factor inside the square bracket goes to zero linearly, and completely removes the logarithmic factor from the cosine integral.

The conclusion which one may draw from Eqs. (A23) and (A25) is that the integral $b_{\mu,\mu'\pm1}(E,E')$ is finite. Together with what we have obtained for $a_{\mu,\mu'\pm1}(E,E')$, the integral **I**, therefore, the left side of Eq. (38), is finite.

Using Eq. (38), we have the overlap integral,

$$\mathbf{II} \equiv \int d^3x \Psi_\alpha^\dagger \nabla_{x_0} \Psi_{\alpha'}(x) = \frac{\mathbf{I}}{E_{\alpha'} - E_\alpha}. \tag{A26}$$

Since **I** is finite for the case of $\mu' = \mu \pm 1$ when $E_{\alpha'} - E_\alpha \to 0$, **II** diverges as $1/(E_{\alpha'} - E_\alpha)$. Because asymptotically from the vortex core the wave functions Ψ_α always approaches a Bessel function, this diverging behavior may be directly deduced from the right-hand side of Eq. (38) with the aid of the recurrence relations of the Bessel functions. The advantage of the demonstration here is that the right-hand side of Eq. (49) is finite when two energies are exactly equal, without the explicit consideration of the diverging behavior of the overlap integral.

There are two comments worthwhile to make.

(1) The existence of the limit at the left-hand side of Eq. (38) in the zero-energy difference indicates that the spectral function of Eq. (49) is a smooth function for small frequencies, and it may be characterized by a power of the frequency.

(2) This limiting behavior also removes the paradox that the frictional coefficient involves inelastic processes, but it may be obtained by calculating the elastic-scattering crossection of quasiparticles implied in the thermodynamic limiting procedure.

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