Critical specific heats of the *N*-vector spin models on the simple cubic and bcc lattices

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We have computed through order β^{21} the high-temperature expansions for the nearest neighbor spin correlation function $G(N,\beta)$ of the classical N-vector model, with general N, on the simple cubic and on the body centered cubic lattices. For this model, also known in quantum field theory as the lattice $O(N)$ nonlinear σ model, we have presented in previous papers extended expansions of the susceptibility, of its second field derivative, and of the second moment of the correlation function. Here we study the internal specific energy and the specific heat $C(N,\beta)$, obtaining updated estimates of the critical parameters and therefore a more accurate direct test of the hyperscaling relation $d\nu(N)=2-\alpha(N)$ on a range of values of the spin dimensionality N, including $N=0$ (the self-avoiding walk model), $N=1$ (the Ising spin 1/2 model), $N=2$ (the XY model), $N=3$ (the classical Heisenberg model). By the newly extended series we also compute the universal combination of critical amplitudes usually denoted by $R_\xi^+(N)$, in fair agreement with renormalization group estimates. [S0163-1829(99)04633-0]

I. INTRODUCTION

We continue in this paper the analysis of recently extended[1–3] high-temperature (HT) expansions for the N-vector model[4] with general spin dimensionality N. Our computation is concerned with the d-dimensional bipartite lattices, namely, the simple-cubic (sc) lattice, the body-centered-cubic (bcc) lattice, and their d-dimensional generalizations.

In previous papers we tabulated (i) the HT series for the zero field susceptibility $\chi(N,\beta)$ and for the second moment of the correlation function $\mu_2(N,\beta)$ through order β^{21}, (ii) the HT series for the second field derivative of the susceptibility $\chi_4(N,\beta)$ through order β^{17}, and have analyzed their critical behavior in the $d=2$ case[1] and in the $d=3$ case.[2,3] We have performed the computation using the (vertex-renormalized) linked cluster expansion method[5] and have produced tables of series coefficients written as explicit functions of the spin dimensionality N with an extension independent of the structure and dimensionality of the lattice. More details on the derivation of the series, and on the checks of validity of our results can be found in our previous papers.[1–3]

In this paper we examine the series expansions of the nearest neighbor correlation function $G(N,\beta)$ through order β^{21}, in order to update, on a range of values of the spin dimensionality N, the direct estimates of the parameters describing the behavior of the specific heat $C(\beta,N)$ on the HT side of the critical point $\beta_c(N)$. We also update direct tests of the hyperscaling relation $d\nu(N)=2-\alpha(N)$ and estimate a related universal combination of critical amplitudes introduced by Stauffer, Ferer, and Wortis[6] and later denoted by $R_\xi^+(N)$.[7–10] Here $\alpha(N)$ is the critical exponent of the specific heat and $\nu(N)$ is the critical exponent of the correlation length $\xi(N,\beta)$. Estimates of $\alpha(N)$ are also obtained by studying the behavior of the extended series for the susceptibility $\chi(\beta,N)$ and for the second moment of the correlation function $\mu_2(\beta,N)$ at the symmetrically placed antiferromag-

netic singular point $\beta_c^{af}(N)=-\beta_c(N)$.[11]

In order to put our work into a proper perspective, it is convenient to list the HT expansions of $G(N,\beta)$ for the sc, bcc, and face-centered-cubic (fcc) lattices, which were published before our extension. As is well known, for $N=0$, the N-vector model reduces[12,13] to the self-avoiding walk (SAW) model, and the expansion of the correlation function $G(0,\beta)$, simply related to the enumeration of the self-avoiding rings (or polygons) had already been computed in Ref. 14 up to order β^{19} for the sc lattice, up to order β^{15} for the bcc lattice,[14] and up to β^{13} for the fcc lattice. In the $N=1$ case, which corresponds to the spin-1/2 Ising model, an expansion of $G(1,\beta)$ for the sc lattice was obtained a few years ago by Enting and Guttmann[15] up to order β^{21} using finite lattice methods. More recently, within the same approach, this computation has been pushed to order β^{23} in Ref. 16 and then to order β^{25} in Ref. 17. Also an approximate determination of the coefficient of β^{27} was reported in Ref. 17. An expansion through order β^{15} for the bcc lattice, and one for the fcc lattice up to order β^{12} have been tabulated in Ref. 18. For $N=2$ (the XY model) the available series[19] for the bcc lattice reached the order β^{11}. In the $N=3$ case (the classical Heisenberg model), the series for the bcc lattice, known only up to order β^9, was reported in Ref. 20.

Finally, let us cite an expansion of $G(N,\beta)$, valid for general N and for all loosely packed lattices, tabulated (with some misprints) in Ref. 4 up to order β^9, which has been later extended to models with general anisotropic pair interaction in Ref. 21. The expansion of $G(N,\beta)$ has been recently pushed[22] to order β^{15} in the case of the sc lattice, but no comparable effort has been devoted to the bcc lattice. In Ref. 23, an expansion to order β^{11}, valid for general N, had been tabulated for the fcc lattice.

We should finally call the readers' attention to the valuable reviews in Refs. 9,10,24 and to the accurate recalculation, within the renormalization group (RG) approach, of the universal critical parameters of the N-vector model performed by Guida and Zinn-Justin.[25] This work is based on

the recently extended field theoretic expansions of Ref. 26 and is also accompanied by an extensive review of the available numerical and experimental data.

The paper is organized as follows. In Sec. II we set our notation and define the quantities we shall study. In Sec. III we discuss briefly the numerical tools used for our estimates and present the results of our analysis of the series. These results are compared with experimental data, with earlier work on shorter HT series, with measures performed in stochastic simulations and with RG estimates, obtained either by the fixed dimension (FD) perturbative technique[10,25–31] or by the Fisher-Wilson[32] ϵ-expansion approach.[10,25,29–31,33,34] Our conclusions are briefly summarized in Sec. IV.

The HT series expansion coefficients of the nearest neighbor correlation function $G(N,\beta)$ expressed in closed form as functions of the spin dimensionality N, for the sc and the bcc lattices, have been tabulated in the appendixes in order to make each step of our work completely reproducible. For convenience of the reader, we also have explicitly evaluated the series coefficients for $N=0$ (the SAW model), $N=1$ (the Ising spin-1/2 model), $N=2$ (the XY model), and $N=3$ (the classical Heisenberg model).

II. DEFINITIONS AND NOTATION

We study the N-vector model with Hamiltonian

$$H\{v\} = -\frac{1}{2} \sum_{\langle \vec{x},\vec{x}'\rangle} v(\vec{x}) \cdot v(\vec{x}'), \qquad (1)$$

where the variable $v(\vec{x})$ represents a N-component classical spin of unit length at the lattice site with position vector $\vec{x}$, and the sum extends to all nearest neighbor pairs of sites. The basic observables are the spin correlation functions. Here we shall be interested in the connected correlation functions $\langle v(0) \cdot v(\vec{x})\rangle_c$ between the spin at the origin and the spin at the site $\vec{x}$. In particular, the nearest neighbor spin correlation function is defined by

$$G^{\#}(N,\beta) = \langle v(0) \cdot v(\vec{\delta})\rangle_c = \sum_{r=0}^{\infty} a_r^{\#}(N)\beta^r, \qquad (2)$$

where $\vec{\delta}$ is a nearest neighbor lattice vector and # stands for either sc or bcc, as appropriate.

Due to the bipartite structure of the sc and the bcc lattices, the connected correlations $\langle v(0) \cdot v(\vec{x})\rangle_c$ are functions of β with the same parity as the lattice distance between the spins and hence alternate expansion coefficients vanish identically: in particular in our expansions of $G^{\#}(N,\beta)$ to order β^{21} only eleven coefficients are nonvanishing. This is the reason why most analyses in the literature have focused on series for the nonbipartite fcc lattice which have no such symmetry. The specific internal energy is defined by

$$U^{\#}(N,\beta) = -\frac{q}{2} G^{\#}(N,\beta), \qquad (3)$$

where q is the lattice coordination number. If we denote the reduced inverse temperature by $\tau^{\#}(N) = 1 - \beta/\beta_c^{\#}(N)$, then $U^{\#}(N,\beta)$ is expected to behave as[9]

$$U^{\#}(N,\beta) \simeq U_{\text{reg}}^{\#}(N,\beta) + A_U^{\#}(N)[\tau^{\#}(N)]^{1-\alpha(N)}$$
$$\times \{1 + a_U^{\#}(N)[\tau^{\#}(N)]^{\theta(N)} + \cdots\} \qquad (4)$$

when $\tau^{\#}(N) \downarrow 0$.

As is customary, in writing the asymptotic form Eq. (4), we have explicitly indicated the presence of the nonsingular background $U_{\text{reg}}^{\#}(N,\beta)$, because the critical singularities of the specific energy are known to be generally very weak. Here $A_U^{\#}(N)$ denotes the critical amplitude of the specific energy, $a_U^{\#}(N)$ is the amplitude of the leading singular correction[35] to scaling, and $\theta(N)$ is the exponent of this correction also called confluent singularity exponent. The ellipses represent higher order singular or analytic correction terms. Unlike the critical exponent $\alpha(N)$, which is universal, the critical amplitudes $A_U^{\#}(N), a_U^{\#}(N)$, etc., are expected to depend on the parameters of the Hamiltonian and on the lattice structure, i.e., they are nonuniversal. Similar considerations apply to the other thermodynamic quantities listed below, which have different critical exponents and different critical amplitudes, but the same leading confluent exponent $\theta(N)$. It is known that $\theta(N) \simeq 0.5$ for small values of N.[10] Having clearly indicated which quantities are universal, we shall often drop the generic superscript # (or its determination) in order to avoid overburdening the notation. Notice also that, since there is no chance of confusion, we have generally omitted the superscript + usually adopted in the literature for the amplitudes which characterize the high-temperature side of the critical point.

The specific heat per site, at fixed magnetic field H, is defined as the temperature derivative of the specific internal energy

$$C_H(N,\beta) = \frac{d}{dT} U(N,\beta) = \frac{q}{2}\beta^2 \frac{d}{d\beta} G(N,\beta), \qquad (5)$$

where T is the temperature. As $\tau(N) \downarrow 0$, the critical behavior of $C_H(N,\beta)$ is described by

$$C_H(N,\beta) \simeq C_H^{\text{reg}}(N,\beta) + A_C(N)[\tau(N)]^{-\alpha(N)}$$
$$\times \{1 + a_C(N)[\tau(N)]^{\theta(N)} + \cdots\} \qquad (6)$$

with $A_C(N) = [1-\alpha(N)]\beta_c(N)A_U(N)$ and $a_C(N) = \{1 + \theta(N)/[1-\alpha(N)]\}a_U(N)$. Notice that our definition of $a_C(N)$ conforms to general usage, but differs by a factor $\alpha(N)$ from Eq. (1.4) of Ref. 31.

We have also examined the susceptibility

$$\chi(N,\beta) = \sum_x \langle v(0) \cdot v(\vec{x})\rangle_c, \qquad (7)$$

the second moment of the correlation function

$$\mu_2(N,\beta) = \sum_x \vec{x}^2 \langle v(0) \cdot v(\vec{x})\rangle_c, \qquad (8)$$

and the second-moment correlation length ξ defined,[36] in terms of χ and μ_2, by

$$\xi^2(N,\beta) = \frac{\mu_2(N,\beta)}{6\chi(N,\beta)}. \qquad (9)$$

The susceptibility $\chi(N,\beta)$ is expected to behave as

$$\chi(N,\beta) \simeq A_\chi(N)[\tau(N)]^{-\gamma(N)}\{1 + a_\chi(N)[\tau(N)]^{\theta(N)} + \cdots\} \quad (10)$$

as $\tau(N) \downarrow 0$. In the case of bipartite lattices $\chi(N,\beta)$ has also an antiferromagnetic singularity at $\beta_c^{\mathrm{AF}}(N) = -\beta_c(N)$, and, in terms of the reduced variable $\tilde{\tau}(N) = 1 - \beta/\beta_c^{\mathrm{AF}}(N)$, we should observe the energylike behavior

$$\chi(N,\beta) \simeq \chi_{\mathrm{reg}}(N,\beta) + B_\chi(N)[\tilde{\tau}(N)]^{1-\alpha(N)} + \cdots \quad (11)$$

as $\tilde{\tau}(N) \downarrow 0$.

The second moment of the correlation function is expected to behave as

$$\mu_2(N,\beta) \simeq A_\mu(N)[\tau(N)]^{-\gamma(N)-2\nu(N)}\{1 + a_\mu(N)[\tau(N)]^{\theta(N)}$$
$$+ \cdots\} \quad (12)$$

as $\tau(N) \downarrow 0$. At the antiferromagnetic singularity, the behavior is completely similar to that of the susceptibility

$$\mu_2(N,\beta) \simeq \mu_2^{\mathrm{reg}}(N,\beta) + B_\mu(N)[\tilde{\tau}(N)]^{1-\alpha(N)} + \cdots \quad (13)$$

as $\tilde{\tau}(N) \downarrow 0$.

For the correlation length we have

$$\xi(N,\beta) \simeq A_\xi(N)[\tau(N)]^{-\nu(N)}\{1 + a_\xi(N)[\tau(N)]^{\theta(N)} + \cdots\} \quad (14)$$

as $\tau(N) \downarrow 0$, and also in this case we expect the energylike behavior

$$\xi(N,\beta) \simeq \xi^{\mathrm{reg}}(N,\beta) + B_\xi(N)[\tilde{\tau}(N)]^{1-\alpha(N)} + \cdots \quad (15)$$

as $\tilde{\tau}(N) \downarrow 0$.

The validity of the hyperscaling relation

$$d\nu(N) = 2 - \alpha(N) \quad (16)$$

first derived by Gunton and Buckingham[37] as an inequality (with the $=$ sign replaced by $\geqslant$), translates into the universality of the amplitude combination[6]

$$R_\xi^+(N) \equiv [g\,\alpha(N)A_C(N)]^{1/d}A_\xi(N), \quad (17)$$

where g is a geometric factor defined by $g = a^d/v_0$, with v_0 the volume per lattice site and a the lattice spacing. For the sc lattice one has $g = 1$, while for the bcc lattice $g = 3\sqrt{3}/4$.

Finally, it is useful to recall[38] that, in the large N limit, $R_\xi^+(N) \approx (N/4\pi)^{1/3}$ and that Bervillier and Godrèche[39] proposed a simple approximate extension of this relationship to small nonzero values of N in the form $R_\xi^+(N) \approx \nu(N)(N/4\pi)^{1/3}$.

III. COMMENTS ON THE ANALYSIS OF THE SERIES

A. Estimates of the specific heat exponents

The main difficulty in computing the specific heat exponents is that $\alpha(N)$ is small for $N \leqslant 1$ and it becomes negative for $N \geqslant 2$. Therefore the specific heat is very weakly divergent for $N \leqslant 1$, whereas it has only a finite cusp for $N \geqslant 2$. The simplest Padé approximant (PA) techniques for estimating the critical parameters are thus expected to be inefficient in the former cases and completely inadequate in the latter.

Moreover, it is not particularly helpful to differentiate the present specific heat series with respect to β in order to sharpen the singularity, because the extrapolations become more sensitive to nonasymptotic or confluent singularity effects. In principle, the inhomogeneous differential approximants (DA) (thoroughly described in Refs. 40) should perform much better than the PA's since they are able to detect even weak singularities and might allow, to some extent, for the confluent corrections to scaling. However, even after our extension of the HT series, the nonzero expansion coefficients are not sufficiently many that these numerical tools can be used effectively. In order to improve the precision of our estimates, we have mainly used simple first order DA's and have biased them with the critical temperatures reliably known from our previous study of the strongly divergent susceptibility series[2] or from other sources.[41,42] In the particular case of the sc spin-1/2 Ising model, we have taken advantage in our analysis also of the two additional series coefficients provided by Ref. 15.

An accurate measure of the scaling correction amplitudes of the specific heat presently seems beyond reach, although their qualitative behavior as functions of N is clear and completely analogous to that of $a_\chi(N)$ and of $a_\xi(N)$. More precisely, $a_C^\#(N)$ is small and negative for $N < 2$, while it is positive and increasing for $N > 2$. Let us recall that, for small values of N, RG computations[9,10] indicate that the universal ratios $a_C(N)/a_\chi(N)$ and $a_C(N)/a_\xi(N)$ are of the order of the unity. On the other hand, our HT analysis of $\chi(\beta,N)$ and $\xi(\beta,N)$ suggested that $a_\chi(N)$ and $a_\xi(N)$ are small (negative for $N < 2$ and positive otherwise), therefore it is reasonable to neglect the corrections to scaling at the present level of accuracy in the specific heat series analysis. We also recall that it was convincigly inferred in Ref. 43 that a_C is negative in the sc, bcc, and fcc spin-1/2 Ising models and, in the sc case, it was suggested in Refs. 15,16 that a_C is very small.

Our direct estimates of $\alpha(N)$ from the specific heat series for the sc and the bcc lattices have been reported in Table I. We have also included in this table the values of $\alpha(N)$ obtained by studying the energylike behavior of the susceptibility Eq. (11) at the antiferromagnetic singularity. The study of the second correlation moment Eq. (13) does not produce results of comparable quality. In this computation, we have found most convenient to analyze the derivative of χ by second order DA's biased with the singularities at $\beta_c(N)$ and $\beta_c^{af}(N)$. Although the expansion of χ is effectively longer than that of the specific heat, it is not easier to measure accurately the exponent of the very weak antiferromagnetic singularity. Therefore the estimates of $\alpha(N)$ so obtained are consistent with, but not more accurate than the others. In particular, we agree with the earlier estimates $\alpha(1) = 0.105(7)$ and $\alpha(1) = 0.11(2)$ obtained by studies of the susceptibility for the Ising model on the bcc lattice in Refs. 11,42.

In recent studies of the $N = 1$ case,[16,17] it has been suggested that the behavior of the specific heat series coefficients as functions of their order is sufficiently smooth that the traditional (biased) ratio techniques can be practically as accurate as the DA procedures. This remains true only for not too large values of N, since an asymptotic regime seems to set in later for larger N. Moreover, for $N > 4$, the ratio sequences show an increasing curvature indicating that the

TABLE I. In the first six lines we have reported the direct HT estimates of the critical exponents $\alpha(N)$ obtained in this work by various routes: by first order DA's of the specific heat biased with $\beta_c(N)$; by similarly biased extrapolation of ratios of the specific heat series coefficients and by second order DA's of $d\chi/d\beta$ biased with $\beta_c(N)$ and $\beta_c^{af}(N)$. We have then reported earlier direct estimates from shorter HT series, some direct MC determinations, and a few experimental measures. For each value of N, our estimates of $\alpha(N)$ have to be compared with the quantity 2-3$\nu(N)$ reported in the last four lines and obtained either from our previous HT study of the correlation length series or from RG estimates via ϵ-expansion and via FD perturbative expansion.

N	0	1	2	3	4
$C_H^{sc}(N,\beta)$ DA	0.24(1)	0.103(8)	$-0.014(9)$	$-0.11(2)$	$-0.22(4)$
$C_H^{bcc}(N,\beta)$ DA	0.23(1)	0.105(9)	$-0.019(8)$	$-0.13(2)$	$-0.25(3)$
$C_H^{sc}(N,\beta)$ Ratio ext.	0.236(8)	0.104(6)	$-0.020(8)$	$-0.15(^{+4}_{-1})$	$-0.27(^{+6}_{-1})$
$C_H^{bcc}(N,\beta)$ Ratio ext.	0.233(8)	0.106(6)	$-0.022(6)$	$-0.16(^{+3}_{-1})$	$-0.29(^{+6}_{-1})$
$\dfrac{d\chi^{sc}}{d\beta}(N,\beta)$ DA	0.239(8)	0.13(3)	0.02(3)	$-0.13(3)$	$-0.24(3)$
$\dfrac{d\chi^{bcc}}{d\beta}(N,\beta)$ DA	0.233(6)	0.107(8)	$-0.01(2)$	$-0.138(8)$	$-0.23(2)$
$\chi^{bcc}(1,\beta)$ DA Ref. 11		0.105(7)			
$\chi^{bcc}(1,\beta)$ DA Ref. 42		0.105(7)			
$C_H(N,\beta)$ DA Ref. 53		0.125(25)	$-0.02(3)$	$-0.22(4)$	
MC Refs. 45,41,46	0.275(15)	0.125(23)		$-0.23(16)$	
Exper. Refs. 25,41,47–50		0.107-0.112	$-0.01285(38)$	$-0.135(2)$	$-0.20(5)$
2-3$\nu(N)$ DA sc Ref. 2	0.2366(18)	0.1055(24)	$-0.025(6)$	$-0.148(6)$	$-0.277(9)$
2-3$\nu(N)$ DA bcc Ref. 2	0.2363(18)	0.1076(15)	$-0.022(6)$	$-0.142(6)$	$-0.268(9)$
2-3$\nu(N)$ ϵ-expans. Ref. 25	0.2375(54)	0.1121(78)	$-0.0055(120)$	$-0.115(16)$	$-0.211(24)$
2-3$\nu(N)$ FD-pert. Ref. 26	0.235(3)	0.109(4)	$-0.011(4)$	$-0.122(10)$	$-0.223(18)$

confluent corrections to scaling cannot be neglected anymore and therefore longer series are needed for a reliable analysis.

We shall use the simplest ratio formulas, since the more elaborate variants proposed in Ref. 44 do not presently make much difference. If we set $C_H(N,\beta)=\Sigma_{n=1}^{\infty}c_n(N)\beta^{2n}$, and allow for the dominant corrections to scaling with exponent $\theta(N)$, the ratio of the successive coefficients of the specific heat expansion in powers of β^2 is expected to behave as

$$r_n=\frac{c_n}{c_{n+1}}=\beta_c^2\left[1+\frac{1-\alpha}{n}+\frac{b}{n^{1+\theta}}+O\left(\frac{1}{n^2}\right)\right]. \quad (18)$$

Therefore α can be estimated from the sequence

$$\alpha_n=1-\left(\frac{r_n}{\beta_c^2}-1\right)n=\alpha+\frac{b}{n^{\theta}}+O\left(\frac{1}{n}\right). \quad (19)$$

The extrapolation of these estimators to $n\rightarrow\infty$ is the main difficulty with this procedure. For $N\leqslant4$, the estimators, when plotted versus $1/n$, show only a small curvature. Therefore we have neglected the scaling correction b/n^{θ} and have simply taken the linear extrapolant $n\alpha_n-(n-1)\alpha_{n-1}$ of the last two estimators as our final estimate of $\alpha(N)$. We have then assigned very conservative uncertainties to these results [also allowing for the errors in $\beta_c(N)$] and, for $N>2$, we have indicated by asymmetric errors the effects of some curvature in the estimator plots.

In Table I, we have also included the results of a few recent direct studies of the specific heat by stochastic meth-

ods. These studies are subject to difficulties analogous to those met in HT analyses. As a consequence, for instance, the Monte Carlo (MC) determination[45] of $\alpha(0)$ on the sc lattice is approximately three standard deviations away from the other quoted values. (We have summed the systematic and the statistical errors separately reported in Ref. 45.)

Also the value of $\alpha(1)$ emerging from a most accurate (see Ref. 41) MC study of the sc lattice Ising model performed by a dedicated processor, shows a considerable uncertainty. The central value, but not the error, is somewhat improved [$\alpha(1)=0.113\pm0.023$] by turning to a particular spin-1 Ising model designed to have small corrections to scaling.

For $N\geqslant2$, it is even harder to determine the exponent $\alpha(N)$ in MC simulations, because of the ambiguity in the separation of the nondivergent singular part of the specific heat from the regular background, as argued in Ref. 46. We have also reported a few experimental measurements of the specific heat exponent[47–50] available for $N=1,2,3,4$.

In order to show quantitatively the validity of the hyperscaling relation Eq. (16), our direct estimates of $\alpha(N)$ have been compared with the quantity $2-3\nu(N)$ also reported in Table I and computed either from our extended HT expansions of the correlation length[2] for the sc and the bcc lattices, or from the estimates of $\nu(N)$ obtained in the RG approach by fifth order ϵ expansion and by seventh order FD perturbation expansion.[25,26] In conclusion, the hyperscaling relation $d\nu(N)=2-\alpha(N)$ appears to be reasonably well verified within the uncertainties of the data.

TABLE II. Estimates of the hyperuniversal quantity $R_\xi^+(N)$. The results of our HT series computation are compared with RG estimates via ϵ expansion or via fixed-dimension perturbative expansion, with a heuristic approximate formula and with experimental measures.

N	0	1	2	3	4
HT sc (this work)	0.258(3)	0.273(4)	0.361(4)	0.431(5)	0.497(6)
HT bcc (this work)	0.258(3)	0.272(4)	0.362(4)	0.433(5)	0.500(6)
RG ϵ-expans. Ref. 8		0.27	0.36	0.42	
RG FD-pert. Refs. 9,39		0.270(1)	0.361(2)	0.435(2)	
HT Refs. 51,19,6		0.2659(7)	0.36(1)	0.42	
MC Ref. 52		0.2685(60)			
$\nu(N)\left(\dfrac{N}{4\pi}\right)^{1/3}$ Ref. 39		0.271(1)	0.363(1)	0.439(2)	0.506(4)
Exper. Refs. 6,39,24		0.25-0.32		0.40-0.45	

B. Estimates of $R_\xi^+(N)$

We have computed the hyperuniversal combination of critical amplitudes $R_\xi^+(N)$ by two methods. In the first procedure, we evaluate the HT expansion of the quantity

$$
F(N,\beta) = 4gq\nu(N)^3[\beta_c(N)]^{9/2}\left(\frac{\xi^2}{\beta}\right)^{9/2}
$$

$$
\times \left(\frac{d\xi^2}{d\beta}\right)^{-3}\frac{d^2G(N,\beta)}{d\beta^2}
$$

$$
= R_\xi^+(N)^3\tau(N)^{2-\alpha(N)-3\nu(N)}\{1+O[\tau(N)^{\theta(N)}]\}
$$

(20)

at the critical temperature. This computation also provides a good test of hyperscaling: indeed $F[N,\beta_c(N)]=R_\xi^+(N)^3$, if Eq. (16) holds. Here we have found convenient to use the ''simplified'' first order DA's, biased with $\beta_c^\#(N)$ and $\theta(N)$, as described in Ref. 3, and have taken the estimates of $\theta(N)$ from Ref. 25. We have reported in Table II only the results obtained by this method which is very stable and seems to be fairly accurate. In this case, our error estimates have to allow only for the spread of the approximants as well as for the uncertainties of $\beta_c(N)$, $\nu(N)$, and $\theta(N)$. The errors quoted mainly derive from the uncertainties in $\theta(N)$, assumed to be generally of the order of 10% and from the uncertainties of $\nu(N)$. The estimates of $R_\xi^+(N)$ obtained by PA's of $F[N,\beta_c(N)]$ are systematically smaller by $\approx 5\%$, indicating, in our opinion, that the ''simplified'' DA's are likely to allow more accurately for the sizable negative amplitude corrections to scaling. The usual first order DA's biased with $\beta_c(N)$ also seem to lead to less accurate estimates.

In the second approach, we obtain $R_\xi^+(N)$ from Eq. (17), after computing separately $A_C(N)$ and $A_\xi(N)$ from the specific heat and the correlation length series respectively, by DA's biased with the critical temperatures and exponents. This second method leads to results systematically smaller (by $\approx 1-2\%$), than those reported in Table II and it is subject to a larger uncertainty, due to the necessity of biasing the direct computation of $A_C(N)$ also with the exponents $\alpha(N)$, whose relative error may be considerable.

In the same table we have also reported the values of $R_\xi^+(N)$ computed via RG (Ref. 9) either to second order in

the ϵ expansion[7] or to fifth order in the FD perturbation expansion.[39] We have also included earlier estimates obtained in Refs. 6,19,51 from the analysis of shorter HT series, by the second above mentioned method.

A recent MC simulation[52] of the Ising model on the sc lattice has determined the universal quantity $f_s^{sc}(1)[A_\xi^{sc}(1)]^3$ which is closely related to $R_\xi^+(1)$. Here $f_s^{sc}(1)$ denotes the amplitude of the singular part of the free energy. For convenience, we have translated this result into the estimate of $R_\xi^+(1)$ reported in Table II, by using the value $\alpha(1)=0.1076(30)$, obtained in Ref. 52 from the hyperscaling Eq. (16).

The values from the approximate formula of Bervillier and Godrèche have been obtained assuming for $\nu(N)$ the FD perturbative results of Ref. 25. We are unable to give sensible error estimates in this case, but it interesting to quote at least the uncertainties deriving from those of $\nu(N)$. Finally, we should mention that, to our knowledge, no other evaluations of $R_\xi^+(N)$ for $N=0$ and $N=4$ are quoted in the literature.

C. Estimates of nonuniversal critical parameters

In Table III, we have reported our estimates of some nonuniversal critical parameters, for various values of N. The inverse critical temperatures $\beta_c^\#(N)$, which have been always used in the biased analyses of this paper were determined in Ref. 2 or taken from Refs. 41,42. The critical amplitudes $A_\xi^\#(N)$ of the second-moment correlation length were determined in Ref. 3.

The critical specific energies $U^\#(N,\beta_c)$ and the critical values of the regular part of the specific heat $C_{reg}^\#(N,\beta_c)$ have been obtained by first order DA's biased with $\beta_c(N)$. Also these data are compatible with the earlier estimates.[52,51]

We have computed the critical amplitudes of the specific heat $A_C^\#(N)$ in two ways: either indirectly, namely, from our estimates of $R_\xi^+(N)$ by using the knowledge of $A_\xi^\#(N)$ and of $\alpha(N)$, or directly, from the specific heat by DA's biased with $\beta_c(N)$ and $\alpha(N)$. The two methods yield compatible results. We have chosen to report in Table III the results of the first approach. Therefore the relatively large errors of $A_C^\#(N)$ mainly reflect the uncertainty of $\alpha(N)$, which, for $N=2$, is so considerable that it is not useful to report any estimates in this case. [For the same reason we have not reported esti-

TABLE III. Estimates of nonuniversal parameters. We report the critical inverse temperatures $\beta_c^\#(N)$ always used in our biased procedures, the critical amplitudes $A_\xi^\#(N)$ and $A_C^\#(N)$, the critical specific energies $U^\#(N,\beta_c)$ and the critical values of the regular part of $C_H^\#(N,\beta)$.

N	0	1	2	3	4
$\beta_c^{\mathrm{sc}}(N)$ HT Refs. 2,41	0.213493(3)	0.2216544(3)	0.45419(3)	0.69305(4)	0.93600(4)
$\beta_c^{\mathrm{bcc}}(N)$ HT Refs. 2,42	0.153128(3)	0.157373(2)	0.320427(3)	0.486820(4)	0.65542(3)
$A_\xi^{\mathrm{sc}}(N)$ HT Ref. 3	0.5101(3)	0.5027(3)	0.4814(3)	0.4541(3)	0.4155(3)
$A_\xi^{\mathrm{bcc}}(N)$ HT Ref. 3	0.4846(2)	0.4659(2)	0.4371(2)	0.4072(2)	0.3691(2)
$A_C^{\mathrm{sc}}(N)$ (this work)	0.546(8)	1.49(5)		−6.0(6)	−6.5(3)
$A_C^{\mathrm{bcc}}(N)$ (this work)	0.481(6)	1.43(4)		−6.5(6)	−7.2(3)
$A_C^{\mathrm{sc}}(N)$ MC Ref. 52		1.45(9)			
$A_C^{\mathrm{sc}}(N)$ HT Ref. 51		1.464(90)			
$A_C^{\mathrm{bcc}}(N)$ HT Ref. 51		1.431(80)			
$U^{\mathrm{sc}}(N,\beta_c)$ (this work)	−1.004(3)	−0.991(1)	−0.990(3)	−0.991(3)	−0.994(4)
$U^{\mathrm{bcc}}(N,\beta_c)$ (this work)	−1.0990(2)	−1.0903(6)	−1.0896(8)	−1.0919(4)	−1.0951(2)
$U^{\mathrm{sc}}(N,\beta_c)$ HT Ref. 51		−0.9902(1)			
$U^{\mathrm{sc}}(N,\beta_c)$ MC Ref. 41		−0.9904(8)			
$U^{\mathrm{bcc}}(N,\beta_c)$ HT Ref. 51		−1.0904(1)			
$C_H^{\mathrm{reg}}(N,\beta_c)$ sc (this work)	−0.66(3)	−1.67(3)		4.9(4)	4.2(3)
$C_H^{\mathrm{reg}}(N,\beta_c)$ MC sc Refs. 52,46		−1.64(11)		5.79(12)	
$C_H^{\mathrm{reg}}(N,\beta_c)$ MC sc Ref. 54				5.70(12)	
$C_H^{\mathrm{reg}}(N,\beta_c)$ bcc (this work)	−0.68(2)	−1.64(3)		5.2(4)	4.3(3)
$C_H^{\mathrm{reg}}(N,\beta_c)$ MC bcc Ref. 54				5.54(14)	

mates of $C_{\mathrm{reg}}^\#(2,\beta_c)$.] On the other hand, the uncertainties of the products $\alpha(N)A_C^\#(N)$ are more modest and therefore it can be of some interest to quote our estimates for $N=2$, namely, $\alpha(2)A_C^{\mathrm{sc}}(2)=0.42(1)$ and $\alpha(2)A_C^{\mathrm{bcc}}(2)=0.44(1)$.

We should stress that here the meaning of the errors for $R_\xi^+(N)$, $A_C^\#(N)$, etc., is not the same as in earlier studies, where the errors describe the spread of the estimates in computations performed at sharply fixed values of $\alpha(N)$ and $\beta_c(N)$. If, in those computations, we allowed also for the uncertainty of $\alpha(N)$, then the estimates and the errors of $R_\xi^+(N)$, $A_C^\#(N)$, etc., would become completely compatible with our results. Therefore, for instance, we have reported in Table III the central values of the estimates of $A_C^\#(1)$ from Ref. 51, based on the assignments $\alpha(1)=0.104$, $\beta_c^{\mathrm{sc}}(1)=0.221630$, and $\beta_c^{\mathrm{bcc}}(1)=0.157368$, but we have taken the liberty of suggesting much larger errors, which correspond to an indicative 5% uncertainty of $\alpha(1)$.

Finally, it is interesting to notice that the product $\alpha(N)A_C^\#(N)$, which is derived with good accuracy from $R_\xi^+(N)$, remains positive in the range of N examined here. Therefore, when $\alpha(N)$ changes its sign for $N=\bar{N}\lesssim 2$, the same must happen for $A_C^\#(N)$. Analogously $C_{\mathrm{reg}}^\#(N,\beta_c)$, which is negative for $N=0,1$, has to change sign for $N\gtrsim\bar{N}$, in order that the maximum of the specific heat stays positive.

IV. CONCLUSIONS

We have analyzed our extended HT expansion of $G(N,\beta)$ for the sc and bcc lattices in order to update the direct estimates of the critical exponent $\alpha(N)$ and of the hyperuniversal combination of amplitudes $R_\xi^+(N)$, over a range of values of N. Due to the smallness of $\alpha(N)$ and to the limited effective length of the series, the relative accuracy of our extrapolations is still generally inferior to that already achieved in our recent HT studies of the susceptibility and of the correlation length critical exponents.[2] However, within the error limits, the main predictions of universality, hyperuniversality, and hyperscaling appear to be well verified and the overall agreement between the HT and the RG estimates of the universal observables is good.

APPENDIX A: THE NEAREST NEIGHBOR CORRELATION FUNCTION ON THE SC LATTICE

The HT expansion coefficients of the nearest neighbor correlation function on the sc lattice are

$$a_1(N) = \frac{1}{N},$$

$$a_3(N) = \frac{8 + 3N}{N^3(2+N)},$$

$$a_5(N) = \frac{352 + 168N + 22N^2}{N^5(2+N)(4+N)},$$

$$a_7(N) = \frac{105984 + 154752N + 85056N^2 + 21960N^3 + 2754N^4 + 135N^5}{N^7(2+N)^3(4+N)(6+N)}.$$

For the coefficients which follow, it is typographically more convenient to set $a_r(N) = P_r(N)/Q_r(N)$ and to tabulate separately the numerator polynomial $P_r(N)$ and the denominator polynomial $Q_r(N)$,

$$P_9(N) = 12349440 + 17871360N + 10010240N^2 + 2751680N^3 + 405776N^4 + 30876N^5 + 954N^6,$$

$$Q_9(N) = N^9(2+N)^3(4+N)(6+N)(8+N),$$

$$P_{11}(N) = 124861808640 + 364560318464N + 467027804160N^2 + 345395589120N^3 + 163465120768N^4 + 51937662976N^5$$
$$+ 11315941120N^6 + 1694683328N^7 + 171418048N^8 + 11171800N^9 + 422520N^{10} + 7026N^{11},$$

$$Q_{11}(N) = N^{11}(2+N)^5(4+N)^3(6+N)(8+N)(10+N),$$

$$P_{13}(N) = 24917940633600 + 71794651299840N + 91099400634368N^2 + 67066306363392N^3 + 31821500096512N^4$$
$$+ 10242128590848N^5 + 2295320471552N^6 + 361789563776N^7 + 39924856512N^8 + 3014946464N^9$$
$$+ 148081312N^{10} + 4249712N^{11} + 53892N^{12},$$

$$Q_{13}(N) = N^{13}(2+N)^5(4+N)^3(6+N)(8+N)(10+N)(12+N),$$

$$P_{15}(N) = 8676547211191910 40 + 3616829986427633664N + 6891583739428601856N^2 + 7957383254837821440N^3$$
$$+ 6225913571872604160N^4 + 3498334649912000512N^5 + 1460523056889888768N^6 + 462563223592566784N^7$$
$$+ 112521154820349952N^8 + 21154253531684864N^9 + 3076240360587264N^{10} + 344376491174400N^{11}$$
$$+ 29339259414560N^{12} + 1863409665456N^{13} + 85223778256N^{14} + 2644451768N^{15} + 49679114N^{16} + 425007N^{17},$$

$$Q_{15}(N) = N^{15}(2+N)^7(4+N)^3(6+N)^3(8+N)(10+N)(12+N)(14+N),$$

$$P_{17}(N) = 3948322260048528015360 + 1822659825968732543 3856N + 38988021723789936033792N^2$$
$$+ 51323869690127645147136N^3 + 46583550742458833829888N^4 + 30960681462370651865088N^5$$
$$+ 15623251635335279411200N^6 + 6126114771192359944192N^7 + 1895134340075627937792N^8$$
$$+ 467022808981231222784N^9 + 92186181864442351616N^{10} + 14603683596490825728N^{11}$$
$$+ 1853863098715137024N^{12} + 187606064202660864N^{13} + 14988669525495552N^{14} + 930810012214464N^{15}$$
$$+ 43862323328864N^{16} + 1510537882592N^{17} + 35726075472N^{18} + 516586876N^{19} + 3426610N^{20},$$

$$Q_{17}(N) = N^{17}(2+N)^7(4+N)^5(6+N)^3(8+N)(10+N)(12+N)(14+N)(16+N),$$

$$P_{19}(N) = 330768394077031316324352000 + 192148949280682946183823 3600N + 5244352748560893054120099840N^2$$
$$+ 8943585498141047607892377600N^3 + 10692857932684576404138885120N^4$$
$$+ 9533746112508667703922262016N^5 + 6584053425730588600199806976N^6$$
$$+ 3611602580377927390173593600N^7 + 1601207739146215800698830848N^8$$

$$+ 58089223140562801843083673 6N^9 + 173958231237568098749120512N^{10} + 432634818212640258592604 16N^{11}$$

$$+ 8970159797560936461959168N^{12} + 155356339879016842831462 4N^{13} + 22478949742051157996339 2N^{14}$$

$$+ 2712771709143852673433 6N^{15} + 2720424198488158732288N^{16} + 2253266792765734184 96N^{17}$$

$$+ 15276530595902585344N^{18} + 836884471109722496N^{19} + 36371584704297344N^{20} + 1221231831603552N^{21}$$

$$+ 30441535564576N^{22} + 528211335752N^{23} + 5666057752N^{24} + 28113366N^{25}.$$

$$Q_{19}(N) = N^{19}(2+N)^9(4+N)^5(6+N)^3(8+N)^3(10+N)(12+N)(14+N)(16+N)(18+N),$$

$$P_{21}(N) = 122469323387965953278371430400 + 7039681357139968749683186073 60N$$

$$+ 190201877440148237292538114867 2N^2 + 32128331430094756215860011991 04N^3$$

$$+ 3807421545781558189263566143488N^4 + 33677732917117004125208313856 00N^5$$

$$+ 2309856527527444083001140445184N^6 + 12600258116295110348045560053 76N^7$$

$$+ 5564347967448682315291515944 96N^8 + 2014626193173915173739099586 56N^9$$

$$+ 60352333626138484849753718784N^{10} + 150572129813681896217892618 24N^{11}$$

$$+ 31424582320403625319623557 12N^{12} + 5500720004891682442809507 84N^{13}$$

$$+ 80841983595825376237305856N^{14} + 996925374242258103647436 8N^{15} + 1029177452713566097747968N^{16}$$

$$+ 88565456715317390606336N^{17} + 6311161666273241710592N^{18} + 368858480778492513280N^{19}$$

$$+ 17443565419605911296N^{20} + 654851921510017152N^{21} + 18987446839217536N^{22} + 408099174234848N^{23}$$

$$+ 6085840871680N^{24} + 55773063792N^{25} + 233966556N^{26},$$

$$Q_{21}(N) = N^{21}(2+N)^9(4+N)^5(6+N)^3(8+N)^3(10+N)(12+N)(14+N)(16+N)(18+N)(20+N).$$

In particular, for $N = 0$ (the SAW model), we have (in terms of the variable $\widetilde{\beta} = \beta/N$)

$$G(0,\widetilde{\beta}) = \widetilde{\beta} + 4\widetilde{\beta}^3 + 44\widetilde{\beta}^5 + 552\widetilde{\beta}^7 + 8040\widetilde{\beta}^9 + 127016\widetilde{\beta}^{11} + 2112320\widetilde{\beta}^{13} + 36484128\widetilde{\beta}^{15} + 648529392\widetilde{\beta}^{17} + 11790401800\widetilde{\beta}^{19}$$

$$+ 218273957968\widetilde{\beta}^{21} + \cdots.$$

For $N = 1$ (the spin-1/2 Ising model), we have

$$G(1,\beta) = \beta + 11/3\beta^3 + 542/15\beta^5 + 123547/315\beta^7 + 14473442/2835\beta^9 + 11336607022/155925\beta^{11} + 605337636044/552825\beta^{13}$$

$$+ 10976336338579019/638512875\beta^{15} + 3022947654230404442/10854718875\beta^{17}$$

$$+ 8582760723898537620322/1856156927625\beta^{19} + 15262009695163033631128084/194896477400625\beta^{21} + \cdots.$$

For $N = 2$ (the XY model), we have

$$G(2,\beta) = 1/2\beta + 7/16\beta^3 + 97/96\beta^5 + 5103/2048\beta^7 + 459719/61440\beta^9 + 218788559/8847360\beta^{11} + 3579816967/41287680\beta^{13}$$

$$+ 20154248931151/63417876480\beta^{15} + 4126827327908711/3424565329920\beta^{17}$$

$$+ 2142771095208749011/456608710656000\beta^{19} + 562665453010146198199/30136174903296000\beta^{21} + \cdots.$$

For $N = 3$ (the classical Heisenberg model), we have

$$G(3,\beta) = 1/3\beta + 17/135\beta^3 + 1054/8505\beta^5 + 80909/637875\beta^7 + 95738/601425\beta^9 + 5992817408726/27152760009375\beta^{11}$$

$$+ 11357358327572/34910691440625\beta^{13} + 156550175755271443/311577921107578125\beta^{15}$$

$$+ 190956190202826883834/237337400087308828125\beta^{17}$$

$$+ 5653569082372034770691255 8/4264597073468808678 1640625\beta^{19}$$

$$+ 3587525942092046754605047 16/1605039262196442538872656 25\beta^{21} + \cdots.$$

APPENDIX B: THE NEAREST NEIGHBOR CORRELATION FUNCTION ON THE BCC LATTICE

The HT expansion coefficients of the nearest neighbor correlation function on the bcc lattice are

$$a_1(N) = \frac{1}{N},$$

$$a_3(N) = \frac{24 + 11N}{N^3(2+N)},$$

$$a_5(N) = \frac{1776 + 1044N + 152N^2}{N^5(2+N)(4+N)},$$

$$a_7(N) = \frac{1050624 + 1713024N + 1062432N^2 + 312600N^3 + 44090N^4 + 2395N^5}{N^7(2+N)^3(4+N)(6+N)}.$$

For the coefficients which follow, it is typographically more convenient to set $a_r(N) = P_r(N)/Q_r(N)$ and to tabulate separately the numerator polynomial $P_r(N)$ and the denominator polynomial $Q_r(N)$,

$$P_9(N) = 237680640 + 391630080N + 251136960N^2 + 79995360N^3 + 13572456N^4 + 1175956N^5 + 40904N^6,$$

$$Q_9(N) = N^9(2+N)^3(4+N)(6+N)(8+N),$$

$$P_{11}(N) = 4657615994880 + 14662439436288N + 20306810757120N^2 + 16297064577024N^3 + 8408736450048N^4$$
$$+ 2927305709568N^5 + 701958299776N^6 + 116098602304N^7 + 13001482080N^8 + 940546304N^9 + 39618896N^{10}$$
$$+ 737112N^{11},$$

$$Q_{11}(N) = N^{11}(2+N)^5(4+N)^3(6+N)(8+N)(10+N),$$

$$P_{13}(N) = 1804392176025600 + 5660420904714240N + 7838369893122048N^2 + 6320116029308928N^3$$
$$+ 3299174287417344N^4 + 1173899872406016N^5 + 292101988094976N^6 + 51294669578688N^7$$
$$+ 6322737698272N^8 + 534749498288N^9 + 29520640808N^{10} + 956957440N^{11} + 13799232N^{12},$$

$$Q_{13}(N) = N^{13}(2+N)^5(4+N)^3(6+N)(8+N)(10+N)(12+N),$$

$$P_{15}(N) = 122002510248369192960 + 543062014542747795456N + 1106788272284626845696N^2$$
$$+ 1369759313298192334848N^3 + 1151523649799700086784N^4 + 697153536634263011328N^5$$
$$+ 314532460294909476864N^6 + 107981066371807617024N^7 + 28558819799096193024N^8$$
$$+ 5854384426156062720N^9 + 930798833517987840N^{10} + 114224602657806848N^{11} + 10696888031248800N^{12}$$
$$+ 749147616393328N^{13} + 37927213609168N^{14} + 1309142853624N^{15} + 27530444114N^{16} + 265776699N^{17},$$

$$Q_{15}(N) = N^{15}(2+N)^7(4+N)^3(6+N)^3(8+N)(10+N)(12+N)(14+N),$$

$$P_{17}(N) = 10781766577667486357913360 + 53110878203882890655170056N + 12134778449671899486093312N^2$$
$$+ 17084121119131816471560192N^3 + 16608988472236389552881664N^4 + 11844719796924403830226944N^5$$
$$+ 6426005983019854016544768N^6 + 2714739161543420662382592N^7 + 906815315440850230886400N^8$$
$$+ 241845902161952176398336N^9 + 51782172208072962975744N^{10} + 8918298897939850518528N^{11}$$
$$+ 1233687735206166823424N^{12} + 136374954925264681472N^{13} + 11933805821587623936N^{14}$$
$$+ 814292868353822272N^{15} + 42326351689562720N^{16} + 1615854325367776N^{17} + 42636630380712N^{18}$$
$$+ 693565371332N^{19} + 5232689960N^{20},$$

$$Q_{17}(N) = N^{17}(2+N)^7(4+N)^5(6+N)^3(8+N)(10+N)(12+N)(14+N)(16+N),$$

$$P_{19}(N) = 17543383833845276297259909120 0 + 10782038082652171498075403059 20N$$

$$+ 3116173607652516436321480212480 N^2 + 5633274990887456528054857236480 N^3$$

$$+ 714776484516302233554848317440 0N^4 + 6772123960394212475971911548928 N^5$$

$$+ 49766977099654171328478110023 68N^6 + 29091969062780578229684640153 60N^7$$

$$+ 1376618167955575605597824876544 N^8 + 53387724218234165703782223052 8N^9$$

$$+ 171186415133933732406244147200 N^{10} + 4565892984308707774106252083 2N^{11}$$

$$+ 10169112743416038967608705024 N^{12} + 189489372930387564659500646 4N^{13}$$

$$+ 2954611557119889025158348 80N^{14} + 3848735122401788477233971 2N^{15} + 41732834047160496155504 64N^{16}$$

$$+ 37448377633816599421644 8N^{17} + 2756845301579765972582 4N^{18} + 164449178556957019187 2N^{19}$$

$$+ 78098433373280888576 N^{20} + 2878619676002249280 N^{21} + 79247785021379008 N^{22} + 1531012629840624 N^{23}$$

$$+ 18487394459632 N^{24} + 104841714952 N^{25},$$

$$Q_{19}(N) = N^{19}(2+N)^9(4+N)^5(6+N)^3(8+N)^3(10+N)(12+N)(14+N)(16+N)(18+N),$$

$$P_{21}(N) = (1261755008880393488192080183296 00 + 7711820119738452952935689512550 40N$$

$$+ 2217488378116952516842355989413888 N^2 + 3990433665646539615566917678399488 N^3$$

$$+ 5043598010921343492445826797535232 N^4 + 4763919940749294277936547050291200 N^5$$

$$+ 3493715661682982610876998803783680 N^6 + 2040596945264227996384033763229696 N^7$$

$$+ 966223362153625194801269330411520 N^8 + 375629302981996871057940890517504 N^9$$

$$+ 120996637012851297773150580768768 N^{10} + 32504072855072819274267570733056 N^{11}$$

$$+ 7314027149319445947024024403968 N^{12} + 1382157562600983080851995279360 N^{13}$$

$$+ 219565426180384061435124916224 N^{14} + 29302944828757838601074274304 N^{15}$$

$$+ 3278043851101943224116459520 N^{16} + 306114536101034343439527936 N^{17}$$

$$+ 23712149887809097730532352 N^{18} + 1509817181912122244980224 N^{19} + 78017870011349442092288 N^{20}$$

$$+ 3213483265751133834688 N^{21} + 102816420657321623712 N^{22} + 2458334158401005552 N^{23}$$

$$+ 41257373964220632 N^{24} + 432739125346952 N^{25} + 2130772922816 N^{26},$$

$$Q_{21}(N) = N^{21}(2+N)^9(4+N)^5(6+N)^3(8+N)^3(10+N)(12+N)(14+N)(16+N)(18+N)(20+N).$$

In particular, for $N=0$ (the SAW model), we have (in terms of the variable $\widetilde{\beta} = \beta/N$)

$$G(0,\widetilde{\beta}) = \widetilde{\beta} + 12\widetilde{\beta}^3 + 222\widetilde{\beta}^5 + 5472\widetilde{\beta}^7 + 154740\widetilde{\beta}^9 + 4737972\widetilde{\beta}^{11} + 152960220\widetilde{\beta}^{13} + 5130099672\widetilde{\beta}^{15} + 177095284092\widetilde{\beta}^{17}$$

$$+ 6253425298080\widetilde{\beta}^{19} + 224879383796232\widetilde{\beta}^{21} + \cdots.$$

For $N=1$ (the spin-1/2 Ising model), we have

$$G(1,\beta) = \beta + 35/3\beta^3 + 2972/15\beta^5 + 279011/63\beta^7 + 46439636/405\beta^9 + 100877055128/31185\beta^{11}$$

$$+ 587703506650264/6081075\beta^{13} + 10981652882712713/3648645\beta^{15}$$

$$+ 1049923978894758374012/10854718875\beta^{17} + 1182698210781462071363672/371231385525\beta^{19}$$

$$+ 29800599277476233215348 51312/27842353914375\beta^{21} + \cdots.$$

For $N=2$ (the XY model), we have

$$G(2,\beta) = 1/2\beta + 23/16\beta^3 + 559/96\beta^5 + 187645/6144\beta^7 + 11417419/61440\beta^9 + 10934199853/8847360\beta^{11}$$

$$+ 1081218105839/123863040\beta^{13} + 4085878131871327/63417876480\beta^{15}$$

$$+ 1683908448367350071/3424565329920\beta^{17} + 753925204192677068291/195689447424000\beta^{19}$$

$$+ 929152049503798552678997/30136174903296000\beta^{21} + \cdots.$$

For $N=3$ (the classical Heisenberg model), we have

$$G(3,\beta) = 1/3\beta + 19/45\beta^3 + 2092/2835\beta^5 + 349939/212625\beta^7 + 2147444/505197\beta^9 + 108732988464808/9050920003125\beta^{11}$$

$$+ 9339742669288/258597714375\beta^{13} + 35412600932786885263/311577921107578125\beta^{15}$$

$$+ 3581664537534537147792/96693014850385078125\beta^{17}$$

$$+ 58805684489449008439435277849/4738441192743120753515625\beta^{19}$$

$$+ 113737149591413683781160445344/267506543699407089812109375\beta^{21} + \cdots.$$

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