Universality, frustration, and conformal invariance in two-dimensional random Ising magnets

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We consider long, finite-width strips of Ising spins with randomly distributed couplings. Frustration is introduced by allowing both ferromagnetic and antiferromagnetic interactions. Free energy and spin-spin correlation functions are calculated by transfer-matrix methods. Numerical derivatives and finite-size scaling concepts allow estimates of the usual critical exponents γ/ν, α/ν, and ν to be obtained, whenever a second-order transition is present. Low-temperature ordering persists for suitably small concentrations of frustrated bonds, with a transition governed by pure-Ising exponents. Contrary to the unfrustrated case, subdominant terms do not fit a simple, logarithmic-enhancement form. Our analysis also suggests a vertical critical line at and below the Nishimori point. Approaching this point along either the temperature axis or the Nishimori line, one finds nondiverging specific heats. A percolationlike ratio γ/ν is found upon analysis of the uniform susceptibility at the Nishimori point. Our data are also consistent with frustration inducing a breakdown of the relationship between correlation-length amplitude and critical exponents, predicted by conformal invariance for pure systems. [S0163-1829(99)07533-5]

I. INTRODUCTION

Frustration induced by quenched randomness may have rather complex effects on the behavior of spin systems.[1] The Edwards-Anderson model[2] for spin glasses is one on which a great deal of theoretical effort has been concentrated, mostly on account of the basic simplicity of its formulation. In the version that will be of direct interest to us, one has Ising spins interacting through couplings of the same strength and random sign ($\pm J$ Ising spin glass model). For the symmetric case of equal concentrations of + and − signs, a two-dimensional ($d=2$) system with only first-neighbor interactions is paramagnetic at all temperatures.[3,4] Recent results, according to which addition of a suitable set of second-neighbor couplings would stabilize a low-temperature spin-glass phase,[5] have been disputed on grounds that the ordering thus observed is a finite-size effect.[6]

However, for *asymmetric* distributions of ferromagnetic and antiferromagnetic bonds one can have long-range order in $d=2$ with only first-neighbor interactions (see Ref. 7 for references to early work) on a square lattice, for suitably small concentrations of frustrated plaquettes.[7] Experimental work on frustrated magnets $X_2Cu_xCo_{1-x}F_4$, $X=$K (Ref. 8) or Rb (Ref. 9), has shown the existence of an ordered state in such conditions, though direct quantitative comparison with theoretical calculations for $\pm J$ Ising systems is not straightforward because (1) ferromagnetic (Cu-Cu) bonds are weaker than antiferromagnetic (Co-Co) by a factor of ~ 8 and (2) the magnetism of Cu is Heisenberg-like with a small Ising anisotropy, which induces additional complications such as transverse freezing.

Estimates of the location of, and properties along, the critical line of the $\pm J$ model have been produced.[10,11] Further, since the pioneering work of Nishimori[12] asymmetric Ising spin glasses have been shown to display quite unique features, in all space dimensionalities. Among these is the so-called *Nishimori line* (NL) along which the internal energy can be calculated exactly, and which roughly separates the regions in the temperature-randomness parameter space where either ferromagnetic or spin-glass correlations dominate. Accordingly, the intersection of the NL with the para-ferromagnetic boundary is of special significance, even in $d=2$ where no spin-glass phase is expected at nonzero temperature ($T\neq 0$). It has been proposed[13] that such an intersection, to be referred to as a *Nishimori point* (NP), coincides with the multicritical point expected to exist along the boundary. Very recently, the NP has been investigated in $d=2$ and 4 by series analysis,[14] and in $d=2$ also by a variant of the Chalker-Coddington[15] network model.[16] Numerical values of critical exponents thus obtained are very close to those of percolation[17] in $d=2$ [though not in $d=4$ (Ref. 14)].

On the other hand, $d=2$ unfrustrated random Ising systems, such as the random-bond (i.e., bonds being J or rJ, $0<r<1$), and the diluted model, have been the subject of renewed interest over the past years, for two main reasons. First, different scenarios regarding the universality class of these systems have been proposed: *weak universality* versus *logarithmic ''corrections.''* In the former, critical exponents are distinct from those in the pure case, but their ratios to ν, the correlation length exponent, remain the same as in the pure case;[18] in the latter, the pure-system power-law critical behavior is reinforced by logarithmic divergences.[19] Secondly, the applicability of conformal invariance to random spin systems has not been put on grounds as firm as those for pure systems,[20] thus (for instance) the corresponding relationship between critical exponents and correlation length amplitudes needs to be checked in each case.

Thus, a systematic study of the asymmetric $\pm J$ Ising model on a square lattice is of interest, not only in relation to specific quantitative questions (such as the shape of the critical line and its intersection with the NP), but also in relation to the broader context of universality classes in disordered systems, singling out the effects of frustration. With this in

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mind, here we will focus on the following main questions. (i) To what extent, if any, do logarithmic corrections to pure-system behavior describe criticality for small degrees of frustration? (ii) Does the connection between critical exponents and correlation length amplitudes hold in the case? (iii) Can we provide evidence for (or against) the conjecture[14] that the critical behavior at the $d=2$ Nishimori point is percolation-like? These issues are addressed through the calculation of free energies and spin-spin correlation functions on long, finite-width strips of a square lattice. The rate of decay of correlation functions determines correlation lengths along the strip. We have already shown how averaged values of such quantities, and their numerically calculated field and temperature derivatives, enable one to extract critical properties of unfrustrated disordered models.[21,22] For the latter class of systems in particular, we gave numerical evidence in favor of the logarithmic corrections scenario; we also showed that the relationship between critical exponents and correlation length amplitudes, predicted by conformal invariance,[20] remains valid provided one uses averaged correlation lengths.[21,23] The validity of conformal invariance ideas for (unfrustrated) disordered q-state Potts models has also been verified.[24,25]

This paper is organized as follows. In Sec. II we outline numerical aspects of our calculational procedures, as applied to the asymmetric $\pm J$ Ising model. Results for the phase boundary and critical behavior above the Nishimori line are discussed in Sec. III, while the Nishimori point and the region below it are discussed separately in Sec. IV. Our findings are then summarized in Sec. V.

II. CALCULATIONAL METHOD

We have used long strips of a square lattice, of width $4 \leqslant L \leqslant 14$ sites with periodic boundary conditions across the strip. Only even widths were used, in order to accommodate possibly occurring unfrustrated antiferromagnetic ground states. We compute spin-spin correlation functions along the ''infinite'' direction by transfer-matrix methods,[26–28] extracting averaged correlation lengths. By the same methods we numerically obtain the free energy and its second derivatives with respect to (i) a uniform external field, which are used in connection with finite-size scaling (FSS) for estimating γ/ν and (ii) temperature, again used with FSS concepts for estimating α/ν. In order to provide samples that are sufficiently representative of disorder, we iterated the transfer matrix[28] typically along 10^7 (10^8 near the NP) lattice spacings.

At each step, the respective vertical and horizontal bonds between first-neighbor spins i and j were drawn from a probability distribution

$$P(J_{ij}) = p\,\delta(J_{ij}-J_0) + (1-p)\,\delta(J_{ij}+J_0). \tag{1}$$

For a square lattice the phase diagram in the $T-p$ plane is invariant with respect to the symmetry $p \leftrightarrow 1-p$; thus we shall restrict ourselves to $0.5 < p \leqslant 1$, meaning that bulk antiferromagnetic order will play no part in what follows.

The direct calculation of correlation functions $\langle \sigma_0 \sigma_R \rangle$, goes according to Sec. 1.4 of Ref. 28, with the corresponding adaptations for an inhomogeneous system.[23] For fixed distances up to $R=50$, and for strips with lengths as given above, the correlation functions are averaged over an ensemble of 10^5–10^6 different estimates to yield $\overline{\langle \sigma_0 \sigma_R \rangle}$. The average correlation length ξ^{av} (which carries a dependence on T, p and strip width L), is in turn defined by

$$\overline{\langle \sigma_0 \sigma_R \rangle} \sim \exp(-R/\xi^{\mathrm{av}}), \tag{2}$$

and is calculated from least-squares fits of straight lines to semilog plots of the average correlation function as a function of distance, in the range $10 \leqslant R \leqslant 50$. Finally, ξ^{av} is itself averaged over the different realizations of disordered bonds. In this context it must be recalled that, although in-sample fluctuations of correlation functions do not die out as strip length is increased, averaged values converge satisfactorily;[29] as done before,[21] here we make use of this fact to calculate error bars of related quantities.

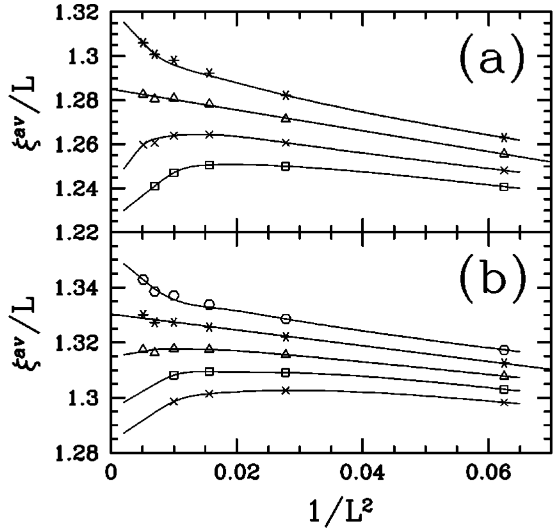

FIG. 1. Size-scaled correlation length versus $1/L^2$ (L is the strip width) at different inverse temperatures, $K \equiv J_0/k_BT$ in the low frustration regime: (a) $p=0.95$ and $K=0.535$ (stars), 0.534 (triangles), 0.533 (crosses), and 0.532 (squares); (b) $p=0.92$ and $K=0.637$ (hexagons), 0.636 (stars), 0.635 (triangles), 0.634 (squares), and 0.633 (crosses). Lines drawn through the points are guides to the eye. See Table I for corresponding estimates of K_c.

In contrast with the unfrustrated disordered models considered earlier,[21,22] here the exact critical temperature is not known as a function of p, so our first step was to use averaged correlation lengths together with FSS ideas[26–28] to obtain an approximate critical curve $T_c(p)$. This approach is safe because the only underlying assumption is that a second-order phase transition occurs, without further hypotheses on its universality class. In the usual phenomenological renormalization recipe, used for pure systems, one looks for the fixed point T^* of $\xi^{\mathrm{av}}(L,T^*,p)/L = \xi^{\mathrm{av}}(L',T^*,p)/L'$ (in the case one would use $L'=L-2$). For disordered systems it should be stressed that, even if logarithmic corrections are present in the bulk limit, the (averaged) correlation length at the critical point should still scale linearly with the strip width L, to leading order.[21] Thus, here we produce estimates of $T_c(p)$ by scanning a range of temperatures for fixed p, and bracketing the interval for which ξ^{av}/L appears to approach a finite value as $L \to \infty$. The width of such temperature interval gives the respective error bar, as illustrated in Fig. 1. Two remarks are in order in relation to this approach. First, this is more convenient here than the standard fixed-point search, since (1) intrinsic uncertainties associated to the individual $\xi^{\rm av}$ are amplified when estimating $T^*(L,L')$ and (2) it is the extrapolation as $L,L'\to\infty$ that matters in the end. As the range of available strip widths is not very broad, it is important that, for given T, the sequence of data from individual L's has one more point, and also slightly smaller error bars, than that of $T^*(L,L')$. Second, with our procedure one already gains an insight into corrections to scaling: by varying the power of $1/L$ against which $\xi^{\rm av}/L$ is plotted, one can check how better to produce (inclined) straight lines within the bracketed temperature range singled out by the initial scan. It must be stressed that the location and width of the bracketed range itself, separating the (high-temperature) regime in which one is certain that $\xi^{\rm av}/L\to 0$ and that (low-temperature) in which $\xi^{\rm av}/L$ diverges, are practically insensitive to the choice of power. Indeed, though in Fig. 1 we plotted $\xi^{\rm av}/L$ vs L^{-2}, inspired by results for pure[30] and unfrustrated random[23] systems, we have also checked that using $\xi^{\rm av}/L$ vs L^{-1}, $L/\xi^{\rm av}$ vs L^{-1} or L^{-2}, changes no significant digits of our extrapolated estimates.

Once, for fixed p, one has an estimate of $T_c(p)$, the next step is to calculate the critical free energy and its appropriate derivatives. This is done by evaluating the largest Lyapunov exponent Λ_L^0 for strips of width L and length $N\gg 1$.[31,32] The average free energy per site is $f_L^{\rm av}(T)=-(k_BT/J_0L)\Lambda_L^0$, in units of J_0. The initial susceptibility and specific heat of a strip are then given by

$$\chi_L(T)=\left.\frac{\partial^2 f_L^{\rm av}(T)}{\partial h^2}\right|_{h=0}, \quad C_L(T)=\left.\frac{\partial^2 f_L^{\rm av}(T)}{\partial T^2}\right|_{h=0}, \qquad (3)$$

where h is an external field coupling to the order parameter; the size dependence of these quantities will be discussed below. We shall take h as uniform (ferromagnetic order), which is reasonable for low frustration; at the Nishimori point, this choice implies singling out one of the two scaling directions (more on this below). An extensive discussion of calculational details is given, for the specific heat, in Ref. 22. Here we recall that, since the derivatives are numerically obtained by calculating, e.g., $2f_L(T_c)-f_L(T_c+\delta T)-f_L(T_c-\delta T)$ with $\delta T=10^{-3}T_c$, sample-to-sample fluctuations are roughly as large as the difference between free energies at these three temperatures; thus one must ensure that *the same configuration* of bonds (that is, the same sequence of pseudorandom numbers) is used in the comparison of different temperatures: free energies of the same bond geometry have to be subtracted. The probable errors for the free energy differences are then much smaller than those for the free energies themselves. Similar procedures were used in a transfer-matrix study of interface energies in random systems.[33] The same argument applies for the susceptibilities, substituting δh (typically of order 10^{-4} in units of J) for δT.

Finally, one should have in mind that the inverse of $\xi^{\rm av}$ is, in principle, distinct from the difference between the two leading Lyapunov exponents, which gives the decay of the most probable, or *typical* (as opposed to averaged) correlation function.[21,23,29,32,34] Nonetheless, for $d=2$ unfrustrated disordered Ising systems they have turned out to be numerically very close,[23,29] at least at the critical point (see below for remarks on low-temperature behavior in the present case); significant differences arise only in the corrections to scaling, which are relevant for extrapolation to the thermodynamic limit.[34] In Refs. 10,11, the model considered here was studied with the aid of typical correlation lengths $\xi^{\rm typ}$ also calculated on strip geometries, but disregarding corrections to scaling; below, we will comment on some of the differences between our results and theirs.

TABLE I. Inverse critical temperatures for low frustration. The pure-system value of ξ/L is $4/\pi=1.2732\ldots$, from conformal invariance.

p	$K_c(p)$	$\lim_{L\to\infty}\xi/L$
0.99	0.4555 ± 0.0005	1.275 ± 0.015
0.95	0.534 ± 0.001	1.285 ± 0.015
0.92	0.6360 ± 0.0015	1.325 ± 0.015

III. ABOVE THE NISHIMORI POINT

We start by applying the above-described procedure to scale correlation lengths, for p close to 1. Figure 1 displays ξ/L vs $1/L^2$ at different temperatures, for (a) $p=0.95$ and (b) $p=0.92$. The corresponding estimates for the critical temperatures are shown in Table I, and compare rather well with those of Refs. 10 and 11. Using exact and approximate data, respectively at $p=1$ and 0.99, the reduced slope of the critical curve at the pure point is estimated to be $[1/T_c(1)]\times(dT_c/dp)|_{p=1}=3.25\pm 0.11$, which compares very well with the exact result, 3.2091.[7]

We now turn to the correlation-length exponent, ν. Since ν does not appear explicitly in the expression for $\xi_L^{\rm av}(T_c)/L$, one resorts to the temperature derivative of the correlation length, which can also be cast in a similar scaling form[21]

$$\mu_L\equiv\frac{d\xi_L^{\rm av}}{dt}=L^{1+1/\nu}\mathcal{G}(z), \quad z\equiv\frac{\xi_\infty(t)}{L}, \qquad (4)$$

with $t\equiv(T-T_c)/T_c$, and $\mathcal{G}$ is a finite-size scaling function. Assuming a simple power-law divergence $\xi_\infty\sim t^{-\nu}$, i.e., ignoring, for the time being, less-divergent terms such as logarithmic corrections, we obtain the estimates for systems of sizes L and $L-2$:

$$\frac{1}{\nu_L}=\frac{\ln(\mu_L/\mu_{L-2})_{T=T_c}}{\ln(L/L-2)}-1, \qquad (5)$$

where the derivatives are calculated numerically at the extrapolated (i.e., $L\to\infty$) value of T_c. For fixed L and $T_c(p)$, we obtain one estimate of ν_L for each disorder configuration; these estimates are then averaged over different disorder configurations to yield the data shown in Fig. 2, for $L=6-14$. The trend displayed in Fig. 2 is dramatically different from the one observed in the case of unfrustrated disorder:[21] All curves (for different values of p) show a distinct downturn (as L increases), and a limiting value $\nu_\infty=1$, common to all values of p considered, becomes more likely.

This should be contrasted with the case of unfrustrated disorder, for which no downturn was observed, and the extrapolations indeed seemed to indicate a steady convergence

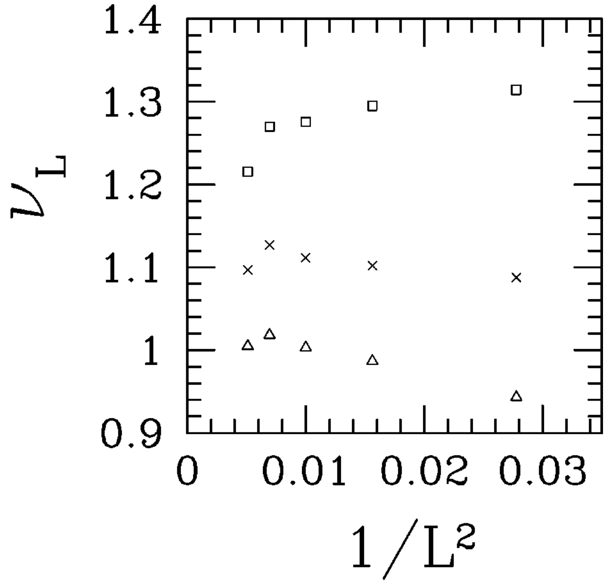

FIG. 2. Size dependence of ν_L as given by Eq. (5), for $p=0.99$ (triangles), 0.95 (crosses), and 0.92 (squares). Error bars are smaller than data points.

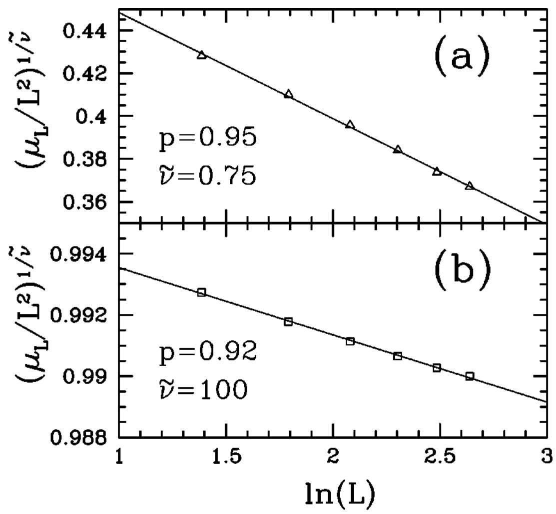

FIG. 3. Fits of Eq. (7) to determine $\tilde{\nu}$, for $p=0.95$ (triangles) and 0.92 (squares). Error bars are smaller than data points.

towards a disorder dependent exponent $\nu=\nu(p)\geqslant\nu_{\rm pure}=1$; cf., Ref. 21. In that case, available theories[19] pointed towards describing those apparent exponents as resulting from power-law divergences with pure-system exponents, enhanced by multiplicative logarithmic terms; such expectations were later confirmed through transfer matrix calculations on strips.[21,22] Though in the present case the downturn in the trend may be taken as indicative that these corrections are absent, considerable insight should be gained by trying to fit the data along similar lines.

The forms of logarithmic corrections in random systems have been derived within a field-theoretic approach,[19] which does not explicitly account for frustration effects. Nonetheless, inspired by our experience with unfrustrated disorder, we decided to check whether in the present case such corrections also arise. The theory[19] which successfully accounts for unfrustrated disorder predicts that the correlation length of the disordered Ising model, near the critical point, is given by

$$\xi_\infty \sim t^{-\nu}[1+C\ln(1/t)]^{\tilde{\nu}}, \tag{6}$$

for the infinite system, where $\nu=1$, C is a disorder-dependent positive constant, and $\tilde{\nu}=1/2$; for $C=0$ one recovers pure-system behavior. As discussed in Ref. 21, logarithmic corrections do not show up in the correlation length for finite systems, but in its temperature derivatives; at criticality, i.e., $t(p)=0$, Eq. (4) becomes

$$\frac{\mu_L}{L^2}\sim(1-A\ln L)^{\tilde{\nu}}, \tag{7}$$

where A is some disorder-dependent constant. While in the Dotsenko-Shalaev theory,[19] $\tilde{\nu}$ was predicted to be 1/2, here we allow it to be determined from an analysis of the data: it is chosen in such a way that, for fixed p, data for $[\mu_L/L^2]^{1/\tilde{\nu}}$ versus $\ln L$ lie on a straight line, for the largest system sizes. Figure 3 shows our results for $p=0.95$ and 0.92: we see that an attempt to fit our data to these logarithmic corrections would require an unlikely variation from $\tilde{\nu}=0.75$ for $p=0.95$ to $\tilde{\nu}=100$ for $p=0.92$. Attempts to fit the data to other forms, such as powers of $\ln L$, turned out to be equally unsuccessful. The conclusion is that logarithmic enhancements play no role in the bulk correlation length for frustrated disorder (at least in a simple, clearly defined way as in the unfrustrated case).

Turning now to the specific heat behavior, we recall that in the Dotsenko-Shalaev theory, the singular part of the bulk specific heat per particle for the disordered Ising model, near the critical point, is given by

$$C_\infty(t)\simeq(1/C_0)\ln[1+C_0\ln(1/t)], \tag{8}$$

where C_0 is proportional to the strength of disorder, and the pure-system simple logarithmic divergence is recovered as $C_0\to 0$. For $C_0\neq 0$ and $t\ll 1$ a double-logarithmic singularity arises, whose amplitude Eq. (8) predicts to decrease as disorder increases. For a finite system, the usual FSS theory applied to this case yields[19]

$$C_L(t=0)\simeq C_1+a\ln(1+b\ln L), \tag{9}$$

where, similarly to Eq. (8), $b\to 0$ for vanishing disorder. We tried to check whether such forms had any relevance in the present case. Our results are displayed in Fig. 4, and a trend similar to unfrustrated randomness is observed: for low disorder, the specific heat increases with system size faster than in a double-logarithmic fashion (e.g., with $\ln L$); as disorder increases ($p=0.92$), the best fit of the data crosses over to double-logarithmic behavior. Though this may be interpreted as signalling the existence of logarithmic corrections, such a discussion is rather subtle.[22] At any rate, an inequivocal conclusion to be drawn from our data is that the specific heat diverges as $L\to\infty$. Accordingly, this enables us to set $\alpha\geqslant 0$ in the hyperscaling relation $d\nu=2-\alpha$, to obtain the condition $\nu\leqslant 1$. This condition, together with the absence of logarithmic corrections for ξ, and the downturn in the sequence of estimates for ν, lead to a scenario of $\nu(p)=1$, as in the pure case.

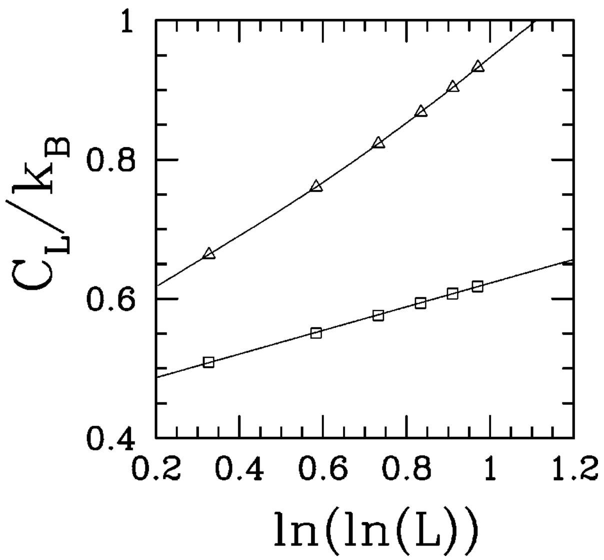

FIG. 4. Specific heat as a function of ln ln L, for $p=0.95$ (triangles) and 0.92 (squares). Error bars are smaller than data points.

In order to build up a fuller picture of the low-frustration regime, we turned to an alternate quantity, the susceptibility. The ratio γ/ν can be obtained in the usual way,

$$\left(\frac{\gamma}{\nu}\right)_L = \frac{\ln[\chi_L/\chi_{L-2}]_{T_c}}{\ln[L/L-2]}, \tag{10}$$

where T_c is understood to be the extrapolated value. We checked for self-consistency of critical-point locations and properties in the following way. First, the procedure we used above to obtain $T_c(p)$ from extrapolations of $\xi_L^{\rm av}$ can be repeated for $\xi^{\rm typ}$; this yields a slightly different extrapolated value $T_c(\xi^{\rm typ})$. For $p=0.95$, for instance, one has $K_c(\xi^{\rm typ})=0.531\pm0.001$, while $K_c(\xi^{\rm av})=0.534\pm0.001$. The sequence of susceptibilities calculated at these estimates of K_c gives rise, through Eq. (10), to the data shown in Fig. 5. One clearly sees that the Ising value $\gamma/\nu=7/4$ is compatible with

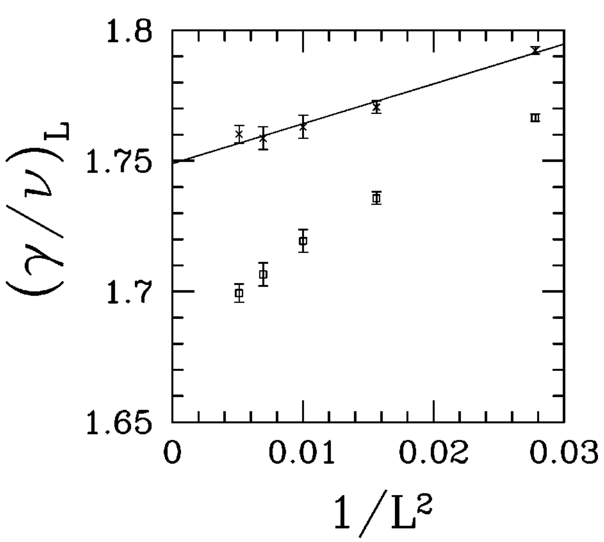

FIG. 5. Ratio of critical exponents γ/ν as a function of $1/L^2$, for $p=0.95$, from Eq. (10), calculated at T_c determined through the scaling of $\xi^{\rm av}$ (crosses) and of $\xi^{\rm typ}$ (squares).

extrapolation of data calculated at $T_c(\xi^{\rm av})$, and *not* with those coming from $T_c(\xi^{\rm typ})$. We take this to mean that (i) from susceptibility, specific-heat, and correlation-length data the most likely self-consistent picture is one in which the critical behavior is pure-Ising for low frustration, (ii) though very likely $\xi^{\rm av}$ and $\xi^{\rm typ}$ will eventually scale in a similar way, higher-order corrections still produce sizeable distortions in the accessible range of strip widths, and (iii) further, it seems that, for not very large widths, $\xi^{\rm av}$ behaves more reliably.

The above analysis, together with the scaling law $\eta=2-(\gamma/\nu)$, predicts that, for low disorder, the exponent describing the decay of correlations at criticality sticks to the pure system value, $\eta=\frac{1}{4}$. Thus, if the exponent-amplitude relationship of conformal invariance remains valid in the case of frustrated disorder, we should have $\lim_{L\to\infty}\xi_L[T_c(p)]/L=1/\pi\eta=1.273\ldots$. We obtain an estimate of $\lim_{L\to\infty}\xi_L[T_c(p)]/L$ by observing the trend followed by the sequences of points calculated at $K_c(p)$ and at $K_c(p)\pm0.001$, to determine the central estimate and its error bars (see Fig. 1); the outcome is shown in the last column of Table I. In spite of the arbitrariness of this approach, the error bars thus obtained are certainly overestimated. Nonetheless, even with such generous allowances, the data for $p=0.92$ show that the conformal invariance prediction is definitely not satisfied, since it lies way outside the range of the error bars. As the critical behavior should be the same along the critical line (at least within the low disorder region), we are led to conclude that, unlike the case of unfrustrated disorder, the exponent-amplitude relationship of conformal invariance breaks down in the case of frustrated disorder.

At $p\simeq0.89$, the transition vanishes abruptly, meaning that we do not find any temperature at which correlation lengths scale linearly with strip width. In Ref. 10, it is found that the typical correlation lengths $\xi^{\rm typ}$ still scale linearly with L at suitably low T for a broader range of p variation, along a line that significantly departs from the vertical on a p-T diagram; however, they find a maximum in $\xi^{\rm typ}$ as a function of temperature for finite values of T. This unexpected behavior has indeed been observed in studies of $\xi^{\rm typ}$ for (unfrustrated) disordered and random-field Ising systems.[35] No similar peak structure occurs when we investigate *average* correlation lengths; instead, these vary monotonically and diverge only as $T\to0$, consistent with the fact that a strip is essentially one dimensional. Though the authors of Ref. 10 acknowledge that such maxima at finite T are unphysical, they assume that their data still are reliable above the peak temperatures, and interpret the corresponding part of their low-temperature results as marking the boundary between a random-antiphase state and the paramagnetic regime, extending as far as $p\simeq0.8$. We have not found any evidence for this phase from our treatment. Strictly speaking, this means only that the expected signature of the corresponding second-order phase transition does not show up when averaged correlation lengths are considered. At present we are unaware of why it should be so, and whether it means that the random-antiphase state is not present at all.

IV. AT AND BELOW THE NISHIMORI POINT

The Nishimori line is given by[12]

$$\exp(2J_0/T)=p/(1-p), \tag{11}$$

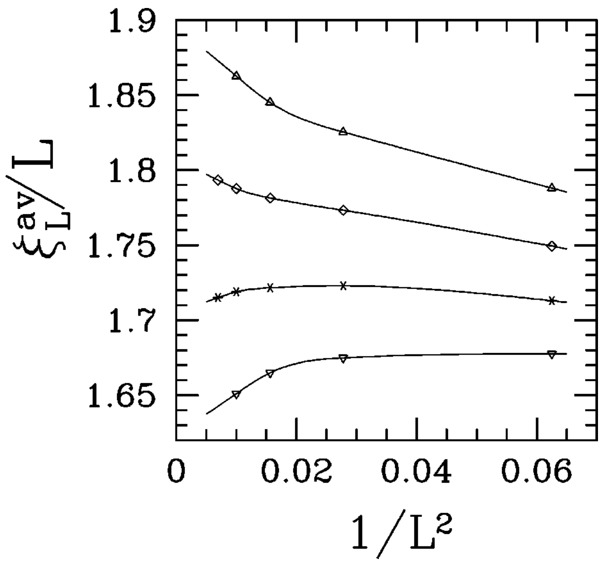

FIG. 6. Size-scaled averaged correlation length versus $1/L^2$ along the Nishimori line: $p=0.892$ (up triangles), 0.891 (diamonds), 0.890 (stars), and 0.889 (down triangles). For each p, the corresponding temperature is given by Eq. (11). Error bars are smaller than data points.

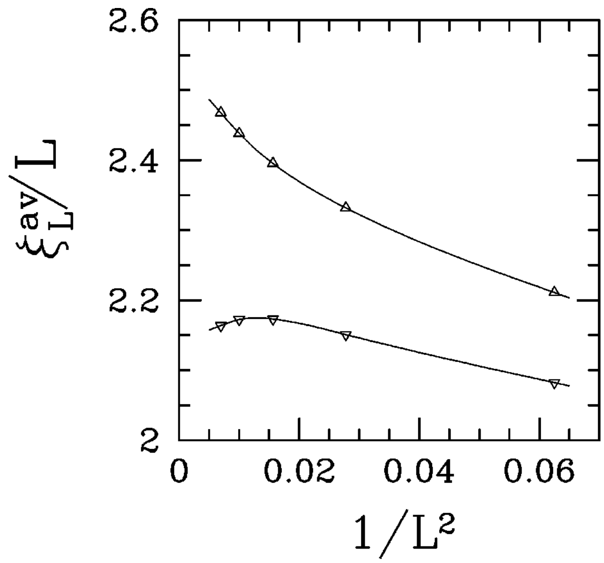

FIG. 7. Size-scaled averaged correlation length versus $1/L^2$ below the Nishimori line: $p=0.892$ (up triangles) and 0.889 (down triangles) at $T=0.4$. Error bars are smaller than, or of similar size to, data points.

and our first task is to determine its intersect with the critical curve, which is done as follows. For each T, we extract $p_{\rm trial}$ from Eq. (11), and calculate $\xi^{\rm av}(p_{\rm trial})$; this procedure is repeated for different system sizes, so that a sequence of estimates, ξ_L/L, is produced. Figure 6 shows our data thus obtained, and two different trends can be clearly observed: curves for $p=0.892$ and 0.891 display an upward curvature, whereas those for $p=0.890$ and 0.889 are bent downwards. Assuming a monotonic behavior (as $L\to\infty$) of ξ/L, any curve outside the interval [0.8900, 0.8910] will certainly not stabilize to a constant value for larger L. Our central estimate for the NP is therefore just the midpoint along the confidence interval (or, one might say, along the complementary of the nonconfidence domain):

$$p_N=0.8905\pm0.0005, \quad T_N=0.954\pm0.002, \qquad (12)$$

where T_N follows from Eq. (11). Our estimate for the location of the NP should be compared with those coming from: series work on the NL,[14] giving $p_N=0.886\pm0.003$, $T_N=0.975\pm0.006$; zero-temperature calculations, together with a no-reentrance assumption, giving $p_N=0.896\pm0.001$ or $p_N=0.894\pm0.002$ (depending on details of the fit);[38] exact combinatorial work[39] $p_N\simeq0.885$ [error bar presumably $\simeq0.005$ (our estimate)]; and Monte Carlo analysis of nonequilibrium relaxation,[40] $p_N=0.8872\pm0.0008$.

Once the Nishimori point has been accurately determined, we can check the region below for reentrant behavior. Evidence has been presented very recently (for Potts spinglasses on hierarchical lattices) that there appears to be no fundamental reason why reentrances should be ruled out in thermodynamic systems,[36] thus this is a matter worthy of consideration. We examine the size dependence of $\xi^{\rm av}$ at a temperature $T=0.4<T_N/2$ and at concentrations slightly away from p_N, $p=0.889$, and 0.892; the results are displayed in Fig. 7. The curve corresponding to $p>p_N$ seems to diverge as $L\to\infty$, indicating that the (T,p) point lies within the ferromagnetic region; the one corresponding to $p<p_N$ seems to vanish as $L\to\infty$, indicating that the corresponding (T,p) point lies outside the ordered phase. Given that both values of p are very close to p_N, we interpret this as an indication of absence of reentrant behavior, and that the critical line $T_c(p)$ is vertical at and below the Nishimori point. This is consistent with theoretical considerations[12,37] and with extensive numerical work[10,11,38] specifically aimed at the two-dimensional $\pm J$ Ising model. As a result, we assume that the scaling directions at the NP are, respectively, tangent to the critical curve (thus, purely temperaturelike) and along the Nishimori line.[13,14,16]

In order to discuss critical exponents, we note that the numerical evaluation of temperature derivatives in Eq. (3) implies that $\delta T\lesssim0.001$ at the NP; since this is of the same order as the estimated error bars in T_N, we shall sit at our own central estimates, Eq. (12), and measure the corresponding δT from there. While for scaling along the tangent (i.e., pure temperaturelike) axis, the considerations on the need to subtract free energies of the same bond geometry[22] are identical to those quoted above, a subtlety arises when considering variations along the Nishimori line, where a temperature change implies a change in p as well. From Eq. (11), one has $(dp/dT)_{p_N,T_N}\simeq0.21$. For the free energy calculation on what is supposed to be a given sample, the use of the same pseudorandom number sequence for T and $T\pm\delta T$ with the typical $\delta T=0.007$ (to be explained below) means that roughly 14 bonds in 10 000 will reverse sign. We have assumed that this is the meaning of ''using the same sample'' along the Nishimori line. While in principle the bond reversals tend to increase in-sample fluctuations, results are manageable (albeit with relative error bars ~ three orders of magnitude larger than those for derivatives along the pure-T direction), no doubt owing partly to dp/dT being small at the Nishimori point. We used a relatively large δT, compared with scaling

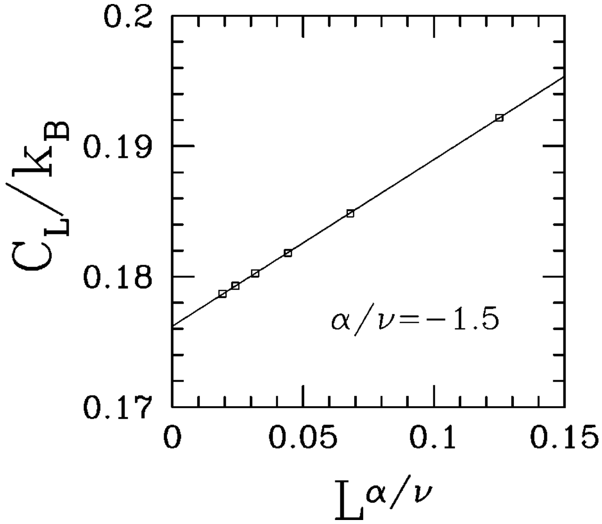

FIG. 8. Size dependence of the specific heat at the Nishimori point, resulting from the best fit of $\alpha/\nu=-1.5$ in Eq. (13). The intersect with the vertical axis is at a finite value, 0.1762. Error bars are smaller than data points.

along the pure-T axis, because of the need to compromise between *fluctuations* coming from the in-sample analysis described above, and the corresponding actual *variations* of the free energy, used to approximate the derivative.

We have tentatively interpreted the derivatives along the NL as specific-heat–like. Accordingly, we have applied FSS to the finite-size specific heats along both scaling directions in order to find estimates of (α/ν); in both cases the specific heat clearly does not diverge as $L\rightarrow\infty$. Tangent to the boundary line we have found that attempts to fit our data to the form

$$C_L=C_\infty+aL^{(\alpha/\nu)_{\rm trial}}, \tag{13}$$

yield a much smaller (four orders of magnitude) chi square for $(\alpha/\nu)_{\rm trial}=-1.5$ than for $(\alpha/\nu)_2\simeq-1.1$ (the latter is extracted from $\nu\simeq2.2$ of Ref. 16 plus the hyperscaling relation $d\nu=2-\alpha$). Figure 8 shows that our data fit neatly into a single-power form, i.e., corrections to scaling seem of little relevance in the case. Along the Nishimori line our data do not give a satisfactory behavior of the chi square for any sensible fitting to Eq. (13): varying α/ν between -2 and -0.5 does not change the chi square significantly, and this persists even when corrections to scaling are accounted for. Thus we are not in a position to compare these data to the percolation value[17] $(\alpha/\nu)_p=-1/2$. In Table II we display our raw data, so readers can reproduce the analysis quoted above, and try alternative procedures of their own devising.

We have thus turned to calculating the uniform susceptibility; as it couples to a ferromagnetic order parameter, the corresponding value of γ/ν is related to criticality upon crossing of the ferro-paramagnetic boundary (i.e., along the Nishimori line). In Fig. 9 we show $(\gamma/\nu)_L$, calculated from Eq. (10), with $T_c\equiv T_N$, as a function of $1/L^2$. The extrapolated value 1.80 ± 0.02 (where the estimated error bars are subjective, but certainly conservative) compares favorably with $(\gamma/\nu)_p=43/24=1.7917\cdots$ of percolation. Series work[14] gives $\gamma=2.37\pm0.05$ and $\nu=1.32\pm0.08$, which yield $\gamma/\nu=1.80\pm0.09$.

TABLE II. Second derivatives of free energy at the NP, for strip widths $L=4-14$. T: along the temperature axis; NL: along the Nisihmori line (see text). Uncertainties in last digits are given in parenthesis.

L	T	NL
4	0.192183(7)	1.558(38)
6	0.184854(10)	1.704(44)
8	0.181824(13)	1.772(39)
10	0.180245(7)	1.846(30)
12	0.179288(25)	1.846(67)
14	0.178667(20)	1.859(73)

Finally, correlation lengths obtained from the decay of spin-spin correlations (which therefore couple to a ferromagnetic order parameter) give an extrapolation of $\xi^{\rm av}/L$ to 1.75 ± 0.05 at the NP (see Fig. 6), rather different from the percolation $(\pi\eta_p)^{-1}=1.5279\cdots$. From experience for low frustration, as described above, we interpret this as signalling a breakdown of the exponent-amplitude relationship, rather than indicating that the transition is not in the percolation universality class.

V. CONCLUSIONS

We have studied the asymmetric $\pm J$ Ising model on a square lattice, by means of transfer matrix calculations of several quantities on long, finite-width strips. First, use has been made of a configurationally *averaged* correlation length, which is distinct from the *typical* (or most probable) correlation length, used in previous transfer matrix studies of the same model. We have shown that an intrinsically self-consistent picture can be obtained by the use of the former

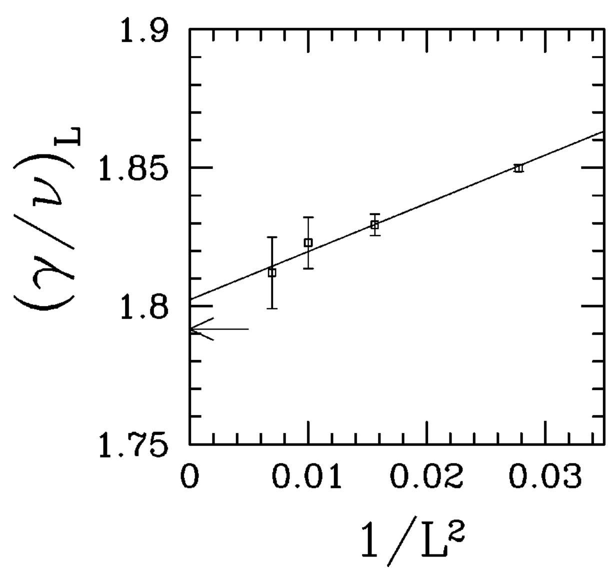

FIG. 9. Ratio of critical exponents, γ/ν, as a function of $1/L^2$, at the Nishimori point, from Eq. (10). The arrow points to the percolation value (see text).

quantity, while (at least for the strip widths within reach) it seems that higher-order corrections to scaling may distort analyses based on the latter. Points on the critical curve $T_c(p)$ were then obtained as those for which $\xi^{\rm av}/L$ approached a finite value as $L\to\infty$; to the best of our knowledge, the estimates for T_c at $p=0.99$, 0.95, and 0.92 thus obtained are the most accurate to date. Secondly, the critical behavior along the critical line has been discussed through the analysis of $d\xi^{\rm av}/dT$, as well as in terms of other quantities, such as the specific heat and the zero-field susceptibility.

The following picture emerged from our analysis. Above the Nishimori line, the correlation length and the susceptibility appear to diverge with power laws, with the same exponents as in the pure case, $\nu=1$ and $\gamma=7/4$; logarithmic corrections (i.e., enhancements) do not seem to play a role in the behavior of these quantities. We were also able to establish that the specific heat diverges, though at most logarithmically. Further, the exponent-amplitude relationship of conformal invariance breaks down as a result of frustration. These results are in marked contrast with the case of unfrustrated disorder, for which logarithmic enhancements were needed in order to explain an apparent disorder dependence on estimates for ν, and the conformal invariance prediction applies.

The intersection of the Nishimori line (NL) with the critical curve (T_N, p_N) has been determined, near which the critical behavior was analyzed; our estimates for (T_N, p_N) are also the most accurate to date. Approaching this Nishimori point either along a temperaturelike direction or along the NL, one finds nondiverging specific heats; while for the former we were able to extract $(\alpha/\nu)_T \simeq -1.5$, for the latter we could not find reliable fits. However, analysis of the uniform susceptibility, which probes the phase transition along the Nishimori line, showed percolationlike behavior, in the sense that γ/ν is very close to the percolation value. Conformal invariance is also absent at the Nishimori point. Further work is clearly necessary in order to fully elucidate all the subtleties related to the multicritical behavior at the Nishimori point. Finally, we found no signature of a reentrance in the phase diagram below the Nishimori point; instead, the critical curve below this point seems to be parallel to the temperature direction.

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