Exclusion statistics: A resolution of the problem of negative weights

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We give a formulation of the single-particle occupation probabilities for a system of identical particles obeying the fractional exclusion statistics of Haldane. We first derive a set of constraints using an exactly solvable model which describes an ideal exclusion statistics system and deduce the general counting rules for occupancy of states obeyed by these particles. We show that the problem of negative probabilities may be avoided with these counting rules. [S0163-1829(99)04933-4]

I. INTRODUCTION

A few years ago, Haldane[1] proposed a generalized exclusion principle in which adding particles into a system leads to a change in the dimension of the single-particle space. Specifically, the generalized exclusion principle envisages systems in which the addition of one particle blocks off g single-particle states for the others, where g, the fractional exclusion statistics (FES) parameter, is an arbitrary number. Obviously $g=0$ for bosons and $g=1$ for fermions. This leads to the following formula for $D_N(g,d)$, the dimension of the N-particle Hilbert space, if the dimension of the single-particle space is d,

$$D_N(g,d) = \frac{[d+(1-g)(N-1)]!}{N![d-1-g(N-1)]!}, \qquad (1)$$

which reduces to the familiar expressions for Bose and Fermi statistics for $g=0$ and $g=1$, respectively.

The thermodynamic properties of an ideal gas of exclusion particles have been investigated widely. Specifically, a definition of an ideal gas of particles with nontrivial exclusion statistics was given in Refs. 2 and 3. In this definition it was assumed that if there were d levels of energy ϵ, then the dimension of the Hilbert space with N particles of energy ϵ is given by Eq. (1). The distribution function may then be computed[2-5] and is given by

$$n(\epsilon) = \frac{1}{w(\epsilon)+g}, \qquad (2)$$

where $w(\epsilon)$ is the solution of the equation[2,5]

$$w(\epsilon)^g [1+w(\epsilon)]^{(1-g)} = e^{\beta\epsilon} \qquad (3)$$

and β is as usual the inverse temperature. If we attempt to interpret this distribution function as arising from the statistical mechanics of a single mode with statistical weight $\rho_n e^{-n\beta\epsilon}$ for the mode to be occupied by n particles, then it was found[6,7] that some of the ρ_n's are invariably negative if g is different from 0 and 1. This raises the problem of interpreting these negative probabilities. It has also been speculated that these negative probabilities are an essential feature of nontrivial exclusion statistics.[7]

A crucial property of exclusion statistical interactions is that they should cause shifts in single particle energies at all

scales.[8] This property is realized by a large class of one-dimensional models of interacting fermions where Fermi liquid theory breaks down.[9,10] In fact it has been shown exactly that quasiparticles with nontrivial exclusion statistics exist in a class of models that are solved by the Bethe ansatz.[3,11] In particular the quasiparticles of the Calogero-Sutherland model (CSM) behave like an ideal exclusion statistics system.[3,11-13] A feature of the exclusion statistics as gleaned from the analysis of various models is that the exclusion acts across a set of levels unlike in the case of Fermi or Bose statistics where the exclusion principle is stated with a single level in mind. It is this crucial difference that results in the occurrence of negative probabilities. We will show that the particles obeying fractional exclusion statistics can be characterized by constraints on the sets of occupation numbers. There are no negative probabilities if these constraints are obeyed. If these constraints are relaxed, then negative weights arise in order to compensate for the resulting over-counting. Indeed this is the way we encounter negative probabilities in other systems in physics—for example, in gauge theories, they arise in the ghost sectors. Ghosts come from the Jacobian associated with nonlinear gauges which essentially ensure the correct counting of states. Another example is that of the Wigner distribution function in quantum mechanics which is not positive definite precisely because some constraints are relaxed. A formulation based on the variable number of single-particle states, which depends on the total number of particles in the system, has been discussed by Isakov[14] as a way to avoid the problem of negative weights. Recently, a microscopic interpretation of exclusion statistics systems has been advanced by Chaturvedi and Srinivasan[15] where they show how this problem of negative probabilities may be solved for semions, $g=1/2$. They have also indicated how their method may be generalized to other values of g.

In this paper we first discuss the origin of negative probabilities in exclusion statistics particle systems. To do this we have chosen an unusual starting point in an equation and its solution given by Ramanujan (discussed in Ref. 16). This starting point makes precise the statements about the occurrence of negative probabilities. We then formulate a counting principle based on the set of constraints which reproduces the Haldane dimension formula. We first extract the counting rules starting from an exactly solvable model of interacting particles and state them in the form of counting rules for arbitrary systems obeying exclusion statistics. This method

not only avoids negative probabilities, but with minimal modification reproduces the results derived by Chaturvedi and Srinivasan for the semion. The counting principle is, however, not restricted to semions alone.

II. PROBLEM OF NEGATIVE WEIGHTS

The problem of negative probabilities was first pointed out by Nayak and Wilczek[6] and elaborated by Polychronakos in a recent paper.[7] In order to clarify the origin of negative probabilities or weights, we first discuss an equation and its solution due to Ramanujan.[16] Ramanujan considered the following equation:

$$aqX^p - X^q + 1 = 0, \tag{4}$$

where a may be complex and p,q are positive. The general solution for X^d is

$$X^d = \sum_{N=0}^{\infty} C_N(p,q,d)a^N, \tag{5}$$

where $C_0(p,q,d)=1$ and $C_1(p,q,d)=d$ and

$$C_N(p,q,d) = \frac{d}{N!} \prod_{j=1}^{N-1} (d+Np-jq), \quad N \geq 2.$$

To make a connection with the result obtained by Polychronakos,[7] which is a particular case of the general solution given by Ramanujan, we now put $p=(1-g)$ and $q=1$; then,

$$C_N(1-g,1,d) = d\frac{[d+(1-g)N-1]!}{N!(d-gN)!}, \tag{6}$$

which is clearly different from the dimension formula of Haldane. However, it correctly reproduces the bosonic and fermionic dimension formula for $g=0$ and $g=1$, respectively. This dimension formula was derived independently by Polychronakos[7] with the restriction that any two particles be at least g sites apart when placed on a periodic lattice. One can also derive the Haldane dimension formula with the restriction that any two particles be g sites apart but without the restriction of periodicity.

Further, if we put $X=(1+w^{-1})$ and $a=e^{-\beta\epsilon}$ in Eq. (4), we immediately obtain Eq. (3) derived earlier by Wu.[2] The important point to notice here is that the dimension formula that precisely leads to the distribution function derived earlier[2–5] is given by C_N and not the Haldane dimension formula. In the limit $d \gg 1$, however, it is easy to see that

$$C_N(1-g,1,d) = D_N(g,d) + O\left(\frac{1}{d}\right).$$

Therefore in the continuum limit, the C_N and D_N are approximately the same.

The grand canonical partition function of the system may be written as

$$Z = (1+w^{-1})^d = \sum_{N=0}^{\infty} C_N(1-g,1,d)e^{-\beta N\epsilon}, \tag{7}$$

where w satisfies Eq. (3). We have also assumed that all the energy levels are degenerate with energy given by ϵ. Note that this is an exact expression and no assumption is required on the single particle dimension d. The negative weights arise[6,7] when one insists on expanding $1+w^{-1}$ in powers of $e^{-\beta\epsilon}$. From Eq. (5) and the definitions following the equation, it follows that

$$1+w^{-1} = \sum_{n=0}^{\infty} C_n(1-g,1,1)e^{-\beta n\epsilon}. \tag{8}$$

The weights

$$C_n(1-g,1,1) = p_n = \prod_{m=2}^{n} \left(1 - \frac{gn}{m}\right) \tag{9}$$

are always negative for $gn>m$ for some m.[7] This is indeed the problem of negative weights associated with exclusion statistics and is claimed to be inherent in the exclusion statistics. There are, however, a few points to note: The negative probabilities arise because of our insistence on the factorization[15] implied in Eq. (7). For example, combining Eqs. (7) and (8) we have

$$Z = \sum_{\{n_j\}} \left(\prod_j C_{n_j}(1-g,1,1) \right) \exp\left(-\beta\epsilon \sum_j n_j\right), \tag{10}$$

where the sum is an unconstrained one over all sets of occupation numbers. The overcounting resulting from this unconstrained sum is compensated for by the occurrence of negative weights. We next derive the precise counting rules which impose constraints on this sum and avoid this problem.

III. REALIZATION IN AN INTERACTING SYSTEM AND COUNTING RULES

Any realization of fractional exclusion statistics must have its origins in systems of interacting particles. The expectation is that under certain conditions systems of interacting particles which obey Fermi or Bose statistics may be described in terms of quasiparticles (or quasiholes) which obey fractional statistics. The quasiparticles of the CSM behave like ideal exclusion statistics particles. The main feature of the CSM is that the total energy of the many-body system can be written in terms of single-quasiparticle energies which involve shifted momenta and these shifts contain information about the exclusion statistics of the quasiparticles. In this section we analyze these shifted momenta and make an explicit connection with the formula in Eq. (1). We then use them to obtain constraints on the allowed set of occupation numbers. These are what we refer to as the counting rules that reproduce the formula in Eq. (1). The statistical mechanics of the system obeying these constraints is then the same as that defined by Wu[2] and all statistical weights are positive.

We begin with the trigonometric Sutherland model[17] of an N-particle system on a ring of unit radius. The Hamiltonian is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{j<i} \frac{2g(g-1)}{\sin^2[(x_i-x_j)/2]}, \qquad (11)$$

where g is the interaction parameter. We will soon identify this with the statistical parameter of the exclusion statistics. While the model can be applied to both interacting bosons and fermions, we choose to work in the fermionic basis hereafter. The energy of an N-fermion state may be written in terms of shifted momenta as

$$E = \sum_{i=1}^{\infty} k_i^2 n_i, \qquad (12)$$

where $n_i = 0,1$ and the shifted momenta k_i (also called pseudomomenta in Ref. 13) are given by

$$k_i = m_i - (1-g)\frac{(N_i^- - N_i^+)}{2}, \qquad (13)$$

where m_i are distinct integers and $N_i^{-(+)}$ are the number of particles with shifted momenta less (greater) than k_i. Note that we could have also started with the Calogero-Sutherland model with harmonic confinement. The results below follow analogously with the proviso that we have shifted energies instead of shifted momenta.

First we establish the relationship between the shifted momenta given above and the Haldane's dimension formula (1). Consider the above system with an upper and lower cutoff on the momenta, k_{max} and k_{min}, respectively. We divide this range of momenta into cells of unit length (the first and last cells could be smaller) and define the occupancy of the jth cell, n_j, to be the number of particles with momenta k_i, such that $j+1 > k_i \geq j$. We identify single-particle space dimension d with the number of cells in the range, i.e., $d = k_{max} - k_{min}$, where d may be fractional. If we now denote the range of the m_i's by d_F, we have

$$d_F = m_{max} - m_{min} = d + (1-g)(N-1). \qquad (14)$$

Since there exists an m_i for every k_i, the total number of states in the range $k_{max} - k_{min}$ is the same as that between $m_{max} - m_{min}$. The total number of states is then the number of ways N distinct integers can be picked from d_F distinct integers, i.e., $^{d_F}C_N$, as in a fermionic description. Substituting for d_F from the above expression we immediately reproduce the Haldane dimension formula in Eq. (1).[8]

In order to obtain the counting rules we will first derive three properties of the set of momenta $\{k_i\}$. If k_i are ordered such that they increase with increasing i, then we have $k_{i+1} - k_i = m_{i+1} - m_i - (1-g)$. If $g<1$, then it follows that $m_{i+1} > m_i$. Further, if $m_{i+1} - m_i = 1$, then $k_{i+1} - k_i = g$, and if $m_{i+1} - m_i > 1$, then $k_{i+1} - k_i > (1+g) > 1$ because m_i's are integers.

We can then draw the following three conclusions from the properties of the shifted momenta k_i.

(1) The ordering in k_i's is the same as the ordering in m_i's.

(2) "Close-packed" m_i's with unit spacing correspond to "close-packed" k_i's with spacing g.

(3) The gaps between any two non-close-packed k_i's are greater than 1. Therefore all the k_is in any cell are close packed.

We now come to the question as to what are the constraints on the sets of occupation numbers $\{k_i\}$. For example, if $g=0$, there are no constraints as in the bosonic case. If $g=1$, the constraints are $n_j \leq 1$ as in the case of fermions. For any other g, one obvious constraint come from the second property derived above, namely, the occupancy of the jth cell $n_j \leq 1/g$ which specifies the maximum occupancy of a given cell assumed to be of unit spacing. This is the same constraint one derives from the distribution function of Wu, Eq. (3). An important departure from the usual bosonic and fermionic case is that the cell size is important and cannot be arbitrarily taken to zero as in the case of bosons and fermions.[6]

There are further constraints on the occupancy. To formulate them we use the third property. Let k_L be the lowest momentum in the jth cell. Then from the second and third properties, it follows that

$$k_L + g(n_j - 1) < j + 1. \qquad (15)$$

We can write k_L as $k_L = j + f(k_L)$, where $f(k_L)$ denotes the fractional part of k_L, that is, $0 \leq f(k_L) < 1$. We then have

$$f(k_L) + g(n_j - 1) < 1. \qquad (16)$$

From Eq. (13), we can express $f(k_L)$ as a function of the occupation numbers,

$$f(k_L) = f\left[-(1-g)\frac{[N_{cj}^- - N_{cj}^+ - (n_j - 1)]}{2}\right], \qquad (17)$$

where $N_{cj}^- = \Sigma_{l<j} n_l$ and $N_{cj}^+ = \Sigma_{l>j} n_l$. Equations (16) and (17) then constitute a set of constraints on the occupation numbers.

We will now show that these form a complete set of constraints. Namely, given any set of occupation numbers, $\{n_j\}$, that satisfies the constraints, there exists a set of momenta, $\{k_i\}$, that realizes it. To do this, consider a set $\{n_j\}$, where $j_{min} \leq j \leq j_{max}$. The lowest value of the momentum in the jth cell is uniquely determined by the occupation numbers through Eq. (17). Because of the third property, all the other momenta are also uniquely determined. Hence we have shown that there are no more constraints. Equations (16) and (17) form a complete set of constraints. Note also that the above logic implies that there is a one-to-one correspondence between the sets of occupation numbers, $\{n_j\}$, that satisfy the constraints (16) and (17) and the sets of momenta, $\{k_i\}$, that satisfy Eq. (13).

We can now remove the scaffolding of the Sutherland model that we started with and *define* an exclusion statistics system by the above constraints. The connection to the dimension formula in Eq. (1) established earlier implies that

$$\sum_{\{n_i\}} F(\{n_i\}) = D_N(g,d), \qquad (18)$$

where $N = \Sigma_j n_j$ and $F(\{n_i\}) = 1$ if $\{n_i\}$ satisfy the constraints and zero otherwise. Note that the weights now are positive definite. There are no negative weights once the constraints are imposed.

Next, we construct some simple examples from the above counting rules. For simplicity we look at occupation numbers for special values of $g = 1/m$ where m is an integer. The rules formulated above for the occupation number of exclusion particles may be combined and restated thus: Let $m = 1/g$, and let N_i be the number of particles in the occupied states below some ith level, $N_i = \Sigma_{j<i} n_j$. Then an occupation $n_i (n_i \leq m)$ is allowed if $(N_i \bmod m) \leq (m - n_i)$. This rule now includes all the three constraints stated above.

To see how this rule is implemented, consider a system of N particles spread over d states. In order that D_N be an integer, we choose $N = mp + 1$, where p is an integer. Since $N \leq md$, we have $p < d$. We shall divide these d states into cells. An allowed configuration may be represented as a string of numbers $(n_1, n_2, n_3, \ldots)$, where each $n_i \leq m$ denotes the occupancy of levels ordered from left to right. Instead of dealing with a configuration where all N particles are spread over d states (some which may be empty), we can simplify the discussion by considering one cell at a time. Each cell may now have a partition of m. This allows us to fill the subsequent cells without reference to the previous cell according to the counting rules since $N_i \bmod m = 0$. We now fill each cell with a partition of m which is allowed by the rules given above. This then generates all possible allowed configurations whose sum is given by D_N.

If, in particular, we are interested in expectation values of symmetric functions of n_i, we can work with symmetrized weights. Consider a symmetric operator $O(\{n_i\})$. The expectation value of this operator may be written as

$$\langle O(\{n_i\}) \rangle = \frac{\sum_{\{n_i\}} F_s(\{n_i\}) O(\{n_i\})}{\sum_{\{n_i\}} F_s(\{n_i\})}, \quad (19)$$

where

$$F_s(\{n_i\}) = \frac{1}{d!} \sum_p F(p_{n_i}). \quad (20)$$

Here p stands for all permutations of the allowed configurations. Every allowed configuration in $\{n_i\}$ may be characterized by the multiplicities q_n; namely, a given allowed configuration may be written as a string, $m^{q_m}(m-1)^{q_{m-1}} \cdots 1^{q_1}$, where $q_1 + 2q_2 + \cdots + mq_m = N$. We may now also allow any permutation of these occupancies (with zeros added to make up d states). The dimension of the N-particle space may then be written as

$$D_N(g,d) = \sum_{\{q_n\}} f_m^N(q_1, q_2, \ldots)^d C_q, \quad (21)$$

where $q = \Sigma_{n=1}^m q_n$. The new weights f are defined as

$$f_m^N(q_1, q_2, \ldots, q_m) = \frac{M_a(q_1, q_2, \ldots, q_m)}{M_t(q_1, q_2, \ldots, q_m)}, \quad (22)$$

where there are M_a allowed configurations after symmetrizing and M_t is the total number of configurations for a given set of q's which define a configuration. We shall clarify this now with specific examples.

A. Case of semion ($g = 1/2, m = 2$)

The maximal occupancy of a state in this case is 2. Hence the allowed occupancy of a state is 2 or 1. Zeros may occur anywhere without changing the rules. Let us implement this in the specific case of $d = 4, N = 5$, say. In this case the allowed configurations are given by the strings (2210),(2111),(1121). In the first configuration, zero can be anywhere and therefore there are four configurations. Notice that a string of the form (1211) or (1112) violates the counting rules. Therefore counting all the allowed configurations we obtain $D_5(1/2,4) = 6$. This is exactly what one gets from the Haldane formula.

Further, if we symmetrize each of these allowed configurations, then the new weights may be computed using Eq. (22). In the specific case of $m = 2$, we have

$$M_t(q_1, q_2) = {}^{q_1+q_2}C_{q_2}, \quad M_a(q_1, q_2) = {}^p C_{q_2}, \quad (23)$$

where p is defined through the equation $N = 2p + 1$. The corresponding f is therefore given by

$$f_2^N = \frac{{}^p C_{q_2}}{{}^{q_1+q_2}C_{q_2}}. \quad (24)$$

Note that these weights, whether in the symmetrized form or unsymmetrized form, are positive definite. Further, this is exactly the formula derived by Chaturvedi and Srinivasan[15] in their microscopic analysis of Haldane statistics for semions.

It is important to stress the differences in these two approaches—in their analysis Chaturvedi and Srinivasan start from a formulation of the statistical mechanics of a system by removing factorizability of the weights as a criterion. They derive the expression for the weights in Eq. (24) by imposing the conditions positivity and the requirement of symmetry (all configurations which are permutations of each other carry the same weight). Our starting point is the Sutherland model. We derive our rules from the properties of shifted momenta. After removing this scaffolding, we obtain not only positive definite weights for each configuration but when symmetrized they reproduce the results of Chaturvedi and Srinivasan.

B. Case with $g = 1/3$ or $m = 3$

The maximal occupancy of a state in this case is 3. The allowed configurations for each cell are (3),(21),(12),(111). That is, we can form a string of allowed configurations with any of these cells in any order to make up N particles. Any number of zeros may be added in between to make up a total of d states.

As in the semion case we may consider expectation values of symmetric functions of n_i. Following the same procedure we can derive the symmetrized weights f_3^N defined in Eq. (22). Since $m = 3$, we have

$$M_t(q_1, q_2, q_3) = {}^{q_1+q_2+q_3}C_{q_3} {}^{q_1+q_2}C_{q_2},$$

$$M_a(q_1, q_2, q_3) = {}^p C_{q_3} {}^{p-q_3}C_{q_2}(2)^{q_2}, \quad (25)$$

where p, as before, is defined through the equation $N=3p+1$. The corresponding weight f is therefore given by

$$f_3^N = \frac{{}^pC_{q_3}\, {}^{p-q_3}C_{q_2}(2)^{q_2}}{{}^{q_1+q_2+q_3}C_{q_3}\, {}^{q_1+q_2}C_{q_2}}. \tag{26}$$

These weights are again positive definite. Chaturvedi and Srinivasan[15] also suggest how their method may be extended beyond the semion case which they considered in detail. However, this extension requires additional conditions which are not imposed in the semion case. In contrast, our rules as derived from the point of view of an exactly solvable model are completely specified independent of the actual value of g (or m). There is an algorithm to derive f_m^N for arbitrary m though this gets complicated for larger m.

IV. SUMMARY

To summarize, we have analyzed the origin of negative probabilities in exclusion statistics systems. To do this we have chosen an unusual starting point in an equation and its solution given by Ramanujan. This starting point makes precise the statements about the occurrence of negative probabilities. Further, we have formulated a counting principle which reproduces the Haldane dimension formula. It can therefore be used to define exclusion statistics purely in terms of state counting. The negative probabilities discussed in the literature can be understood as arising when the system constrained by the counting rules is replaced by an unconstrained one. The negative weights then compensate for the introduction of unphysical configurations. This is therefore exactly analogous to other situations in physics where negative probabilities arise—for example, the ghosts and negative norm states in gauge theories or as in the case of the Wigner distribution in quantum mechanics.

[1] F.D.M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).

[2] Yong Shi Wu, Phys. Rev. Lett. **73**, 922 (1994).

[3] S.B. Isakov, Phys. Rev. Lett. **73**, 2150 (1994).

[4] A.K. Rajagopal, Phys. Rev. Lett. **74**, 1048 (1995).

[5] A. Dasniére de Veigy and S. Ouvry, Phys. Rev. Lett. **72**, 600 (1994).

[6] C. Nayak and F. Wilczek, Phys. Rev. Lett. **73**, 2740 (1994).

[7] A.P. Polychronakos, Phys. Lett. B **365**, 202 (1996).

[8] M.V.N. Murthy and R. Shankar, Phys. Rev. Lett. **72**, 3629 (1994).

[9] P.W. Anderson, Prog. Theor. Phys. Suppl. **107**, 45 (1992).

[10] M.V.N. Murthy and R. Shankar, Physica B **212**, 315 (1995).

[11] D. Bernard and Yong Shi Wu, cond-mat/9404025 (unpublished).

[12] M.V.N. Murthy and R. Shankar, Phys. Rev. Lett. **73**, 3331 (1994); **75**, 353 (1995).

[13] Z.N.C. Ha, Phys. Rev. Lett. **73**, 1574 (1994).

[14] Serguei B. Isakov, Phys. Rev. B **53**, 6585 (1996).

[15] S. Chaturvedi and V. Srinivasan, Phys. Rev. Lett. **78**, 4316 (1997).

[16] B.C. Berndt, R.J. Emery, and B.M. Wilson, Adv. Math. **49**, 123 (1983).

[17] B. Sutherland, Phys. Rev. A **5**, 1372 (1972).