

Giant Shapiro steps for two-dimensional Josephson-junction arrays with time-dependent Ginzburg-Landau dynamics

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Two-dimensional Josephson junction arrays at zero temperature are investigated numerically within the resistively shunted junction (RSJ) model and the time-dependent Ginzburg-Landau (TDGL) model with global conservation of current implemented through the fluctuating twist boundary condition (FTBC). Fractional giant Shapiro steps are found for *both* the RSJ and TDGL cases. This implies that the local current conservation, on which the RSJ model is based, can be relaxed to the TDGL dynamics with only global current conservation, without changing the sequence of Shapiro steps. However, when the maximum widths of the steps are compared for the two models some qualitative differences are found at higher frequencies. The critical current is also calculated and comparisons with earlier results are made. It is found that the FTBC is a more adequate boundary condition than the conventional uniform current injection method because it minimizes the influence of the boundary. [S0163-1829(99)00625-6]

Two-dimensional (2D) Josephson junction arrays (JJA's) have been the subject of much current interest because of their phase transitions and nonequilibrium transport properties.¹ In the presence of an external uniform magnetic field, the frustration f , defined by the number of flux quanta per plaquette, plays an important role reflected, e.g., in the value of the critical current.² Furthermore, when an $L \times L$ square array with $f = p/q$ (p, q are integers) is driven by combined direct and alternating currents $I_{\text{ext}}(t) = I_d + I_a \sin \omega t$, fractional giant Shapiro steps at voltages

$$\langle V \rangle = n \left(\frac{\hbar \omega}{2e} \right) \left(\frac{L}{q} \right), \quad (1)$$

where $\langle \dots \rangle$ is the time average and n is an integer, have been observed both in experiments³ and in computer simulations.⁴⁻⁷ Qualitative arguments have been proposed to explain these fractional steps in terms of vortex motion^{4,5} and topological invariance.^{8,9} In addition to the fractional steps, a series of small subharmonic steps has been found.¹⁰

Two slightly different models have been used to catch the essential properties of a JJA: the resistively -shunted junction (RSJ) model and the time-dependent Ginzburg-Landau (TDGL) model.¹¹ The RSJ model is based on the assumption that all the current goes through the array and that the current is conserved locally at each instant. The TDGL model in the absence of an external current describes either a situation where all the current goes through the array, but where the local current conservation is relaxed, or a situation where not all the current goes through the array (leakage to the ground) and the current is conserved at each instant. The former view means that the TDGL model can be regarded as a simplified version of the RSJ model and at the same time as a less restrictive model of a JJA. The latter view has led to the suggestion that a JJA with local damping is a possible realization of the TDGL model.^{12,13} In the presence of an external current the physics of the TDGL model depends on the choice of the boundary condition. We use here a boundary condition corresponding to the case when the normal current flow is through the array just as in the RSJ case. However, current is only conserved globally and not locally. In this

way, we can compare the effects caused by the difference in local current conserved dynamics as in the RSJ case with the TDGL dynamics which only has global current conservation.¹⁴

Both dynamic models are equivalent as far as static equilibrium properties are concerned, since they have the same equilibrium Boltzmann distribution.¹⁵ On the other hand, for dynamic quantities such as the dynamic dielectric function,^{15,16} flux-noise spectrum,^{13,17} and current-voltage (I - V) characteristics,^{15,18} the equivalence is not guaranteed. It has recently been suggested that the TDGL model could describe the flux-noise experiment for a JJA better than the RSJ model.¹³ However, a somewhat different conclusion was reached in Ref. 15 where properties like the linear response and nonlinear I - V characteristics were found to be the same for the two models.

In Ref. 15 a novel boundary condition [the fluctuating twist boundary condition (FTBC)] based on global current conservation was introduced.¹⁹ We show in this work that the very same Shapiro steps are found in the TDGL and RSJ models when we employ the FTBC as the boundary condition.¹⁴ This suggests that the existence of the steps does not depend on the details of the dynamic models: This robustness can be explained by the topological nature of steps where the ground state degeneracy (both models are equivalent in this respect) has been shown to play an important role.⁹ The widths of Shapiro steps as a function of I_a and ω have also been a subject of much interest. For example, the maximum width of the integer and the fractional steps have been shown to have a different frequency dependence for larger frequencies.^{6,20,21} We find that the maximum width of the half-integer step for the TDGL dynamics has a different frequency dependence than for the RSJ case. This offers an experimental possibility (similar to the experiment on an $f = 0$ array in Ref. 22) to investigate whether a JJA could sometimes be better described by the TDGL dynamics.

In the presence of applied direct currents, the critical currents $I_c(f)$, beyond which the voltage takes nonzero values, have been measured in experiments²³ and simulations.^{4,23} Although theoretical predictions^{2,23} and experiments²³ for f

$=1/2$ give the result $I_c(f=1/2) = \sqrt{2} - 1 \approx 0.414$ in units of $I_c(f=0)$, computer simulations with the conventional method of uniform current injection gives the value 0.35 ± 0.01 .^{4-7,24} In Ref. 5 it has been argued that this discrepancy is due to the boundary condition and that the conventional method destroys the translational symmetry of the ground state. On the other hand, we find in this work that the FTBC gives the value 0.4142(1) for both the RSJ and TDGL models, which suggests that the FTBC is a more adequate boundary condition since it conserves translational symmetry.

We start by introducing the equations of motion for the RSJ and TDGL models with the FTBC (see Ref. 15 for details). In the FTBC the twist variable $\Delta \equiv (\Delta_x, \Delta_y)$ is introduced and the gauge-invariant phase difference is changed into²⁵

$$\phi_{ij} = \theta_i - \theta_j - A_{ij} - \mathbf{r}_{ij} \cdot \Delta, \quad (2)$$

where θ_i is the phase of the superconducting order parameter at site i , $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$ is a unit vector from site i to j , and $A_{ij} \equiv (2\pi/\Phi_0) \int_i^j \mathbf{A} \cdot d\mathbf{l}$ with the magnetic vector potential \mathbf{A} and the flux quantum Φ_0 for Cooper pairs.

In the RSJ model, the equations of motion for phase variables are determined by the *local* current conservation at each site (see, for example, Refs. 4 and 10):

$$\dot{\theta}_i = - \sum_j G_{ij} \sum_k' \sin \phi_{jk}, \quad (3)$$

where the primed summation is over four nearest neighbors of j , G_{ij} is the square lattice Green function, and the unit of time is $\hbar/2eRI_c$ with shunt resistance R and critical current I_c of the single junction. In this work we only consider the array at zero temperature and accordingly the thermal noise terms are disregarded (see Ref. 15 for finite temperatures). For the TDGL model the equations of motion are given by^{15,16}

$$\dot{\theta}_i = - \sum_j' \sin \phi_{ij}, \quad (4)$$

where t is in units of $2e/I_c$.

In the FTBC case, the periodicities of the phase variables are preserved in both directions, i.e., $\theta_i = \theta_{i+L\hat{x}} = \theta_{i+L\hat{y}}$, and thus the voltage drop in each direction across the whole array is given by

$$V_x = - \frac{\hbar L}{2e} \Delta_x, \quad V_y = - \frac{\hbar L}{2e} \Delta_y, \quad (5)$$

from the Josephson relation. From the condition of *global* current conservation in each direction, we obtain the equations of motion for the twist variables of the array driven by an external current I_{ext} (in units of I_c) in the x direction:¹⁵

$$I_{\text{ext}} = - \frac{d\Delta_x}{dt} + \frac{1}{L^2} \sum_{\langle ij \rangle_x} \sin \phi_{ij}, \quad (6)$$

$$0 = - \frac{d\Delta_y}{dt} + \frac{1}{L^2} \sum_{\langle ij \rangle_y} \sin \phi_{ij}, \quad (7)$$

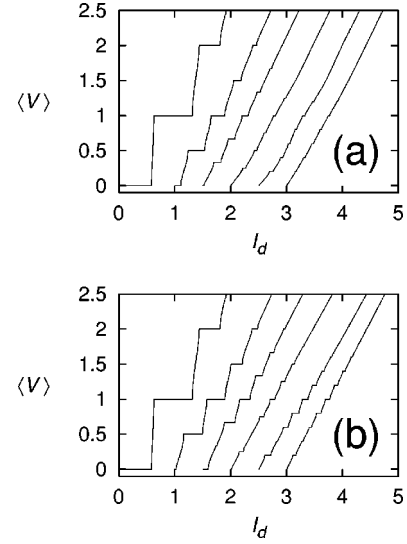


FIG. 1. Time-averaged voltages $\langle V \rangle$ in units of $L\hbar\omega/2e$ versus direct current I_d for (a) the RSJ and (b) the TDGL models in an $L \times L$ Josephson junction array in case of $f=0, 1/2, 1/3, 1/4, 1/5$, and $2/5$ (from the left to the right). The sizes of arrays are $L=4$ (for $f=0$), 8 ($f=1/2$ and $1/4$), 9 ($f=1/3$), and 10 ($f=1/5$ and $2/5$), and we have used the fluctuating twist boundary condition together with the condition of global current conservation and applied external currents $I_{\text{ext}} = I_d + I_a \sin \omega t$ with $I_a = I_c$ and $\omega/2\pi = 0.1$ in units of $2eRI_c/\hbar$ for the RSJ model and $I_c/2e$ for the TDGL model, respectively. Fractional giant Shapiro steps are clearly shown for the TDGL as well as for the RSJ model. All curves except $f=0$ are horizontally displaced for clarity.

where $\sum_{\langle ij \rangle_x}$ ($\sum_{\langle ij \rangle_y}$) is the summation over all nearest-neighbor pairs in the x (y) direction.

It should be observed that we are here considering the TDGL model [as specified by the dynamical equation (4)] with globally conserved current [as specified by Eqs. (6) and (7)]. One can also consider the TDGL model [as specified by Eq. (4)] without global current conservation within the plane [i.e., without Eqs. (6) and (7)] and interpret it as a model with leakage to the ground.^{12,13} In this case an applied external current only leads to dissipation to the ground at the boundaries where the current is injected and extracted. Consequently there exist no giant Shapiro steps for the TDGL model without globally conserved current within the plane.²⁶

We first consider an array with an external current $I_{\text{ext}} = I_d + I_a \sin \omega t$. We use the Euler algorithm with discrete time step $\Delta t = 0.05$ to integrate the equations of motion [Eqs. (3), (6), and (7) for the RSJ model and Eqs. (4), (6), and (7) for the TDGL model], and the time-averaged voltages $\langle V \rangle$ in units of $L\hbar\omega/2e$ are calculated from Eq. (5). We adopt the simulated annealing Monte Carlo method to find the ground states of the array and then use them as initial conditions of phase variables together with $\Delta_x(t=0) = \Delta_y(0) = 0$. Figure 1 shows the fractional giant Shapiro steps in the I - V characteristics for (a) the RSJ model and (b) the TDGL model. Although we have found quantitative differences in the Shapiro steps for the TDGL model (e.g., small step sizes at $\langle V \rangle = 1/3$ and $1/2$ for $f = 1/3$ and $1/4$, respectively), it is clear that with the FTBC not only the RSJ but also the TDGL model generates the integer and fractional steps. We have also observed weak subharmonic steps for both models as in Ref. 10.

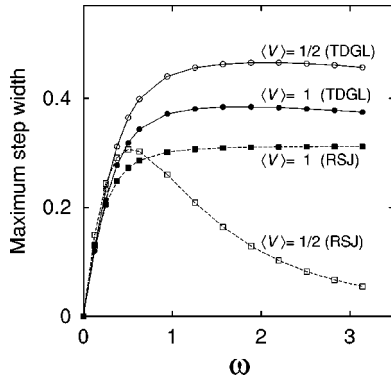


FIG. 2. Frequency dependence of the maximum widths of the Shapiro steps at $\langle V \rangle = 1/2$ and 1 for the RSJ and TDGL models with the FTBC in case of $f = 1/2$. Our results for the RSJ model are in good agreement with the analytic results. (Ref. 20). The high-frequency behavior of the 1/2 step for the TDGL model is shown to differ from the RSJ model. The lines are guides to the eye.

Figure 2 displays the maximum widths^{20,22} of the steps $\langle V \rangle = 1/2$ and 1 versus ω for both models. Although both show the same qualitative behavior in the low-frequency regime, it is apparent from the figure that the high-frequency behaviors of the half-integer steps are different for the two models. Since the frequency dependence of the maximum width can be measured for a JJA,²² this offers the possibility of experimentally distinguishing between the two types of dynamics.

We have also performed computer simulations applying a constant direct current $I_{\text{ext}}(t) = I_d$ for three different cases and obtained the critical currents $I_c(f)$: One case is the RSJ model with the conventional method of uniform current injection which employs the periodic boundary condition (the free boundary condition) in the direction perpendicular (parallel) to the applied currents. The other two are the RSJ and TDGL models with the FTBC. We present in Table I a comparison of these three cases, which reveals that the FTBC gives correct values for both the RSJ and TDGL models. For the conventional current injection method we obtained different values, e.g., $I_c(f = 1/2) = 0.35(1)$, as was also found in Ref. 5. We checked the system size dependence for $L = 4, 8, \dots, 128$ and found no change. Nevertheless, these smaller values are caused by the boundary condition which destroys the translational symmetry of the ground state.⁵ In Ref. 5 this problem was circumvented by a nonuniform injection method which matched the translational symmetry of the ground state and the correct value 0.414 was found. In

TABLE I. Comparison of critical currents at $f = 0, 1/2$, and $1/3$ for the the RSJ model with conventional method (I_c^{conv}), the RSJ model with the fluctuating twist boundary condition (I_c^{RSJ}), the TDGL with the FTBC (I_c^{TDGL}), and the analytic results in Ref. 23 (I_c^{anal}). All values are in units of $I_c(f = 0)$ and the numbers in parentheses are numerical errors in the last digits.

f	I_c^{conv}	I_c^{RSJ}	I_c^{TDGL}	I_c^{anal}
0	1.00(1)	1.0000(1)	1.0000(1)	1
1/2	0.35(1)	0.4142(1)	0.4142(1)	0.41421
1/3	0.14(1)	0.2679(1)	0.2679(1)	0.26789

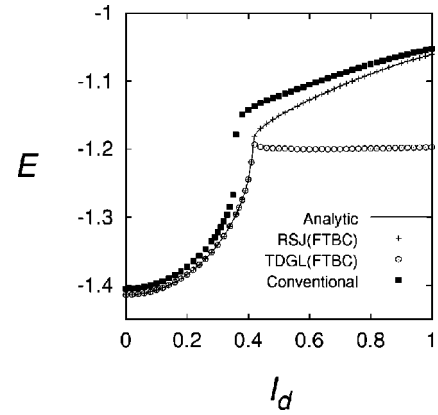


FIG. 3. Energy per site, $E \equiv -\langle \sum_{\langle ij \rangle} \cos \phi_{ij} \rangle / L^2$, for an $L \times L$ array in case of $f = 1/2$ with an applied direct current I_d . The results from the analytic equations (solid curve) in Ref. 20 are in perfect agreement with our results for the RSJ model using the FTBC. The TDGL model gives the same E below the critical current. However, the RSJ model with the conventional method has a cusp structure at a different value of I_d .

the FTBC case the translational symmetry of the ground state is automatically preserved and consequently this boundary condition directly yields the correct result. We also calculated the critical current with the busbar geometry¹⁸ and obtained an even smaller value of $I_c(f = 1/2)$ than for the conventional uniform injection method, as was already noticed in Refs. 5 and 24. From these comparisons we conclude that the FTBC has an advantage over other commonly used boundary conditions.

In Fig. 3 we show the average energy defined by $E \equiv -\langle \sum_{\langle ij \rangle} \cos \phi_{ij}(t) \rangle / L^2$ as a function of I_d for several cases. Our results for the RSJ model with the FTBC are in perfect agreement with the results from the analytic equations given in Ref. 20, which suggests that our results contain no boundary or finite-size effects. The TDGL model is found to give the same E for currents less than the critical value. Beyond the critical current, the TDGL model gives a lower energy, implying that the array is closer to the ground state than the

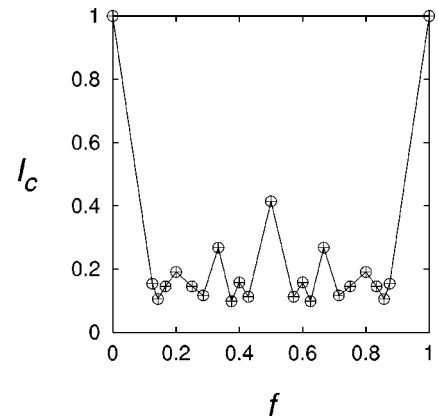


FIG. 4. Critical currents $I_c(f)$ as a function of f for the RSJ (denoted by +) and the TDGL (O) at $f = p/q$ with $q = 1, 2, \dots, 8$. The fluctuating twist boundary condition is used together with the condition of global current conservation. For all values of f tested in this work, the RSJ and TDGL models give the same value of $I_c(f)$ within numerical accuracy. The line is a guide to the eye.

RSJ model. We believe that this explains the robustness of the $1/2$ step of the TDGL model at high frequency (see Fig. 2), since it is expected that the ground state and its vortex superlattice structure plays an important role in creating the half-integer steps.⁶ We find in all cases that $E(I_d)$ has a cusp structure at the critical current. One may also note in Fig. 3 that the conventional uniform current injection method leads to a result which differs from the exact analytical result. Figure 4 gives the critical currents $I_c(f)$ at $f=p/q$ with $q=1, 2, \dots, 8$ (for comparisons with previous works, see Ref. 2). For all values of f , we obtain identical values of $I_c(f)$ for the TDGL and RSJ models.

In conclusion, we have performed simulations for the RSJ and TDGL models subject to the FTBC. Fractional giant Shapiro steps are obtained for *both* models, which suggests that the existence of the steps does not depend crucially on the condition of instantaneous local current conservation. However, the maximum width of the half-integer step at $f=1/2$ has a qualitatively different high-frequency behavior for the two models. The critical currents of the array with direct applied currents were also calculated for both the mod-

els subject to the FTBC and compared with the results obtained for the RSJ model with the conventional method of uniform current injection. It was concluded that the FTBC for both models gives values in agreement with experiments and analytic calculations, while the conventional method fails in this respect.

The present calculation supports the conclusion reached in Ref. 15 that the TDGL and RSJ models with the FTBC are qualitatively equivalent for low frequencies (compare Fig. 2) and small currents (compare Fig. 3) whereas for larger frequencies and larger currents there exist qualitative differences. The fact that both the models have qualitatively similar sequences of giant Shapiro steps suggests that the existence of these steps is strongly linked to an equilibrium property like the ground state degeneracy.⁹

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