Dependence of diffusion and multifractality at the quantum Hall transition on the effective dimension of the sample

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The time evolution of wave packets at the transition point in the two-dimensional (2D) Chalker-Coddington model is investigated numerically. As a function of the shape of the sample, characterized by the aspect ratio, it is found that for metallic diffusive systems there is a time scale beyond which the return probability is affected drastically, leading to a rapid change in the exponent of its time decay and reflecting the change from 2D to 1D behavior. While for quantum systems, at the critical point, the change in the aspect ratio is found to lead toward localization which, on the other hand, leads the critical level-spacing distribution $P(s)$ to approach the Poissonian one. [S0163-1829(99)09927-0]

It is well established that spectral statistics of disordered conductors and insulators are governed by universal distribution functions. For energies small compared to the Thouless energy, the correlations between energy levels of metals are described by random-matrix theory, leading to the Wigner-Dyson distribution for level spacing.[1] In the insulating regime, the energy levels are completely random and their statistics are governed by the Poissonian statistical laws of uncorrelated variables. In both cases, deviations from these universal distributions decrease with increasing system size and they vanish in the limit of infinite systems, provided the system is not exactly at the metal-insulator transition point.[2] This difference in the behavior between metals and insulators is mainly due to the strong overlap of the extended states in the metallic case, which is negligible in the insulating case where the eigenfunctions are localized. On the other hand, the spectral statistics are intimately related to the time evolution of wave packets. If a wave packet spreads diffusively for long times the system is a metal. In contrast, if a wave packet does not spread and stays around its origin for long times the system is an insulator.

During the past few years, much attention has been focused on the critical properties near the localization-delocalization transition (LDT) in disordered systems.[3–11] Intensive studies made by different groups have shown that exactly at the transition point the system exhibits a third universal behavior intermediate between those corresponding to the metallic and insulating regimes.

However, recently the universality of the critical distribution was questioned by numerical investigations of the three- and two-dimensional Anderson models at the transition point.[12] Their results lead to the conclusion that the critical statistics might be strongly affected by the boundary conditions of the system. The situation is more complicated in the case of the two-dimensional (2D) Chalker-Coddington network model (QHE).[13] In QHE systems all states are localized with a localization length diverging at the critical energy $E_c=0$. At LDT the eigenstates are multifractal[14,15] and the extended metallic phase is absent. For finite systems, the introduction of Dirichlet boundary conditions affects strongly the location of the critical region and this effect should be taken into account for the determination of the critical $P(s)$ distribution.[16] More recently, it has been suggested that the statistical properties at the critical point of the three-dimensional Anderson model depend on the shape of the sample while the critical disorder W_c is unchanged.[17]

In this paper we present the results of numerical investigations of both statistical properties of the quasienergy levels and the time evolution of wave packets near the critical point of the 2D Chalker-Coddington network model.[13] This model describes the localization-delocalization transition in the integer quantum Hall effect, which is similar to the three-dimensional Anderson transition in the absence of time-reversal symmetry. Using this model, we studied the effect of the formal invariance[18] on both the critical statistical properties and the time evolution of wave packets. The two-dimensionality of the model considered here allows for the study of large systems. We have concentrated our investigations on the level distribution $P(s)$ at small level spacings s up to the order of the mean level spacing Δ and long-time evolution of a wave packet because the short-range energy regime, or equivalently long-time limit, is affected by the long-range structure of the eigenstates, which, on the other hand, is expected to be most sensitive to the shape of the sample. The shape of the sample is characterized by the aspect ratio q, defined as $q=L_x/L_y$, where L_x and L_y are the longitudinal and transversal lengths of the system, respectively. For QHE systems with periodic boundary conditions, it is found that the scale-independent critical distribution $P(s)$, as well as the evolution of a wave packet, is strongly affected as a function of the aspect ratio of the sample while the critical energy remains unchanged, $E_c=0$. In fact, an increase of the aspect ratio leads to localization of the eigenstates with a localization length of the order of the width of the sample.[13] This effect is shown to decrease the repulsion between energy levels described by a $P(s)$ distribution closer to the Poissonian one (Fig. 1). This result is shown to be strongly related to a change in the dynamical properties of the system from two-dimensional behavior to one-dimensional behavior.

We start with a short description of the Chalker-Coddington network model. It is based on ideas developed in

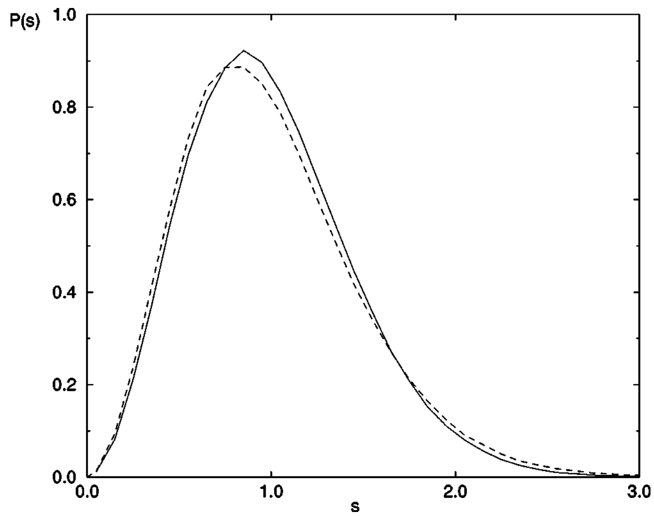

FIG. 1. The critical level-spacing distribution $P(s)$ at energy $E=0$ for a system of linear size $L=30$ and aspect ratio $q=1$ (solid line) shows significant deviations from the $P(s)$ distribution for $q=2$ (dashed line).

(Ref. 19) for the description of the Anderson transition in terms of scattering theory. It consists of 2×2 scattering matrices representing saddle points of a smooth random potential that are arranged on a square lattice and connected by one-dimensional unidirectional channels, called links, corresponding to equipotential lines between the saddle points.[13] The scattering matrices S_j describe the transitions from electron states of incoming links ψ_k, ψ_l to outgoing link states ψ_m, ψ_n (see Fig. 2),

$$\begin{pmatrix} \psi_m \\ \psi_n \end{pmatrix} = S_j \begin{pmatrix} \psi_k \\ \psi_l \end{pmatrix}, \quad S_j = \begin{pmatrix} t_{mk} & t_{ml} \\ t_{nk} & t_{nl} \end{pmatrix}. \tag{1}$$

By traversing a link l an electron acquires an additional phase ϕ_l, randomly distributed in $[0,2\pi]$ but constant for each link, which we absorb in the complex, energy-dependent scattering coefficients $t(E)_{mk}, t(E)_{nk}$. The transmission probabilities $T_+ = |t_{ml}|^2 = |t_{nk}|^2$, $T_- = |t_{mk}|^2 = |t_{nl}|^2$ are determined by the difference between electron energy E and saddle-point energy u_j, $T_+^{-1} = 1 + \exp[(u_j - E)/E_t]$, $T_- = 1 - T_+$.[20] For our calculations we choose $u_j \equiv 0$ and set

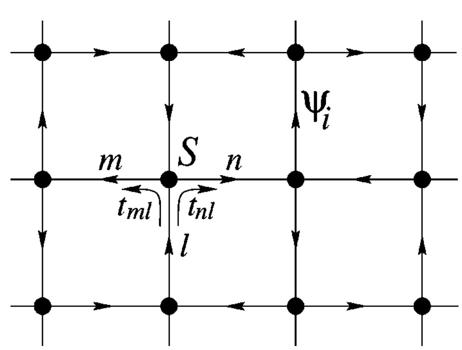

FIG. 2. The Chalker-Coddington network. At each saddle point a scattering matrix S describes the transition from incoming to outgoing states. The operator U maps each incoming link amplitude to the two outgoing links according to the transmission coefficients $t_{ml}, t_{nl}, \ldots$.

the tunneling energy $E_t = 1$. Different disorder configurations are realized by taking different sets of random phases ϕ_l.

A network state is given by the set of complex amplitudes ψ_l on each link, $\Psi = \{\psi_l\}$. A unitary operator $U(E)$ acting on these states, according to the scattering coefficients t_{ml}, is defined by

$$U(E)\mathbf{e}_l = t(E)_{ml}\mathbf{e}_m + t(E)_{nl}\mathbf{e}_n, \tag{2}$$

where $\mathbf{e}_k = \{\delta_{kl}\}_l$ denotes a unit vector at link l, whose components are zero except at link k.[14] $U(E)$ is unitary due to the current conservation at each node and can be interpreted as a discrete time-evolution operator for network states, $\Psi(t+\tau) \equiv U(E)\Psi(t)$.[15] The eigenvectors $\phi(E)_l$ of $U(E)$ correspond to eigenstates of the real system and the phases or quasienergies $\omega(E)_l$ of their complex eigenvalues $\exp[i\omega_l(E)]$ to energy levels in the vicinity of the energy E.[21] At the transition point, the eigenvectors $\phi(E_c)$ of $U(E)$ exhibit a multifractal structure similar to the metal-insulator transition.[14]

The model shows a quantum-Hall-effect–type transition from localized to delocalized states at a singular critical energy E_c. For an infinite sample and for energies E far away from the critical point, $|E-E_c| \gg E_t$, the eigenstates of $U(E)$ are strongly localized, with localization length ξ_E of order of the link length. When E approaches the critical energy E_c, $\xi(E)$ diverges with $\xi \propto |E-E_c|^{-\nu}$, where $\nu \approx 2.3$ is the critical exponent of the integer quantum Hall transition.[13] Note that the localization length $\xi(E)$ varies with E, but not with the quasienergy ω. Therefore, the level statistics does not change within one quasispectrum $\omega_l(E)$. For this property, which allows as to focus on a very narrow energy regime, the network model is especially suitable for the study of critical spectral statistics. In the strongly localized regime, $|E-E_c| \gg E_t$, the levels $\omega_l(E)$ are independent and obey Poisson statistics. Approaching E_c, level repulsion emerges, reflected by distribution of the quasilevel spacings $s_l = (\omega_{l+1} - \omega_l)/\Delta$;[20] close to a Wigner-Dyson type.

Here, we analyzed the spectra for systems of different lengths L_x and widths L_y with periodic boundaries in all directions. All data were obtained by averaging over at least 5000 disorder configurations. Due to the symmetry of QHE systems with periodic boundary conditions, the critical energy is exactly located at $E_c = 0$. We have studied the level-spacing distributions $P_c(s)$ for different values of the aspect ratio q. In Fig. 1 we have plotted the results for two values of the aspect ratio $q=1$ and $q=2$ at energy $E=0$. In agreement with the result obtained in Ref. 17, the $P_c(s)$ distribution gets closer to the Poissonian behavior as the aspect ratio q increases.

To give a more quantitative explanation of this behavior, we considered the metallic behavior of a wave packet in a 2D system given by

$$Q(r,t) = \frac{\exp[-r^2/(4Dt)]}{4\pi Dt}, \tag{3}$$

at time t and distance r from the origin, where D is the diffusion constant. It is easy to show that for a 2D system with $L_x > L_y$ and periodic boundary conditions there exists a time scale τ such that the system will behave like a 2D system when $t < \tau$ and will exhibit 1D behavior when $t > \tau$. The

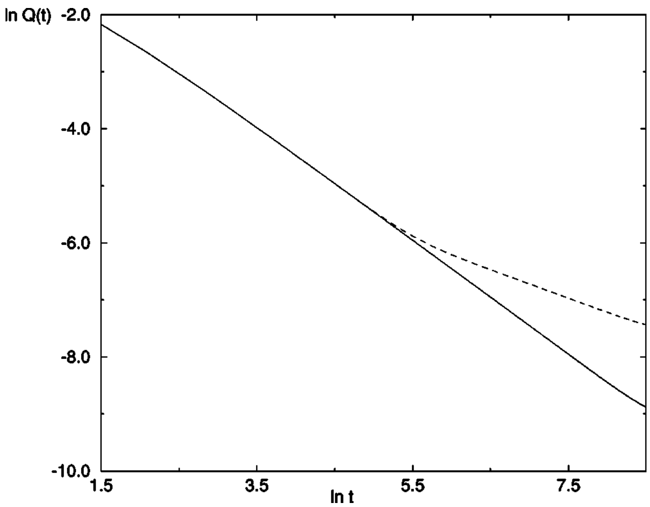

FIG. 3. Metallic behavior of the return probability for a wave packet at the critical point $E=0$ for systems with aspect ratios $q=1$ (solid line) and $q=3$ (dashed line).

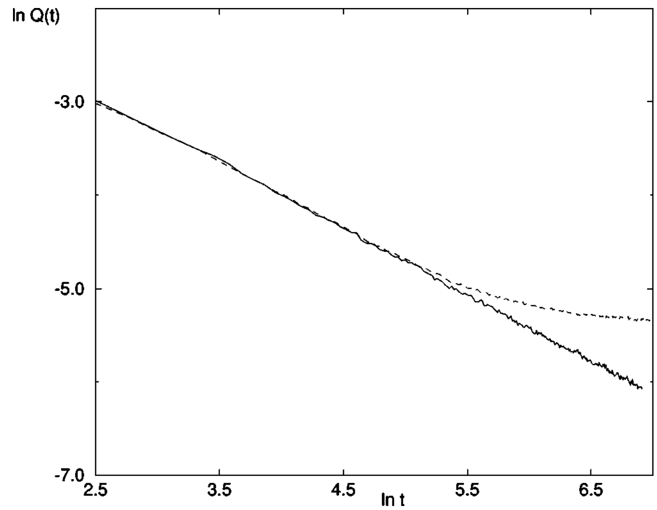

FIG. 4. Return probability for a wave packet at the critical energy $E=0$ for $q=1$ (solid line) shows a significant deviation from $Q(t)$ obtained at the same energy for a system with $q=3$ (dashed line).

time scale τ corresponds to the time needed by the wave packet to diffuse along the transversal direction of length L_y. In fact, for $t<\tau$ the return probability $Q(r=0,t)$ scales like t^{-1}. While due to saturation along the transversal direction (y axis), for $t>\tau$, $Q(r,t)$ becomes independent of the transversal coordinate y, and the evolution of the wave packet reduces to that of a one-dimensional system, where the return probability scales like $t^{-1/2}$. Similar arguments were previously used by Imry *et al.*[22] to obtain scaling laws relating the critical indices for different dimensionalities. To check this point we used the classical version of QHE systems by replacing the scattering coefficients t_{ml} by the transmission probabilities $T=|t_{ml}|^2$ in the definition of the operator $U(E)$. Then, the discrete time evolution of a wave packet is obtained by repeatedly acting with the operator U on a wave packet initially peaked at a single link, taken as the origin e_0. The return probability, at time t, is defined by

$$Q(0,t)=|\langle e0|U^t|e0\rangle|^2, \tag{4}$$

with integer t. From Fig. 3 it is clear that for large values of q there is a time scale τ after which the value of the exponent of the return probability changes from -1 to $-\frac{1}{2}$, thus demonstrating the reduction of the effective dimensionality of the system from 2 to 1. It is also important to notice that the crossover between the two regimes is very sharp. In the case of QHE systems, the situation is extremely different due to the interference effects. Consequently, the return probability saturates almost exactly at time τ as can be seen in Fig. 4 for $q=3$, thus reflecting the one-dimensional behavior of the system for $t>\tau$. This effect can be understood if we take into account the fact that for systems with one finite dimension L_y, the critical localization length $\xi(E)$ is of the order of the width of the sample L_y,[13] so that for large values of q and large times the shape of the wave packet exhibits an exponential decay from its origin, as represented in Fig. 5. In Fig. 4 we have also plotted the return probability for $q=1$. In contrast to the previous case, the wave packet spreads over all the samples with a time decay exponent $\delta=0.74$, which is in good agreement with previous calculations. As mentioned above, the $P(s)$ distribution and the evolution of a wave packet are strongly correlated. For large values of q corresponding to a quasi-one-dimensional system of width L_y, the localization length $\xi(E)$ of the eigenstates at the center of the band is of the order of the width L_y. As a result, the overlap between eigenstates whose centers are separated in space by distances larger than L_y (along the x axis) decreases exponentially, leading to a decrease of the repulsion between the corresponding eigenvalues as can be seen in Fig. 2. So the question whether there is a shape dependence of the critical level-spacing distribution remains open. This question could be studied using other shapes, such as spheres and discs in three and two dimensions, respectively. These geometries, unlike the torus one, do not introduce a time scale after which the behavior of the sample changes from d-dimensional to $(d-1)$-dimensional behavior. Finally we should mention that this result is not in contradiction with a recent analytical work[23] where the localization effects are negligible.

The question of the dependence of the level-spacing distribution and the time evolution of a wave packet at the transition point on the aspect ratio is investigated numerically, using the 2D Chalker-Coddington model. The main

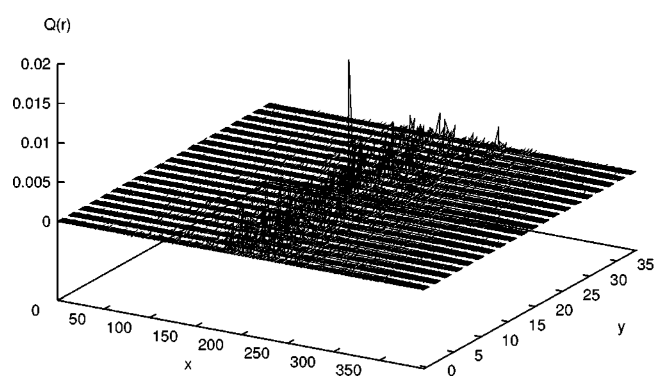

FIG. 5. Spread of a wave packet along the x axis after a long time at energy $E=0$ for $L_x=200$ and $q=10$.

conclusions that can be drawn from our results are the following: we give a detailed explanation of the behavior of the $P(s)$ distribution as a function of the aspect ratio q. The observed evolution of the $P(s)$ toward the Poissonian distribution, when we increase the value of q, is mainly due to the localization effects as can be deduced from the time evolution of a wave packet. We also demonstrate that for $q > 1$, there exist a characteristic time scale τ after which the dynamical properties of the sample changes from two-dimensional behavior to one-dimensional behavior.

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