Effective analysis of the O(N) antiferromagnet: Low-temperature expansion of the order parameter

Christoph P. Hofmann*
Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland
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We investigate the low-energy properties of Lorentz-invariant theories with a spontaneously broken rotation symmetry $\mathrm{O}(N)\rightarrow\mathrm{O}(N-1)$. The leading coefficients of the low-temperature expansion for the partition function are calculated up to and including three loops. Emphasis is put into the special case $N=3$: it describes the antiferromagnet which has been extensively studied. Our results obtained within the framework of the effective Lagrangian technique are compared with the literature. In particular, we show that, at order T^7 for the heat capacity and T^6 for the order parameter, respectively, logarithmic terms appear in the low-temperature expansion, which have been overlooked so far. [S0163-1829(99)02325-5]

I. INTRODUCTION

In the present paper, we investigate the low-temperature properties of spontaneously broken O(N) symmetric theories. More precisely, we consider the specific case where the symmetry $G=\mathrm{O}(N)$ of the Lagrangian is spontaneously broken to $H=\mathrm{O}(N-1)$—we then have $N-1$ Goldstone bosons in the broken phase $(N\geqslant 2)$. If the perturbations, which break the rotation symmetry explicitly, are small, these excitations remain light and dominate the low-energy behavior of the system. Moreover, the Goldstone particles interact weakly at low energies and a systematic perturbative expansion can be advised. In particular, the partition function can be evaluated in this manner, amounting to low-temperature theorems for quantities of physical interest.

A very efficient tool to analyze the low-energy structure of a system which exhibits spontaneous symmetry breaking is provided by the effective Lagrangian technique. The method applies to any system where the Goldstone bosons are the only excitations without energy gap. The essential point here is that the properties of these degrees of freedom and their mutual interaction are strongly constrained by the symmetry inherent in the underlying model—the specific nature of the underlying model itself, however, is not important. For a pedagogic outline of the effective Lagrangian method with applications to nonrelativistic systems, the reader may consult Ref. 1, which is written in a condensed matter perspective.[2–4]

Within the framework of effective Lagrangians, the low-temperature behavior of chiral theories, where the respective groups are $G=\mathrm{SU}(n)_L\times\mathrm{SU}(n)_R$ and $H=\mathrm{SU}(n)_V$, has been analyzed in detail.[5,6] In particular, the low-temperature expansion for the quark condensate, which represents the most prominent order parameter in quantum chromodynamics (QCD), has been evaluated to three loops. In the present work, we repeat this analysis for Lorentz-invariant theories displaying a spontaneously broken rotation symmetry, $\mathrm{O}(N)\rightarrow\mathrm{O}(N-1)$—the system may then be referred to as O(N) antiferromagnet.

Apart from this general analysis, our interest is devoted to the special case $N=3$: the results obtained then describe the O(3) antiferromagnet, where the spin waves or magnons represent the corresponding Goldstone degrees of freedom. This system has been widely studied in condensed matter physics and it is instructive to compare our results with the findings derived within the microscopic Heisenberg model. The lattice structure of a solid singles out preferred directions, such that the effective Lagrangian is not invariant under space rotations. In the case of a cubic lattice, the anisotropy, however, only shows up at higher orders of the derivative expansion[7]—the discrete symmetry of the lattice thus implies space rotation symmetry. Hence, the leading order effective Lagrangian of an antiferromagnet is Lorentz invariant:[8] antiferromagnetic spin-wave excitations exhibit relativistic kinematics, with the velocity of light being replaced by the spin-wave velocity.

In a Lorentz-invariant theory, the following invariance theorem holds: up to a Wess-Zumino term, the effective Lagrangian may be brought to a form which is manifestly invariant with respect to the internal symmetry of the underlying theory.[9] The procedure of constructing the corresponding effective Lagrangian is thus straightforward: one writes down the most general expression consistent with Lorentz symmetry and the internal symmetry G of the underlying model in terms of Goldstone fields $U^a(x)$, $a=1,\ldots,$ $\dim(G)$-$\dim(H)$—the effective Lagrangian then consists of a string of terms involving an increasing number of derivatives or, equivalently, amounts to an expansion in powers of the momentum. Moreover, the effective method allows us to systematically take into account interactions which explicitly break the symmetry G of the underlying model, provided that they can be treated as perturbations.[10,11]

In the particular case we are considering, the symmetry O(N) is broken by an external field: it is convenient to collect the $(N-1)$ Goldstone fields U^a in a N-dimensional vector $U^i=(U^0,U^a)$ of unit length,

$$U^i(x)U^i(x)=1, \tag{1.1}$$

and to take the constant external field along the zeroth axis, $H^i=(H,0,\ldots,0)$. The Euclidean form of the effective Lagrangian up to and including order p^4 then reads[12]

$$\mathcal{L}_{\rm eff}=\tfrac{1}{2}F^2\,\partial_\mu U^i\,\partial_\mu U^i-\Sigma_s H^i U^i$$
$$-e_1(\partial_\mu U^i\,\partial_\mu U^i)^2-e_2(\partial_\mu U^i\,\partial_\nu U^i)^2+k_1\frac{\Sigma_s}{F^2}(H^iU^i)$$
$$\times(\partial_\mu U^k\,\partial_\mu U^k)-k_2\frac{\Sigma_s^2}{F^4}(H^iU^i)^2-k_3\frac{\Sigma_s^2}{F^4}H^iH^i. \quad (1.2)$$

In the power counting scheme, the field $U(x)$ counts as a quantity of order 1. Derivatives correspond to one power of the momentum $\partial_\mu \propto p$, whereas the external field H counts as a term of order p^2. Hence, at leading order ($\propto p^2$) two coupling constants F and Σ_s occur, at next-to-leading order ($\propto p^4$) we have five such constants, e_1, e_2, k_1, k_2, k_3. Note that these couplings are not fixed by symmetry—they parametrize the physics of the underlying theory.

The effective Lagrangian method provides us with a simultaneous expansion in powers of the momenta and of the external field. To a given order in the low-energy expansion only a finite number of coupling constants and a finite number of graphs contribute. Consider, e.g., scattering processes between Goldstone particles. The leading contributions to the scattering amplitudes stem from the tree graphs. Graphs involving l loops are suppressed by l powers of p^2/F^2 and do therefore not affect the leading terms.

If the scattering amplitudes are thus needed to accuracy p^4, the effective Lagrangian must be known up to and including $\mathcal{L}^4_{\rm eff}$. There are two types of contributions at this order of the low-energy expansion: one-loop graphs of $\mathcal{L}^2_{\rm eff}$ and tree graphs containing one vertex from $\mathcal{L}^4_{\rm eff}$. Similarly, at order p^6, two-loop graphs of $\mathcal{L}^2_{\rm eff}$ as well as one-loop graphs involving one vertex from $\mathcal{L}^4_{\rm eff}$ and tree graphs with two vertices from $\mathcal{L}^4_{\rm eff}$ or one vertex from $\mathcal{L}^6_{\rm eff}$ contribute, etc.

It is convenient to use dimensional regularization of the loop integrals. In this scheme, the ultraviolet divergences of the one-loop graphs of $\mathcal{L}^2_{\rm eff}$ are absorbed in a renormalization of the coupling constants which occur in $\mathcal{L}^4_{\rm eff}$—the constants F and Σ_s in $\mathcal{L}^2_{\rm eff}$ are not renormalized. More generally, dimensional regularization ensures that the ultraviolet divergences which occur in the *sum* of all graphs of order p^{2n} are removed by a suitable renormalization of the coupling constants in $\mathcal{L}^{2n}_{\rm eff}$—the lower order Lagrangians $\mathcal{L}^2_{\rm eff}, \ldots, \mathcal{L}^{2n-2}_{\rm eff}$ remain untouched.

II. FINITE TEMPERATURE

The effective Lagrangian method can readily be extended to finite temperature. In the partition function, contributions of massive particles are suppressed exponentially, such that the Goldstone bosons dominate the properties of the system at low temperatures. Hence, the low-energy theorems for scattering amplitudes, e.g., are converted into temperature theorems for the partition function. In the power counting rules, the role of the external momenta is taken over by the temperature, which is treated as a small quantity of order p. The interaction among the Goldstone degrees of freedom now generates perturbations of order $p^2/F^2 \propto T^2/F^2$.

In the effective Lagrangian framework, the partition function is represented as a Euclidean functional integral[13–15]

$$\mathrm{Tr}[\exp(-\mathcal{H}/T)]=\int [dU]\exp\left(-\int_T d^4x\,\mathcal{L}_{\rm eff}\right). \quad (2.1)$$

The integration is performed over all field configurations which are periodic in the Euclidean time direction $U(\vec{x},x_4+\beta)=U(\vec{x},x_4)$, with $\beta\equiv 1/T$. The low-temperature expansion of the partition function is obtained by considering the fluctuations of the field U around the ground state $V=(1,0,\ldots,0)$, i.e., by expanding U^0 in powers of U^a, $U^0=\sqrt{1-U^aU^a}$. The leading contribution (order p^2) contains a term quadratic in U^a which describes free Goldstone bosons of mass $M^2=\Sigma_s H/F^2$. The remainder of the effective Lagrangian is treated as a perturbation. Evaluating the Gaussian integrals in the standard manner, one arrives at a set of Feynman rules which differ from the conventional rules of the effective Lagrangian method only in one respect: the periodicity condition imposed on the Goldstone field modifies the propagator. At finite temperature, the propagator is given by

$$G(x)=\sum_{n=-\infty}^{\infty}\Delta(\vec{x},x_4+n\beta), \quad (2.2)$$

where $\Delta(x)$ is the Euclidean propagator at zero temperature. We restrict ourselves to the infinite volume limit and evaluate the free energy density z, defined by

$$z=-T\lim_{L\to\infty}L^{-3}\ln[\mathrm{Tr}\exp(-\mathcal{H}/T)]. \quad (2.3)$$

Temperature thus produces remarkably little change: to obtain the partition function, one simply restricts the manifold on which the fields are living to a torus in Euclidean space. The effective Lagrangian remains unaffected—the coupling constants $F, \Sigma_s, e_1, \ldots,$ are temperature independent.

To evaluate the graphs of the effective theory, it is convenient to use dimensional regularization, where the zero-temperature propagator reads

$$\Delta(x)=(2\pi)^{-d}\int d^dp\, e^{ipx}(M^2+p^2)^{-1}$$
$$=\int_0^\infty d\rho(4\pi\rho)^{-d/2}e^{-\rho M^2-x^2/4\rho}. \quad (2.4)$$

We perform the calculation up to and including terms of order p^8 in the free energy density of the system. To this order in the momenta, contributions to the effective Lagrangian involving at most eight derivatives enter and the perturbative expansion requires the evaluation of graphs containing at most three loops.

The renormalization procedure is identical with the one used in connection with chiral effective theories: the same graphs have to be evaluated and, up to Clebsch-Gordan coefficients specific to the group O(N), the same ultraviolet divergent expressions occur in the loop-integrals—for the explicit form of these quantities and the construction of the respective counterterms, the reader is thus referred to Ref. 6. We just mention the fact that the contributions to the effective Lagrangian of order p^6 and of order p^8 merely renormalize the mass of the Goldstone bosons and the vacuum energy. At the order in the low-temperature expansion we are considering here, the values of the coupling constants occurring in these pieces of the effective Lagrangian are therefore irrelevant and the renormalized mass takes the form

$$M_\pi^2 = \frac{\Sigma_s H}{F^2} + \frac{N-3}{32\pi^2}\frac{(\Sigma_s H)^2}{F^6}\ln\frac{H}{H_M} + \mathcal{O}(H^3). \tag{2.5}$$

The logarithmic scale H_M is determined by k_1 and k_2, i.e., by two next-to-leading order coupling constants (order p^4)—a brief discussion can be found in the Appendix.

In the limit of a zero external field, the low-temperature expansion of the free energy density is a power series of the type

$$z = \sum_{m,n=0,1,\ldots} c_{mn}(T^2)^m (T^2 \ln T)^n, \tag{2.6}$$

and the evaluation of the graphs of the effective theory corresponds to a calculation of the coefficients in this series, which are pure numbers.

If the external field H is different from zero, the low-temperature expansion is not a simple power series in T and $\ln T$. The free energy density then involves nontrivial functions of the ratio M_π/T. To analyze the behavior of the system at temperatures of the order of M_π, we treat both T and M_π as small quantities compared to the scale of the underlying theory,[16] allowing the ratio M_π/T to have any value (simultaneous expansion in powers of T and of M_π at *fixed* ratio M_π/T). The infrared singularities involving negative powers of T are thus removed by reordering, i.e., writing the series in terms of the two variables T and M_π/T and ordering powers of T. In this generalized sense, the low-temperature expansion of the free energy density is a power series of the form (2.6) even for nonzero external field; the symmetry breaking merely affects the coefficients c_{mn} which now become nontrivial functions of the ratio M_π/T.

III. RESULTS

In this section, we are going to discuss the low-temperature properties of the O(N) antiferromagnet, described by the effective Lagrangian (1.2). Since the system is homogeneous, the pressure is given by the temperature dependent part of the free energy density

$$P = \varepsilon_0 - z. \tag{3.1}$$

To begin with, let us consider the thermodynamic quantities in the limit of a zero external field $H \to 0$. For those quantities we are interested in, we need all contributions to the pressure which are at most linear in the external field.

The energy density of the vacuum then reads

$$\varepsilon_0 = -\Sigma_s H + \mathcal{O}(H^2). \tag{3.2}$$

The term quadratic in H involves a logarithm, which depends on a scale determined by next-to-leading order coupling constants.

The formula for the pressure takes the form

$$P = \tfrac{1}{2}(N-1)g_0 + 4\pi a (g_1)^2 + \pi g\left[b - \frac{j}{\pi^3 F^4}\right] + \mathcal{O}(p^{10}). \tag{3.3}$$

The dependence of the quantity P on temperature is contained in the functions $g_r(M_\pi,T)$ and $j(M_\pi,T)$, which are defined in the Appendix.

In the limit $H \to 0$ ($\Leftrightarrow T \gg M_\pi$) we are interested in, the functions g_0, g_1, and g are given by[6]

$$g_0 = \tfrac{1}{45}\pi^2 T^4\left[1 - \frac{15}{4\pi^2}\frac{M_\pi^2}{T^2} + \mathcal{O}\left(\frac{M_\pi}{T}\right)^3\right],$$

$$g_1 = \tfrac{1}{12}T^2\left[1 - \frac{3}{\pi}\frac{M_\pi}{T} + \mathcal{O}\left(\frac{M_\pi^2}{T^2}\ln\frac{M_\pi}{T}\right)\right],$$

$$g = \tfrac{1}{675}\pi^4 T^8\left[1 - \frac{15}{4\pi^2}\frac{M_\pi^2}{T^2} + \mathcal{O}\left(\frac{M_\pi}{T}\right)^3\right], \tag{3.4}$$

while j diverges logarithmically,

$$j = \nu \ln\frac{T}{M_\pi} + j_1 + j_2\frac{M_\pi^2}{T^2} + \mathcal{O}\left(\frac{M_\pi}{T}\right)^3,$$

$$\nu \equiv \frac{5(N-1)(N-2)}{48}. \tag{3.5}$$

The quantities j_1 and j_2 are real numbers, determined by the group O(N).

The constant a is linear in the external field, whereas b depends logarithmically on H and involves a scale H_b,

$$a = -\frac{(N-1)(N-3)}{32\pi}\frac{\Sigma_s H}{F^4},$$

$$b = -\frac{5(N-1)(N-2)}{96\pi^3 F^4}\ln\frac{H}{H_b}. \tag{3.6}$$

The scale H_b is related to the coupling constants e_1 and e_2 of order p^4 (see the Appendix).

Equipped with the above formulas, the low-temperature expansion of the pressure amounts to

$$P = \tfrac{1}{90}\pi^2(N-1)T^4\left[1 + \frac{N-2}{72}\frac{T^4}{F^4}\ln\frac{T_p}{T} + \mathcal{O}(T^6)\right]. \tag{3.7}$$

The first contribution represents the free Bose gas term which originates from a one-loop graph, whereas the effective interaction among the Goldstone bosons, remarkably, only manifests itself through a term of order T^8. This contribution contains a logarithm, characteristic of the effective Lagrangian method, which involves a scale T_p related to H_b (see the Appendix).

It is instructive to compare this formula for the pressure with the analogous relation occurring in theories with a spontaneously broken chiral symmetry, i.e., for $G = SU(n)_R \times SU(n)_L \to H = SU(n)_V$:[17]

$$P = \tfrac{1}{90}\pi^2(n^2-1)T^4\left[1 + \frac{n^2}{144}\frac{T^4}{F_\chi^4}\ln\frac{T_p^\chi}{T} + \mathcal{O}(T^6)\right].$$

An immediate consistency check of these two results is provided by the particular case $N=4 \Leftrightarrow n=2$: since the two groups O(4) and O(3) are locally isomorphic to $SU(2)\times SU(2)$ and SU(2), respectively, the above three-loop representations for the pressure have to coincide—the formula referring to the O(4) antiferromagnet in zero external field has to be identical with the one for QCD with two flavors ($n=2$) in the chiral limit (zero quark mass). Indeed, this is the case.

The corresponding expressions for the energy density u, for the entropy density s and for the heat capacity c_V are readily worked out from the thermodynamic relations

$$s=\frac{\partial P}{\partial T}, \quad u=Ts-P, \quad c_V=\frac{\partial u}{\partial T}=T\frac{\partial s}{\partial T}, \tag{3.8}$$

with the result

$$u=\tfrac{1}{30}\pi^2(N-1)T^4\left[1+\frac{N-2}{216}\frac{T^4}{F^4}\left(7\ln\frac{T_p}{T}-1\right)+\mathcal{O}(T^6)\right],$$

$$s=\tfrac{2}{45}\pi^2(N-1)T^3\left[1+\frac{N-2}{288}\frac{T^4}{F^4}\left(8\ln\frac{T_p}{T}-1\right)+\mathcal{O}(T^6)\right],$$

$$c_V=\tfrac{2}{15}\pi^2(N-1)T^3\left[1+\frac{N-2}{864}\frac{T^4}{F^4}\left(56\ln\frac{T_p}{T}-15\right)+\mathcal{O}(T^6)\right]. \tag{3.9}$$

The order parameter is given by the logarithmic derivative of the partition function with respect to the external field

$$\Sigma_s(T)=-\frac{\partial \varepsilon_0}{\partial H}+\frac{\partial P}{\partial H}. \tag{3.10}$$

This leads to

$$\Sigma_s(T)=\Sigma_s\left\{1-\frac{N-1}{24}\frac{T^2}{F^2}-\frac{(N-1)(N-3)}{1152}\frac{T^4}{F^4} -\frac{(N-1)(N-2)}{1728}\frac{T^6}{F^6}\ln\frac{T_\Sigma}{T}+\mathcal{O}(T^8)\right\}. \tag{3.11}$$

The terms of order T^0, T^2, T^4, and T^6 arise from tree, one-loop, two-loop, and three-loop graphs, respectively. Up to and including T^4, the coefficients are determined by the constant F which thus sets the scale of the expansion. The logarithm only shows up at order T^6: the scale T_Σ involves next-to-leading order coupling constants (see the Appendix).

As expected, the order parameter gradually melts as the temperature rises. The effective method, however, has its limitations: the low-temperature expansion can only be trusted at low temperatures—the curly bracket in Eq. (3.11) represents a correction. In particular, the critical temperature cannot be accurately determined by setting Eq. (3.11) equal to zero.

For nonzero external field, as we have seen before, the low-temperature representations of the thermodynamic quantities and the order parameter retain their form, except that the coefficients now become functions of M_π/T. In the region $T \gg M_\pi$ one recovers the results of the theory for zero external field, whereas in the opposite limit, $T \ll M_\pi$, even the Goldstone bosons freeze. The properties of the system are therefore very sensitive to the value of the ratio M_π/T. Take, e.g., the pressure: in the limit $H\to 0$, a contribution of order p^6, as we have seen, does not occur. This is no longer the case for an approximate symmetry ($H\neq 0$): remarkably, the corresponding term of order p^6 ($\propto HT^4$) turns out to be negative ($N\neq 2$), signaling an attractive interaction between the Goldstone degrees of freedom. Note that, with respect to the limit $H\to 0$, the sign of the effective interaction has changed: there, the first nonleading term (order p^8, $\propto T^8\ln[T_p/T]$) is positive and the interaction thus repulsive.

IV. O(3) ANTIFERROMAGNET

Lorentz invariance is a crucial ingredient of our analysis: in Lorentz-noninvariant theories, the effective Lagrangian picks up additional terms. It is therefore not legitimate *a priori* to transfer the above results to nonrelativistic systems displaying a spontaneously broken rotation symmetry.

In particular, for $N=3$, the above low-temperature theorems do not in general hold for systems exhibiting collective magnetic behavior, where the spin waves or magnons are the relevant Goldstone excitations: in the leading order effective Lagrangian, a term of topological nature appears, which is O(3) invariant only up to a total derivative and violates Lorentz symmetry.[8,18] However, this contribution is proportional to the spontaneous magnetization, such that, for the O(3) *anti*ferromagnet, it does not occur—the leading term of the effective Lagrangian for this system thus coincides with the leading contribution in the relativistic expression (1.2). Note that, in this analogy, the velocity of light has been replaced by the spin-wave velocity.

At nonleading order, however, additional terms occur in the effective Lagrangian, which spoil the formal relativistic invariance. As we have seen in the preceding section, effective coupling constants of order p^4 only show up in logarithmic scales—the coefficients in front of the logarithms exclusively involve leading order coupling constants. Hence, in what follows, we neglect the complication arising from non-invariant terms of order p^4, and discuss the O(3) antiferromagnet in the framework of the Lagrangian (1.2).

Let us first consider the low-temperature expansion of the order parameter—for the O(3) antiferromagnet, this quantity is referred to as staggered magnetization. Remarkably, for $N=3$, the T^4 term in formula (3.11) drops out, and we end up with

$$\Sigma_s(T)=\Sigma_s\left\{1-\frac{1}{12}\frac{(k_BT)^2}{\hbar v F^2}-\frac{1}{864}\frac{(k_BT)^6}{\hbar^3 v^3 F^6}\ln\frac{T_\Sigma}{T}+\mathcal{O}(T^8)\right\}. \tag{4.1}$$

Note that, for later convenience, we have restored the dimensions: k_B is Boltzmann's constant and v is the spin-wave velocity. The low-energy constant F already appears in the leading T coefficient. By comparing the above result for the staggered magnetization with the expression derived in condensed matter physics, we are thus able to identify the effective coupling constant F in terms of microscopic quantities.

So let us briefly recall how the O(3) antiferromagnet is described within the microscopic theory. In the Heisenberg model, the exchange Hamiltonian $\mathcal{H}_0$,

$$\mathcal{H}_0 = -2J \sum_{n.n.} \vec{S}_m \cdot \vec{S}_n , \quad J = \text{const}, \tag{4.2}$$

formulates the dynamics in terms of spin operators $\vec{S}_m$, attached to lattice sites m. Note that the summation only extends over nearest neighbor pairs and, moreover, the isotropic interaction is assumed to be the same for any two adjacent lattice sites. If the sign of the exchange integral J is negative, antiparallel spin alignment is favored, such that we end up with antiferromagnetic behavior. Clearly, the Hamiltonian is invariant under rotations of the spin directions, generated by

$$\vec{Q} = \sum_n \vec{S}_n . \tag{4.3}$$

The ground state of the antiferromagnet, however, does not exhibit this O(3) symmetry, and its microscopic description is highly nontrivial: in our analysis, we take it for granted that it spontaneously breaks the symmetry down to the group O(2).

Moreover, the antiferromagnet is commonly discussed within the following idealized picture: the system is considered as composed of two sublattices a and b, where a and b spins are of equal magnitude and the arrangement is such that all nearest neighbors of an a spin are b spins and vice versa. Furthermore, let us assume that the structure of the lattice is simple cubic. Although the ground state of the antiferromagnet does not exhibit spontaneous magnetization, the sublattice magnetization itself is not zero. And it is this latter quantity which is extensively discussed in the literature: indeed, at leading order in the temperature expansion, a T^2 decrease of the sublattice magnetization has been predicted by many authors[19–23]

$$\frac{M_a(0) - M_a(T)}{g\mu_B} = \frac{1}{2} \frac{V}{a^3} \frac{1}{2\pi^2 \sqrt{2z}} \left(\frac{k_B T}{2|J|S} \right)^2 \zeta(2). \tag{4.4}$$

The expression involves the following quantities: the exchange integral (J), the highest eigenvalue of the spin operator $S_n^3(S)$, the number of nearest neighbors of a given lattice site (z), the entire volume of the system (V), the length of the unit cell (a), the Landé factor (g), the Bohr magneton (μ_B), and the Riemann zeta function.

The sublattice magnetization at zero temperature

$$M_a(0) = \frac{1}{2} \frac{V}{a^3} g\mu_B (S - \sigma) \tag{4.5}$$

involves the quantity σ, a small number which depends on the structure of the lattice: in the case of a simple cubic lattice it takes the value $\sigma = 0.078$.[24] This relation reflects the well-known fact that the ground state of the antiferromagnet is highly nontrivial: in particular, the naive picture where the spin vectors of the two sublattices point in mutually opposite directions (''Néel state''), i.e., $\sigma = 0$, only represents an approximation.

In order to compare the expression (4.4) referring to the sublattice magnetization with our result (4.1) for the staggered magnetization, we observe that the two quantities are related via

$$\Sigma_s(T) = \frac{2M_a(T)}{V} = \frac{g\mu_B}{a^3} \left\{ (S-\sigma) - \frac{1}{2\pi^2\sqrt{2z}} \left(\frac{k_B T}{2|J|S} \right)^2 \zeta(2) + \cdots \right\}. \tag{4.6}$$

The above microscopic expression agrees with the effective expansion (4.1) up to order T^2, provided that the two coupling constants F and Σ_s are identified as

$$F^2 = \frac{S-\sigma}{\sqrt{2z}} \frac{\hbar v}{a^2} = 2S(S-\sigma) \frac{|J|}{a}, \quad \Sigma_s = \frac{g\mu_B(S-\sigma)}{a^3}. \tag{4.7}$$

Note that the spin-wave velocity v is given by the following combination of microscopic quantities:

$$v = 2|J|S\sqrt{2z}\, a/\hbar. \tag{4.8}$$

The scale of the low-temperature expansion is set by $F\sqrt{\hbar v}$—let us briefly estimate its value. Written in terms of the exchange integral J, we obtain

$$F\sqrt{\hbar v} = 2|J|S\sqrt{(S-\sigma)\sqrt{2z}}. \tag{4.9}$$

Now, for a simple cubic lattice ($z = 6, \sigma = 0.078$) and for $S = 1/2$, the double square root on the right hand side is approximately equal to 1, such that we end up with $F\sqrt{\hbar v} \approx |J|$. Typically, the exchange integral for antiferromagnets is around $|J| \approx 10^{-3}$ eV,[25,26] and the scale $F\sqrt{\hbar v}$ thus of the same order of magnitude. This is to be contrasted with the situation in QCD, where the relevant quantity, $F_\chi \sqrt{\hbar c}$, takes the value 92 MeV—the respective scales in the two theories thus differ in about eleven orders of magnitude.

As far as subleading terms in the expansion of the staggered magnetization are concerned, it is well known that a T^4 contribution is absent:[20,21,23,27] the spin-wave interaction only manifests itself at higher orders. Although these authors predict a term proportional to six powers of the temperature, in agreement with our result (4.1), they do not find the logarithmic dependence on the temperature. We conclude that it is extremely difficult to calculate the corrections of order T^6 in the framework of a microscopic theory.

As a second comparison, let us now discuss the heat capacity. Setting $N = 3$ in the effective expansion (3.9), we end up with

$$c_V = \tfrac{4}{15} \pi^2 \frac{k_B^4 T^3}{\hbar^3 v^3} \left[1 + \frac{1}{864} \frac{(k_B T)^4}{\hbar^2 v^2 F^4} \left(56 \ln \frac{T_p}{T} - 15 \right) + \mathcal{O}(T^6) \right]. \tag{4.10}$$

Replacing v according to Eq. (4.8), the leading contribution amounts to

$$c_V=\frac{4\pi^2 k_B}{15a^3}\left(\frac{k_B T}{2|J|S\sqrt{2z}}\right)^3. \tag{4.11}$$

This expression perfectly agrees with the leading term obtained from a microscopic or phenomenological analysis of the antiferromagnet.[19,20,28–30]

As far as corrections to the free Bose gas term are concerned, it is also known that the spin-wave interaction does not manifest itself through a T^5 term in the expansion for the heat capacity.[20,31] However, there is again a disagreement with respect to the structure of this correction: in Ref. 20, a simple T^7 contribution is predicted—a logarithmic dependence on the temperature is not found.

Nevertheless, it is instructive to compare the two expansions for the heat capacity at the T^7 level. Inserting the microscopic expression (4.7) for F into the effective expansion, we obtain

$$c_V=\tfrac{4}{15}\pi^2\frac{k_B^4T^3}{\hbar^3v^3}\left[1+\frac{1}{864}\frac{2za^4}{(S-\sigma)^2}\frac{(k_BT)^4}{\hbar^4v^4}\times\left(56\ln\frac{T_p}{T}-15\right)+\mathcal{O}(T^6)\right].$$

Now, in order for the nonleading term to be consistent with the corresponding microscopic result of order T^7,[20]

$$\frac{28\pi^4}{225\sqrt{3}}\frac{1}{S}k_B\frac{a^4}{(\hbar v)^7}(k_BT)^7, \tag{4.12}$$

the logarithm must have the following numerical value ($S=1/2$, simple cubic lattice): $\ln(T_p/T)\approx 1.5$. Hence, for that specific value of the temperature, $T_0=0.23T_p$, the two results coincide.

Let us consider the analogous situation in QCD: there, the value of T_p^χ can be extracted from experiment ($\pi\pi$ scattering, for details see, e.g., Ref. 6), yielding $T_p^\chi\approx 275$ MeV. Accordingly, with the above value of the logarithm, we get $T_0^\chi\approx 60$ MeV. The critical temperature is estimated to be around 170 MeV—so we see that the values for T_0^χ, T_c^χ, and T_p^χ are of the same order of magnitude. Note that the respective scales for the antiferromagnet differ in about ten orders of magnitude: with a critical temperature (Néel temperature) of $T_N\approx\mathcal{O}(0.01\ \text{eV})$, we end up with $T_p\approx\mathcal{O}(0.01\ \text{eV})$.

V. SUMMARY AND OUTLOOK

The presence of states with small excitation energies affects the behavior of the system in a very specific manner, controlled by the symmetries of the underlying theory. These symmetries unambiguously fix the values of the coefficients in the low-temperature expansion of the order parameter and the thermodynamic quantities up to two leading order coupling constants F and Σ_s. Symmetry, however, does not determine the logarithmic scales T_p and T_Σ, which occur in the temperature expansion and involve next-to-leading order coupling constants.

The low-temperature theorems for the order parameter and the thermodynamic quantities are exact up to and including three loops: independently of the specific underlying model, they are valid for any Lorentz-invariant theory where an O(N) symmetry is spontaneously broken to O($N-1$). For $N=4$, the expansion (3.11) for the order parameter not only describes the quark condensate of QCD with two flavors in the chiral limit, but also, e.g., describes the Higgs condensate in the standard model of elementary particle physics. The only difference between the two representations concerns the numerical values of the coupling constants occurring therein—the low-temperature description turns out to be universal.

Another interesting case, which we have discussed in detail, is given by $N=3$: the O(3) antiferromagnet. The low-temperature theorems for the staggered magnetization and the heat capacity both agree with the results given in the literature up to and including two loops. At the three-loop level, however, the results no longer coincide: to my knowledge, the logarithmic temperature dependence at order T^7 in the expansion for the heat capacity and order T^6 for the staggered magnetization, has been overlooked so far. The effective Lagrangian method not only proves to be more efficient than the complicated microscopic analysis, but also addresses the problem from a unified point of view based on symmetry—at large wavelengths, the microscopic structure of the system only manifests itself in the numerical values of a few coupling constants.

Although Lorentz invariance is a crucial ingredient of the present analysis, the effective Lagrangian method is not restricted to this domain: ferromagnets, e.g., where the spin waves follow a quadratic dispersion law, may be analyzed within the framework of nonrelativistic effective Lagrangians.[1,8,11] In particular, the low-temperature expansion for the order parameter of a ferromagnet, its spontaneous magnetization, has been calculated to three loops—the results, which go beyond Dyson's pioneering microscopic analysis,[32] will be presented in a forthcoming paper.[33]

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APPENDIX

In this appendix, we discuss the various quantities showing up in the formula for the pressure

$$P=\tfrac{1}{2}(N-1)g_0+4\pi a(g_1)^2+\pi bg-\frac{1}{F^4}I+\mathcal{O}(p^{10}). \tag{A1}$$

Unlike in Sec. III, we do not restrict ourselves to contributions at most linear in the external field. In the second part of the appendix, we briefly comment on the logarithmic scales T_p and T_Σ.

Let us first consider the renormalized Goldstone-boson mass. The calculation yields

$$M_\pi^2=\frac{\Sigma_s H}{F^2}+[2(k_2-k_1)+(N-3)\lambda]\frac{(\Sigma_s H)^2}{F^6}+c\frac{(\Sigma_s H)^3}{F^{10}}+\mathcal{O}(H^4), \tag{A2}$$

where the constant c involves the relevant coupling constants of order p^6. The quantity λ contains a pole at $d=4$,

$$\begin{aligned}\lambda&=\tfrac{1}{2}(4\pi)^{-d/2}\Gamma(1-\tfrac{1}{2}d)M^{d-4}\\&=\frac{M^{d-4}}{16\pi^2}\left[\frac{1}{d-4}-\tfrac{1}{2}\{\ln 4\pi+\Gamma'(1)+1\}+\mathcal{O}(d-4)\right].\end{aligned} \tag{A3}$$

This singularity is absorbed in a renormalization of the combination k_2-k_1 of coupling constants of order p^4. One ends up with a logarithm depending on H and a term independent thereof. The latter can be absorbed in a new scale H_M, such that the expression for the renormalized mass takes the form displayed in Eq. (2.5),

$$M_\pi^2=\frac{\Sigma_s H}{F^2}+\frac{N-3}{32\pi^2}\frac{(\Sigma_s H)^2}{F^6}\ln\frac{H}{H_M}+\mathcal{O}(H^3). \tag{A4}$$

Now, the functions g_0, g_1, g, and I occurring in the formula (A1) for the pressure, depend in a nontrivial manner on M_π and T. The quantities g_r are associated with the d-dimensional noninteracting Bose gas

$$g_r(M_\pi,T)=2\int_0^\infty\frac{d\rho}{(4\pi\rho)^{d/2}}\rho^{r-1}\exp(-\rho M_\pi^2)\times\sum_{n=1}^{\infty}\exp(-n^2/4\rho T^2). \tag{A5}$$

The function g is the following combination thereof:

$$g=3g_0(g_0+M_\pi^2 g_1). \tag{A6}$$

The expression for the three-loop integral I is more complicated:

$$\begin{aligned}I=&\tfrac{1}{48}(N-1)(N-3)M_\pi^4\overline{J}_1-\tfrac{1}{4}(N-1)(N-2)\overline{J}_2\\&-\tfrac{1}{16}(N-1)(N-3)^2M_\pi^4(g_1)^2g_2\\&+\tfrac{1}{48}(N-1)(N-3)(3N-7)M_\pi^2(g_1)^3.\end{aligned} \tag{A7}$$

The quantities $\overline{J}_1$ and $\overline{J}_2$,

$$\begin{aligned}\overline{J}_1&=J_1-c_1-c_2g_1+6(d-2)\lambda(g_1)^2,\\\overline{J}_2&=J_2-c_3-c_4g_1+\tfrac{1}{3}(d+6)(d-2)\lambda(\overline{G}_{\mu\nu})^2\\&\quad+\tfrac{2}{3}(d-2)\lambda M_\pi^4(g_1)^2,\\\overline{G}_{\mu\nu}&=-\tfrac{1}{2}\delta_{\mu\nu}g_0+\delta_\mu^4\delta_\nu^4(\tfrac{1}{2}dg_0+M_\pi^2g_1),\end{aligned} \tag{A8}$$

remove the singularities of the loop integrals J_1 and J_2, respectively:

$$\begin{aligned}J_1&=\int_T d^dx[G(x)]^4,\\J_2&=\int_T d^dx[\partial_\mu G(x)\partial_\mu G(x)]^2.\end{aligned} \tag{A9}$$

For the explicit structure of the temperature independent counterterms $c_1\cdots c_4$, the reader is referred to Ref. 6.

The connection between the formula (3.3) for the pressure given in Sec. III,

$$P=\tfrac{1}{2}(N-1)g_0+4\pi a(g_1)^2+\pi g\left[b-\frac{j}{\pi^3F^4}\right]+\mathcal{O}(p^{10}),$$

and Eq. (A1) is established by splitting off a factor g from the expression I,

$$I=\frac{1}{\pi^2}gj. \tag{A10}$$

This relation defines the function j.

The constants a and b in Eq. (A1) contain the various coupling constants which occur in the effective Lagrangian[34]

$$\begin{aligned}a=&-\frac{(N-1)(N-3)}{32\pi}\frac{\Sigma_sH}{F^4}+\frac{N-1}{4\pi}\frac{(\Sigma_sH)^2}{F^8}\\&\times\Bigg\{[(N+1)(e_1+e_2)+k_2-k_1]\\&-\frac{(N-1)^2}{2}\lambda-\frac{3N^2+32N-67}{768\pi^2}\Bigg\},\\b=&\frac{N-1}{\pi F^4}\left\{[2e_1+Ne_2]-\frac{5(N-2)}{3}\lambda-\frac{N-2}{16\pi^2}\right\}.\end{aligned} \tag{A11}$$

Repeating the steps which led from Eq. (A2) to Eq. (A4), the constants a and b may be conveniently written as

$$\begin{aligned}a&=-\frac{(N-1)(N-3)}{32\pi}\frac{\Sigma_sH}{F^4}-\frac{(N-1)^3}{256\pi^3}\frac{(\Sigma_sH)^2}{F^8}\ln\frac{H}{H_a},\\b&=-\frac{5(N-1)(N-2)}{96\pi^3F^4}\ln\frac{H}{H_b}.\end{aligned} \tag{A12}$$

Finally, let us consider the logarithmic scales T_p and T_Σ, showing up in the low-temperature expansion of the thermodynamic quantities and the order parameter. They are both related to the scale Λ_b,

$$T_p=\Lambda_b\exp(-j_1/\nu),\qquad T_\Sigma=\Lambda_b\exp\left(-j_1/\nu+\frac{4\pi^2}{15\nu}j_2\right), \tag{A13}$$

where

$$\Lambda_b = \frac{\sqrt{\Sigma_s H_b}}{F}, \quad \nu \equiv \frac{5(N-1)(N-2)}{48}. \tag{A14}$$

Note that T_p and T_Σ depend on the constants j_1 and j_2 occurring in the expansion of the function j (in the limit $T \gg M_\pi$),

$$j = \nu \ln \frac{T}{M_\pi} + j_1 + j_2 \frac{M_\pi^2}{T^2} + \mathcal{O}\left(\frac{M_\pi}{T}\right)^3. \tag{A15}$$

The formulas given in this appendix can readily be checked by setting $N=4$: using $M^2 = \Sigma_s H/F^2$ and identifying the respective coupling constants as $e_1 = l_1$, $e_2 = l_2$, and $k_2 - k_1 = l_3$, the results for two-flavor QCD are reproduced.[6,12]

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*Present address: Department of Physics, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093.

[1] C. P. Hofmann, preceding paper, Phys. Rev. B **60**, 388 (1999).

[2] Reference 3 also addresses the issues of applying techniques developed for particle physics to condensed matter systems with Goldstone bosons. A specific application concerning SO(5) invariance and high-T_c superconductivity can be found in Ref. 4.

[3] C. P. Burgess, hep-th/9808176 (unpublished).

[4] C. P. Burgess and C. A. Lutken, Phys. Rev. B **57**, 8642 (1998).

[5] J. Gasser and H. Leutwyler, Phys. Lett. B **184**, 83 (1987).

[6] P. Gerber and H. Leutwyler, Nucl. Phys. B **321**, 387 (1989).

[7] P. Hasenfratz and F. Niedermayer, Z. Phys. B **92**, 91 (1993).

[8] H. Leutwyler, Phys. Rev. D **49**, 3033 (1994).

[9] H. Leutwyler, Ann. Phys. (N.Y.) **235**, 165 (1994).

[10] An example related to condensed matter physics concerns systems exhibiting collective magnetic behavior. Ferromagnets and antiferromagnets are described within the O(3)-invariant Heisenberg model. Interactions which explicitly break the rotation symmetry, such as external magnetic and anisotropy fields (see Ref. 1) or magnetic dipole and spin-orbit interactions (see Ref. 11), can easily be incorporated into the effective Lagrangian machinery.

[11] J. M. Roman and J. Soto, cond-mat/9709298 (unpublished).

[12] P. Hasenfratz and H. Leutwyler, Nucl. Phys. B **343**, 241 (1990).

[13] For a review of the effective Lagrangian method at nonzero temperature, see Refs. 14. For a general review of field theory at finite temperature, see Ref. 15.

[14] H. Leutwyler, Nucl. Phys. B, Proc. Suppl. **4**, 248 (1988); H. Leutwyler, in *Warsaw International Symposium on Elementary Particle Physics, Kazimierz, 1988 – New Theories in Physics*, edited by Z. Ajduk, S. Pokorski, and A. Trautman (World Scientific, Singapore, 1989), p. 116; also in *Summer Institute in Theoretical Physics, Kingston, 1988—Symmetry Violations in Subatomic Physics*, edited by B. Castel and P. J. O'Donnell (World Scientific, Singapore, 1989), p. 57.

[15] N. P. Landsman and C. G. van Weert, Phys. Rep. **145**, 141 (1987); J. I. Kapusta, *Finite-Temperature Field Theory* (Cambridge University Press, Cambridge, 1989).

[16] In quantum chromodynamics this scale is given by Λ_{QCD}, the renormalization group invariant scale of QCD; in connection with antiferromagnets, the relevant scale may be identified with the exchange integral J in the Heisenberg model.

[17] H. Leutwyler, *Low Temperature Expansion in QCD*, International Europhysics Conference on High Energy Physics, Uppsala, 1987, edited by Olga Botner (European Physics Society, Petit-Lancy, 1987), p. 320.

[18] E. Fradkin, *Field Theories of Condensed Matter Systems*, Frontiers in Physics Vol. 82 (Addison-Wesley, New York, 1991).

[19] R. Kubo, Phys. Rev. **87**, 568 (1952).

[20] T. Oguchi, Phys. Rev. **117**, 117 (1960).

[21] F. Keffer and R. Loudon, J. Appl. Phys. (Suppl.) **32**, 2 (1961).

[22] W. Ludwig, *Festkörperphysik II* (Akademische Verlagsgesellschaft, Frankfurt am Main, 1970).

[23] T. Oguchi and A. Honma, J. Appl. Phys. **34**, 1153 (1963).

[24] P. W. Anderson, Phys. Rev. **86**, 694 (1952).

[25] F. Keffer, *Spin Waves*, in *Encyclopedia of Physics—Ferromagnetism*, edited by S. Flügge and H. P. J. Wijn (Springer, Berlin, 1966), Vol. 18-2, p. 1.

[26] J. S. Smart, in *Magnetism*, edited by G. T. Rado and H. Suhl (Academic Press, New York, 1963), Vol. II, p. 63.

[27] If the discreteness of the lattice is taken into account, one ends up with corrections proportional to $T^4, T^6, \ldots$ (Ref. 20).

[28] J. van Kranendonk and J. H. van Vleck, Rev. Mod. Phys. **30**, 1 (1958).

[29] A. I. Akhiezer, V. G. Baryakhtar, and M. I. Kaganov, Sov. Phys. Usp. **3**, 567 (1961).

[30] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Course of Theoretical Physics Vol. 9, edited by E. M. Lifshitz and L. P. Pitajewski (Pergamon, London, 1981), Pt. 2.

[31] We are not considering effects due to the discreteness of the lattice, which yield additional terms proportional to $T^5, T^7, \ldots$ (Ref. 20).

[32] F. J. Dyson, Phys. Rev. **102**, 1217 (1956); **102**, 1230 (1956).

[33] C. P. Hofmann (unpublished).

[34] Note that, with respect to the last terms in the brace ($\propto \pi^{-2}$), our definition differs from the convention used in Ref. 6: the authors split these terms between Eq. (A11) and the function I (A7), such that, for $N=4$ (strong interaction) the constants a and b may be identified with sums of scattering lengths and effective ranges (pion-pion scattering).