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Level correlations in disordered metals: The replica σ model

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We compute energy-level correlations in weakly disordered metallic grains using the fermionic replica method. We use the standard σ-model approach and show that nontrivial saddle points, which break replica symmetry, must be included in the calculation to reproduce the oscillatory behavior of the correlations. We calculate the correlation functions in all three classical ensembles Gaussian-orthogonal, -unitary, and -sympletic, including the finite-dimensional gradient corrections. Our results coincide with those obtained by the supersymmetric σ model and the semiclassical trace formula. [S0163-1829(99)01230-8]

I. INTRODUCTION

The statistics of energy levels of electrons in disordered metals has attracted much attention in the past decades. Gor'kov and Eliashberg[1] conjectured that it obeys Wigner-Dyson laws derived for random matrices.[2] This conjecture received a strong support almost 20 years later when Efetov[3] introduced the supersymmetric (SUSY) σ model. It appears that the zero-dimensional version of the σ model gives exactly the random matrix theory statistics of Wigner and Dyson. The use of the SUSY formulation seemed crucial since the alternative replica theory,[4,5] when applied to the pure random matrix problem, seemed unable to reproduce the correct oscillatory behavior of the level correlation function.[6] Later Altshuler and Shklovskii[7] realized that in a finite-dimensional system, the correlation function is modified with respect to the universal random matrix level statistics: this modification appears when the energy difference is of the order of the Thouless energy, E_c (equal to $\hbar$ over the diffusion time through the sample), and the corrections depend on the dimensionality d, conductance g, and shape of the sample. They used diagrammatic perturbation theory and consequently could trace only the modifications of the nonoscillatory part of the correlation functions. Finite dimensional modifications of the oscillatory part by the gradient terms were calculated in Ref. 8, and more generally in Ref. 9, using the SUSY technique. The result were subsequently reproduced using a semiclassical trace formula approach.[10] It followed from these works that a power-law decay of the oscillatory correlations crosses over to an exponential decay at the scale E_c. The precise behavior of this crossover depends both on the dimensionality and the symmetry class of problem. The essential feature of these results is that *all* nonuniversal terms may be expressed through the spectral determinant of a single classical differential operator. For the case of a disordered metal grain it turns out to be the diffusion operator in the corresponding geometry.

In a recent paper[11] we have shown how the fermionic replica method may be used to calculate the level statistics of the Gaussian unitary ensemble (GUE). The calculations of Ref. 11 were specific to the GUE and essentially used the Itzykson-Zuber[12] integral for the unitary group. The purpose of the present paper is to present a more general approach to the fermionic replica calculations of the level statistics, which is not based on the peculiarities of the unitary ensemble, and uses rather the standard path of the σ model. We shall present the calculations of the level correlations in disordered metals for all three classical symmetry ensembles: orthogonal (GOE), unitary, and symplectic (GSE). We also include the effects of gradient terms on the level statistics reproducing exactly the results of Refs. 8–10. Our colleagues I. V. Yurkevich and I. V. Lerner have independently been developing a nonlinear σ-model approach with replica symmetry, using a complementary approach to ours.[13]

Our strategy is as follows: we deal with the standard fermionic replica σ model[5] with an action written in terms of the $(n_1+n_2)\times(n_1+n_2)$-dimensional $\hat{Q}$ matrix, where $n_{1,2}\to 0$ are numbers of replicas. The symmetry group of the action $G(n_1+n_2)$ is broken down to the exact $G(n_1)\times G(n_2)$ by a finite-energy difference $\omega=\epsilon_1-\epsilon_2$ of the correlation function (G is a symmetry group of the $\hat{Q}$ matrix, which depends on a symmetry class of the problem). Based on the experience of the GUE solution,[11] we consider all possible saddle points of the σ model both replica symmetric and replica nonsymmetric. The latter spontaneously break the exact symmetry of each sub-block $G(n)$ down to $G(p)\times G(n-p)$ with $0\leqslant p\leqslant n$ (here $n=n_1,n_2$ and $p=p_1,p_2$). The corresponding manifold of the Goldstone modes has an exact degeneracy for space-independent ($\mathbf{q}=0$) modes. The contribution of such saddle-point manifolds [the coset space $G(n)/G(p)G(n-p)$] to the partition (generating) function is proportional to their volume. The volumes of the coset spaces play a central role in our analysis, since after the analytical continuation $n\to 0$ they determine which of the saddle points contribute to the generating function. It turns out that in addition to the replica-symmetric (perturbative) saddle point ($p=0$) there is only one additional saddle-point manifold (in each block) with $p=1$ in the GOE and GUE cases and two manifolds $p=1$ and $p=2$ in the GSE case.

These replica-nonsymmetric saddle points give rise to an oscillatory part of the correlation function (since their action remains finite, and imaginary, in the $n \to 0$ limit). One thus gets the correct oscillatory behaviors of the correlations, with one oscillation frequency in the GOE and GUE and two oscillation frequencies in the GSE. One should notice that the effect of the replica symmetry-breaking saddle points are not limited to the correlation of levels. In the random matrix limit, they are known to describe the finite-size oscillatory correction to the density of states inside the asymptotic support of the spectrum,[11] and the exponentially small tails outside.[14]

We then calculate the fluctuations around each of the saddle-point manifolds in the Gaussian approximation. This is legitimate at relatively large energy $\omega \gg \Delta$ and for a good metal, $g \equiv E_c/\Delta \gg 1$ (Δ is the mean level spacing and g is dimensionless conductance). No relation between ω and E_c is assumed. As a result, one obtains the energy dependent amplitudes of the oscillatory parts, as well as those of the smooth parts, of the correlation functions, in the asymptotic regime $\omega \gg \Delta$. For small energy, $\omega \ll E_c$, the correlation coincides with the random matrix theory predictions, whereas for larger energy $\omega > E_c$ it gets modified in the nonuniversal (dimensionality and g dependent) way in agreement with Refs. 8–10.

The structure of the paper is as follows. In Sec. II we introduce notations and present a general discussion of the matrix field theory which allows to compute the energy-level correlations. We compute the saddle points of this action and the quadratic fluctuations around them. We also comment on the connection of our approach to the usual nonlinear σ model. In Sec. III we apply the theory to the three classical ensembles. Finally in Sec. IV we briefly discuss the results, their range of validity, and the possible further developments. The Appendix contains the computations of the volumes of the relevant coset spaces, for each of the three symmetry classes.

II. THE REPLICA MATRIX MODEL AND ITS SADDLE POINTS

A. Preliminaries

We shall discuss the correlation functions of the density of states (DOS), which is defined as

$$\nu(\epsilon) \equiv V^{-1}\,\mathrm{Tr}\,\delta(\mu+\epsilon-H), \tag{1}$$

where V is the volume and $H=H_0+U_{dis}$ is the Hamiltonian of the system. Here H_0 is the Hamiltonian of the corresponding regular (clean) system, and U_{dis} is a random disorder potential. We are interested in the large energy behavior and we thus measure all energies from the large positive chemical potential μ: the deviation ϵ from μ is supposed to scale as the mean level spacing Δ.

The retarded/advanced Green functions $G^{\pm}(\epsilon)$ are defined as

$$G^{\pm}(\epsilon)=(\mu+\epsilon-H\pm i\eta)^{-1} \tag{2}$$

with η infinitesimal. The density of states $\nu(\epsilon)$ is thus equal to the small η limit of $(G^-(\epsilon)-G^+(\epsilon))/(2\pi i)$. The average DOS at large enough μ is a featureless smooth function, which we shall approximate by a constant, $\langle \nu(\epsilon)\rangle \equiv (\Delta V)^{-1}$. Hereafter the angular brackets stand for the averaging over the ensemble of random disorder potentials, which we assume to be Gaussian and short-range correlated with zero mean and a variance given by

$$\langle U_{dis}(r)U_{dis}(r')\rangle=(2\pi\nu\tau)^{-1}\delta(r-r'), \tag{3}$$

where τ is an elastic scattering mean free time.

The main object of our study is the connected two-point correlation function of energy levels, defined as

$$R(\epsilon_1,\epsilon_2)\equiv \nu^{-2}\langle \nu(\epsilon_1)\nu(\epsilon_2)\rangle-1. \tag{4}$$

Using the fact that $\langle G^{\pm}G^{\pm}\rangle=\langle G^{\pm}\rangle\langle G^{\pm}\rangle=-(\pi V\nu)^2$, one finds

$$R(\epsilon_1,\epsilon_2)=\frac{1}{2\pi^2}[\mathrm{Re}S(\epsilon_1,\epsilon_2)-\pi^2];$$

$$S(\epsilon_1,\epsilon_2)\equiv \Delta^2\langle G^+(\epsilon_1)G^-(\epsilon_2)\rangle. \tag{5}$$

With the replica trick the two-point function S may be written as[5,6,11]

$$S(\epsilon_1,\epsilon_2)=\lim_{n_{1,2}\to 0}\frac{\Delta^2}{n_1 n_2}\frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\langle Z^{(n_1,n_2)}(\hat{E})\rangle, \tag{6}$$

where we have introduced the diagonal $(n_1+n_2)\times(n_1+n_2)$ matrix $E_{ji}=\delta_{ji}E_j$ with

$$E_j=\begin{cases}\mu+\epsilon_1+i\eta & j=1,\dots,n_1;\\ \mu+\epsilon_2-i\eta & j=n_1+1,\dots,n_1+n_2.\end{cases} \tag{7}$$

The generating function $Z^{(n_1,n_2)}$ may be written as a functional integral over $2(n_1+n_2)$ fermionic fields. Getting from such a fermionic vector field theory to a matrix formulation is a standard procedure[5,6] which we shall not repeat in detail. Performing the Gaussian averaging over U_{dis}, introducing a $(n_1+n_2)\times(n_1+n_2)$ Hubbard-Stratonovich matrix field $\hat{Q}(r)$ and integrating finally over the fermionic degrees of freedom, one obtains for the generating function[5,6]

$$Z^{(n_1,n_2)}(\hat{E})=\int d[\hat{Q}]\exp\{-A[\hat{Q},\hat{E}]\}, \tag{8}$$

where the action $A[\hat{Q},\hat{E}]$ is given by

$$A[\hat{Q},\hat{E}]=\frac{\pi\nu}{4\tau}\mathrm{Tr}\,\hat{Q}^2-\mathrm{Tr}\,\ln\left(\hat{E}+\frac{\nabla^2}{2m}+\frac{i}{2\tau}\hat{Q}\right). \tag{9}$$

The symmetry of $\hat{Q}$ and the integration measure $d[\hat{Q}]$ depend on the symmetry class of the problem and will be discussed in Sec. III separately for each ensemble. The trace operation includes both the replica indices and the spatial variables.

B. Saddle points

We shall evaluate the integral in Eq. (8) by the saddle-point method, and check *a posteriori* that such an evaluation is indeed justified in the limit of a weak disorder. A space independent solution of the saddle-point equation satisfies:

$$\hat{Q}_{s.p.} = \frac{i}{\pi\nu} \sum_{\mathbf{p}} \left(\hat{E} - \frac{\mathbf{p}^2}{2m}\mathbb{1} + \frac{i}{2\tau}\hat{Q}_{s.p.} \right)^{-1}, \tag{10}$$

where the sum over $\mathbf{p}$ runs over the set of eigenmodes of the pure Hamiltonian H_0 with some appropriate boundary conditions. The saddle-point matrix $\hat{Q}_{s.p.}$ may be diagonalized by a transformation: $\hat{Q}_{s.p.} = U^{-1}\hat{\Lambda}U$, where U is an element of the symmetry group G of the problem, $U \in G(n_1+n_2)$, and $\hat{\Lambda}$ is diagonal, $\Lambda_{ji} = \delta_{ji}\lambda_j$. The saddle-point Eq. (10) then implies that U takes the form

$$U = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}, \quad V_1 \in G(n_1), \quad V_2 \in G(n_2), \tag{11}$$

while each eigenvalue is a solution of the equation

$$\lambda_j = \frac{i}{\pi\nu} \sum_{\mathbf{p}} \frac{1}{E_j - (\mathbf{p}^2/2m) + (i/2\tau)\lambda_j} \approx \operatorname{sgn}(\lambda_j). \tag{12}$$

Here we have used the standard approximation, valid at large $\mu\tau$, where one substitutes the sum over modes by an integral over $\epsilon_p = \mathbf{p}^2/2m - \mu$, and neglects variations of $\nu(\epsilon)$ in a vicinity of $\epsilon = 0$. We thus find that the eigenvalues of $\hat{Q}_{s.p.}$ take the values

$$\lambda_j = \pm 1, \quad j \in 1, \ldots, n_1+n_2. \tag{13}$$

There exist $(n_1+1)(n_2+1)$ saddle-point manifolds. Each such manifold $\mathcal{M}_{(p_1p_2)}$ may be indexed by two integers $p_1 \in \{0, \ldots, n_1\}$ and $p_2 \in \{0, \ldots, n_2\}$, and is generated by the block-diagonal symmetry transformations U of the type (11) applied to the diagonal matrix

$$\hat{\Lambda}_{(p_1p_2)} = \operatorname{diag}\{\underbrace{-1, \ldots -1}_{p_1}, \underbrace{+1, \ldots +1}_{n_1-p_1}, \underbrace{+1, \ldots +1}_{p_2}, \underbrace{-1, \ldots -1}_{n_2-p_2}\}. \tag{14}$$

In order to scan the manifold $\mathcal{M}_{(p_1p_2)}$ in a nonredundant way, one must restrict the symmetry transformations U to the coset space

$$\mathcal{M}_{(p_1p_2)} = \frac{G(n_1)}{G(p_1)G(n_1-p_1)} \times \frac{G(n_2)}{G(p_2)G(n_2-p_2)}. \tag{15}$$

It is useful to define the free propagator which is the diagonal matrix with eigenvalues

$$G_{jj} \equiv \left(E_j + \frac{\nabla^2}{2m} + \frac{i}{2\tau}\lambda_j \right)^{-1}. \tag{16}$$

The eigenvalues take four different values depending on the value of the index j. One can characterize them by two binary indices (α,σ), where $\alpha = 1,2$ designates two replica blocks with the energies ϵ_α and $\sigma = \operatorname{sgn}(\lambda_j)$. For each energy $(\alpha = 1,2)$, we have a retarded and an advanced propagator, $G_\alpha^{\pm}$, defined by

$$G_\alpha^{\pm} \equiv \left(\mu + \frac{\nabla^2}{2m} + \epsilon_\alpha \pm \frac{i}{2\tau} \right)^{-1}; \quad \alpha = 1,2. \tag{17}$$

In these notations the free propagator takes the form

$$\hat{G} = \operatorname{diag}\{\underbrace{G_1^-, \ldots G_1^-}_{p_1}, \underbrace{G_1^+, \ldots G_1^+}_{n_1-p_1}, \underbrace{G_2^+, \ldots G_2^+}_{p_2}, \underbrace{G_2^-, \ldots G_2^-}_{n_2-p_2}\}. \tag{18}$$

C. Saddle-point action

On the manifold $\mathcal{M}_{(p_1p_2)}$, the saddle-point action is given by

$$A_{(p_1p_2)} = \frac{\pi\nu}{4\tau} \operatorname{Tr} \hat{\Lambda}^2_{(p_1p_2)} - \operatorname{Tr} \ln\left(\hat{E} + \frac{\nabla^2}{2m} + \frac{i}{2\tau}\hat{\Lambda}_{(p_1p_2)} \right), \tag{19}$$

where the trace involves both space and replica indices. In terms of the free propagators G_α^σ defined in Eq. (17) it reads

$$A_{(p_1p_2)} = \frac{\pi(n_1+n_2)}{4\tau\Delta} - p_1 \operatorname{Tr}\ln G_1^- - (n_1-p_1)\operatorname{Tr}\ln G_1^+ - p_2 \operatorname{Tr}\ln G_2^+ - (n_2-p_2)\operatorname{Tr}\ln G_2^-, \tag{20}$$

where the traces involve only spatial variables. Expanding to the first order in $\tau\epsilon_\alpha \ll 1$, and omitting unessential constant factors, one finds

$$\operatorname{Tr}\ln G_\alpha^{\pm} \approx \mp i\pi\epsilon_\alpha/\Delta. \tag{21}$$

Finally, neglecting all constants, which vanish in the limit $n_{1,2} \to 0$, one obtains for the saddle-point action

$$A_{(p_1p_2)} = \frac{i\pi}{\Delta}(n_1\epsilon_1 - n_2\epsilon_2 - 2p_1\epsilon_1 + 2p_2\epsilon_2). \tag{22}$$

D. Quadratic fluctuations

Let us expand around the saddle point $\hat{Q}_{s.p.} = U^{-1}\hat{\Lambda}_{(p_1p_2)}U$, writing $\hat{Q}(r) = \hat{Q}_{s.p.} + U^{-1}\delta\hat{Q}(r)U$. The action expanded to the second order is diagonalized in terms of the Fourier components $\delta Q_{ij}(\mathbf{q})$ of the fluctuations:

$$A[\hat{Q},\hat{E}] \approx A_{(p_1 p_2)} + \frac{1}{2} \sum_{i,j=1}^{n_1+n_2} \sum_{\mathbf{q}} M_{ij}(\mathbf{q}) \delta Q_{ij}(\mathbf{q}) \delta Q_{ji}(-\mathbf{q}), \tag{23}$$

where the eigenvalue of the eigenmode $(i,j,\mathbf{q})$, $M_{ij}(\mathbf{q})$, is given by

$$M_{ij}(\mathbf{q}) = \frac{\pi}{2\tau\Delta} - \frac{1}{4\tau^2} \sum_{\mathbf{p}} G_{ii}\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right) G_{jj}\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right). \tag{24}$$

There exist *a priori* sixteen different fluctuation eigenvalues for each momentum mode $\mathbf{q}$. It is convenient to index them according to the binary decomposition introduced after Eq. (16). Each index $j \in 1,\ldots,n+n'$ is associated with a pair of indices (α,σ), where $\alpha=1,2$ characterizes the energy, ϵ_α, and $\sigma=\mathrm{sgn}(\lambda_j)$ characterizes the retarded/advanced nature of the propagator. The 16 different fluctuation eigenvalues are then

$$M_{(\alpha\sigma)(\alpha'\sigma')}(\mathbf{q}) = \frac{\pi}{2\tau\Delta} - \frac{1}{4\tau^2} \sum_{\mathbf{p}} G_{\alpha}^{\sigma}\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right) G_{\alpha'}^{\sigma'}\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right). \tag{25}$$

The corresponding momentum sums are easily computed, resulting in

$$\sum_{\mathbf{p}} G_{\alpha}^{\pm}\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right) G_{\alpha'}^{\pm}\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right) \approx 0;$$

$$\sum_{\mathbf{p}} G_{\alpha}^{\pm}\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right) G_{\alpha'}^{\mp}\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right) \approx \frac{2\pi\tau}{\Delta}[1 - D\mathbf{q}^2\tau \pm i(\epsilon_\alpha - \epsilon_{\alpha'})\tau]. \tag{26}$$

In the last expression we have expanded the sum to first order in the small parameters $D\mathbf{q}^2\tau \ll 1$ and $|\epsilon_1-\epsilon_2|\tau \ll 1$ where D is the diffusion constant defined as $D = 2\mu\tau/(md)$

We obtain eventually the following list of eigenvalues for each spatial mode $\mathbf{q}$:

• When $\sigma=\sigma'$ the eigenvalue is

$$M_{(\alpha\sigma)(\alpha'\sigma)} = \frac{\pi}{2\Delta}\frac{1}{\tau}. \tag{27}$$

We shall call the corresponding modes ''massive'' and denote their number for each spatial mode $\mathbf{q}$ as $\mathcal{N}_m$.

• When $\sigma\neq\sigma'$ and $\alpha\neq\alpha'$ the eigenvalues are

$$M_{(1+)(2-)} = M_{(2-)(1+)} = \frac{\pi}{2\Delta}(D\mathbf{q}^2 - i\omega)$$
$$M_{(1-)(2+)} = M_{(2+)(1-)} = \frac{\pi}{2\Delta}(D\mathbf{q}^2 + i\omega), \tag{28}$$

where $\omega = \epsilon_1 - \epsilon_2$. These are standard diffusive modes associated with the $G(n_1+n_2)$ symmetry of the action, which is explicitly broken by a nonzero ω. We shall call them ''soft'' modes and denote their number for each spatial mode $\mathbf{q}$ as $\mathcal{N}_{s-}$ and $\mathcal{N}_{s+}$ correspondingly.

• When $\sigma\neq\sigma'$ and $\alpha=\alpha'$ the eigenvalue is

$$M_{(\alpha+)(\alpha-)} = M_{(\alpha-)(\alpha+)} = \frac{\pi}{2\Delta} D\mathbf{q}^2. \tag{29}$$

These are the Goldstone modes associated with the spontaneous breaking of the exact $G(n_1)\times G(n_2)$ symmetry by replica nonsymmetric saddle points. They exist only for the manifolds with nonzero p_1 or p_2. We shall call the corresponding modes ''zero'' modes and denote their number for each spatial mode $\mathbf{q}$ as $\mathcal{N}_z$.

Such a separation of modes into massive, soft, and zero is well justified in the limit where $\Delta \ll \omega$; $D\mathbf{q}^2 \ll 1/\tau$. This specifies the regime where our methods and results are applicable. The number of modes depends on the number of independent degrees of freedom of the $\hat{Q}$ matrix and should be specified separately for each of the ensembles. We can perform now the Gaussian integrals over $\delta\hat{Q}$ fluctuations. Each eigenmode with eigenvalue $M_{(\alpha\sigma)(\alpha'\sigma')}(\mathbf{q})$ contributes a factor

$$\sqrt{\frac{\pi}{M_{(\alpha\sigma)(\alpha'\sigma')}(\mathbf{q})}}, \tag{30}$$

to the generating function $\langle Z^{(n_1,n_2)}\rangle$. The exception is the zero mode in the space independent, $\mathbf{q}=0$, sector.[15] This mode has identically zero mass, originating from the exact degeneracy of the $\mathcal{M}_{(p_1 p_2)}$ saddle-point manifold. Therefore in the $\mathbf{q}=0$ sector the integral over the zero mode results in the volume $\mathcal{V}_{(n_1 n_2)}^{(p_1 p_2)}$ of the coset space (15). These volumes are calculated in the Appendix for each of the three classical symmetry ensembles.

E. Generating function

Finally putting all the factors together one finds for the average generating function

$$\langle Z^{(n_1,n_2)}(\hat{E})\rangle = \sum_{p_1,p_2=0}^{n_1,n_2} e^{-A_{(p_1,p_2)}} \mathcal{V}_{(n_1 n_2)}^{(p_1 p_2)} \left(\frac{2\Delta}{i\omega}\right)^{\mathcal{N}_{s+}/2} \left(\frac{2\Delta}{-i\omega}\right)^{\mathcal{N}_{s-}/2} (2\Delta\tau)^{\mathcal{N}_m/2}$$
$$\times \prod_{\mathbf{q}\neq 0}\left[\left(\frac{2\Delta}{D\mathbf{q}^2}\right)^{\mathcal{N}_z/2} \left(\frac{2\Delta}{D\mathbf{q}^2+i\omega}\right)^{\mathcal{N}_{s+}/2} \left(\frac{2\Delta}{D\mathbf{q}^2-i\omega}\right)^{\mathcal{N}_{s-}/2} (2\Delta\tau)^{\mathcal{N}_m/2}\right], \tag{31}$$

where the saddle-point action, $A_{(p_1,p_2)}$ is given by Eq. (22). The first line in this expression represents the saddle-point action and the fluctuations in the $\mathbf{q}=0$ sector, whereas the second line originates from the Gaussian fluctuations of $\mathbf{q}\neq 0$ modes. In Sec. III we shall evaluate the coset space volumes, $\mathcal{V}_{(n_1 n_2)}^{(p_1 p_2)}$ and the number of modes, $\mathcal{N}_{z,s,m}=\mathcal{N}_{z,s,m}(p_1,p_2)$, for each of the classical symmetry classes. Hereafter we shall put $\Delta=1$, implying that all energies are measured in units of the mean level spacing, Δ.

F. The nonlinear sigma model

Let us briefly comment on the connection to the usual formulation of the problem in terms of the nonlinear σ model. For simplicity we discuss only the unitary case. This basically amounts to a reorganization of the computation we did above, which uses the strong hierarchy of masses (the massive modes are much more massive than the soft ones). Assuming first that $1/(\tau\Delta)$ is large, one finds that the saddle points of Eq. (8) are given by the set of matrices with $\hat{Q}^2=1$. This set is actually an ensemble of n_1+n_2+1 disconnected manifolds $\mathcal{S}_r$, corresponding to all possible values of $r=p_1-p_2$ (or equivalently of the trace of $\hat{Q}$). It is easily seen that all the modes which move away from these manifolds are massive, with a mass $\pi/(2\Delta\tau)$. These massive modes correspond to perturbing the matrix $\hat{Q}$ by a $\delta\hat{Q}$ such that $\hat{Q}\delta\hat{Q}+\delta\hat{Q}\hat{Q}\neq 0$, and the number of such massive modes is: $n_1^2+n_2^2+2r(r-n_1+n_2)$. Performing the integration over the massive modes one can write (up to irrelevant constants)

$$Z^{(n_1,n_2)}(\hat{E})\simeq\sum_{r=-n_2}^{n_1}\left(\sqrt{\frac{2\Delta\tau}{\pi}}\right)^{[n_1^2+n_2^2+2r(r-n_1+n_2)]K}\times\int_{\mathcal{S}_r}d[\hat{Q}]\exp\{-A[\hat{Q},\hat{E}]\},\tag{32}$$

where K is the number of different $\mathbf{q}$ modes. Notice that the manifold $\mathcal{S}_r$ is characterized by $\hat{Q}^2=1$, $\mathrm{Tr}\,\hat{Q}=n_1-n_2-2r$. It thus contains all matrices of the type $\hat{Q}=U^{-1}\hat{\Lambda}_{(p_1,p_2)}U$ with $U\in G(n_1+n_2)$, and $\hat{\Lambda}_{(p_1,p_2)}$ defined in Eq. (14), with $p_1-p_2=r$.

Expanding the action for slow spatial variations of $\hat{Q}$ on the manifold $\mathcal{S}_r$, one gets to first order in ω the standard sigma model:

$$Z^{(n_1,n_2)}(\hat{E})\simeq\sum_{r=-n_2}^{n_1}\left(\sqrt{\frac{2\Delta\tau}{\pi}}\right)^{[n_1^2+n_2^2+2r(r-n_1+n_2)]K}\times\int_{\mathcal{S}_r}d[\hat{Q}]\exp\left\{-\frac{\pi\nu D}{4}(\nabla\hat{Q})^2-\frac{i\pi\nu\omega}{2}\mathrm{Tr}\,(\hat{\Lambda}\hat{Q})\right\},\tag{33}$$

where $\hat{\Lambda}\equiv\hat{\Lambda}_{(0,0)}$.

For large ω, one can study the sigma model by a saddle-point approximation. The generic variations around a point $\hat{Q}$ of $\mathcal{S}_r$, staying on $\mathcal{S}_r$, are of the type $\delta\hat{Q}=[\hat{Q},W]$, with an arbitrary matrix $W(r)$. The stationarity of the action imposes $\mathrm{Tr}(\hat{\Lambda}[\hat{Q},W])=0$, which implies that the saddle points $\hat{Q}_{s.p.}$ commute with $\hat{\Lambda}$. One easily deduces that, on $\mathcal{S}_r$, the saddle-point submanifolds are exactly the submanifolds $\mathcal{M}_{p_1,p_2}$ with $p_1-p_2=r$. This approach basically reorganizes our previous computation by grouping together all the submanifolds with a fixed value of $\mathrm{Tr}\,\hat{Q}$ (or p_1-p_2). As we shall see below, for $n_1,n_2\to 0$, the only saddle points which contribute to leading order in $\Delta\tau$ are the ones with $p_1=p_2$, which are all located on the same manifold with $r=0$. Hence to the leading order one can approximate the generating function in Eq. (33) by an integral over the single manifold $\mathcal{S}_0$, which is what is usually done in the σ-model approach. We control this result well at large ω because we can do the sums over the p_1, p_2 and control the analytic continuation. But we believe that it is probably correct also for any ω. The reason is the following: in Eq. (33) one may extend the sum over r to a sum going from $-\infty$ to ∞, because when r is outside of the interval $\{-n_2,\ldots,n_1\}$ the volume of the integration space vanishes (this can be checked, e.g., in the limit $\omega\to 0$). One may then take the limit $n_1,n_2\to 0$ at fixed r. It is clear that the leading term comes from $r=0$, which minimizes the number of massive modes. So the usual σ-model formulation, with an integral over $\mathcal{S}_0$ only and the action given by Eq. (33), seems to be correct. However, one must keep in mind that, on this manifold there are, for large ω, several saddle-point submanifolds, which lead to the oscillations in the correlation functions. In the random matrix case, at least (without the gradient term), one may also try to perform the integration over the entire manifold $\mathcal{S}_0$, without resorting to the saddle-point method. If the nonlinear σ model, formulated on $\mathcal{S}_0$ only, is indeed correct, this should give the exact result, not restricted to $\omega\gg 1$. This procedure was attempted in Ref. 6, but the analytical continuation of the expressions emerging from these calculations still remains to be studied.

III. CORRELATION FUNCTIONS

A. Unitary ensemble

In the presence of a weak magnetic field the Hubbard-Stratonovich matrix $\hat{Q}$ is Hermitian.[5] The corresponding symmetry group is the unitary group, $G=U$. The measure of the functional integral over Hermitian matrices $Q_{ij}(\mathbf{q})$ in Eq. (8) is given by

$$d[\hat{Q}]=\prod_{\mathbf{q}}\left[\prod_j dQ_{jj}(\mathbf{q})\prod_{i<j}d\,\mathrm{Re}\,Q_{ij}(\mathbf{q})d\,\mathrm{Im}\,Q_{ij}(\mathbf{q})\right].\tag{34}$$

There are $(n_1+n_2)^2$ degrees of freedom for each spatial mode $\mathbf{q}$. Looking at the classification of modes, Eqs. (27)–(29), one finds that the number of massive, soft, and zero modes is

$$\begin{aligned}\mathcal{N}_m&=p_1^2+(n_1-p_1)^2+p_2^2+(n_2-p_2)^2+2p_1(n_2-p_2)\\&\quad+2p_2(n_1-p_1)\\&=n_1^2+n_2^2+2(p_1-p_2)(p_1-p_2-n_1+n_2);\\ \mathcal{N}_{s+}&=2p_1p_2;\\ \mathcal{N}_{s-}&=2(n_1-p_1)(n_2-p_2);\\ \mathcal{N}_z&=2p_1(n_1-p_1)+2p_2(n_2-p_2),\end{aligned}\tag{35}$$

which add up to $(n_1+n_2)^2$ as they should. Notice that the number of zero modes $\mathcal{N}_z$ coincides with the number of dimensions of the degenerate coset space manifold $U(n_1)/(U(p_1)U(n_1-p_1))\times U(n_2)/(U(p_2)U(n_2-p_2))$. The volume of this coset space is calculated in the Appendix and is given by

$$\mathcal{V}^{(p_1p_2)}_{(n_1n_2)}=(4\pi)^{p_1(n_1-p_1)+p_2(n_2-p_2)}F^{p_1}_{n_1}F^{p_2}_{n_2},\tag{36}$$

where

$$F^p_n\equiv\frac{\Gamma(1+n)}{\Gamma(1+p)\Gamma(1+n-p)}\prod_{j=1}^{p}\frac{\Gamma(1+j)}{\Gamma[1+(n-j+1)]}.\tag{37}$$

Since $F^{p>n}_n=0$ the sums over p_1 and p_2 in Eq. (31) may be extended up to infinity. The resulting expression may be then continued analytically to $n_{1,2}\to 0$, using the procedure which was detailed in our previous paper.[11] The analytical continuation, $n\to 0$ at fixed p, of the F^p_n symbol, Eq. (37), shows that

$$F^0_{n\to 0}=1;\quad F^1_{n\to 0}=n;\quad F^{p\geqslant 2}_{n\to 0}=O(n^p).\tag{38}$$

Therefore only the terms with $p_{1,2}=0,1$ may contribute to the correlation function S, Eq. (6). The number of massive modes, in the limit $n_{1,2}\to 0$ at fixed p_1,p_2, is $\mathcal{N}_m\to 2(p_1-p_2)^2$. Therefore the terms with $p_1\neq p_2$ can be neglected to leading order in the parameter $\Delta\tau\ll 1$. One thus ends up with the two contribution to the generating function: $p_1=p_2=0$ and $p_1=p_2=1$.

The first piece with $p_1=p_2=0$ is the usual replica symmetric contribution. Using Eqs. (31), (35), and (38) one finds

$$\langle Z^{(n_1,n_2)}(\hat{E})\rangle|_{p_1=p_2=0}=e^{\pi i(n_2\epsilon_2-n_1\epsilon_1)}\prod_{\mathbf{q}}\left(\frac{1}{D\mathbf{q}^2-i\omega}\right)^{n_1n_2}.\tag{39}$$

Using Eq. (6), one finds for the corresponding contribution to the correlation function

$$S(\omega)|_{p_1=p_2=0}=\pi^2+\sum_{\mathbf{q}}\frac{1}{(D\mathbf{q}^2-i\omega)^2}.\tag{40}$$

This is the well-known perturbative contribution.[7] The replica nonsymmetric manifold with $p_1=p_2=1$ gives

$$\langle Z^{(n_1,n_2)}(\hat{E})\rangle|_{p_1=p_2=1}=n_1n_2\frac{e^{2\pi i\omega}}{4\pi^2\omega^2}\prod_{\mathbf{q}\neq 0}\left(\frac{(D\mathbf{q}^2)^2}{(D\mathbf{q}^2)^2+\omega^2}\right).\tag{41}$$

Differentiating over ϵ_1 and ϵ_2 according to Eq. (6) and keeping only the leading contribution in $\omega/\Delta\gg 1$, one obtains for the corresponding contribution to the correlation function

$$S(\omega)|_{p_1=p_2=1}=\frac{e^{2\pi i\omega}}{\omega^2}\mathcal{D}(\omega),\tag{42}$$

where we have introduced the spectral determinant of the diffusion operator $\mathcal{D}(\omega)$ defined as

$$\mathcal{D}(\omega)\equiv\prod_{\mathbf{q}\neq 0}\left[1+\left(\frac{\omega}{D\mathbf{q}^2}\right)^2\right]^{-1}.\tag{43}$$

Finally using Eq. (5) one finds

$$R(\omega)=\frac{1}{2\pi^2}\,\mathrm{Re}\sum_{\mathbf{q}}\frac{1}{(D\mathbf{q}^2-i\omega)^2}+\frac{\cos(2\pi\omega)}{2\pi^2\omega^2}\mathcal{D}(\omega),\tag{44}$$

in agreement with Refs. 9 and 10.

One may wonder why only the saddle points with $p=0,1$ contribute to the result, while seemingly equivalent ones with $p=n,n-1$ do not. Indeed, for integer n all the expressions [e.g., Eq. (31)] are symmetric with respect to the interchange $p\leftrightarrow n-p$. However, after the analytical continuation $n\to 0$ this symmetry appears to be broken. After extending summations in Eq. (31) up to infinity and taking noninteger n, one faces highly divergent series, which should be regularized in a proper way. The way suggested in Ref. 11 is to consider an integral equation for a function $g(z,n)=\sum_{p=0}^{n}F^p_nz^p$ and study its solutions for small noninteger n. The symmetry $p\leftrightarrow n-p$ would manifest itself in a simple relation between $g(z,n)$ and $g(1/z,n)$. As was argued in Ref. 11, the solution of the integral equation for noninteger n exhibits a singularity at the unit circle $|z|=1$. As a result the initial series Eq. (31) is an asymptotic representation of the true solution only for $|z|<1$, which breaks the symmetry $p\leftrightarrow n-p$. Note that it is the existence of a positive infinitesimal imaginary part of ω [cf. Eq. (7)] which dictates that in our case $|z|<1$. For a negative infinitesimal imaginary part of ω this procedure would select instead the saddle points with $p=n,n-1$.

B. Orthogonal ensemble

If the time-reversal symmetry is not broken the Hubbard-Stratonovich matrix $\hat{Q}$ appears to be a self-dual real-quaternion matrix.[5] This means that each element Q_{ij} may be written as

$$Q_{ij}=\sum_{a=0}^{3}Q^a_{ij}\tau_a\tag{45}$$

with real Q^a_{ij}, where

$$\tau_0=\begin{pmatrix}1&0\\0&1\end{pmatrix};\quad\tau_1=\begin{pmatrix}0&-i\\-i&0\end{pmatrix};$$
$$\tau_2=\begin{pmatrix}0&-1\\1&0\end{pmatrix};\quad\tau_3=\begin{pmatrix}-i&0\\0&i\end{pmatrix}.\tag{46}$$

Moreover, $Q_{ji}=(Q_{ij})^\dagger$, where conjugation operation acts on the τ matrices. Such matrices may be diagonalized by rotations from the *symplectic* group,[2] $G=Sp(n_1+n_2)$, which is the relevant symmetry group for the GOE. Diagonal elements of the $\hat{Q}$ matrix, Q_{ii} are characterized by a single real number Q^0_{ii}, whereas off-diagonal ones $Q_{i<j}$ are parametrized by four numbers, Q^a_{ij}, $a=0,\ldots,3$. Altogether there are $2(n_1+n_2)^2-(n_1+n_2)$ degrees of freedom for each spatial mode $\mathbf{q}$. The measure of the functional integral in Eq. (8) is given by

$$d[\hat{Q}]=\prod_{\mathbf{q}}\left[\prod_j dQ^0_{jj}(\mathbf{q})\prod_{i<j}\prod_{a=0}^{3} dQ^a_{ij}(\mathbf{q})\right]. \tag{47}$$

One easily finds that the number of massive, soft, and zero modes is

$$\begin{aligned}\mathcal{N}_m&=2p_1^2-p_1+2(n_1-p_1)^2-(n_1-p_1)+2p_2^2-p_2\\&\quad+2(n_2-p_2)^2-(n_2-p_2)\\&\quad+4p_1(n_2-p_2)+4p_2(n_1-p_1)\\&=2n_1^2+2n_2^2-(n_1+n_2)+4(p_1-p_2)(p_1-p_2-n_1+n_2);\\ \mathcal{N}_{s+}&=4p_1p_2;\\ \mathcal{N}_{s-}&=4(n_1-p_1)(n_2-p_2);\\ \mathcal{N}_z&=4p_1(n_1-p_1)+4p_2(n_2-p_2),\end{aligned} \tag{48}$$

which add up to $2(n_1+n_2)^2-(n_1+n_2)$. The number of zero modes $\mathcal{N}_z$ coincides with the number of dimensions of the degenerate coset space manifold $Sp(n_1)/[Sp(p_1)Sp(n_1-p_1)]\times Sp(n_2)/[Sp(p_2)Sp(n_2-p_2)]$. The volume of this coset space is calculated in the appendix and is given by

$$\mathcal{V}^{(p_1p_2)}_{(n_1n_2)}=(4\pi)^{2p_1(n_1-p_1)+2p_2(n_2-p_2)}E^{p_1}_{n_1}E^{p_2}_{n_2}, \tag{49}$$

where

$$E^p_n\equiv\frac{\Gamma(1+n)}{\Gamma(1+p)\Gamma(1+n-p)}\prod_{j=1}^{p}\frac{\Gamma(1+2j)}{\Gamma[1+2(n-j+1)]}. \tag{50}$$

Since $E^{p>n}_n=0$ the sums over p_1 and p_2 in Eq. (31) may be extended up to infinity. The resulting expression may be then continued analytically to $n_{1,2}\to 0$ (cf. Ref. 11). In the limit $n\to 0$ the E^p_n symbol, Eq. (50), is given by

$$E^0_{n\to 0}=1;\quad E^1_{n\to 0}=2n;\quad E^{p\geq 2}_{n\to 0}=O(n^p). \tag{51}$$

Therefore only the terms with $p_{1,2}=0,1$ may contribute to the correlation function $S(\omega)$, Eq. (6). The number of massive modes, in the limit where $n_{1,2}\to 0$ at fixed p_1,p_2, is $\mathcal{N}_m\to 4(p_1-p_2)^2$. Therefore the terms with $p_1\neq p_2$ may be neglected to leading order in the parameter $\Delta\tau\ll 1$. As in the unitary case, only two terms with $p_1=p_2=0$ and $p_1=p_2=1$ contribute to the generating function.

The replica symmetric contribution $p_1=p_2=0$ is very similar to the one of the unitary ensemble. Using Eqs. (31), (48), and (51) one finds

$$\langle Z^{(n_1,n_2)}(\hat{E})\rangle|_{p_1=p_2=0}=e^{\pi i(n_2\epsilon_2-n_1\epsilon_1)}\prod_{\mathbf{q}}\left(\frac{1}{D\mathbf{q}^2-i\omega}\right)^{2n_1n_2}. \tag{52}$$

Using Eq. (6), one finds for the corresponding contribution to the correlation function

$$S(\omega)|_{p_1=p_2=0}=\pi^2+2\sum_{\mathbf{q}}\frac{1}{(D\mathbf{q}^2-i\omega)^2}, \tag{53}$$

in agreement with the known perturbative calculations.[7] The replica nonsymmetric saddle-point manifold, $p_1=p_2=1$, contribute as

$$\langle Z^{(n_1,n_2)}(\hat{E})\rangle|_{p_1=p_2=1}=n_1n_2\frac{e^{2\pi i\omega}}{4\pi^4\omega^4}\prod_{\mathbf{q}\neq 0}\left(\frac{(D\mathbf{q}^2)^2}{(D\mathbf{q}^2)^2+\omega^2}\right)^2. \tag{54}$$

Differentiating over ϵ_1 and ϵ_2 according to Eq. (6) and keeping only the leading contribution in $\omega/\Delta\gg 1$, one obtains for the corresponding term in the correlation function

$$S(\omega)|_{p_1=p_2=1}=\frac{e^{2\pi i\omega}}{\pi^2\omega^4}\mathcal{D}^2(\omega), \tag{55}$$

where the spectral determinant $\mathcal{D}(\omega)$ is defined by Eq. (43). Finally, from Eq. (5) one finds

$$R(\omega)=\frac{1}{\pi^2}\,\mathrm{Re}\sum_{\mathbf{q}}\frac{1}{(D\mathbf{q}^2-i\omega)^2}+\frac{\cos(2\pi\omega)}{2\pi^4\omega^4}\mathcal{D}^2(\omega), \tag{56}$$

again in agreement with Refs. 9 and 10.

C. Symplectic ensemble

If the spin of electrons is taken into account and the strong spin-orbit scattering is assumed the Hamiltonian of the system acquires a quaternion (symplectic) structure.[5] The corresponding symmetry of the $\hat{Q}$ matrix is the *orthogonal* one, $G=O(n_1+n_2)$. The $\hat{Q}$ is a real symmetric matrix and the integration measure in Eq. (8) is

$$d[\hat{Q}]=\prod_{\mathbf{q}}\left[\prod_{i\leq j} dQ_{ij}(\mathbf{q})\right]. \tag{57}$$

There are $[(n_1+n_2)^2+(n_1+n_2)]/2$ real degrees of freedom for each spatial mode $\mathbf{q}$. The number of massive, soft, and zero modes is

$$\begin{aligned}\mathcal{N}_m&=\frac{1}{2}[p_1^2+p_1]+\frac{1}{2}[(n_1-p_1)^2+(n_1-p_1)]+\frac{1}{2}[p_2^2+p_2]\\&\quad+\frac{1}{2}[(n_2-p_2)^2+(n_2-p_2)]\\&\quad+p_1(n_2-p_2)+p_2(n_1-p_1)\\&=\frac{1}{2}[n_1^2+n_2^2+n_1+n_2]+(p_1-p_2)(p_1-p_2-n_1+n_2);\\ \mathcal{N}_{s+}&=p_1p_2;\end{aligned} \tag{58}$$

$$\mathcal{N}_{s-}=(n_1-p_1)(n_2-p_2);$$

$$\mathcal{N}_z=p_1(n_1-p_1)+p_2(n_2-p_2),$$

which correctly add up to $[(n_1+n_2)^2+(n_1+n_2)]/2$. The number of zero modes $\mathcal{N}_z$ coincides with the number of dimensions of the degenerate coset space manifold $O(n_1)/[O(p_1)O(n_1-p_1)]\times O(n_2)/[O(p_2)O(n_2-p_2)]$. The volume of this coset space is calculated in the Appendix and is given by

$$\mathcal{V}_{(n_1n_2)}^{(p_1p_2)}=(2\sqrt{\pi})^{p_1(n_1-p_1)+p_2(n_2-p_2)}G_{n_1}^{p_1}G_{n_2}^{p_2}, \tag{59}$$

where

$$G_n^p\equiv\frac{\Gamma(1+n)}{\Gamma(1+p)\Gamma(1+n-p)}\prod_{j=1}^{p}\frac{\Gamma(1+j/2)}{\Gamma[1+(n-j+1)/2]}. \tag{60}$$

Since $G_n^{p>n}=0$ the sums over p_1 and p_2 in Eq. (31) may be extended up to infinity. The resulting expression may be continued analytically to $n_{1,2}\to 0$ (cf. Ref. 11). The G_n^p symbol, Eq. (60), in the limit $n\to 0$ is

$$G_{n\to 0}^0=1;\quad G_{n\to 0}^1=\frac{\sqrt{\pi}}{2}n;\quad G_{n\to 0}^2=-\frac{1}{4}n;$$

$$G_{n\to 0}^{p\geqslant 3}=O(n^{[(p+1)/2]}), \tag{61}$$

where $[x]$ denotes integer part of x. Therefore only the terms with $p_{1,2}=0,1,2$ contribute to the correlation function $S(\omega)$, Eq. (6). The number of massive modes, in the limit where $n_{1,2}\to 0$ at fixed p_1,p_2, is $\mathcal{N}_m\to(p_1-p_2)^2$, making terms with $p_1\neq p_2$ small in the parameter $\Delta\tau\ll 1$. One therefore finds three relevant contributions to the generating function: $p_1=p_2=0$, $p_1=p_2=1$, and $p_1=p_2=2$.

The replica symmetric contribution $p_1=p_2=0$ comes almost without changes. Employing Eqs. (31), (58), and (61) one finds

$$\langle Z^{(n_1,n_2)}(\hat{E})\rangle|_{p_1=p_2=0}=e^{\pi i(n_2\epsilon_2-n_1\epsilon_1)}\prod_{\mathbf{q}}\left(\frac{1}{D\mathbf{q}^2-i\omega}\right)^{n_1n_2/2}. \tag{62}$$

From Eq. (6), one finds for the corresponding contribution to the correlation function

$$S(\omega)|_{p_1=p_2=0}=\pi^2+\frac{1}{2}\sum_{\mathbf{q}}\frac{1}{(D\mathbf{q}^2-i\omega)^2}, \tag{63}$$

in agreement with Ref. 7. The first replica nonsymmetric manifold, $p_1=p_2=1$, results in

$$\langle Z^{(n_1,n_2)}(\hat{E})\rangle|_{p_1=p_2=1}=n_1n_2\frac{e^{2\pi i\omega}}{8\omega}\prod_{\mathbf{q}\neq 0}\left(\frac{(D\mathbf{q}^2)^2}{(D\mathbf{q}^2)^2+\omega^2}\right)^{1/2}. \tag{64}$$

Differentiating over ϵ_1 and ϵ_2 according to Eq. (6) and keeping only the leading contribution in $\omega/\Delta\gg 1$, one obtains for the corresponding contribution to the correlation function

$$S(\omega)|_{p_1=p_2=1}=\frac{\pi^2e^{2\pi i\omega}}{2\omega}\sqrt{\mathcal{D}(\omega)}, \tag{65}$$

where the spectral determinant $\mathcal{D}(\omega)$ is defined by Eq. (43). Finally, the second replica nonsymmetric manifold, $p_1=p_2=2$, gives

$$\langle Z^{(n_1,n_2)}(\hat{E})\rangle|_{p_1=p_2=2}=n_1n_2\frac{e^{4\pi i\omega}}{(4\pi)^4\omega^4}\prod_{\mathbf{q}\neq 0}\left(\frac{(D\mathbf{q}^2)^2}{(D\mathbf{q}^2)^2+\omega^2}\right)^2, \tag{66}$$

and consequently

$$S(\omega)|_{p_1=p_2=2}=\frac{e^{4\pi i\omega}}{16\pi^2\omega^4}\mathcal{D}^2(\omega). \tag{67}$$

Using Eq. (5) one finds

$$R(\omega)=\frac{1}{4\pi^2}\,\mathrm{Re}\sum_{\mathbf{q}}\frac{1}{(D\mathbf{q}^2-i\omega)^2}+\frac{\cos(2\pi\omega)}{4\omega}\sqrt{\mathcal{D}(\omega)}+\frac{\cos(4\pi\omega)}{32\pi^4\omega^4}\mathcal{D}^2(\omega), \tag{68}$$

again in agreement with Refs. 9 and 10.

IV. DISCUSSION OF THE RESULTS

Let us briefly discuss the energy scales, the approximations involved in the calculations, and their range of validity. There are four important energy scales: the mean level spacing Δ; the Thouless energy $E_c=\hbar D/L^2$ (L is the system size); the inverse scattering time $\hbar/\tau$; and the chemical potential μ. In the calculations above, the following hierarchy was assumed: $\Delta\ll E_c\ll\hbar/\tau\ll\mu$. The condition $\hbar/\tau\ll\mu$, which means that the disorder is weak enough, was used to evaluate momentum sums by contour integration. The inequality $E_c\ll\hbar/\tau$, which is equivalent to L much larger than the mean free path l, tells that the system is in the diffusive regime. It was used to derive the diffusive dispersion law in Eq. (26). Finally, $g=E_c/\Delta$ is the dimensionless conductance, and the condition that $g\gg 1$ means that the system is metallic. This condition was used to calculate integrals over zero modes with $\mathbf{q}\neq 0$ in the saddle-point approximation. One more inequality was assumed in our derivation, the fact that the difference in energies ω is much larger than the level spacing Δ. This is a technical assumption, which allowed us to evaluate soft modes integrals by the saddle-point technique. It would be interesting to perform the calculations without this last assumption, extending thus the results to arbitrarily small ω.

Our calculations give the correlation as functions of ω in the form of a finite sum of oscillating harmonics (two in the GOE and GUE and three in the GSE), with ω dependent amplitudes. The set of harmonics is exact and has to do only with the symmetry of the problem, specifically with the volumes of the relevant coset spaces. The amplitudes, on another hand, were obtained in the saddle-point approximation only. Using our formulation, one may develop a perturbation theory near the replica nonsymmetric saddle points, much in the same way as it was done near the replica symmetric one, see, e.g., Ref. 16. From such a perturbation theory one may obtain a systematic expansion of the amplitudes of the oscillatory terms, in powers of $\Delta/\omega < 1$

We would like to point out striking similarities between our replica approach and the SUSY one of Ref. 9 which was also based on the saddle-point calculations. In particular the list of modes is the same. In the SUSY case, the zero modes and soft modes are, respectively, associated with the rotations inside the fermionic block and between fermions and bosons. In some sense our p and $n-p$ replica blocks are similar to the bosonic and fermionic blocks of the SUSY theory. To appreciate better this analogy, one would need a more detailed understanding of the mathematical structure of the theory. In particular, one would like to define the unitary (or other) group, $U(n)$, for noninteger n and trace its relation to the graded symmetry. Another interesting problem is to appreciate better connections to the semiclassical method of Ref. 10.

The existence of the replica nonsymmetric saddle points opens two very important questions. One concerns their relevance to the renormalization-group treatment of the localization problem for one electron. Another, even more challenging one is to extend the replica theory of interacting electrons[17] to account for new saddle points.

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APPENDIX: ZERO MODES AND VOLUME OF THE COSET SPACE

1. Unitary case

The manifold $\mathcal{M}_{(p_1p_2)}$ of saddle-point matrices $\hat{Q}$ is generated by unitary transformations U of $U(n_1)\times U(n_2)$, applied to the diagonal matrix $\hat{\Lambda}_{(p_1p_2)}$, cf. Eq. (11). We must first find which choices of V_1, V_2 actually change the $\hat{Q}$ matrix. A general unitary transformation V of $U(n_1)$ can be written as a product RW where the matrix W has the block diagonal structure

$$W=\begin{pmatrix} W_{p_1} & 0 \\ 0 & W_{n_1-p_1}\end{pmatrix}, \tag{A1}$$

where W_{p_1} and $W_{n_1-p_1}$ are unitary matrices of size p_1 and n_1-p_1, respectively. The matrix W belongs to the subgroup $U(p_1)\times U(n_1-p_1)$ of $U(n_1)$ which leaves the saddle-point matrix invariant. The ''proper'' V_1 transformations which change the matrix $\hat{Q}$ while staying on the saddle-point manifolds are thus the elements R_1 of the coset space $U(n_1)/U(p_1)U(n_1-p_1)$, and similarly the proper V_2 transformations are elements R_2 which belong to $U(n_2)/U(p_2)U(n_2-p_2)$.

To compute the volume of the set of proper R transformations in $U(n)$ (here n stands for either n_1 or n_2), we start from the usual decomposition[2] of the integral over the group of $n\times n$ Hermitian matrices X in terms of the n eigenvalues x_j, and the unitary transformation V such that $X=V^{-1}(\mathrm{diag}\{x_1,\ldots,x_n\})V$:

$$d\rho_n(X)=\prod_{j=1}^{n} dx_j \prod_{j=1}^{n-1}\theta(x_{j+1}-x_j) \times \prod_{1\leqslant j<k\leqslant n}(x_j-x_k)^2 d\rho_n(V). \tag{A2}$$

In this integral we have ordered the eigenvalues (the θ function is Heavyside's step function), in such a way that the integral over V scans the whole set of allowed unitary transformations. We can compute the normalization of the ''angular'' measure for instance by integrating a Gaussian function:

$$I\equiv\int d\rho_n(X)\exp\left(-\frac{1}{2}\mathrm{Tr}\,X^2\right)=\pi^{n^2/2}2^{n/2} \tag{A3}$$

which can be computed using the above decomposition and the Selberg's integral:[2]

$$\begin{aligned} I&=\frac{1}{n!}\left[\int d\rho_n(V)\right]\int\prod_{j=1}^{n}dx_j\prod_{j<k}(x_j-x_k)^2 \\ &\quad\times\exp\left(-\frac{1}{2}\sum_j x_j^2\right) \\ &=\frac{1}{n!}\left[\int d\rho_n(V)\right](2\pi)^{n/2}\prod_{j=1}^{n}\Gamma(j+1). \end{aligned} \tag{A4}$$

Therefore one gets the normalization of the integral over the angular measure:

$$\mathcal{V}_n^U\equiv\int d\rho_n(V)=\pi^{(n^2-n)/2}\frac{n!}{\prod_{j=1}^{n}\Gamma(j+1)}. \tag{A5}$$

This result is easily checked by a direct counting argument: the choice of V is a choice of a Hermitian basis. The first vector of the basis is an arbitrary unit vector, the corresponding volume of integration is thus $S_{2n}/2\pi$ where $S_d=d\pi^{d/2}/\Gamma(1+d/2)$ is the volume of the d-dimensional unit sphere, and the division by 2π deals with a global phase choice. The second unit vector of the basis must be orthogonal to the first one, which fixes two real conditions, and its volume is thus $S_{2n-2}/2\pi$. After iterating this construction, one gets the result (A5).

We now decompose $V=RW$, and the angular integral $d\rho_n(V)$ as

$$d\rho_n(V)=d\rho_p(W_p)d\rho_{n-p}(W_{n-p})d\rho_{n,p}(R). \tag{A6}$$

This defines the measure $d\rho_{n,p}$ in the $2p(n-p)$ space of the proper transformations R. The normalization of this measure is

$$\int d\rho_{n,p}(R)=\frac{\mathcal{V}_n^U}{\mathcal{V}_p^U\mathcal{V}_{n-p}^U}=\pi^{p(n-p)}F_n^p, \tag{A7}$$

where we have introduced the symbol F_n^p defined by

$$F_n^p\equiv\frac{n!}{p!(n-p)!}\frac{\Pi_{j=1}^p\Gamma(j+1)\Pi_{j=1}^{n-p}\Gamma(j+1)}{\Pi_{j=1}^n\Gamma(j+1)}. \tag{A8}$$

We can now go on to the exact evaluation of the zero mode integrals. We keep within the subspace of the space independent $\hat{Q}$ matrices ($\mathbf{q}=0$ modes) which are the only modes having a zero eigenvalue sector. Clearly the zero modes integrals factorize into two independent pieces, associated with each of the two coset spaces $U(n_1)/U(p_1)U(n_1-p_1)$ and $U(n_2)/U(p_2)U(n_2-p_2)$. We can compute each such piece by working with an $n\times n$ Hermitian matrix X and computing

$$Z^{(n)}\equiv\int d\rho_n(X)\exp\left[-\frac{\pi\nu}{4\tau}\mathrm{Tr}\,X^2+\mathrm{Tr}\ln\left(E+\frac{i}{2\tau}X\right)\right]. \tag{A9}$$

We expand around the saddle-point manifold generated by $X=\Lambda_p\equiv\mathrm{diag}\{\lambda_1\ldots\lambda_n\}=\mathrm{diag}\{-1\ldots,-1,1,\ldots 1\}$ by writing

$$X=R\Lambda_pR^{-1}+\delta X,\quad \delta X=RW(\mathrm{diag}\{x_1,\ldots,x_n\})W^{-1}R^{-1}, \tag{A10}$$

where W, as above, is in $U(p)\times U(n-p)$ and R is a proper transformation. Using the decompositions of the measure defined in Eqs. (A2) and (A6), one obtains

$$\begin{aligned}Z^{(n)}=&\int dx_1\ldots dx_p\prod_{1\leqslant i<j\leqslant p}(x_i-x_j)^2\prod_{j=1}^{p-1}\theta(x_{j+1}-x_j)\int d\rho_p(W_p)\int dx_{p+1}\ldots dx_n\prod_{p+1\leqslant i<j\leqslant n}(x_i-x_j)^2\\ &\times\prod_{j=p+1}^{n-1}\theta(x_{j+1}-x_j)\int d\rho_{n-p}(W_{n-p})\int d\rho_{n,p}(R)\prod_{i=1}^{p}\prod_{j=p+1}^{n}(-2+x_i-x_j)^2\theta(2+x_{p+1}-x_p)\\ &\times\exp\left\{-A_p+\frac{\pi\nu}{4\tau}\sum_{j=1}^n x_j^2+\frac{1}{8\tau^2}\mathrm{Tr}[(E+i\Lambda_p/2\tau)^{-1}\delta X(E+i\Lambda_p/2\tau)^{-1}\delta X]\right\},\end{aligned} \tag{A11}$$

where A_p is the saddle-point action

$$A_p=\frac{\pi\nu}{4\tau}\mathrm{Tr}\,\Lambda_p^2+\mathrm{Tr}\ln((E+i\Lambda_p/2\tau)). \tag{A12}$$

The integral in Eq. (A11) can be simplified by the following observations: the integrals over x_i are all massive modes, and thus one can assume that $|x_i-x_j|\ll 1$. Therefore the third line of Eq. (A11) is just a constant, equal to $\pi^{p(n-p)}F_n^p2^{2p(n-p)}$. Apart from this constant, the rest of Eq. (A11) is nothing but the integrals over the massive modes.

What we have shown here is that, in the sector $\mathbf{q}=0$ of uniform fluctuations, the exact integral over the saddle-point manifold (the zero mode directions) gives a factor

$$(4\pi)^{p_1(n_1-p_1)+p_2(n_2-p_2)}F_{n_1}^{p_1}F_{n_2}^{p_2}. \tag{A13}$$

2. Orthogonal case

We shall not repeat here all the steps of the previous computation, they run in exactly the same way. We just give the main modifications. The integral over the symplectic group, generalizing Eq. (A4) is equal to

$$\begin{aligned}\mathcal{V}_n^S=&\int d\rho_n(V)=n!\left(\int d\rho_n(X)\exp\left(-\frac{X^2}{2}\right)\right)\\ &\times\left(\int\prod_{j=1}^n dx_j\prod_{j<k}(x_j-x_k)^4\exp\left[-\frac{1}{2}\sum_j x_j^2\right]\right)^{-1}\end{aligned} \tag{A14}$$

and the computation of Selberg's integral gives the volume

$$\mathcal{V}_n^S=n!\frac{\pi^{2n^2-3n/2}2^n}{\Pi_{j=1}^n\Gamma(1+2j)}. \tag{A15}$$

The ratio of volumes is

$$\frac{\mathcal{V}_n^S}{\mathcal{V}_p^S\mathcal{V}_{n-p}^S}=\pi^{2p(n-p)}E_n^p, \tag{A16}$$

where

$$E_n^p\equiv\frac{n!}{p!(n-p)!}\frac{\Pi_{j=1}^p\Gamma(2j+1)\Pi_{j=1}^{n-p}\Gamma(2j+1)}{\Pi_{j=1}^n\Gamma(2j+1)}. \tag{A17}$$

Finally the factor coupling the eigenvalues $j\leqslant p$ to those $j>p$ in the analog of Eq. (A11) becomes

$$\prod_{i=1}^p\prod_{j=p+1}^n(-2+x_i-x_j)^4 \tag{A18}$$

so that the final integral over the zero mode manifold is

$$(4\pi)^{2[p_1(n_1-p_1)+p_2(n_2-p_2)]}E_{n_1}^{p_1}E_{n_2}^{p_2}. \tag{A19}$$

3. Symplectic case

We just give again the main modifications. The integral over the orthogonal group, generalizing Eq. (A4), is equal to

$$\mathcal{V}_n^O = \int d\rho_n(V) = n! \left(\int d\rho_n(X) \exp\left(-\frac{X^2}{2} \right) \right) \times \left(\int \prod_{j=1}^{n} dx_j \prod_{j<k} |x_j - x_k| \exp\left[-\frac{1}{2} \sum_j x_j^2 \right] \right)^{-1}, \tag{A20}$$

where X is a real symmetric matrix. The computation of Selberg's integral gives the volume

$$\mathcal{V}_n^O = n! \frac{\pi^{n^2/4+(n/4)} 2^{n^2/4-5n/4}}{\Pi_{j=1}^n \Gamma(1+j/2)}. \tag{A21}$$

The ratio of volumes is

$$\frac{\mathcal{V}_n^O}{\mathcal{V}_p^O \mathcal{V}_{n-p}^O} = (2\pi)^{p(n-p)/2} G_n^p, \tag{A22}$$

where

$$G_n^p \equiv \frac{n!}{p!(n-p)!} \frac{\Pi_{j=1}^p \Gamma(j/2+1) \Pi_{j=1}^{n-p} \Gamma(j/2+1)}{\Pi_{j=1}^n \Gamma(j/2+1)}. \tag{A23}$$

Finally the factor coupling the eigenvalues $j \leqslant p$ to those $j > p$ in the analog of Eq. (A11) becomes

$$\prod_{i=1}^{p} \prod_{j=p+1}^{n} |-2 + x_i - x_j| \tag{A24}$$

so that the final integral over the zero mode manifold is

$$(4\pi)^{[p_1(n_1-p_1)+p_2(n_2-p_2)]/2} G_{n_1}^{p_1} G_{n_2}^{p_2}. \tag{A25}$$

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