Electron-electron interactions in disordered metals: Keldysh formalism

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We develop a field theory formalism for the disordered interacting electron liquid in the dynamical Keldysh formulation. This formalism is an alternative to the previously used replica technique. In addition, it naturally allows for the treatment of nonequilibrium effects. Employing the gauge invariance of the theory and carefully choosing the saddle point in the Q-matrix manifold, we separate purely phase effects of the fluctuating potential from the ones that change quasiparticle dynamics. As a result, the cancellation of super-divergent diagrams (double logarithms in $d=2$) is automatically built into the formalism. As a by-product we derive a nonperturbative expression for the single-particle density of states. The remaining low-energy σ model describes the quantum fluctuations of the electron distribution function. Its saddle-point equation appears to be the quantum kinetic equation with the appropriate collision integral along with collisionless terms. The Altshuler-Aronov corrections to the conductivity are shown to arise from the one-loop quantum fluctuation effects. [S0163-1829(99)03227-0]

I. INTRODUCTION

The physics of weakly disordered interacting electron systems at low temperatures has been a subject of considerable theoretical and experimental interest over the past years (for review see Refs. 1 and 2). Although significant progress has been made in this direction, many thermodynamic and transport properties of such systems are not completely understood and continue to stimulate both experimental and theoretical research. The latest revival of interest in the problem was prompted by the experimental discovery of a possible metal-insulator transition originally in clean Si metal-oxide semiconductor field-effect transistors[3] and later in p-type GaAs.[4]

The low-temperature behavior of the conductivity of a metal is mainly determined by the quantum (weak localization)[5,6] and interaction[7] corrections to the classical Drude result. These corrections are especially strong in low-dimensional ($d\leqslant 2$) systems. In two dimensions, for example, both the lowest order weak-localization correction[5] and the lowest order interaction correction[7] diverge logarithmically at low temperatures. The ultimate faith of the low-temperature phase is determined by the interplay between these corrections.

According to the scaling theory of localization,[8] in the absence of electron-electron (e-e) interactions (and with no spin-orbit scattering), quantum corrections lead to localization of all single-particle states in dimensions $d\leqslant 2$, and thus to insulating behavior for arbitrarily weak disorder (weak localization). Wegner[9] proposed a replicated σ model to study this problem. With the coupling constant corresponding to the dimensionless conductance g, this σ model provided justification for the one-parameter scaling theory of localization.[8] Later, Efetov[10] introduced a supersymmetric version of the σ model which obviated the need to take the tricky[11,12] zero replica number limit.

Finkel'stein[13] developed a replicated σ-model approach for interacting disordered electron systems, which was further developed in Refs. 14 and 15. He demonstrated further its renormalizability in the one-loop approximation, and obtained the one-loop renormalization-group flow equations. From these equations it followed that the weak-coupling fixed point corresponding to a noninteracting metal is unstable. The need for introducing replicas in Finkel'stein's approach follows from the fact that the ensemble-averaged observables are obtained as derivatives of the *averaged logarithm* of the partition function. The formalism in Refs. 13–15 utilizes the Matsubara representation, and is therefore restricted to the equilibrium situation.

Later it was suggested that the Keldysh-type field theory, originally developed for the treatment of nonequilibrium systems,[16] may be an alternative to the replica technique.[17–19] The point is that the use of the Keldysh closed contour in the time direction leads to an automatically normalized (disorder independent) partition function. This circumvents the need to introduce replicas. A similar situation exists in the theory of spin glasses, where in addition to the replica approach[20] the Martin-Siggia-Rose formalism,[21] analogous to the Keldysh approach,[21–23] has been used. This formalism provided insight complementary to that gained from the replica approach. Horbach and Schön[24] developed a σ model for noninteracting electrons in the Keldysh formalism. Although our treatment differs from theirs in many important details, we have benefitted much from their work. In a parallel and independent work[25] Chamon, Ludwig, and Nayak applied similar ideas to the treatment of interacting electrons. They derived Finkelsteins's[13] renormalization-group equations in the framework of the Keldysh formalism. Their work provides a useful application of the technique, which is to a large extent complementary to the one presented in this paper.

Here we apply the Keldysh formalism to disordered inter-

acting systems. We restrict ourselves to a consideration of spinless electrons in the presence of a weak magnetic field (unitary ensemble), and leave the considerations of the spin and Cooper channels for future work. Another important distinction of the present theory from the previous ones[13–15] is the different choice of a saddle point of the functional integral on the Q-matrix manifold. The saddle point in our formalism explicitly depends on a fluctuating potential in the system (the Hubbard-Stratonovich field, which decouples the electron-electron interaction). This choice of the saddle point allows us to separate pure phase effects of the fluctuating potential and to present the first, to our knowledge, clear derivation of the tunneling density of states (DOS) in a metal film obtained earlier by Finkel'stein[13] and Levitov and Shytov[26] by different means. Another advantage of this choice of the saddle point is that the perturbative expressions for gauge-invariant quantities contain only single logarithms of temperature or frequency (in $d=2$). The diagrams containing double logarithms which appear in the standard diagrammatic expansion[27] or in Finkel'stein's formalism,[13] and cancel each other for any gauge-invariant quantity, do not appear in our formulation at all. This significantly reduces the number of diagrams in each order of the perturbation theory. We then obtain a low-energy theory in the form of a σ model. The advantage of the Keldysh formulation is that it allows for a clear physical interpretation of the effective degrees of freedom. These turn out to be the quantum fluctuations of the electron distribution function. The saddle-point equation on the massless manifold is just the quantum kinetic equation with an appropriate collision integral. The one-loop fluctuations on top of this saddle point lead to corrections to various observables, and in the case of conductivity can be identified with the Altshuler-Aronov corrections.[7,28]

The paper is organized as follows: In Sec. II we present the functional integral representation of the Keldysh partition function for disordered interacting electron systems. Section III is devoted to the choice of an interaction-dependent saddle point and the derivation of an effective σ model as the massless fluctuations around this saddle point. We discuss some applications of the theory, like the derivation of the nonperturbative expression for the single-particle Green function, in Sec. IV. Quantum fluctuations and Altshuler-Aronov corrections to the conductivity are the subjects of Sec. V. In Sec. VI we derive the quantum kinetic equation as the saddle-point equation on the massless manifold, and discuss the corresponding collision relaxation times along with the collisionless terms. Finally, in Sec. VII we briefly discuss the obtained results and the future perspectives.

II. FUNCTIONAL INTEGRAL FORMULATION

A. Keldysh formalism

Consider a unitary evolution of a system along a closed contour $\mathcal{C}$ in the time direction which consists of the propagation from $t=-\infty$ to $t=+\infty$, and then back from $t=+\infty$ to $t=-\infty$. All external time-dependent fields are assumed to be *exactly* the same during the forward and backward evolution processes. As a result, at the end of such evolution the system must find itself precisely in the original state. We thus conclude that the evolution operator

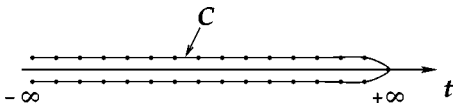

FIG. 1. Schematic representation of the discretization of the time contour $\mathcal{C}$. The dots on the upper and lower branches of the contour denote the discretized time points.

$$\hat{U}_{\mathcal{C}} \equiv 1. \tag{1}$$

Let us consider next the partition function defined as

$$Z = \mathrm{Tr}\{\rho_0 \hat{U}_{\mathcal{C}}\}/\mathrm{Tr}\{\rho_0\} = 1, \tag{2}$$

where ρ_0 is a density matrix of the system at the initial time $t=-\infty$ before the interactions and disorder are adiabatically switched on. A more informative object is the generating functional, which is obtained by introducing source fields. It is clear that to have a generating functional not identically equal to unity, the source fields should have a different behavior on the forward and backward parts of the contour. To shorten the subsequent expressions we shall operate with the partition function [Eq. (2)], and will introduce the generating functional in Sec. IV C.

The next step is to divide the $\mathcal{C}$ contour into $2N+1$ time steps, such as $t_1 = t_{2N+1} = -\infty$ and $t_{N+1} = +\infty$ as shown in Fig. 1. Following the standard route,[29] we obtain the coherent-state functional integral, by introducing a resolution of unity at each time step. Taking the $N\to\infty$ limit, for the partition function we obtain

$$Z = \mathcal{N} \int \mathcal{D}\overline{\psi}\,\psi \exp\{iS[\overline{\psi},\psi]\}, \tag{3}$$

where $\mathcal{N}$ is *disorder-independent* normalization constant[30] and the fermionic action is given by

$$S[\overline{\psi},\psi] = \int_{\mathcal{C}} dt \left\{ \int d\mathbf{r}\, \overline{\psi}[G_0^{-1} - U_{dis}(\mathbf{r})]\psi - \frac{1}{2}\int\int d\mathbf{r}\, d\mathbf{r}'\, \overline{\psi}(\mathbf{r})\overline{\psi}(\mathbf{r}')V_0(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}) \right\}. \tag{4}$$

Here the inverse bare Green function is a shorthand notation for

$$G_0^{-1} = i\frac{\partial}{\partial t} + \frac{\nabla_r^2}{2m}, \tag{5}$$

where the time derivative is taken along the contour $\mathcal{C}$. Notation (5) is somewhat symbolic: while inverting this operator it is necessary to invert its discretized version first, and only then take the limit $N\to\infty$.[29]

We then divide the fermionic field $\psi(r,t)$ into the two components $\psi_1(r,t)$ and $\psi_2(r,t)$ which reside on the forward and backward parts of the time contour, respectively. Since the interaction part of the action is strictly local in time, it may be rewritten as $S_{int}[\psi_1] - S_{int}[\psi_2]$ (the minus sign comes from the opposite direction of the time integral on the backward part of the contour)

$$S_{int}[\psi_i]=-\frac{1}{2}\int dt\, d\mathbf{r}\, d\mathbf{r}'\,\overline{\psi}_i(\mathbf{r})\overline{\psi}_i(\mathbf{r}')V_0(\mathbf{r}-\mathbf{r}')\times\psi_i(\mathbf{r}')\psi_i(\mathbf{r}). \tag{6}$$

Here $i=1$ and 2, and $V_0(\mathbf{r}-\mathbf{r}')$ is a bare interaction potential. We now introduce two independent auxiliary bosonic fields $\hat{\phi}_{1(2)}(\mathbf{r},t)$ to decouple the two interaction terms by the Hubbard-Stratonovich transformation. As a result, for the partition function one obtains

$$Z=\tilde{\mathcal{N}}\int \mathcal{D}\hat{\Phi}\, e^{(i/2)\mathrm{Tr}\{\hat{\Phi}^T V_0^{-1}\sigma_3\hat{\Phi}\}}\int \mathcal{D}\overline{\Psi}\,\Psi e^{iS[\overline{\Psi},\Psi,\hat{\Phi}]}, \tag{7a}$$

$$S[\overline{\Psi},\Psi,\hat{\Phi}]=\mathrm{Tr}\{\overline{\Psi}[\hat{G}_0^{-1}-U_{dis}\sigma_3+\hat{\phi}_\alpha\hat{\gamma}^\alpha]\Psi\}. \tag{7b}$$

Here we have introduced the following vector notations for the fermionic doublet Ψ, the bosonic doublet $\hat{\Phi}$, and two vertex matrices $\hat{\gamma}^\alpha$:

$$\Psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad \hat{\Phi}=\begin{pmatrix}\hat{\phi}_1\\ \hat{\phi}_2\end{pmatrix}, \tag{8a}$$

$$\hat{\gamma}^1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \hat{\gamma}^2=\begin{pmatrix}0&0\\0&-1\end{pmatrix}. \tag{8b}$$

The inverse matrix Green function stands for

$$\hat{G}_0^{-1}=\begin{pmatrix} i\dfrac{\partial}{\partial t}+\dfrac{\nabla_r^2}{2m} & 0\\ 0 & -i\dfrac{\partial}{\partial t}-\dfrac{\nabla_r^2}{2m}\end{pmatrix}. \tag{9}$$

The trace operation in Eq. (7b) and henceforth is understood to be performed over the 2×2 structure as well as over the time and space variables.

B. Disorder averaging

The great advantage of the Keldysh technique is that the normalization constant $\tilde{\mathcal{N}}$ in Eq. (7a) does not depend on the realization of the disorder potential. Thus the disorder averaging can be performed directly, without the need to resort to the replica trick. Hereafter we employ the simplest model of the Gaussian, δ-correlated disorder

$$\langle\ldots\rangle=\int \mathcal{D}U_{dis}\cdots\exp\left\{-\pi\nu\tau\int d\mathbf{r}\,U_{dis}^2(\mathbf{r})\right\}, \tag{10}$$

where the disorder strength is characterized by the elastic mean free time τ; ν is the bare density of states at the Fermi energy. Next, we perform the Gaussian integration over U_{dis} in Eq. (7a) and decouple the arising (nonlocal in time) quartic interaction by means of the Hubbard-Stratonovich transformation. Doing so, we obtain

$$\langle e^{-i\,\mathrm{Tr}\{\overline{\Psi}U_{dis}\sigma_3\Psi\}}\rangle=\exp\left\{-(4\pi\nu\tau)^{-1}\int d\mathbf{r}\left[\int dt\,\overline{\Psi}(\mathbf{r},t)\sigma_3\Psi(\mathbf{r},t)\right]^2\right\} \tag{11a}$$

$$=\int\mathcal{D}\hat{Q}\exp\left\{-\int d\mathbf{r}\,dt\,dt'\left[\frac{\pi\nu}{4\tau}\mathrm{Tr}\,\hat{Q}_{tt'}(\mathbf{r})\hat{Q}_{t't}(\mathbf{r})+\frac{1}{2\tau}\overline{\Psi}(\mathbf{r},t)\hat{Q}_{tt'}(\mathbf{r})\sigma_3\Psi(\mathbf{r},t')\right]\right\}. \tag{11b}$$

Here we have introduced the Hubbard-Stratonovich field $\hat{Q}$ which is a matrix with indices both in the Keldysh 2×2 space and in the time space. To ensure the convergence of the integral in Eq. (11b), the $\hat{Q}$ matrix is chosen to be Hermitian. After these transformations the fermionic functional integral in Eq. (7a) can be formally performed, leading to

$$\det\left|\hat{G}_0^{-1}+\frac{i}{2\tau}\hat{Q}\sigma_3+\hat{\phi}_\alpha\hat{\gamma}^\alpha\right|. \tag{12}$$

As a result, the disorder averaged partition function takes the forms

$$\langle Z\rangle=\int\mathcal{D}\hat{\Phi}\, e^{(i/2)\mathrm{Tr}\{\hat{\Phi}^T V_0^{-1}\sigma_3\hat{\Phi}\}}\int\mathcal{D}\hat{Q}\, e^{iS[\hat{Q},\hat{\Phi}]}, \tag{13a}$$

$$iS[\hat{Q},\hat{\Phi}]=-\frac{\pi\nu}{4\tau}\mathrm{Tr}\,\hat{Q}^2+\mathrm{Tr}\ln\left[\hat{G}_0^{-1}+\frac{i\hat{Q}\sigma_3}{2\tau}+\hat{\phi}_\alpha\hat{\gamma}^\alpha\right]. \tag{13b}$$

As before the trace operation is understood to be performed over the Keldysh and the time indices as well as over the coordinate space; the unessential normalization constant is omitted.

C. Keldysh rotation

In the notations introduced in Eqs. (8) and (9) the electron Green functions $\hat{G}$ are matrices in the 2×2 Keldysh space. Their components are not independent and satisfy certain general identities.[16,31] This interdependence becomes most transparent if one introduces the rotated Green functions G denoted by the absence of the hat and defined as

$$G\equiv L\sigma_3\hat{G}L^\dagger, \tag{14}$$

where the unitary matrix L is given by

$$L=\frac{1}{\sqrt{2}}(\sigma_0-i\sigma_2)=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}. \tag{15}$$

As follows from the definition of the Keldysh Green function,[16,31] the rotated Green function has the structure

$$G(t,t')=\begin{pmatrix} G^R(t,t') & G^K(t,t') \\ 0 & G^A(t,t') \end{pmatrix}, \tag{16}$$

where $G^{R(A)}(t,t')$ vanish for $t\leqslant t'(t\geqslant t')$. To pass to the rotated representation we introduce a new Hubbard-Stratonovich field Q which is related to the old one, $\hat{Q}$, by the following unitary transformation:

$$Q\equiv L\hat{Q}L^{\dagger}. \tag{17}$$

We also introduce the rotated bare inverse Green function G_0^{-1}, expressed through $\hat{G}_0^{-1}$ of Eq. (9) in a manner consistent with Eq. (14):

$$G_0^{-1}\equiv L\hat{G}_0^{-1}\sigma_3 L^{\dagger}=\left(i\frac{\partial}{\partial t}+\frac{\nabla_r^2}{2m}\right)\sigma_0. \tag{18}$$

It is also convenient to perform a linear transformation of the bosonic doublet $\hat{\Phi}$ by introducing the symmetric and the antisymmetric combinations of the fields residing on the upper and lower branches of the contour $\mathcal{C}$:

$$\phi_1=\tfrac{1}{2}(\hat{\phi}_1+\hat{\phi}_2),\quad \phi_2=\tfrac{1}{2}(\hat{\phi}_1-\hat{\phi}_2). \tag{19}$$

Then the rotated vertex matrices for these new fields are $\gamma^{1(2)}=L(\hat{\gamma}^1\pm\hat{\gamma}^2)\sigma_3 L^{\dagger}$ with the following explicit forms:

$$\gamma^1=\sigma_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \gamma^2=\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{20}$$

Any classical external field takes on identical values on the two branches of the contour and, hence, in the rotated basis has only the first symmetric component. The nonzero antisymmetric component may appear only as a virtual fluctuating field. Below we shall sometimes refer to the first and second components of the bosonic fields as *classical* and *quantum* ones, correspondingly. Since the presence of an external classical field does not change the basic fact that $Z=1$, any auxiliary source field should have a nonvanishing quantum component to generate an observable. We shall return to this observation in Sec. IV C.

Utilizing the cyclic invariance of the trace operation, we obtain the following expressions for the partition function, Eq. (13), through the new variables Φ and Q:

$$\langle Z\rangle=\int \mathcal{D}\Phi\, e^{i\,\mathrm{Tr}\{\Phi^T V_0^{-1}\sigma_1\Phi\}}\int \mathcal{D}Q\, e^{iS[Q,\Phi]}, \tag{21a}$$

$$iS[Q,\Phi]=-\frac{\pi\nu}{4\tau}\mathrm{Tr}\,Q^2+\mathrm{Tr}\ln\left[G_0^{-1}+\frac{i}{2\tau}Q+\phi_\alpha\gamma^\alpha\right]. \tag{21b}$$

III. NONLINEAR σ MODEL

A. Saddle-point equation

We shall look now for a saddle point of the functional integral over the Q matrix in Eq. (21a). The aim is to find a stationary solution for a given realization of the fluctuating bosonic fields Φ, slowly varying in space and time. Calculating a variation of the action [Eq. (21b)] over the Q matrix, one obtains the following equation for the saddle-point matrix $\underline{Q}=\underline{Q}[\Phi]$:

$$\underline{Q}_{t,t'}(r)=\frac{i}{\pi\nu}\left[G_0^{-1}+\frac{i}{2\tau}\underline{Q}+\phi_\alpha\gamma^\alpha\right]^{-1}\Bigg|_{r,r;t,t'}. \tag{22}$$

We are unable to solve this equation exactly; therefore, our goal will be to find its approximate solution, which is as close as possible to the true stationary point of the functional integral in Eq. (21a). To execute this program, we first consider the case where $\Phi=0$. It is easy to check that in this case

$$\Lambda_{t-t'}\equiv\underline{Q}[\Phi=0]=\frac{i}{\pi\nu}\sum_p G(p,t-t'), \tag{23}$$

where the impurity averaged Keldysh Green function is [cf. Eq. (16)]

$$G(p,\epsilon)=\begin{pmatrix} G^R(p,\epsilon) & G^K(p,\epsilon)\\ 0 & G^A(p,\epsilon)\end{pmatrix} =\begin{pmatrix}1&F_\epsilon\\0&-1\end{pmatrix}\begin{pmatrix}G^R(p,\epsilon)&0\\0&G^A(p,\epsilon)\end{pmatrix}\begin{pmatrix}1&F_\epsilon\\0&-1\end{pmatrix}, \tag{24}$$

with

$$G^{R(A)}(p,\epsilon)=[\epsilon-\epsilon_p\pm i/(2\tau)]^{-1}, \tag{25a}$$

$$G^K(p,\epsilon)=G^RF-FG^A. \tag{25b}$$

The function F defined by Eq. (25b) can be expressed through the single-particle distribution function $n(\epsilon)$ as $F(\epsilon)=1-2n(\epsilon)$. In equilibrium at temperature T it is given by

$$F_\epsilon^{eq}=\tanh\frac{\epsilon}{2T}. \tag{26}$$

Substituting Eqs. (24) and (25) into Eq. (23), and performing the momentum summation, for the noninteracting $(\Phi=0)$ saddle point one obtains

$$\Lambda_\epsilon=\begin{pmatrix}\mathbb{1}_\epsilon^R & 2F_\epsilon\\0&-\mathbb{1}_\epsilon^A\end{pmatrix},\quad \Lambda_{t-t'}=\begin{pmatrix}\delta_{t-t'-0}&2F_{t-t'}\\0&-\delta_{t-t'+0}\end{pmatrix}. \tag{27}$$

Here we have introduced the retarded and advanced unities, $\mathbb{1}_\epsilon^{R(A)}$, which should be understood as Fourier transforms of infinitesimally shifted δ functions. This particular form of the Green function is a result of the approximation that the single-particle DOS is independent of the energy ϵ. In reality it does depend on ϵ, and the retarded (advanced) components of Λ_ϵ are analytic functions of energy in the upper (lower) half-plane which do depend on energy on the scale of order of the Fermi energy ϵ_F. Therefore, the infinitesimally shifted δ functions in Eq. (27) should be understood as $\delta_{t\mp0}=f_\pm(t)\Theta(\pm t)$, where $\Theta(t)$ is the Heaviside function, and $f_\pm(t)$ are functions that are highly peaked for $|t|\lesssim\epsilon_F^{-1}$ and satisfy the normalization condition $\int_0^{\pm\infty}dt\,f_\pm(t)=1$. This

high-energy regularization is important to remember in calculations to avoid spurious unphysical constants. In particular, for obvious reasons,

$$\mathbb{1}^R_{t-t'}\mathcal{M}^R_{t',t}=0, \quad \mathbb{1}^A_{t-t'}\mathcal{M}^A_{t',t}=0, \tag{28}$$

where $\mathcal{M}^{R(A)}_{t',t}$ is an arbitrary retarded (advanced) matrix in the time space.

Substituting Eq. (27) into Eq. (22) with $\Phi=0$, it is easy to see that $Q=\Lambda$ solves the noninteracting saddle-point equation for *any* function F_ϵ. [The simplest way to check it is to use the decomposition equation (24).] This is natural, since any distribution function is allowed for the noninteracting electron gas. We shall see below how the interaction effects drive the system toward the equilibrium distribution [Eq. (26)].

Let us now include a finite $\phi_\alpha(r,t)$ into Eq. (22). To this end we notice that this equation can be still solved exactly for the particular case of spatially uniform realizations of the boson field, $\phi_\alpha=\phi_\alpha(t)$. This is obvious since such a field may be gauged out, resulting in

$$\underline{Q}_{t,t'}[\Phi(t)]=\exp[i\textstyle\int^t dt\,\phi_\alpha(t)\gamma^\alpha]\Lambda_{t-t'} \times\exp[-i\textstyle\int^{t'} dt\,\phi_\alpha(t)\gamma^\alpha]. \tag{29}$$

The validity of this solution can be verified by acting with the operator $[G_0^{-1}+i/(2\tau)\underline{Q}+\phi_\alpha(t)\gamma^\alpha]$ on both sides of Eq. (22), and utilizing the fact that $\Lambda_{t-t'}$ solves Eq. (22) with $\Phi=0$. We also rely on the commutativity of the vertex matrices, $[\gamma^1,\gamma^2]=0$, in writing the solution in the form of Eq. (29).

We now consider the case where the bosonic fields ϕ_α are slowly (compared to the mean free path l) varying in space. In analogy with Eq. (29) we shall look for an approximate solution of Eq. (22) in the form of a local (in time and space) gauge transformation of Λ

$$\underline{Q}_{t,t'}(r)=e^{ik_\alpha(r,t)\gamma^\alpha}\Lambda_{t-t'}e^{-ik_\alpha(r,t')\gamma^\alpha}, \tag{30}$$

where $k_\alpha=k_\alpha[\Phi]$ is a certain *linear* functional of the fields Φ, whose specific form is to be determined to satisfy Eq. (22) in the best possible way.

To proceed we introduce a new Hubbard-Stratonovich field $\widetilde{Q}$ which is related to the old one, Q, by the gauge transformation

$$Q_{t,t'}(r)=e^{ik_\alpha(r,t)\gamma^\alpha}\widetilde{Q}_{t,t'}(r)e^{-ik_\alpha(r,t')\gamma^\alpha}. \tag{31}$$

Substituting this definition into the action [Eq. (21b)], and using the invariance of the trace under a cyclic permutation of operators, we can rewrite the action as

$$iS[\widetilde{Q},\Phi]=-\frac{\pi\nu}{4\tau}\mathrm{Tr}\,\widetilde{Q}^2 +\mathrm{Tr}\ln\left[G_0^{-1}+C-\frac{\nabla k_\alpha\nabla k_\beta\gamma^\alpha\gamma^\beta}{2m}+\frac{i}{2\tau}\widetilde{Q}\right], \tag{32}$$

where we have introduced the notation

$$C(r,t)\equiv(\phi_\alpha-\partial_t k_\alpha-\mathbf{v}_F\nabla k_\alpha)\gamma^\alpha, \tag{33}$$

with the Fermi velocity, $\mathbf{v}_F=-i\nabla_r/m$. To find the approximate saddle point of Eq. (30), we substitute $\widetilde{Q}=\Lambda+\delta\widetilde{Q}$ into Eq. (32) and require terms linear in $\delta\widetilde{Q}$ to vanish. In doing so we neglect the diamagnetic term $\nabla k_\alpha\nabla k_\beta\gamma^\alpha\gamma^\beta/2m$, since it is quadratic in k_α (and hence in Φ) and is also smaller than C in the parameter $q/p_F\ll 1$, where q is the characteristic momentum scale of variation of Φ. As a result we obtain the equation

$$-\pi\nu\Lambda_{t',t}+i[G-GCG+GCGCG-\cdots]_{t',t}(r,r)=0. \tag{34}$$

The first two terms in this expression cancel, according to Eq. (23). The freedom of choosing the $K[\Phi]$ functional is not sufficient to cancel all the terms in this expansion. We thus concentrate on the term which is linear in Φ and K:

$$\sum_p G(p_+,\epsilon_+)C(q,\omega)G(p_-,\epsilon_-) =\pi\nu\tau[(\phi_\alpha+i\omega k_\alpha)(\gamma^\alpha-\Lambda_+\gamma^\alpha\Lambda_-) -Dq^2k_\alpha(\Lambda_+\gamma^\alpha-\gamma^\alpha\Lambda_-)], \tag{35}$$

where $p_\pm=p\pm q/2$, $\epsilon_\pm=\epsilon\pm\omega/2$ and $\Lambda_\pm=\Lambda_{\epsilon_\pm}$. To derive Eq. (35) one may employ the following useful representation of the Keldysh Green function:

$$G(p,\epsilon)\equiv\left[G_0^{-1}+\frac{i}{2\tau}\Lambda_\epsilon\right]^{-1} =\frac{1}{2}G^R(p,\epsilon)(\sigma_0+\Lambda_\epsilon)+\frac{1}{2}G^A(p,\epsilon)(\sigma_0-\Lambda_\epsilon). \tag{36}$$

Only $\Sigma_p G^R G^A$ and $\Sigma_p G^R\mathbf{v}_F G^A$ contribute to Eq. (35). Multiplying Eq. (35) by Λ_{ϵ_+} from the left, one obtains the following matrix condition for the vanishing of the linear term in Eq. (34):

$$Dq^2k_\alpha(\Lambda_+\gamma^\alpha\Lambda_--\gamma^\alpha)+(\phi_\alpha+i\omega k_\alpha)(\Lambda_+\gamma^\alpha-\gamma^\alpha\Lambda_-)=0. \tag{37}$$

To cancel (1,1), (2,2), and (2,1) components of the matrix on the left-hand side of this equation, the functional K should satisfy

$$(Dq^2+i\omega)k_2(q,\omega)+\phi_2(q,\omega)=0. \tag{38}$$

Provided this equality is obeyed, the condition to cancel the Keldysh (2,1) component on the left-hand side of Eq. (37) is

$$(Dq^2-i\omega)k_1-\phi_1=-2Dq^2k_2\frac{1-F_{\epsilon_+}F_{\epsilon_-}}{F_{\epsilon_+}-F_{\epsilon_-}}. \tag{39}$$

This equation cannot be in general satisfied, since its right-hand side contains an explicit dependence on ϵ, whereas the left-hand side is supposed to be ϵ independent. This happens because the trial saddle-point solution (30), is too restrictive. In particular, we demanded that it may be obtained from Λ by the rotation, which is local in time. As a result, the functional K depends only on differential energy ω and not on ''center-of-mass'' energy ϵ. This restriction is in an apparent

contradiction with Eq. (39). There is, however, an important particular case when Eq. (39) may be solved. This is the case of thermal equilibrium, where the fermionic distribution function is given by Eq. (26). As a result,

$$\frac{1-F^{eq}_{\epsilon_+}F^{eq}_{\epsilon_-}}{F^{eq}_{\epsilon_+}-F^{eq}_{\epsilon_-}}=\coth\frac{\omega}{2T}\equiv B^{eq}_{\omega}, \tag{40}$$

where B^{eq}_{ω} stands for the equilibrium bosonic distribution function. Thus the matrix equation (37) for the functional $K[\Phi]$ can be resolved in equilibrium. The result may be written in a short form as

$$\mathcal{D}^{-1}(q,\omega)K(q,\omega)=\Pi_{\omega}^{-1}\Phi(q,\omega). \tag{41}$$

Here we have introduced the following bosonic matrix propagators in the 2×2 Keldysh space:

$$\mathcal{D}(q,\omega)=\begin{pmatrix}\mathcal{D}^K(q,\omega) & \mathcal{D}^R(q,\omega)\\ \mathcal{D}^A(q,\omega) & 0\end{pmatrix}, \tag{42}$$

with

$$\mathcal{D}^{R(A)}(q,\omega)=(Dq^2\mp i\omega)^{-1}, \tag{43a}$$

$$\mathcal{D}^K(q,\omega)=B_{\omega}[\mathcal{D}^R(q,\omega)-\mathcal{D}^A(q,\omega)], \tag{43b}$$

and

$$\Pi_{\omega}=-i\omega\mathcal{D}(q=0,\omega)=\begin{pmatrix}2B_{\omega} & \mathbb{1}^R_{\omega}\\ -\mathbb{1}^A_{\omega} & 0\end{pmatrix}. \tag{44}$$

The superscript ''*eq*'' denoting equilibrium in the bosonic distribution has been omitted for brevity.

Equations (30) and (41) complete the task of finding the approximate saddle point $\underline{Q}=\underline{Q}[\Phi]$ for any given realization of fields Φ. On this solution we are able to cancel only the term linear in Φ in the expansion [Eq. (34)]. This guarantees only that terms like $\Phi\delta\tilde{Q}$ will not appear in the expansion of the action around the saddle point given by Eqs. (30) and (41). Terms like $\Phi^2\delta\tilde{Q}$ may (and will) arise in such expansion. We shall see later that it is precisely these terms that are responsible for the divergent Altshuler–Aronov corrections to conductivity.[28] The ability to avoid $\Phi\delta\tilde{Q}$ terms is, strictly speaking, limited only to the thermal equilibrium. For an out-of-equilibrium situation such terms reappear and require some care (see Sec. VI).

The influence of the external potential Φ on the electron dynamics (and hence on the Green function) is twofold[32]: (i) it changes the particle trajectory, and (ii) it changes the phase of the electron wave function. The first effect is proportional to the electric field $\mathbf{E}=-\nabla\Phi$, and is small for the long-wavelength spatial configurations of Φ. The second effect, however, requires no actual electric fields. It is proportional to Φ itself, rather than $\nabla\Phi$, and is akin to the Aharonov-Bohm effect. It changes the phase but not the amplitude of the wave function, and can be taken into account in the Eikonal approximation. The second effect exceeds the first one for the long-wavelength fluctuations of the potential; therefore, it is especially important in the presence of the long-range Coulomb interactions. The approximation to the saddle point [Eq. (30)] is similar to the Eikonal approximation. It is designed to account for the phase effect of the slow fluctuations of the potential Φ. Note that the phase K enters the saddle-point equation only through its total time derivative along the trajectory of a particle, $d/dt=\partial_t+\mathbf{v}_F\nabla$ [cf. Eq. (33)]. If we demand that C vanish, we obtain the standard Eikonal equation[32] for the action K of the particle moving with a given velocity $\mathbf{v}$ in an external field Φ. Unfortunately, the ansatz [Eq. (30)] is too restrictive to nullify C for particles of every velocity $\mathbf{v}$. Eventually all the particles in the Fermi sea interfere to produce the Green function $\underline{Q}_{t,t'}(r)$. Equation (30) approximately accounts for the phase interference between particles moving along different trajectories. Since the particle dynamics is diffusive this leads to the diffusive relations (41)–(43) between the external potential Φ and the phase K. As will be clear below the choice of the saddle point in Eq. (30) considerably simplifies the subsequent calculations. In particular, it eliminates completely the family of superdivergent diagrams which cancel in the traditional treatment[13] after sometimes tedious calculations.

B. Effective action

To formulate an effective low-energy theory in terms of the fluctuating fields $\tilde{Q}$ and Φ, we need to examine the fluctuations around the saddle point. The fluctuations of $\tilde{Q}$ fall into two general classes:[9,10,13] (i) massive, with the mass $\propto 1/\tau$; and (ii) massless, those on which the action depends only very weakly. The fluctuations along the massive modes can be integrated out in the Gaussian approximation and lead to insignificant renormalization of various parameters in the action. The massless, or Goldstone, modes describe diffusive motion of the electrons. The fluctuations of the $\tilde{Q}$ matrix along these massless modes are not small and should be treated carefully. The Goldstone modes can be parametrized by the $\tilde{Q}$ matrices satisfying a certain nonlinear constraint.[9,10,13]

To identify the relevant Goldstone modes consider the first term in Eq. (32). The saddle point given by Eqs. (30) and (41) satisfies

$$\tilde{Q}^2=\begin{pmatrix}\mathbb{1}^R_{\epsilon} & 0\\ 0 & \mathbb{1}^A_{\epsilon}\end{pmatrix}, \tag{45}$$

and the first term in Eq. (32) vanishes. The fluctuations of $\tilde{Q}$ which do not satisfy Eq. (45) are massive. The massless modes are generated by rotations of the saddle point and can be parametrized as[9,10,13]

$$\tilde{Q}=\mathcal{T}^{-1}\Lambda\mathcal{T}. \tag{46}$$

The parametrization of the rotation matrices $\mathcal{T}$ must ensure the convergence of the functional integration over the matrices given by Eq. (46). Below we only assume that such a parametrization exists, whereas the concrete form of $\mathcal{T}$ is not important for what follows.

One way of parametrizing the rotations is to write $\mathcal{T}=\exp\{W/2\}$, where, without loss of generality, $W\Lambda=-\Lambda W$. Expanding Eq. (32) to the second order in W and neglecting for a moment the term arising due to e-e interac-

tions, it is easy to establish that in the diffusive regime the relevant fluctuations must satisfy the condition

$$W_{\epsilon,\epsilon'} \neq 0 \quad \text{only if } |\epsilon|,|\epsilon'| < 1/\tau. \tag{47}$$

That is, all effective degrees of freedom are concentrated in the narrow energy strip of the width $1/\tau \ll \epsilon_F$ near the Fermi energy. Therefore, the matrices $\mathcal{T}$ differ from unity only in the narrow region of energies defined by Eq. (47). For this reason the gauge transformation $\mathcal{U}_{t,t'}(r) = \exp\{-ik_\alpha(t,r)\gamma_\alpha\}\delta(t-t')$ in Eq. (31) cannot be incorporated into a redefinition of $\mathcal{T}$ and should be carried out explicitly. Indeed, being diagonal in time indices, the matrix $\mathcal{U}_{t,t'}$ spreads over the entire energy space and, thus, cannot be reduced to a disturbance which is close to the Fermi shell. Physically, this describes the fact the low-wave-number scalar potential $\Phi(q,t)$ shifts the entire electronic band and not only the energy strip given by Eq. (47). It is essential to follow the variations of the electron spectrum all the way down to the bottom of the band to respect the charge neutrality imposed by the Coulomb interactions. As we shall see below, once the phase factors in Eq. (31) have been taken into account, the residual interactions may be regarded as being short range without loss of generality.

Substituting Eq. (46) into Eq. (32), and retaining only the universal (τ-dependent) terms in the expansion of the logarithm, for the $\tilde{Q}$ action we obtain

$$iS[\tilde{Q},\Phi] = i\nu \,\mathrm{Tr}\{(\Phi - i\omega K)^T \sigma_1 (\Phi + i\omega K)\} - \frac{\pi\nu}{4} \times [D\,\mathrm{Tr}\{\partial_r \tilde{Q}\}^2 + 4i\,\mathrm{Tr}\{(\epsilon + (\phi_\alpha + i\omega k_\alpha)\gamma^\alpha)\tilde{Q}\}], \tag{48}$$

where we have introduced the long derivative

$$\partial_r \tilde{Q} \equiv \nabla \tilde{Q} + i[\nabla k_\alpha \gamma^\alpha, \tilde{Q}]. \tag{49}$$

A few comments are in order regarding Eq. (48). First, it is restricted to $\tilde{Q}$, which satisfies Eq. (45). The last two terms, containing $\tilde{Q}$, conventionally originate from $\Sigma_p \mathbf{v}_F G^R \mathbf{v}_F G^A$ and $\Sigma_p G^{R(A)}$ combinations in the expansion of the logarithm. On the other hand, the first term on the right-hand side of Eq. (48) originates from $\Sigma_p G^R G^R$ and $\Sigma_p G^A G^A$ combinations. These terms should be retained since, as was mentioned above, the matrix $\phi_\alpha(\epsilon - \epsilon')\gamma^\alpha$ is not restricted to the $1/\tau$ shell near the Fermi energy. To derive this term we employed the fact that for any physical fermionic distribution function

$$F_{\epsilon \to \pm\infty} \to \pm 1. \tag{50}$$

Finally, the terms like $\Sigma_p \mathbf{v}_F G^R \mathbf{v}_F G^R$, although nonvanishing, cancel against the diamagnetic term.

Employing the explicit form of the long derivative, Eq. (49), and the relation between the K and Φ fields [Eq. (41)], for the the partition function one finally obtains

$$\langle Z \rangle = \int \mathcal{D}\Phi \exp\{i\,\mathrm{Tr}[\Phi^T V^{-1} \Phi]\} \times \int \mathcal{D}\tilde{Q} \exp\{iS_0[\tilde{Q}] + iS_1[\tilde{Q},\nabla K] + iS_2[\tilde{Q},\nabla K]\}, \tag{51}$$

where S_l, $l=0$, 1, and 2, contain ∇K in the lth power and are given by

$$iS_0[\tilde{Q}] = -\frac{\pi\nu}{4}[D\,\mathrm{Tr}\{\nabla\tilde{Q}\}^2 + 4i\,\mathrm{Tr}\{\epsilon\tilde{Q}\}], \tag{52a}$$

$$iS_1[\tilde{Q},\nabla K] = -i\pi\nu[D\,\mathrm{Tr}\{\nabla k_\alpha \gamma^\alpha \tilde{Q}\nabla\tilde{Q}\} + \mathrm{Tr}\{(\phi_\alpha + i\omega k_\alpha)\gamma^\alpha \tilde{Q}\}], \tag{52b}$$

$$iS_2[\tilde{Q},\nabla K] = \frac{\pi\nu D}{2}[\mathrm{Tr}\{\nabla k_\alpha \gamma^\alpha \tilde{Q}\nabla k_\beta \gamma^\beta \tilde{Q}\} - \mathrm{Tr}\{\nabla k_\alpha \gamma^\alpha \Lambda \nabla k_\beta \gamma^\beta \Lambda\}]. \tag{52c}$$

The effective interaction matrix V is nothing but the screened interaction in the random-phase approximation (RPA)

$$V(q,\omega) = [V_0^{-1}(q)\sigma_1 + P_0(q,\omega)]^{-1}, \tag{53}$$

where $P_0(q,\omega)$ is the bare density-density correlator. It has a typical form of a bosonic correlator in Keldysh space,

$$P_0(q,\omega) = \begin{pmatrix} 0 & P_0^A(q,\omega) \\ P_0^R(q,\omega) & P_0^K(q,\omega) \end{pmatrix}, \tag{54}$$

with

$$P_0^{R(A)}(q,\omega) = \nu \frac{Dq^2}{Dq^2 \mp i\omega}, \tag{55a}$$

$$P_0^K(q,\omega) = B_\omega[P_0^R(q,\omega) - P_0^A(q,\omega)]. \tag{55b}$$

To derive Eqs. (51)–(55), we had to add and subtract the term $\mathrm{Tr}\{\nabla k_\alpha \gamma^\alpha \Lambda \nabla k_\beta \gamma^\beta \Lambda\}$, and employed the equation

$$\int_{-\infty}^{+\infty} d\epsilon\, \mathrm{Tr}\{\gamma^\alpha \gamma^\beta - \gamma^\alpha \Lambda_{\epsilon_+} \gamma^\beta \Lambda_{\epsilon_-}\} = 4\omega(\Pi_\omega^{-1})^{\alpha\beta}. \tag{56}$$

Here $\epsilon_\pm = \epsilon \pm \omega/2$, and matrices Λ and Π are defined by the Eqs. (27) and (44) correspondingly. Equation (56) is based on the following relations between bosonic and fermionic distribution functions:

$$\int_{-\infty}^{+\infty} d\epsilon\,(F_{\epsilon_+} - F_{\epsilon_-}) = 2\omega, \tag{57}$$

$$\int_{-\infty}^{+\infty} d\epsilon\,(1 - F_{\epsilon_+} F_{\epsilon_-}) = 2\omega B_\omega. \tag{58}$$

The last equation is obviously satisfied in the thermal equilibrium. For a nonequilibrium situation it should be considered as a definition of B_ω.

Equations (51)–(55) together with Eq. (41) constitute an effective nonlinear σ model for interacting disordered electron gas. The model consists of two interacting fields: the

matrix field $\tilde{Q}$, obeying the nonlinear constraint [Eq. (45)], and the bosonic vector field Φ (or equivalently K). As will be apparent later, the $\tilde{Q}$ field describes fluctuations of the quasiparticle distribution function, whereas Φ (or K) represents propagation of electromagnetic fields through the media. The following sections are devoted to an analysis of this model and calculations of various physical quantities on the basis of the model.

IV. APPLICATIONS OF THE FORMALISM

A. Single-particle Green function

In this section we shall show how the developed formalism can be used for a calculation of the average single-particle Green function at coinciding spatial points. This quantity is defined as

$$\hat{\mathcal{G}}_{i,j}(t-t')=-i\langle\langle \psi_i(r,t)\overline{\psi}_j(r,t')\rangle\rangle, \tag{59}$$

where $\langle\langle\cdots\rangle\rangle$ denotes both the quantum and the disorder averaging. It is convenient to apply the Keldysh rotation [Eq. (14)], and define

$$\mathcal{G}(t-t')=L\sigma_3\hat{\mathcal{G}}(t-t')L^{\dagger}. \tag{60}$$

Such a Green function arises, e.g., in calculations of the tunneling DOS, or shot noise power. To evaluate it one may introduce a source term in Eq. (4), directly coupled to a bilinear combination of the fermion operators. Following the same algebra as above one finds that the source field enters into the logarithm in Eq. (21b). Differentiating finally with respect to the source and putting it to zero, for the Green function one obtains

$$\mathcal{G}(t-t')=\int \mathcal{D}\Phi\, e^{i\,\mathrm{Tr}\{\Phi^T V_0^{-1}\sigma_1\Phi\}}\int \mathcal{D}Q\, e^{iS[Q,\Phi]} \times\left[G_0^{-1}+\frac{i}{2\tau}Q+\phi_\alpha\gamma^\alpha\right]^{-1}\Bigg|_{\mathbf{r},\mathbf{r};t,t'}. \tag{61}$$

We shall evaluate the integral over the Q matrix by the saddle-point approximation, neglecting both the massive and the massless fluctuations around the stationary point. Then, according to Eq. (22), the pre-exponential factor is simply $-i\pi\nu Q_{t,t'}$. At the saddle point Q is given by Eq. (30). Transforming the action $S[Q,\Phi]$ as it was done in Sec. III B, one obtains

$$\mathcal{G}=-i\pi\nu\int \mathcal{D}\Phi\, e^{i\,\mathrm{Tr}\{\Phi^T V^{-1}\Phi\}}e^{ik_\alpha(t)\gamma^\alpha}\Lambda_{t-t'}e^{-ik_\alpha(t')\gamma^\alpha}. \tag{62}$$

Since K is the linear functional of Φ, given by Eq. (41), the remaining functional integral is Gaussian. Employing Eqs. (41) and (53)–(55), for the correlator of the K fields (averaged over fluctuations of Φ) one obtains

$$\langle k_\alpha(q,\omega)k_\beta(-q,-\omega)\rangle_\Phi=\frac{i}{2}\mathcal{V}_{\alpha\beta}(q,\omega), \tag{63a}$$

$$\mathcal{V}(q,\omega)=\mathcal{D}(q,\omega)\Pi_\omega^{-1}V(q,\omega)(\Pi_{-\omega}^{-1})^T\mathcal{D}^T(-q,-\omega). \tag{63b}$$

The Keldysh matrix $\mathcal{V}$ has the familiar structure of a bosonic propagator:

$$\mathcal{V}(q,\omega)=\begin{pmatrix}\mathcal{V}^K(q,\omega) & \mathcal{V}^R(q,\omega)\\ \mathcal{V}^A(q,\omega) & 0\end{pmatrix}, \tag{64}$$

with

$$\mathcal{V}^{R(A)}(q,\omega)=\frac{-1}{(Dq^2\mp i\omega)^2}\left(\frac{1}{V_0}+\frac{\nu Dq^2}{Dq^2\mp i\omega}\right)^{-1}, \tag{65a}$$

$$\mathcal{V}^K(q,\omega)=B_\omega[\mathcal{V}^R(q,\omega)-\mathcal{V}^A(q,\omega)]. \tag{65b}$$

One may recognize that this propagator precisely corresponds to the screened Coulomb interaction line dressed by two diffusons at the vertices. Thus, the role of the K field is to take into account automatically both the RPA-screened interactions and its vertex renormalization by diffusons.

To calculate the functional integral [Eq. (62)], we write the phase factors as

$$e^{\pm ik_\alpha\gamma^\alpha}=\tfrac{1}{2}(e^{\pm i(k_1+k_2)}+e^{\pm i(k_1-k_2)})\gamma^1 +\tfrac{1}{2}(e^{\pm i(k_1+k_2)}-e^{\pm i(k_1-k_2)})\gamma^2, \tag{66}$$

and perform the Gaussian integration according to Eq. (63). The result may be conveniently expressed in the form

$$\mathcal{G}(t)=-i\pi\nu\sum_{\alpha\beta=1}^{2}(\gamma^\alpha\Lambda_t\gamma^\beta)\mathcal{B}_{\alpha\beta}(t), \tag{67}$$

where the fictitious propagator $\mathcal{B}$ has the standard bosonic structure [as, e.g., Eq. (64)] with

$$\mathcal{B}^{R(A)}(t)=\tfrac{1}{2}e^{(i/2)[\mathcal{V}^K(t)-\mathcal{V}^K(0)]}(e^{(i/2)\mathcal{V}^{R(A)}(t)}-e^{-(i/2)\mathcal{V}^{R(A)}(t)}), \tag{68a}$$

$$\mathcal{B}^K(t)=\tfrac{1}{2}e^{(i/2)[\mathcal{V}^K(t)-\mathcal{V}^K(0)]}(e^{(i/2)[\mathcal{V}^R(t)-\mathcal{V}^A(t)]} +e^{-(i/2)[\mathcal{V}^R(t)-\mathcal{V}^A(t)]}). \tag{68b}$$

The $\langle KK^T\rangle$ propagator, $\mathcal{V}$, defined by Eqs. (64) and (65), is taken at coinciding spatial points

$$\mathcal{V}(t)=\int\frac{d\omega}{2\pi}e^{-i\omega t}\sum_q\mathcal{V}(q,\omega). \tag{69}$$

The electron Green function must satisfy several important requirements: the tunneling DOS, $\nu(\epsilon)$, which is defined as

$$\nu(\epsilon)=\frac{i}{2\pi}[\mathcal{G}^R(\epsilon)-\mathcal{G}^A(\epsilon)], \tag{70}$$

must be a positive definite quantity. In addition, in thermal equilibrium the R, A, and K components of the bosonic and fermionic propagators are related by the fluctuation-dissipation theorem (FDT). Below, we demonstrate that our approximation [Eqs. (67) and (68)] for the Green function

FIG. 2. Lowest order interaction correction to the single-particle Green function. The wavy line here denotes the RPA-screened Coulomb interaction. The impurity-dressed single-particle Green functions are depicted by solid lines, and the double dashed lines represent diffusons.

satisfies these requirements. For this purpose it is convenient to rewrite identically Eq. (67) in the form

$$\mathcal{G}^{>(<)}(t)=-i\pi\nu\Lambda_t^{>(<)}\mathcal{B}^{>(<)}(t), \tag{71}$$

where

$$\mathcal{B}^R(t)-\mathcal{B}^A(t)=\mathcal{B}^>(t)-\mathcal{B}^<(t), \tag{72a}$$

$$\mathcal{B}^K(t)=\mathcal{B}^>(t)+\mathcal{B}^<(t). \tag{72b}$$

The $>$ and $<$ components of the fermionic Green functions are related to R, A, and K in the same manner. From Eqs. (72) and Eqs. (68), we obtain

$$\mathcal{B}^{>(<)}(\omega)=\frac{1}{2}e^{-(i/2)\mathcal{V}^K(t=0)}\int dt\, e^{i\omega t}\exp\left\{\frac{i}{2}\int\frac{d\omega'}{2\pi}e^{-i\omega' t}\times\sum_q\left[\mathcal{V}^R(q,\omega')-\mathcal{V}^A(q,\omega')\right]\left(\coth\frac{\omega'}{2T}\pm 1\right)\right\}. \tag{73}$$

According to the FDT the equilibrium bosonic and fermionic Green functions in the frequency representation satisfy the relations

$$\mathcal{B}^>(\omega)=\exp\{\omega/T\}\mathcal{B}^<(\omega), \tag{74a}$$

$$\mathcal{G}^>(\epsilon)=-\exp\{\epsilon/T\}\mathcal{G}^<(\epsilon). \tag{74b}$$

It is not difficult to see that if any pair of bosonic Green functions $B^>(t)$ and $B^<(t)$ satisfies Eq. (74a) then for any analytic function $f(z)$ the pair $f^>(t)\equiv f(B^>(t))$ and $f^<(t)\equiv f[B^<(t)]$ also satisfies it. Indeed,

$$f^{>(<)}(\omega)=\int dt\, e^{i\omega t}f\left(\int\frac{d\omega'}{2\pi}B^{>(<)}(\omega')e^{-i\omega' t}\right). \tag{75}$$

Expanding f on the right-hand side of this equation in the Taylor series and performing the t integration, we see that in each order of the expansion $f^>(\omega)=\exp(\omega/T)f^<(\omega)$. One can also check that if $\mathcal{G}^{>(<)}$ and $\mathcal{B}^{>(<)}$ satisfy the FDT [Eq. (74)], then so do the functions $\tilde{\mathcal{G}}^{>(<)}$ defined as

$$\tilde{\mathcal{G}}^{>(<)}(t)=\mathcal{G}^{>(<)}(t)\mathcal{B}^{>(<)}(t). \tag{76}$$

Noting that the arguments in the exponential in Eq. (73) obviously satisfy the FDT [Eq. (74a)], we conclude that $\mathcal{B}^{>(<)}(\omega)$ [Eq. (73)] and the approximate Green function, [Eq. (67)] satisfies it as well.

To establish the positive definiteness of the tunneling density of states [Eq. (70)], we first show that $\mathcal{B}^{>(<)}(\omega)$ in Eq. (73) is positive definite. Indeed, $\exp[-i\mathcal{V}^K(t=0)]$ is real, as can be seen from Eqs. (65). It is also not difficult to see that each Fourier component of the argument of the exponential in Eq. (73) is positive definite. All the coefficients in the Taylor series of the exponential are positive and, since the Fourier transform of a product is given by the convolution of *positively defined* Fourier transforms, we conclude that the left-hand side of Eq. (73) is positive definite. We next use Eqs. (72a) and (74a) to rewrite the tunneling density of states as

$$\nu(\epsilon)=\frac{i}{2\pi}\mathcal{G}^>(\epsilon)(1+e^{-\epsilon/T}), \tag{77}$$

where $\mathcal{G}^>(\epsilon)$ is given by Eq. (71). Since $\Lambda_\epsilon^>\geqslant 0$ we immediately see from Eq. (71) that the tunneling density of states is positive.

In equilibrium, it is convenient to write the DOS through the Keldysh Green function employing the FDT:

$$\nu(\epsilon)=\frac{\nu}{\tanh\epsilon/(2T)}\int dt\, e^{i\epsilon t}F_t\,\mathcal{B}^K(t), \tag{78}$$

As was proven above, Eqs. (77) and (78) are equivalent. One can then express $\mathcal{B}^K(t)$ through $\mathcal{B}^{>(<)}(t)$, where the latter are conveniently rewritten as

$$\mathcal{B}^{>(<)}(t)=\frac{1}{2}\exp\left\{\int\frac{d\omega}{2\pi}\left(\coth\frac{\omega}{2T}(1-\cos\omega t)\pm i\sin\omega t\right)\mathrm{Im}\sum_q\mathcal{V}^R(q,\omega)\right\}. \tag{79}$$

Expanding this expression to the first order in the interaction, $\mathcal{V}$, and substituting into Eq. (78), one recovers the Altshuler-Aronov result for the zero-bias anomaly.[28] This perturbative result corresponds to the diagram drawn in Fig. 2.

We shall restrict ourselves to the analysis of the nonperturbative result [Eqs. (78) and (79)] only at $T=0$. Noting that for $T=0$, $F_t=(i\pi t)^{-1}$, one obtains

$$\nu(\epsilon)=\frac{\nu}{\pi}\int dt\,\frac{\sin|\epsilon|t}{t}\exp\left\{\int_0^\infty\frac{d\omega}{\pi}\mathrm{Im}\sum_q\mathcal{V}^R(\omega)(1-\cos\omega t)\right\}\cos\left\{\int_0^\infty\frac{d\omega}{\pi}\mathrm{Im}\sum_q\mathcal{V}^R(\omega)\sin\omega t\right\}. \tag{80}$$

In the two-dimensional case Eq. (65a) with $V_0(q)=2\pi e^2/q$ leads to

$$\int_0^\infty \frac{d\omega}{\pi}\,\mathrm{Im}\sum_q \mathcal{V}^R(q,\omega)\begin{pmatrix}1-\cos\omega t\\ \sin\omega t\end{pmatrix} = -\frac{1}{8\pi^2 g}\begin{cases}\ln t/\tau \ln t\tau\omega_0^2 + 2\gamma \ln t\omega_0\\ \pi \ln t\omega_0,\end{cases} \tag{81}$$

where $g=\nu D$ is the conductance, $\omega_0=D\kappa^2$, $\kappa=2\pi e^2\nu$ is the inverse screening radius; and $\gamma=0.577\ldots$ is the Euler constant. Since we have neglected the fluctuations of $\tilde{Q}$, we have missed corrections of order $g^{-1}\ln t/\tau$ (in $d=2$); therefore, Eq. (80) can only be trusted for ϵ not too small, such that $(8\pi^2 g)^{-1}\ln(\epsilon\tau)^{-1}\ll 1$. However $\ln^2 t/\tau$ terms have been accounted for correctly by our procedure. If, in addition, $g^{-1}\ln\omega_0\tau\ll 1$, the time integral in Eq. (80) may be performed by the stationary point method, resulting in

$$\nu(\epsilon)=\nu\exp\left\{-\frac{1}{8\pi^2 g}\ln(|\epsilon|\tau)^{-1}\ln\tau\omega_0^2/|\epsilon|\right\}. \tag{82}$$

Theoretically, however, $g^{-1}\ln\omega_0\tau$ need not be small. In that case the stationary point integration should be somewhat modified and terms $\propto \ln t\omega_0$ should be retained.

We have achieved a nonperturbative resummation of anomalously divergent, $\propto \ln^2\epsilon\tau$, terms for a single-particle Green function. The nonperturbative expression for the DOS essentially arises from the gauge noninvariance of the single-particle Green function. The calculations above are in essence the ''Debye-Waller'' factor[33] due to almost pure gauge fluctuations of electric fields, cf. Eq. (62). Gauge-invariant characteristics (such as, e.g., conductivity) do not carry phase factors, and therefore are not affected by the interactions on this level of accuracy (fluctuations of $\tilde{Q}$ should be retained). In perturbation theory this fact is reflected by the cancellation of diagrams without diffusons (apart from those which renormalize vertices).[28] In our formulation such terms never appear since the phase factors cancel along any closed loop diagram.

The gauge physics of anomalous corrections to the DOS was first realized by Finkel'stein,[13,33] who obtained a nonperturbative result similar to ours. Nazarov,[34] and later Levitov and Shytov,[26] obtained the same result (in imaginary time) by semiclassical reasoning. Kopietz[35] recently reinstated it, stressing the role of phase fluctuations. The analogous expression for the zero-dimensional case has also been known for some time.[36–38] We believe that we provide its first consistent derivation using the σ model. Unlike the previous approaches, the Keldysh technique provides the answer directly in real time and at finite temperature. This enables us to circumvent the tedious analytical continuation procedure.

B. Shot noise

In this subsection we shall use the results obtained in Sec. IV A to calculate the power spectrum of current noise through a tunneling contact between a clean metal and a dirty metal film. The power spectrum of current noise is given by

$$S(\omega)=\int dt\, e^{i\omega t}\langle\langle \hat{I}(t)\hat{I}(0)+\hat{I}(0)\hat{I}(t)\rangle\rangle. \tag{83}$$

Here the current operator in the tunneling approximation is given by $\hat{I}(t)=iTa^\dagger(t,r_0)b(t,r_0)-iT^*b^\dagger(t,r_0)a(t,r_0)$, and $b(t,r_0)$ and $a(t,r_0)$ are electron annihilation operators at the position of the contact in the dirty film and in the clean metal, respectively. Below, all the fermion operators and the Green functions are taken at the point of the tunneling contact r_0, and we omit the position argument for brevity. Using the expression for the current operator, we can rewrite Eq. (83) as

$$S(\omega)=|T|^2\int dt\, e^{i\omega t}[\mathcal{G}_a^<(-t)\mathcal{G}_b^>(t)+\mathcal{G}_a^>(t)\mathcal{G}_b^<(-t) +\mathcal{G}_a^<(t)\mathcal{G}_b^>(-t)+\mathcal{G}_a^>(-t)\mathcal{G}_b^<(t)], \tag{84}$$

where $\mathcal{G}_a$ and $\mathcal{G}_b$ are Green functions for the clean metal and for the dirty film, respectively. We assume that the voltage V is applied across the contact. To the lowest order in the tunneling matrix element, the Green functions under these conditions are equilibrium, except that the chemical potentials in the two metals differ by eV. Therefore in the lowest order in the tunneling amplitude we can express the power spectrum of current noise through the equilibrium Green functions. Expressing them through DOS with the aid of the FDT and utilizing the fact that for the clean metal DOS, ν_a, is independent of energy ϵ, one obtains

$$S(\omega)=2\pi\nu_a|T|^2\int d\epsilon\,\nu_b(\epsilon)\{n(\epsilon)[2-n(\epsilon+\omega-eV) -n(\epsilon-\omega-eV)]+[1-n(\epsilon)][n(\epsilon+\omega-eV) +n(\epsilon-\omega-eV)]\}, \tag{85}$$

where $n(\epsilon)=[1+\exp(\epsilon/T)]^{-1}$ is the Fermi function.

Setting $V=0$ in Eq. (85), we obtain the power spectrum of the equilibrium current noise in the contact $S_0(\omega)$. The excess noise is given by the difference $\delta S(\omega)=S(\omega)-S_0(\omega)$. The noise power is a symmetric function of frequency, and at zero temperature reduces to

$$\delta S(\omega>0)=2\pi\nu_a|T|^2\left[\int_{-|\omega-eV|}^{\omega+eV} d\epsilon\,\nu_b(\epsilon)-\int_{-\omega}^{\omega} d\epsilon\,\nu_b(\epsilon)\right]. \tag{86}$$

At zero frequency the shot noise is proportional to the total current. This is natural, since in the lowest order in the tunneling amplitude the electrons pass through the contact extremely rarely and, therefore, can be viewed as noninteracting. The role of interactions reduces to modification of the density of the available states. The cusp present at zero temperature in the noise power spectrum for noninteracting electrons at $\omega=0$ and $\omega=eV$ is washed out because DOS vanishes at $\epsilon=0$.

C. External fields and auxiliary sources

In some sense our previous manipulations leading to Eqs. (51) and (52) were no more than a complicated representation of unity. This is so since, according to the basic idea of the Keldysh technique, the partition function Z is identically

equal to unity. To make the entire construction meaningful one should introduce auxiliary source fields, which enable one to compute various observables. We shall do this in parallel with introducing external classical fields. Since we shall mostly discuss the conductivity, we will use the vector potential $\mathbf{A}(r,t)$ as an example.[39] Other fields (e.g., the scalar potential) may be introduced in a similar way. We introduce a doublet in the rotated Keldysh frame

$$\mathbf{A}(r,t)=\begin{pmatrix}\mathbf{a}_1(r,t)\\ \mathbf{a}_2(r,t)\end{pmatrix}, \tag{87}$$

which is related by the usual transformation [cf. Eq. (19)] with the two fields $\hat{a}_\alpha(r,t)$ residing on the two branches of the time contour. The vector potentials enters the fermionic Hamiltonian through the long spatial derivatives, $\nabla_r \to \nabla_r + i\hat{\mathbf{a}}_\alpha\hat{\gamma}^\alpha$. The classical external vector potential is the same on the two branches of the contour, and hence it is described by the symmetric component $\mathbf{a}_1(r,t)$ only, whereas $\mathbf{a}_2=0$. In this case the generating function is still equal to unity:

$$Z[\mathbf{a}_1,\mathbf{a}_2=0]=1. \tag{88}$$

To obtain a nontrivial generating function, one has to introduce a quantum component of the source field, $\mathbf{a}_2(r,t)$. This component does not have a classical meaning, and thus has to be nullified at the end of the calculations. Its presence however is essential for generating observables. One can easily check that the current density defined as

$$\mathbf{j}=\frac{e}{2mi}\left\langle \frac{1}{2}\sum_{i=1,2}\{\overline{\psi}_i(\nabla+i\mathbf{a}_1)\psi_i-(\nabla-i\mathbf{a}_1)\overline{\psi}_i\psi_i\}\right\rangle_\psi \tag{89}$$

is given by[40]

$$\mathbf{j}(r,t)=-\frac{e}{2i}\frac{\delta Z[\mathbf{A}]}{\delta\mathbf{a}_2(r,t)}\bigg|_{\mathbf{a}_2=0}. \tag{90}$$

We restrict ourselves to the case of longitudinal vector potentials only. Taking into account the fact that the external electric field is given by $-i\omega\mathbf{a}_1(q,\omega)/e$, one obtains that the linear response conductivity is given by the retarded component of the current-current correlator

$$\sigma(q,\omega)=\frac{e^2}{i\omega}\Sigma^{2,1}(q,\omega), \tag{91}$$

where

$$\Sigma^{\alpha,\beta}(q,\omega)=\frac{1}{2i}\frac{\delta^2 Z[\mathbf{A}]}{\delta a_\beta(\mathbf{q},\omega)\delta a_\alpha(-\mathbf{q},-\omega)}\bigg|_{\mathbf{A}=0}. \tag{92}$$

Here we have omitted the vector indices of a using its longitudinal character. In general, any response function is given by the (2,1) component of the appropriate bosonic correlator. The structure of the theory guarantees that this is a retarded function [cf., e.g., Eq. (54)].

In the presence of an external vector potential, $\mathbf{A}$, the trial saddle point [Eq. (30)], is shifted. Noting that $\mathbf{A}$ enters the action always in the combination $\nabla K+\mathbf{A}$, one finds that the condition for the optimal K is given by Eq. (37), with the substitution $Dq^2K \to Dq^2K+iD(qA)$.[41] Solving this equation in a manner that was done in Sec. III A, one obtains that Eq. (41) should be modified as

$$\mathcal{D}^{-1}(q,\omega)K(q,\omega)=\Pi_\omega^{-1}\Phi(q,\omega)-iD\sigma_1(qA(q,\omega)), \tag{93}$$

where bosonic propagators $\mathcal{D}(q,\omega)$ and $\Pi_\omega=-i\omega\mathcal{D}(q=0,\omega)$ are defined by Eqs. (42) and (44). In solving Eq. (37) with the external field we still assumed that the fermionic distribution function is the equilibrium one. This is a legitimate procedure in linear response. The generalization to the nonequilibrium case is discussed in Sec. VI. After disregarding the massive modes and expanding the logarithm, one obtains Eq. (48), with the long derivative modified as

$$\partial_r\tilde{Q}\equiv\nabla_r\tilde{Q}+i[(\nabla k_\alpha+\mathbf{a}_\alpha)\gamma^\alpha,\tilde{Q}], \tag{94}$$

and K given by Eq. (93). Since $\gamma^1=1$, any static external field $\mathbf{a}_1(r)$ appears to be decoupled from $\tilde{Q}$. This reflects the fact that diffusons are not coupled to a static magnetic field. On the other hand, even space- and time-independent quantum component, $\mathbf{a}_2$, is coupled to $\tilde{Q}$. A little algebra shows that

$$\nabla K+\mathbf{A}=-i\mathbf{q}\mathcal{D}\Pi^{-1}[\Phi+(qA)\omega/q^2], \tag{95a}$$

$$\Phi+i\omega K=Dq^2\mathcal{D}\sigma_1[\Phi+(qA)\omega/q^2]. \tag{95b}$$

With these expressions and the long derivative given by Eq. (94), one can rearrange Eq. (48) to obtain the average generating function in the form

$$\langle Z[A]\rangle=\int\mathcal{D}\Phi\,\exp(i\,\mathrm{Tr}\{\Phi^T V_0^{-1}\sigma_1\Phi+[\Phi+(qA)\omega/q^2]^T P_0[\Phi+(qA)\omega/q^2]\})\int\mathcal{D}\tilde{Q}\,\exp\left\{\sum_{l=0}^{2} iS_l[\tilde{Q},\nabla K+\mathbf{A}]\right\}, \tag{96}$$

where the action S_l, $l=0$, 1, and 2 is given by Eqs. (52) and the bare polarization operator $P_0(q,\omega)$ is given by Eq. (55). By virtue of Eqs. (95) the entire action is expressible through the combination $\Phi+(qA)\omega/q^2$, which is proportional to the (gauge invariant) electric field $\nabla\Phi+\partial_t\mathbf{A}$. This fact immediately guarantees that the continuity equation is satisfied to all orders in the perturbation theory. Indeed, one could introduce the external scalar potential φ which enters the action always as $\Phi+\varphi$ (apart from the bare interaction term, V_0). Then, due to the fact that $Z=Z[\varphi+(qA)\omega/q^2]$, the charge

density, $\rho=(2i)^{-1}\delta Z/\delta\varphi_2$, and current density, $\mathbf{j}=-(2i)^{-1}\delta Z/\delta\mathbf{a}_2$, has to be related by

$$(\nabla\mathbf{j})+\partial_t\rho=0. \tag{97}$$

The corresponding variation with respect to the classical components φ_1 and $\mathbf{a}_1$ guarantees continuity at each branch of the contour separately even in the presence of nonzero auxiliary quantum fields. As a result of continuity, an exact relation between current-current and density-density correlators holds:

$$P(q,\omega)=\frac{q^2}{\omega^2}\Sigma(q,\omega). \tag{98}$$

At the saddle point, $\tilde{Q}=\Lambda$, one has $S_l[\Lambda,\nabla K+\mathbf{A}]=0$. Thus, neglecting the fluctuations of $\tilde{Q}$, for the RPA generating function one obtains

$$\langle Z_{RPA}[A]\rangle=\exp\left\{iA^T\frac{\omega^2}{q^2}P_0A\right\}\times\int\mathcal{D}\Phi\exp\left(i\,\mathrm{Tr}\left\{\Phi^TV^{-1}\Phi+2\Phi^T\frac{\omega}{q}P_0A\right\}\right). \tag{99}$$

Finally performing the Gaussian integration, one finds

$$\langle Z_{RPA}[A]\rangle=\exp\left(i\,\mathrm{Tr}\left\{A^T\frac{\omega^2}{q^2}P_{RPA}A\right\}\right), \tag{100}$$

where $P_{RPA}(q,\omega)$ is the RPA screened density-density (polarization) correlator, which is given by

$$P_{RPA}(q,\omega)=P_0-P_0[V_0^{-1}+P_0]^{-1}P_0=[P_0^{-1}+V_0]^{-1}, \tag{101}$$

and has the structure of a bosonic correlator:

$$P(q,\omega)=\begin{pmatrix}0 & P^A(q,\omega)\\ P^R(q,\omega) & P^K(q,\omega)\end{pmatrix}. \tag{102}$$

The ω^2/q^2 factor in Eq. (100) reflects the relation between the density-density and longitudinal components of the current-current correlators [Eq. (98)]. The fact that the (1,1) component of P (as well as of any other bosonic correlator) vanishes is a manifestation of the normalization condition [Eq. (88)]. Employing Eq. (91), for the conductivity in the RPA one obtains

$$\sigma_{RPA}(q,\omega)=e^2\nu D\frac{-i\omega}{Dq^2[1+\nu V_0(q)]-i\omega}, \tag{103}$$

One is usually interested in the *irreducible* part of the density-density (or current-current) correlators, which describes the linear response to the *total* or internal field and not to the external field as discussed above. The relation between the irreducible part P_{irr} and the total P is exactly the same as between the bare P_0, and P_{RPA} [Eq. (101)],

$$P_{irr}(q,\omega)=[P^{-1}(q,\omega)-V_0(q)]^{-1}. \tag{104}$$

Shifting the integration variable $\Phi\to\Phi-(qA)\omega/q^2$ in Eq. (96) and differentiating twice with respect to A, one obtains an exact relation for the polarization operator,

$$P(q,\omega)=V_0^{-1}+2iV_0^{-1}\langle\Phi(q,\omega)\Phi^T(-q,-\omega)\rangle V_0^{-1}, \tag{105}$$

where $\langle\Phi\Phi^T\rangle$ is an exact propagator (averaged with respect to the full action [Eq. (96)]). Employing Eq. (104), one finds

$$P_{irr}=\frac{i}{2}(\langle\Phi\Phi^T\rangle)^{-1}-V_0^{-1}. \tag{106}$$

If one is interested in the response to a uniform external field, $q=0$, the expressions may be further simplified. Noting that for the Coulomb interaction $V_0^{-1}(q=0)=0$ and employing Eq. (98) and relation between Φ and K [Eq. (41)], one obtains

$$\Sigma_{irr}(q=0,\omega)=\frac{i}{2}[\langle\nabla K(\omega)\nabla K^T(-\omega)\rangle]^{-1}. \tag{107}$$

Unlike in the calculations of the single-particle Green function, only ∇K and never K itself appears in calculations of gauge invariant quantities. This allows one to consider a universal limit of strong interactions $V_0^{-1}(q)\to0$. In this limit it is convenient to change the integration variable from Φ to ∇K (although formally it is a vector, it has only a longitudinal component and hence a number of variables is conserved). In the new variables the Gaussian weight is given by $i\,\mathrm{Tr}\{\nabla K^Tq^{-2}\mathcal{V}^{-1}\nabla K\}$, where $\mathcal{V}$ is defined by Eqs. (64) and (65). In the universal limit one has

$$\mathcal{V}^{-1}(q,\omega)\to-\nu Dq^2\mathcal{D}^{-1}(q,\omega), \tag{108}$$

where the diffusion propagator $\mathcal{D}$ is defined by Eqs. (42) and (43). Finally, for the action in terms of ∇K one obtains

$$\langle Z\rangle=\int\mathcal{D}\nabla K\,e^{-i\nu D\,\mathrm{Tr}\{\nabla K^T\mathcal{D}^{-1}\nabla K\}}\times\int\mathcal{D}\tilde{Q}\exp\left(\sum_{l=0}^{2}iS_l[\tilde{Q},\nabla K]\right). \tag{109}$$

Equations (107) and (109) constitute a complete framework for calculations of gauge-invariant response functions. Neglecting $\tilde{Q}$ fluctuations, one finds $\Sigma_{irr}(q=0,\omega)=\nu Di\omega\Pi_\omega^{-1}$, which leads to the Drude conductivity $\sigma=e^2\nu D$. Fluctuations of $\tilde{Q}$ and ∇K lead to weak-localization and interaction corrections. Note that unlike in the case of the DOS (Sec. IV A), fluctuations of ∇K alone, with $\tilde{Q}=\Lambda$, do not lead to any corrections to linear response. This is a direct consequence of gauge invariance of linear response functions. Only combined fluctuations of ∇K and $\tilde{Q}$, discussed in Sec. V, renormalize the Drude conductivity.

V. FLUCTUATION EFFECTS

A. $\tilde{Q}$-matrix parametrization

As discussed in Sec. III B the massless fluctuations of the $\tilde{Q}$ matrix can be parametrized as

$$\tilde{Q} = \exp\{-W/2\}\Lambda \exp\{W/2\}, \tag{110}$$

where

$$W\Lambda + \Lambda W = 0. \tag{111}$$

Employing Eq. (24), one obtains that the general form of W, which satisfies condition (111), is

$$W = \begin{pmatrix} 1 & F \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & w \\ \overline{w} & 0 \end{pmatrix} \begin{pmatrix} 1 & F \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} F\overline{w} & -w + F\overline{w}F \\ -\overline{w} & -\overline{w}F \end{pmatrix} \tag{112}$$

where $\overline{w}$ and w are arbitrary Hermitian matrices in the time space. Below we shall thus understand the functional integration over $\tilde{Q}$ as integration over Hermitian $\overline{w}$ and w. Notice that $\tilde{Q}$ itself (as well as the Green function G) appears to be non-Hermitian. This means that the ''contour'' of integration in the $\tilde{Q}$ space is deformed from being pure Hermitian to pass through the non-Hermitian saddle point. As will be apparent later, the physical meaning of w is a deviation of the fermionic distribution function, F, from its stationary value. At the same time, $\overline{w}$ has no classical interpretation. To a large extent it plays the role of the quantum counterpart of w, which appears only as the internal line in the diagrams.

One may expand now the action [Eqs. (52)] in powers of $\overline{w}$ and w. The expansion of the noninteracting action, $iS_0[\tilde{Q}]$ starts from the second order, which has a familiar diffusive structure

$$iS_0^{(2)}[W] = \frac{\pi\nu}{2} \overline{w}_{\epsilon_1\epsilon_2}[-D\nabla^2 + i(\epsilon_1 - \epsilon_2)] w_{\epsilon_2\epsilon_1}. \tag{113}$$

As a result the bare propagator of the $\tilde{Q}$-matrix fluctuations is given by

$$\langle w_{\epsilon_2\epsilon_1}(q)\overline{w}_{\epsilon_3\epsilon_4}(-q)\rangle_W = -\frac{2}{\pi\nu}\frac{\delta_{\epsilon_1\epsilon_3}\delta_{\epsilon_2\epsilon_4}}{Dq^2 + i(\epsilon_1 - \epsilon_2)} = -\frac{2\delta_{\epsilon_1\epsilon_3}\delta_{\epsilon_2\epsilon_4}}{\pi\nu} D^A(q, \epsilon_1 - \epsilon_2). \tag{114}$$

The higher-order terms describe nonlinear interactions of diffusons with the vertices having the structure of Hikami boxes. One can easily work out this expansion in the Keldysh formalism. We shall not do it here, since our main focus is on the interaction effects. Substituting $\delta\tilde{Q}^{(1)} = [\Lambda, W]/2$ into $iS_1[\tilde{Q}, \nabla K]$, one obtains, in the first order in W,

$$iS_1^{(1)}[W, \nabla K] = -\frac{i\pi\nu}{2}\mathrm{Tr}\{[D\nabla^2 k_\alpha(\Lambda\gamma^\alpha\Lambda - \gamma^\alpha) + (\phi_\alpha + i\omega k_\alpha)(\gamma^\alpha\Lambda - \Lambda\gamma^\alpha)]W\}. \tag{115}$$

In equilibrium $iS_1^{(1)}[W, \nabla K] = 0$. Indeed, the right-hand side of Eq. (115) coincides with Eq. (37), which was used to determine the K functional. In equilibrium we were able to solve Eq. (37) by an appropriate choice of K. This was precisely the motivation behind looking for the saddle point for each realization of the Hubbard-Stratonovich field: to cancel terms linear in W. Since we could not find the exact saddle point, such terms do appear, however, only in the second order in ∇K. For $iS_2[\tilde{Q}, \nabla K]$, part of the action, one obtains

$$\begin{aligned} iS_2^{(1)}[W, \nabla K] &= \frac{\pi\nu D}{2}\nabla k_\alpha(\epsilon_1 - \epsilon_2)\mathrm{Tr}\{[\gamma^\alpha\Lambda_{\epsilon_2}\gamma^\beta\Lambda_{\epsilon_3} - \Lambda_{\epsilon_1}\gamma^\alpha\Lambda_{\epsilon_2}\gamma^\beta]W\}\nabla k_\beta(\epsilon_2 - \epsilon_3) \\ &= \pi\nu D\,\mathrm{Tr}\{\nabla K^T(\epsilon_1 - \epsilon_2)[M^w_{\epsilon_2} w_{\epsilon_3\epsilon_1} + M^{\overline{w}}_{\epsilon_1\epsilon_2\epsilon_3}\overline{w}_{\epsilon_3\epsilon_1}]\nabla K(\epsilon_2 - \epsilon_3)\}, \end{aligned} \tag{116}$$

where we have introduced two vertex matrices in the bosonic Keldysh space:

$$M^w_{\epsilon_2} = \begin{pmatrix} 0 & 1 \\ -1 & -2F_{\epsilon_2} \end{pmatrix},$$

$$M^{\overline{w}}_{\epsilon_1\epsilon_2\epsilon_3} = \begin{pmatrix} 2F_{\epsilon_2} - F_{\epsilon_1} - F_{\epsilon_3} & 1 + F_{\epsilon_1}F_{\epsilon_3} - 2F_{\epsilon_2}F_{\epsilon_3} \\ -1 - F_{\epsilon_1}F_{\epsilon_3} + 2F_{\epsilon_2}F_{\epsilon_1} & F_{\epsilon_1} + F_{\epsilon_3} - 2F_{\epsilon_1}F_{\epsilon_2}F_{\epsilon_3} \end{pmatrix}. \tag{117}$$

$(M^w_{\epsilon_2})^{1,1} = 0$ is a manifestation of the normalization condition $Z = 1$. Indeed, this matrix element connects only the classical components of W and K fields, which alone cannot change the normalization. Being averaged over fluctuations of ∇K with the action Eq. (109), $iS_2^{(1)}[W, \nabla K]$ gives

$$\langle iS_2^{(1)}[W, \nabla K]\rangle_{\nabla K} = 2\pi i\overline{w}_{\epsilon\epsilon}\{(F_{\epsilon+\omega} - F_\epsilon)\mathcal{D}^K(\omega) - (1 - F_{\epsilon+\omega}F_\epsilon)[\mathcal{D}^R(\omega) - \mathcal{D}^A(\omega)]\}. \tag{118}$$

There is no term proportional to the classic component, w. In equilibrium the right-hand side of Eq. (118) is obviously zero. Out of equilibrium, it is this term which is responsible for the standard collision integral; see Sec. VI. As we shall see in Sec. V B, fluctuations described by $iS_2^{(1)}[W, \nabla K]$ are responsible for the Altshuler-Aronov corrections to conductivity. For completeness we write also the second order expansion of $iS_1[W, \nabla K]$

$$\begin{aligned} iS_1^{(2)}[W, \nabla K] = i\nu D\{&\nabla k_1(\epsilon_1 - \epsilon_2) \\ &\times(\nabla w_{\epsilon_2\epsilon_3}\overline{w}_{\epsilon_3\epsilon_1} - \overline{w}_{\epsilon_2\epsilon_3}\nabla w_{\epsilon_3\epsilon_1}) \\ &+ \nabla k_2(\epsilon_1 - \epsilon_2)[-F_{\epsilon_2}\nabla\overline{w}_{\epsilon_2\epsilon_3}w_{\epsilon_3\epsilon_1} \\ &- w_{\epsilon_2\epsilon_3}\nabla\overline{w}_{\epsilon_3\epsilon_1}F_{\epsilon_1} + B_{\epsilon_1 - \epsilon_2}\nabla \\ &\times(w_{\epsilon_2\epsilon_3}\overline{w}_{\epsilon_3\epsilon_1} - \overline{w}_{\epsilon_2\epsilon_3}w_{\epsilon_3\epsilon_1})]\}. \end{aligned} \tag{119}$$

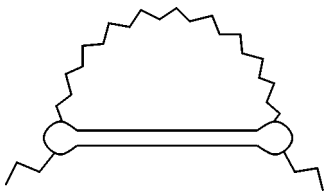

FIG. 3. Lowest order self-energy diagram for $\langle \nabla K \nabla K^T \rangle$ propagator. The zigzag lines represent the bare $\langle \nabla K \nabla K^T \rangle$ propagators, the parallel solid lines denote the $\langle WW \rangle$ propagator, and the open circles with two zigzag and two straight lines emanating from them represent the $\nabla K^T W \nabla K$ vertices.

B. Altshuler-Aronov corrections

Restricting oneself to the lowest nonvanishing terms in the expansion over W [Eqs. (113) and (116)], one obtains a Gaussian theory with respect to the W fluctuations. After integrating out these fluctuations employing Eq. (114), one ends up with the action for the ∇K field only:

$$\begin{aligned} iS[\nabla K] = & -i\nu D \,\mathrm{Tr}\{\nabla K^T_{-\omega}(r)\mathcal{D}^{-1}(r-r',\omega)\nabla K_{\omega}(r')\} \\ & -2\pi\nu D^2 \,\mathrm{Tr}\{\nabla K^T_{\epsilon_1-\epsilon_2}(r) \overline{M^{w}_{\epsilon_1\epsilon_2\epsilon_3}} \nabla K_{\epsilon_2-\epsilon_3}(r)\} \\ & \times D^A(r-r',\epsilon_3-\epsilon_1)\mathrm{Tr}\{\nabla K^T_{\epsilon_3-\epsilon_4}(r') \\ & \times M^{w}_{\epsilon_4}\nabla K_{\epsilon_4-\epsilon_1}(r')\}. \end{aligned} \tag{120}$$

This way the $(\nabla K)^4$ effective vertex is generated. Perturbatively the $(\nabla K)^4$ interaction term may be treated by pairing two fields, say $\nabla K^T \langle \nabla K \nabla K^T \rangle \nabla K$. This results in a renormalization of the bare correlator, $\mathcal{D}^{-1}$. Only pairing of ∇K fields in different spatial points leads to nonvanishing corrections; see Fig. 3. There are four different ways one can pair $\langle \nabla k_\alpha(r) \nabla k_\beta(r') \rangle$. Taking into account all these four possibilities and integrating over an intermediate energy one obtains a correction, e.g., for the retarded component of the $(\langle \nabla K \nabla K^T \rangle)^{-1}$ correlator:

$$\begin{aligned} [\delta\mathcal{D}^{-1}(q,\omega)]^R = & -\frac{4}{d\nu}\sum_{q',\omega'} \{D^R(q+q',2\omega+\omega') \\ & \times [\mathcal{D}(q',\omega+\omega')]^R - D^R(q+q',\omega+\omega') \\ & \times [\mathcal{D}(q',\omega')]^R\}\omega' B_{\omega'}, \end{aligned} \tag{121}$$

where B_ω is defined by [Eq. (58)]; d is the dimensionality. Obviously, the correction preserves the retarded character of the corresponding component. In equilibrium, the correction to the Keldysh component obeys the fluctuation-dissipation relation

$$[\delta\mathcal{D}^{-1}]^K = \coth\frac{\omega}{2T}\{[\delta\mathcal{D}^{-1}]^R - [\delta\mathcal{D}^{-1}]^A\}. \tag{122}$$

Employing Eqs. (91) and (107), for the correction to the $q=0$ conductivity one obtains

$$\begin{aligned} \delta\sigma(\omega) = & -i\frac{4e^2 D}{d\omega}\sum_{q',\omega'} D^R(q',\omega+\omega')[\mathcal{D}(q',\omega')]^R \\ & \times\{(\omega'-\omega)B_{\omega'-\omega} - \omega' B_{\omega'}\}. \end{aligned} \tag{123}$$

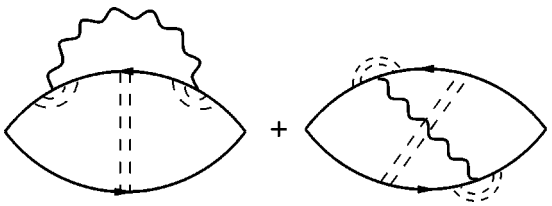

FIG. 4. Lowest order diagrams for the interaction correction to conductivity. Their sum is equivalent to the diagram in Fig. 3 in the present formalism.

In the low frequency limit this reduces to the familiar expression[28]

$$\delta\sigma = i\frac{2\sigma_d}{\pi d\nu}\int_{-\infty}^{\infty} d\omega \frac{\partial}{\partial\omega}\left(\omega\coth\frac{\omega}{2T}\right)\sum_q \frac{1}{(Dq^2 - i\omega)^2}, \tag{124}$$

where $\sigma_d = e^2\nu D$.

Note that this expression is given by the sum of diagrams drawn in Fig. 4. The other diagrams which are presented in Fig. 5 add up to zero. They represent the purely phase correction to the single-particle Green function and therefore do not enter the expression for the conductivity. In the present formalism these diagrams do not appear at all.

In two dimensions expression (124) leads to the logarithmically divergent *negative* correction to the conductivity (or conductance):[7]

$$\frac{\delta\sigma_2}{\sigma_2} = \frac{\delta g}{g} = \frac{e^2}{2\pi^2 g}\ln T\,\tau^{el}, \tag{125}$$

where the elastic mean free time τ^{el} enters as an upper cutoff in the integral over frequency.

To handle this divergence one may try to set up a self-consistent mean-field treatment of the $(\nabla K)^4$ nonlinearity. To this end let us put $\langle \nabla K \nabla K^T \rangle$ propagator on Fig. 3 to be a dressed one, $\tilde{\mathcal{D}}$, where $\tilde{\mathcal{D}}^{-1} = \mathcal{D}^{-1} + \delta\mathcal{D}^{-1}$. Then Eq. (121) may be rewritten as a closed nonlinear equation for, e.g., the retarded component of the propagator, $[\tilde{\mathcal{D}}(q,\omega)]^R$;

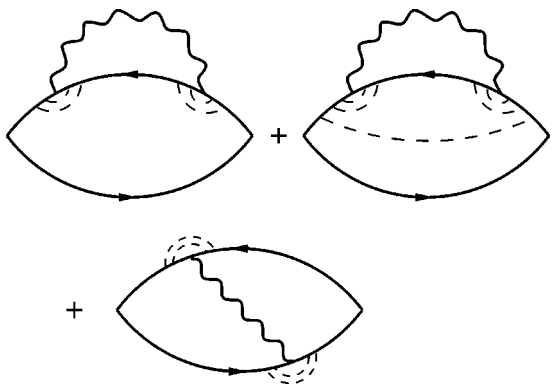

FIG. 5. Diagrams for the interaction corrections to conductivity which add up to zero. These diagrams never appear in our formalism.

$$\left[Dq^2 - i\omega - \frac{4}{d\nu} \sum_{q',\omega'} D^R(q+q',\omega+\omega')[\tilde{\mathcal{D}}(q',\omega')]^R[(\omega'-\omega)B_{\omega'-\omega} - \omega' B_{\omega'}] \right][\tilde{\mathcal{D}}(q,\omega)]^R = 1. \quad (126)$$

The frequency-dependent conductivity is then given by

$$\sigma(\omega) = e^2 \nu D \frac{([\tilde{\mathcal{D}}(q=0,\omega)]^R)^{-1}}{-i\omega}. \quad (127)$$

One may easily check that in the one-loop approximation there are no other corrections to the conductivity. Indeed, possible corrections like $\langle (iS_1^{(2)}[W,\nabla K])^2 \rangle_W = 0$ and $\langle iS_2^{(2)}[W,\nabla K] \rangle_W = 0$ vanish, since they include the energy integration of purely retarded or advanced functions. Being expanded to higher orders in W, these terms yield weak-localization corrections.

VI. KINETIC EQUATION

The aim of this section is to demonstrate how the kinetic equation for the distribution function F appears naturally in the framework of the Keldysh formulation. The kinetic equation is nothing but the saddle-point equation for the effective action on the $\tilde{Q}$ matrix.[42] In the case of interacting electrons it is obtained by integrating out the K (or equivalently Φ) degrees of freedom. Consider the partition function [Eq. (21a)], with the action $S(\tilde{Q},\Phi)$ given by Eq. (48). Let us perform the Φ integration first. As a result for the average partition function we obtain

$$\langle Z \rangle = \int \mathcal{D}\tilde{Q}\, e^{iS_{eff}[\tilde{Q}]}, \quad (128a)$$

$$iS_{eff}[\tilde{Q}] = \ln \int \mathcal{D}\Phi\, e^{i\,\mathrm{Tr}\{\Phi^T V_0^{-1} \sigma_1 \Phi\} + iS[\tilde{Q},\Phi]}. \quad (128b)$$

Since the action $S(\tilde{Q},\Phi)$ [Eq. (48)] is quadratic in Φ (given the linear relation between K and Φ) the integration in the last expression can be carried out explicitly. We find it more convenient, however, to proceed with expression (128b). To obtain a nontrivial kinetic theory one may assume the presence of classical external fields, like, e.g., scalar or vector potentials. These fields can be introduced in the action Eq. (48) the way it was done in Sec. IV C.

We shall now look for the saddle-point equation for $\tilde{Q}$,

$$\frac{\delta S_{eff}[\tilde{Q}]}{\delta \tilde{Q}} = 0, \quad (129)$$

obtained under the condition $\tilde{Q}^2 = 1$. Let us reiterate the logic of the entire procedure. After averaging over disorder and introducing the Q matrix, we found that the low-energy degrees of freedom are described by the Q matrices given by Eq. (31) with $\tilde{Q}^2 = 1$. We then restrict ourselves to this massless manifold and look for a realization of $\tilde{Q}$ which extremizes the effective action. The latter is obtained by integrating out the photon fields originating from e-e interactions. Without any external fields (and/or nontrivial boundary conditions) such an extremal $\tilde{Q}$ is simply given by Λ [Eq. (27)], with the equilibrium F function [Eq. (26)]. If external fields (and/or nontrivial boundary conditions) are present, the stationary $\tilde{Q}$ may deviate from Λ, still being on the massless manifold $\tilde{Q}^2 = 1$. The stationary point is to be found by solving Eq. (129), which turns out to be precisely the kinetic equation with the collision integral term.

Before proceeding along these lines, let us comment on the relation between the phase K, introduced in Sec. III A and the Hubbard-Stratonovich field Φ. The procedure of Sec. III A was based on the property of the equilibrium distribution described by Eq. (40). We need to generalize it for nonequilibrium situations. To this end we note that the equation for the quantum component $k_2(r,t)$ [Eq. (38)] does not contain a distribution function and remains valid for a nonequilibrium case. The equation for the classical component $k_1(r,t)$ [Eq. (39)] cannot be satisfied identically out of equilibrium. Thus the choice of $k_1(r,t)$ allows for a certain arbitrariness. However, as we shall see below, this arbitrariness *does not* affect the form of the kinetic (saddle-point) equation. It would manifest itself in a calculation of fluctuation corrections (cf. Sec. V) to the nonequilibrium saddle point result. We shall not attempt this task here. For our purposes it is sufficient to keep the definition of $K(r,t)$ given by Eq. (41) [or Eq. (93) if external fields are present]. The equilibrium bosonic distribution function used in the definition of the Keldysh component of the propagator $\mathcal{D}(q,\omega)$ [Eq. (43b)] does not show up in the kinetic equation.

Employing Eq. (128b), we rewrite the saddle-point equation (129) as

$$\left\langle \frac{\delta S[\tilde{Q},\Phi]}{\delta \tilde{Q}} \right\rangle_\Phi = 0, \quad (130)$$

where

$$\langle \cdots \rangle_\Phi = \frac{\int \mathcal{D}\Phi\, e^{i\,\mathrm{Tr}\{\Phi^T V_0^{-1} \sigma_1 \Phi\} + iS[\underline{\tilde{Q}},\Phi]} \cdots}{\int \mathcal{D}\Phi\, e^{i\,\mathrm{Tr}\{\Phi^T V_0^{-1} \sigma_1 \Phi\} + iS[\underline{\tilde{Q}},\Phi]}}. \quad (131)$$

Here $\underline{\tilde{Q}}$ is a self-consistent saddle-point solution of Eq. (130). Performing variation of the action $S(\tilde{Q},\Phi)$ given by Eq. (48) under the condition $\tilde{Q}^2 = 1$, one obtains

$$\langle D\partial_r(\underline{\tilde{Q}}\partial_r\underline{\tilde{Q}}) + i[(\epsilon + (\phi_\alpha + i\omega k_\alpha)\gamma^\alpha), \underline{\tilde{Q}}] \rangle_\Phi = 0, \quad (132)$$

where $\underline{\tilde{Q}}^2 = 1$. This equation is analogous to the kinetic equation in the semiclassical theory of disordered superconductors.[43,44] We have derived it here for the case of a normal interacting metal.

We shall seek the solution of Eq. (132) in the classical form, e.g., obeying the condition $\underline{\tilde{Q}}_{21}=0$. A nonzero quantum component at the saddle point would violate causality. Provided $\underline{\tilde{Q}}_{21}=0$ and $\underline{\tilde{Q}}^2=1$ are satisfied, the saddle-point solution assumes the form

$$\underline{\tilde{Q}}_{\epsilon,\epsilon'}=\begin{pmatrix} \mathbb{1}^R_\epsilon \delta(\epsilon-\epsilon') & 2F_{\epsilon,\epsilon'}(r) \\ 0 & -\mathbb{1}^A_\epsilon \delta(\epsilon-\epsilon') \end{pmatrix}, \tag{133}$$

where $F_{\epsilon,\epsilon'}(r)$ is a nonstationary distribution function. Assuming that the saddle point has the form given by Eq. (133), one can easily check that the exponent in the Φ averaging, [Eq. (131)] does not contain linear terms in Φ (or ∇K). Indeed, the terms proportional to ∇k_1 vanish identically, which is a manifestation of the normalization condition $Z=1$. From another hand, terms proportional to ∇k_2 are reduced to the full gradient (the fact that there is no ambiguity in the choice of k_2 is important here), and thus also vanish upon the spatial integration. As a result the terms linear in Φ (or ∇K) in the saddle-point equation (132) do not survive the Φ integration. Therefore, Eq. (132) may be reduced to

$$D\nabla_r(\underline{\tilde{Q}}\nabla_r\underline{\tilde{Q}})+i[\epsilon,\underline{\tilde{Q}}]=D\langle[\nabla k_\alpha \gamma^\alpha \underline{\tilde{Q}}\nabla k_\beta \gamma^\beta,\underline{\tilde{Q}}]\rangle_\Phi . \tag{134}$$

The right-hand side of this equation contains the collision integral term along with the collisionless renormalization of the kinetic part. To evaluate it, one needs to know the propagator $\langle\nabla k_\alpha(r,t)\nabla k_\beta(r',t')\rangle_\Phi$ at $r=r'$, averaged over the nonequilibrium action [Eq. (131)]. To follow the same notations as for the equilibrium case, we shall denote this propagator as

$$\langle\nabla k_\alpha(r,t)\nabla k_\beta(r',t')\rangle_\Phi=-\frac{i}{2\nu D}\mathcal{D}^{\alpha\beta}_{t,t'}(r,r'). \tag{135}$$

The form of the saddle point $\underline{\tilde{Q}}$ given by Eq. (133) guarantees that $\mathcal{D}_{t,t'}$ has the standard retarded/advanced structure of a Keldysh propagator. Employing Eq. (28), one finds that the only nonzero matrix component of Eq. (134) is its Keldysh (1,2) component. The corresponding equation for the distribution function $F_{t,t'}(r)$ takes the form

$$D\nabla_r^2 F_{t,t'}-(\partial_t+\partial_{t'})F_{t,t'}=\frac{i}{\nu}\left[F_{t,t'}\left(\mathcal{D}^K_{t,t'}-\frac{1}{2}[\mathcal{D}^K_{t,t}+\mathcal{D}^K_{t',t'}]\right)+(\mathcal{D}^R_{t,t_1}-\mathcal{D}^A_{t_1,t'})(\delta_{t,t_1}\delta_{t_1,t'}-F_{t,t_1}F_{t_1,t'})\right]. \tag{136}$$

Here all F functions and propagators $\mathcal{D}$ are to be taken at the same spatial point; integration over t_1 is assumed in the last term on the right-hand side. Note that the left-hand side of this equation is a linear diffusion operator acting on $F_{t,t'}(r)$. The subsequent calculations are significantly simplified by passing to the Wigner representation,

$$F_\epsilon(r,\tau)=\int\int dt\,dt' F_{t,t'}(r)e^{i\epsilon(t-t')}\delta\left(\tau-\frac{t+t'}{2}\right). \tag{137}$$

Furthermore we shall assume that $F_\epsilon(r,\tau)$ is a slow function of τ on the scale $1/T$ (or any other inverse characteristic scale of energy, ϵ). With this assumption Eq. (136) may be rewritten as

$$\begin{aligned} &D\nabla_r^2 F_\epsilon(\tau)-\partial_\tau F_\epsilon(\tau)-\partial_\tau\mathcal{R}_\epsilon(\tau)\partial_\epsilon F_\epsilon(\tau)+\partial_\epsilon\mathcal{R}_\epsilon(\tau)\partial_\tau F_\epsilon(\tau) \\ &\quad=\frac{i}{\nu}\sum_\omega\{\mathcal{D}^K_\omega(\tau)[F_{\epsilon-\omega}(\tau)-F_\epsilon(\tau)]+[\mathcal{D}^R_\omega(\tau)-\mathcal{D}^A_\omega(\tau)][1-F_{\epsilon-\omega}(\tau)F_\epsilon(\tau)]\}, \end{aligned} \tag{138}$$

where

$$\mathcal{R}_\epsilon(r,\tau)=\frac{1}{2\nu}\sum_\omega[\mathcal{D}^R_\omega(r,r,\tau)+\mathcal{D}^A_\omega(r,r,\tau)]F_{\epsilon-\omega}(r,\tau). \tag{139}$$

The right-hand side of Eq. (138) represents the collision integral, cf. Eq. (118). If the equilibrium relation [Eq. (43b)] between Keldysh and retarded and advanced components of $\mathcal{D}$ holds, then the equilibrium distribution function [Eq. (26)] nullifies the collision integral. Therefore, Eq. (138) is satisfied in the thermal equilibrium. This, in fact, provides justification for our previous use of Λ with the equilibrium F function as the saddle point. Indeed, without interactions (and hence without collision integral) any stationary function F_ϵ satisfies the saddle-point equation. It is the relaxation processes due to e-e interactions that render the equilibrium solution unique. The terms which contain real part of the self-energy, $\mathcal{R}_\epsilon(r,\tau)$, lead to a collisionless renormalization of the kinetic part; see Sec. VI B.

To proceed further we need an explicit form of the nonequilibrium propagator $\mathcal{D}_\omega(r,r,\tau)$. We shall evaluate it in the universal limit of strong interactions, $V_0^{-1}\to 0$. Substituting the saddle-point $\underline{\tilde{Q}}$ given by Eq. (133) into the action $S[\underline{\tilde{Q}},\Phi]$ [Eq. (48)], and performing the Gaussian integration, one finds [cf. Eq. (56)]

$$\mathcal{D}^{\alpha\beta}_\omega(r,r',\tau)=\left[-D\nabla_r^2\sigma_1^{\alpha\beta}+\delta_{r,r'}\frac{i\pi}{2}\sum_\epsilon \mathrm{Tr}\{\gamma^\alpha\underline{\tilde{Q}}_{\epsilon+\omega/2}(r,\tau)\gamma^\beta\underline{\tilde{Q}}_{\epsilon-\omega/2}(r,\tau)-\gamma^\alpha\gamma^\beta\}\right]^{-1}. \tag{140}$$

The term with ∇_r^2 originates from the term $|\Phi+i\omega K|^2\sigma_1$ in Eq. (48). (It is easy to check that the ambiguity in the choice of k_1, mentioned above, disappears upon the calculation of this term by the symmetry reason.) The local in space term in Eq. (140) originates from $D\,\mathrm{Tr}(\partial_r\tilde{Q})^2$. Assuming that any distortion of the equilibrium distribution is limited to a vicinity of the Fermi energy, e.g., $F_{\epsilon\to\pm\infty}(r,\tau)\to\pm 1$, one finds

$$\mathcal{D}_\omega(r,r',\tau)=\begin{pmatrix} 0 & -D\nabla_r^2+i\omega\delta_{r,r'} \\ -D\nabla_r^2-i\omega\delta_{r,r'} & -2i\omega\delta_{r,r'}B_\omega(r,\tau)\end{pmatrix}^{-1}. \tag{141}$$

By definition, the nonequilibrium bosonic distribution function is given by [cf. Eq. (58)]

$$B_\omega(r,\tau)=\frac{1}{2\omega}\int_{-\infty}^{\infty}d\epsilon[1-F_{\epsilon+\omega/2}(r,\tau)F_{\epsilon-\omega/2}(r,\tau)]. \tag{142}$$

According to Eq. (141) the retarded and advanced components of $\mathcal{D}$ are not modified with respect to their equilibrium value [Eq. (43a)]. As a result, $\mathcal{D}_\omega^{R(A)}(r,r',\tau)=\mathcal{D}_\omega^{R(A)}(r-r')$ even in a nonequilibrium situation. Inverting the operator on the right-hand side of Eq. (141), for the Keldysh component at coinciding spatial points one finds

$$\mathcal{D}_\omega^K(r,r,\tau)=2i\omega\int d^dr'\Big[\mathcal{D}_\omega^R(r-r')B_\omega(r',\tau)\mathcal{D}_\omega^A(r'-r) + \frac{1}{2i}[\mathcal{D}_\omega^R(r-r')\partial_\tau B_\omega(r',\tau)\partial_\omega\mathcal{D}_\omega^A(r'-r)-\partial_\omega\mathcal{D}_\omega^R(r-r')\partial_\tau B_\omega(r',\tau)\mathcal{D}_\omega^A(r'-r)]\Big]. \tag{143}$$

From now on we shall retain only the first term in this expression, which is dominant due to the assumed slowness of the temporal variations of $F_\epsilon(\tau)$. If in addition $B_\omega(r,\tau)$ changes slowly on the spatial scale $L_\omega=\sqrt{D/\omega}$, where $\omega\sim T$, the expression for the Keldysh component acquires the quasiequilibrium form

$$\mathcal{D}_\omega^K(r,r,\tau)=B_\omega(r,\tau)\sum_q[\mathcal{D}_\omega^R(q)-\mathcal{D}_\omega^A(q)]. \tag{144}$$

One can calculate gradient corrections to this expression, which lead to a nonlocal collision integral. Usually such corrections may be safely neglected. Finally in this hydrodynamic regime, the kinetic equation takes the form

$$D\nabla_r^2F_\epsilon(\tau)-[1-\partial_\epsilon\mathcal{R}_\epsilon(\tau)]\partial_\tau F_\epsilon(\tau)-\partial_\tau\mathcal{R}_\epsilon(\tau)\partial_\epsilon F_\epsilon(\tau) = -\sum_\omega\left[\frac{2}{\nu}\mathrm{Im}\sum_q\mathcal{D}_\omega^R(q)\right]\{B_\omega(\tau)[F_{\epsilon-\omega}(\tau)-F_\epsilon(\tau)]+[1-F_{\epsilon-\omega}(\tau)F_\epsilon(\tau)]\}, \tag{145}$$

with

$$\mathcal{R}_\epsilon(r,\tau)=\frac{1}{\nu}\sum_{\omega,q}[\mathrm{Re}\mathcal{D}_\omega^R(q)]F_{\epsilon-\omega}(r,\tau) \tag{146}$$

and $\mathcal{D}_\omega^R(q)=(Dq^2-i\omega)^{-1}$.

A. Collision integral and relaxation time

Using the conventional fermion distribution function $n_\epsilon(r,\tau)=(1-F_\epsilon(r,\tau))/2$, one can rewrite the collision integral in the usual form with ''out'' and ''in'' relaxation terms. Indeed, employing Eqs. (57) and (58), one identically rewrites the right-hand side of Eq. (145) as

$$-\int\int_{-\infty}^{\infty}\frac{d\omega\,d\epsilon'}{\pi}\frac{4\mathrm{Im}\sum_q\mathcal{D}_\omega^R(q)}{\nu\omega}[n_\epsilon n_{\epsilon'-\omega}(1-n_{\epsilon'})(1-n_{\epsilon-\omega})-n_{\epsilon'}n_{\epsilon-\omega}(1-n_\epsilon)(1-n_{\epsilon'-\omega})]. \tag{147}$$

This is precisely the collision term derived by Altshuler[45] and Altshuler and Aronov[47] two decades ago. One can linearize this expression around the equilibrium distribution by the substitution

$$F_\epsilon(r,\tau)=F_\epsilon^{eq}-w_\epsilon(r,\tau)/2, \tag{148}$$

and keeping linear terms in $w_\epsilon(r,\tau)$. This way one derives the familiar results for the relaxation rates.[46,47,28] We shall

not repeat this procedure here. Instead we shall demonstrate how these quantities may be extracted directly from the effective action. To this end we note that the kinetic equation (138) may be written as $(2\pi\nu)^{-1}\delta iS_{eff}/\delta\overline{w}_{\epsilon}(r,\tau)=0$. (As usual, an observable is generated by differentiation with respect to a quantum component.) Thus the linearized version of the kinetic equation is just $-(\pi\nu)^{-1}\delta^2 iS_{eff}/\delta\overline{w}\delta w|_{\tilde{Q}=\Lambda}$. According to Eqs. (128b) and (131),

$$\frac{\delta^2 iS_{eff}}{\delta\overline{w}\delta w}=\left\langle\frac{\delta^2 iS[\tilde{Q},\Phi]}{\delta\overline{w}\delta w}\right\rangle_{\Phi}+\left\langle\frac{\delta iS[\tilde{Q},\Phi]}{\delta\overline{w}}\,\frac{\delta iS[\tilde{Q},\Phi]}{\delta w}\right\rangle_{\Phi}-\left\langle\frac{\delta iS[\tilde{Q},\Phi]}{\delta\overline{w}}\right\rangle_{\Phi}\left\langle\frac{\delta iS[\tilde{Q},\Phi]}{\delta w}\right\rangle_{\Phi},\tag{149}$$

where all the variational derivatives are calculated at $\tilde{Q}=\overline{\tilde{Q}}=\Lambda$. The last term in this expression vanishes identically, since Λ is obviously a solution of the kinetic equation (130). The first term originates from the expansion of $\langle iS_2[W,\nabla K]\rangle_{\nabla K}$ [Eq. (52c)] to the leading order in w and $\overline{w}$. After a little algebra one obtains (there are no terms with $\overline{w}\overline{w}$ in equilibrium)

$$\langle iS_2^{(2)}[W,\nabla K]\rangle_{\nabla K}=\frac{i\pi}{2}(\overline{w}_{\epsilon_+-\omega,\epsilon_--\omega}-\overline{w}_{\epsilon_+,\epsilon_-})\times[\mathcal{D}^R_{\omega}(B^{eq}_{\omega}+F^{eq}_{\epsilon_--\omega})-\mathcal{D}^A_{\omega}(B^{eq}_{\omega}+F^{eq}_{\epsilon_+-\omega})]w_{\epsilon_-,\epsilon_+},\tag{150}$$

where $\epsilon_{\pm}=\epsilon\pm\Omega/2$. Equivalently this expression can be obtained by variation of the $F_{\epsilon}(r,\tau)$ functions in Eq. (136). The terms with $\overline{w}_{\epsilon_+,\epsilon_-}$ and $\overline{w}_{\epsilon_+-\omega,\epsilon_--\omega}$ represent ''out'' and ''in'' relaxation processes, respectively. Their condensed diagrammatic representation is given in Figs. 6(a) and 6(b). The full set of corresponding original diagrams may be found, e.g., in Ref. 48. Restricting ourselves to the diagonal fluctuations $\Omega=0$ only, for the ''out'' relaxation rate we obtain

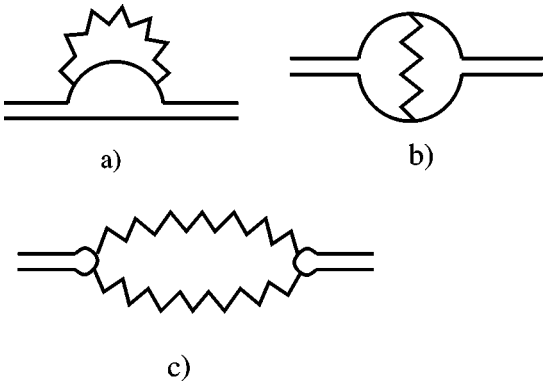

FIG. 6. Diagrammatic representation of the Gaussian part of the effective action, S_{eff} [Eq. (128b)]. (a) and (b) represent ''out'' and ''in'' relaxation terms correspondingly; they originate from the first term in Eq. (149), $\langle iS_2\rangle_{\nabla K}$. The nonlocal term (c) arises from the second term in Eq. (149), $\langle iS_2 iS_2\rangle_{\nabla K}$.

$$\frac{1}{\tau_{out}(\epsilon,T)}=\sum_{\omega q}\left[\frac{2}{\nu}\mathrm{Im}\mathcal{D}^R_{\omega}(q)\right]\left[\coth\frac{\omega}{2T}+\tanh\frac{\epsilon-\omega}{2T}\right].\tag{151}$$

At $T=0$ in two dimensions this leads to the familiar result[28]

$$\frac{1}{\tau^{d=2}_{out}(\epsilon)}=\frac{|\epsilon|}{4\pi g},\tag{152}$$

where $g=\nu D$. Expanding the right-hand side of Eq. (150) in a small Ω, one can also recover the collisionless terms on the left-hand side of Eq. (145).

Finally we concentrate on the second term on the right-hand side of Eq. (149). This term corresponds to the variation of $\mathcal{D}^K_{t,t'}$ in Eq. (136) over a deviation from its equilibrium value [or, equivalently, variation of $B_{\omega}(r,\tau)$ in Eq. (145)]. Its condensed diagrammatic representation is depicted on Fig. 6(c). As has already been mentioned above, this term is generally spatially nonlocal. We take only its local part here. Technically it originates from a connected part of $\frac{1}{2}\langle S_2^{(1)}[W,\nabla K]S_2^{(1)}[W,\nabla K]\rangle_{\nabla K}$, where $iS_2^{(1)}$ is given by Eq. (116). Performing averaging over ∇K and omitting cumbersome $\overline{w}\overline{w}$ terms, one obtains

$$\langle iS_2^{(1)}iS_2^{(1)}\rangle_{\nabla K}=-\frac{\pi^2}{2}\sum_{\epsilon\epsilon'\omega\Omega q}\overline{w}_{\epsilon_+,\epsilon_-}\mathrm{Tr}\{[M^w_{\epsilon'+\omega}+(M^w_{\epsilon'-\omega})^T]\mathcal{D}_{\omega+\Omega/2}(M^{\overline{w}}_{\epsilon_-,\epsilon-\omega,\epsilon_+})^T\mathcal{D}_{\omega-\Omega/2}\}w_{\epsilon'_-,\epsilon'_+}$$
$$=\pi^2\overline{w}_{\epsilon_+,\epsilon_-}[(2F^{eq}_{\epsilon-\omega}-F^{eq}_{\epsilon_-}-F^{eq}_{\epsilon_+})(F^{eq}_{\epsilon'+\omega}+F^{eq}_{\epsilon'-\omega})\mathcal{D}^R_{\omega-(\Omega/2)}\mathcal{D}^A_{\omega+(\Omega/2)}]w_{\epsilon'_-,\epsilon'_+},\tag{153}$$

For $\Omega=0$ this expression coincides with the variation of $B_{\omega}(r,\tau)$ over w in Eq. (145). Expanding to first order in Ω, one obtains the correction terms written in Eq. (143). Equations (149), (150), and (153) along with Eq. (113), complete calculations of $\delta^2 iS_{eff}/\delta\overline{w}\delta w$ on the mean-field level. Let us note for completeness that $\delta^2 iS_{eff}/\delta w\delta w=0$, which is a manifestation of the normalization condition. On the other hand, $\delta^2 iS_{eff}/\delta\overline{w}\delta\overline{w}\neq 0$ originates solely from the second term on the right-hand side of Eq. (149); cf. Fig. 6(c).

B. Collisionless terms

Finally we briefly discuss the physics of the collisionless terms. Collisionless terms originate from the real part of the

self-energy, $\mathcal{R}_\epsilon(r,t)$, and thus already appear in the first order in the bare interaction (unlike the collision integral, which arises only in the second order). For the screened Coulomb interaction, from Eq. (146) one obtains

$$\mathcal{R}_\epsilon(r,\tau)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}F_{\epsilon-\omega}(r,\tau)\int (d^dq)\frac{1}{\nu}\frac{Dq^2}{(Dq^2)^2+\omega^2}. \tag{154}$$

In two dimensions this leads to the logarithmic expression

$$\mathcal{R}_\epsilon(r,\tau)=-\frac{1}{4\pi g}\int_{-1/\tau^{el}}^{1/\tau^{el}}\frac{d\omega}{2\pi}\ln(\tau^{el}|\omega|)F_{\epsilon-\omega}(r,\tau), \tag{155}$$

where we have used the superscript for the elastic mean free time, τ^{el}, to avoid confusion with a physical time τ. If one linearizes the kinetic equation (145) around the equilibrium distribution, its left-hand side acquires the form

$$D\nabla_r^2F_\epsilon(\tau)-\left[1+\frac{\ln(\tau^{el}\max\{T,|\epsilon|\})}{4\pi^2g}\right]\partial_\tau F_\epsilon(\tau)+\frac{\partial_\epsilon F_\epsilon^{eq}}{4\pi g}\int\frac{d\epsilon'}{2\pi}\ln(\tau^{el}|\epsilon-\epsilon'|)\partial_\tau F_{\epsilon'}(\tau). \tag{156}$$

We focus first on the logarithmic renormalization of the coefficient in front of $\partial_\tau F$. This coefficient corresponds to the charge Z in Finkel'stein's terminology.[13] Equation (156) then describes the renormalization of Z (with the correct coefficient). We stress, however, that in our theory renormalization of Z takes place at the level of the saddle-point equation for the effective action, and not as a result of the fluctuation corrections. This distinguishes Z from the conductance g, whose renormalization occurs only at the level of the one-loop correction; see Sec. V B and Eq. (125). Physically renormalization of Z originates from the suppression of the single-particle DOS by the residual short-range interactions. This effect is due to the fact that single-particle Hartree-Fock energies are shifted by the interactions in a way to reduce the DOS near the Fermi energy. We consider it very satisfactory that such a purely mean-field effect is taken into account by the saddle-point equation and not by fluctuation corrections. The important point, however, is to keep the last term of the expression (156) as well. This is to say that only the ''out''-minus-''in'' combination has the physical meaning. Being considered together, as an integral operator acting on $F_{\epsilon'}$, these terms do not lead to divergent corrections.

VII. DISCUSSION

We have developed a field theory for interacting disordered metals using the Keldysh dynamic formulation. The advantages of this technique are twofold: (i) one avoids introduction of the replica trick, and (ii) one naturally gains the ability to deal with nonequilibrium situations. The latter manifests itself in the presence of the nontrivial object $F_{t,t'}(r)$, which plays the role of the fermionic distribution function. The saddle-point equation of the theory turns out to be the kinetic equation which determines this function. No such object is apparent in the replicated Matsubara formulation,[13,15] since by construction it is limited to the equilibrium case. Based on the analogy with spin glasses,[22] one may speculate that nontrivial solutions $F_{t,t'}(r)$ of the saddle-point equation are analogous to the replica symmetry-broken solutions of the saddle-point equation in the replica formulation. We mainly focused our attention on a careful analysis of the saddle equations of the theory. In particular we suggested the following two-step procedure.

(i) In the first step we account for the purely phase effects of the fluctuating electric fields on the single-particle Green function by an appropriately chosen gauge transformation. This enables us to get rid of temporal variations of the Green function which are not related to the quasiparticle dynamics. The remaining temporal fluctuations of the Green functions are associated with the particle dynamics, and can be described in terms of the quasiparticle distribution function $F_{t,t'}(r)$. This formulation ensures an explicit gauge invariance of the kinetic equation, and preserves the continuity relations at every stage. As a by-product of this procedure, we were able to obtain a nonperturbative expression for the DOS—the case where the phase effects give the main contribution. Such phase effects do not contribute to gauge-invariant observables, which are represented diagrammatically by closed loops. In the usual diagram technique this corresponds to a cancellation between certain diagrams (the diagrams containing double logarithms in two dimensions). By explicitly accounting for the phase effects we get rid of these diagrams, which significantly reduces the number of terms in each order of perturbative expansion.

(ii) After the phase effects have been taken into account, we obtain a theory formulated in terms of the $\tilde{Q}$-matrix field. The latter describes quantum fluctuations of the electron distribution function in the close vicinity of the Fermi energy. Restricting ourselves to the manifold of the massless fluctuations given by $\tilde{Q}^2=1$, we obtain the effective σ-model action $S_{eff}[\tilde{Q}]$, [Eq. (128b)]. Searching then for the extremum of this action, we arrive at the kinetic equation of the distribution function. After this two-step saddle-point procedure one should consider quantum fluctuation effects. The Altshuler-Aronov corrections[28] to the conductance, g, turn out to be a manifestation of the one-loop quantum fluctuations.

Although we have obtained renormalization of both parameters g and Z [Eqs. (125) and (156)] we deliberately avoided putting it in the framework of the renormalization group. The point is that after introducing the phase transformation and integrating out the photon fields, the effective action on $\tilde{Q}$ [Eq. (128b)] obtains a complicated form. We cannot prove that this entire form is reproducible after the fast mode elimination. A seemingly better possibility is to perform the renormalization of the action, which contains both $\tilde{Q}$ and ∇K fields [Eq. (48)]. In this case one has to specify how the relation between K and Φ fields changes in the process of renormalization. Since we believe that the introduction of the phase, K, is a vital element of the theory, the more complicated form of the action (compared to the one of Finkel'stein) is justifiable.

We hope that the present formulation will help to shed light on the nature of the low-temperature phase of low-dimensional disordered metals. A few aspects of this theory seem to us very suggestive in this respect. Certain parallels with the spin glasses theory may prove to be useful. Apart from the extremely complicated problems relating to the character of the low-temperature phase, the functional Keldysh formalism may be useful for a description of non-equilibrium effects in disordered metals and superconductors. The extension of this formalism to include the spin and Cooper channels will be a subject of our future work.

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