Smectic liquid crystals in random environments

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We study smectic liquid crystals in random environments, e.g., aerogel. A low-temperature analysis reveals that even arbitrarily weak *quenched* disorder (i.e., arbitrarily low aerogel density) destroys translational (smectic) order, in agreement with recent experimental results. A harmonic approximation to the *elastic* energy suggests that there may be no ''smectic Bragg glass'' phase in this system: even at zero temperature, it is riddled with dislocation loops induced by the quenched disorder. This result would imply the destruction of orientational (nematic) order as well, and that the thermodynamically sharp nematic–smectic-*A* transition is destroyed by disorder. We show, however, that the anharmonic elastic terms neglected in the above approximate treatment *are* important (i.e., are ''relevant'' in the renormalization group sense), and may, indeed, stabilize the smectic Bragg glass and the sharp phase transition into it. However, they do *not* alter our conclusion that translational (smectic) order is always destroyed. In contrast, we expect that *weak annealed* disorder should have no *qualitative* effects on the smectic order. [S0163-1829(99)09121-3]

I. INTRODUCTION

A. Motivation and background

The effect of quenched disorder on condensed matter systems is an important and challenging problem that continues to be actively investigated, because of its relevance to real systems, which always contain some amount of random inhomogeneity. Recently, much attention has focused on the random field XY model as a minimal model of a broad class of systems such as disordered Josephson junction arrays,[1] roughening of crystal surfaces growing on a disordered substrate,[2] and the pinning of Wigner crystals, vortex lattices in superconductors,[3] or charge density waves.[4] Another system that is considerably less well understood is the superfluid transition, e.g., He^4 in the random, ''fractal-like'' environment of aerogel.[5] In apparent contradiction to the Harris criterion,[6] the disorder modifies the critical properties of the transition (including the critical exponents).

The central question addressed in all of the above studies is: does a distinct low temperature ''glass'' phase exist in any of these disordered systems? And if so, what (if any) *static* property distinguishes it from the high temperature, conventional thermally disordered phase? The answer to the second question is clearly subtle: arguments dating back to Larkin[7] show that it is impossible to have long-ranged translational order in three dimensions in a randomly pinned elastic medium. Hence, the glass cannot be distinguished from the high temperature ''liquid'' phase by long-ranged translational order.

In the context of pinned vortex lattices in dirty type II superconductors, Fisher's[3] original argument for the existence a vortex glass phase is based on a two $(1+1)$-dimensional model in which the random pinning is relevant at low temperatures but becomes irrelevant at higher temperatures.[8] The low-temperature pinning relevant phase is then identified as the glass phase, and the temperature at which the pinning becomes irrelevant is the thermodynamically sharp glass transition temperature.

Unfortunately, this simple scenario cannot be directly carried over to three dimensions for vortex lattices (though, as we shall show later, it almost can for smectics in aerogel). In three-dimensional vortex lattices (and other pinned isotropic elastic media listed in the opening paragraph), the aforementioned Larkin argument shows that disorder is *always* relevant in $d=3$. Hence, its relevance cannot be used as a criterion for distinguishing the glass and liquid phases.

So what can? One appealing proposal is the ''Bragg glass''[9–11] picture, in which the glass, while being elastically disordered, is topologically ordered, and is therefore distinguished from the liquid by being free of unbound dislocation loops, which proliferate in the liquid. The glass transition is then identified as an ''unbinding'' of the dislocation loops.

This transition is then, qualitatively, very similar to the melting of the flux lattice in the absence of disorder, which can also be thought of as an unbinding of dislocation loops. The only difference is that in the glass problem, the flux lattice is translationally disordered *both* above and *below* the transition. The absence of defects below the transition, however, means that the low temperature solid phase still has a finite shear modulus, leading to glassy behavior.

Another related system that has been experimentally investigated is nematic liquid crystals in aerogel near and below the pure system's nematic–smectic-*A* (NA) transition temperature.[12] The mean-field theory of a bulk, pure smectic liquid crystal looks similar to that of an XY model, with a complex scalar order parameter ψ characterizing the smectic density wave. As was first noted by de Gennes[13,14] a more precise model must include the coupling of ψ to the nematic director fluctuations $\delta\mathbf{n}$, which takes the model out of the XY universality class.[14–16] Although the agreement between ex-

periments and theory is far from perfect, the *theoretical* consensus is that the scaling near the NA transition *should* cross over from that of a three-dimensional XY transition line (as in a neutral superfluid) to inverted XY-like behavior with anisotropic x-ray correlation length exponents.[16]

The goal of this paper is to investigate the effects of disorder on the liquid crystal phase diagram in the vicinity of and below the NA transition. Does the NA transition survive in the presence of even weak disorder? If so, what is the nature of the low-temperature phase? A condensed report of some of our results, the details and extensions of which are presented in this paper, has appeared in recent publications.[17,18]

Our interest in this problem was stimulated by recent experiments[12,19–22] that aim to answer these very questions. In these experiments, a liquid crystal that exhibits a bulk nematic–smectic-A transition is introduced into an aerogel. X-ray scattering measurements show that this system *never* develops true smectic long-ranged (or even quasi-long-ranged) order: the translational correlation length ξ^X remains finite at all temperatures. This length smoothly increases as temperature is lowered across the bulk NA transition temperature, monotonically and slowly rising to some finite asymptotic low-temperature limit, which does *not* appear to be associated with any natural length scale of the aerogel itself. Our theory predicts this x-ray correlation length as the scale at which the pinning disorder energy begins to dominate over the smectic elastic energy; consistent with these experimental observations, the smectic correlation length is *not* simply ''pore'' size of the aerogel.[12] This behavior is also in sharp contrast to the fast rise of the smectic correlation length to the nominal pore size at the bulk transition temperature that one would expect in more regularly porous materials, which can be understood as a bulklike sharp transition cutoff by the finite pore size. The specific heat in aerogel was also measured and was found to exhibit a broadened but very well defined peak, near but slightly shifted down from the bulk transition temperature $T_{\rm NA}$. These experimental observations suggest the destruction of the NA transition by the disorder imposed by the fractal aerogel environment.[23] However, in contrast, recent experiments,[62] which study the NA transition in even lower density aerosils, appear to show a true resolution limited heat capacity singularity, while displaying a *finite* x-ray smectic correlation length indicative of short-range translational order. These latter experimental findings appear to support the idea that the nematic–smectic-A transition in pure systems, when confined in a low density quenched random structure, is replaced by a new phase transition into a novel thermodynamically stable phase with a finite smectic correlation length in both the high- and low-temperature phases.

B. Summary, interpretation and consequences of the results

The main conclusion of our work is that, consistent with these and many other experiments,[12,19–22] the three-dimensional smectic phase, as defined by the existence of quasi-long-ranged translational order, is unstable to arbitrarily weak *quenched* disorder (i.e., arbitrarily low aerogel or aerosil density). Furthermore, we find that a new *orientationally* ordered low-temperature ''smectic Bragg glass'' (SBG) phase replaces the smectic phase, and a thermodynamically sharp nematic-to-SBG (N-SBG) transition can survive in the presence of arbitrarily weak disorder, if and only if two *universal* positive definite anomalous exponents η_B and η_K satisfy the bounds

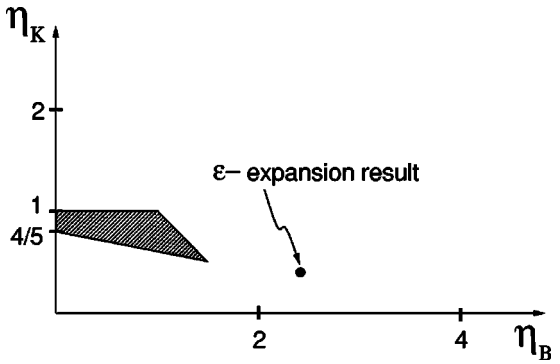

FIG. 1. The region indicates those values of η_B and η_K for which, in three dimensions, the long-range orientationally ordered ''smectic Bragg glass'' phase is stable for sufficiently small disorder (sufficiently low density aerogel).

$$\eta_K+\eta_B<2, \tag{1.1a}$$

$$\eta_K<1, \tag{1.1b}$$

$$\eta_B+5\,\eta_K>4. \tag{1.1c}$$

where the bounds in Eqs. (1.1a),(1.1b) come from the requirement of long-ranged orientational order and the condition for dislocations to remain confined, respectively. The region in the η_B, η_K plane that satisfies these three bounds is illustrated in Fig. 1.

These exponents are *universal*; i.e., the same for *all* smectics in low density *quenched* disordered media. Unfortunately, we have only been able to calculate them in a rather poor approximation: a $d=5-\epsilon$ expansion. Since we are interested in $d=3$, the ostensibly small parameter ϵ in this expansion is $=2$, and so the expansion is expected to be rather poor. However, taking this expansion as our best estimate (since it is our *only* estimate) of the values of η_B and η_K, we obtain, in $d=3$, $\eta_B=12/5$, and $\eta_K=2/5$. These values, which are illustrated by the dot labeled ''ϵ expansion'' in Fig. 1, violate the bounds in Eq. (1.1), and, hence, imply that the smectic Bragg glass phase does *not* exist, and that the thermodynamically sharp NA transition is destroyed by the presence of any disorder, no matter how weak.

However, since the expansion parameter in this calculation $\epsilon=5-d=2$ in the physical case of $d=3$, this argument that the transition and the SBG are destroyed is utterly uncompelling. Fortunately, experiments (which we will describe shortly) well *below* the pure transition temperature $T_{\rm NA}$ can measure the exponents η_B and η_K. With these numbers in hand, the bounds given in Eqs. (1.1) would then provide an unambiguous prediction as to whether or not the transition to the SBG is *always* destroyed by disorder.

The physical significance of these exponents is quite intriguing: they reflect ''anomalous elasticity,'' which, in this context, refers to the fact that the smectic bend modulus K and the layer compression modulus B are *not* constants in the randomly pinned smectic, but, rather, singular functions of

the wave vector $\mathbf{k}$ under consideration. $K(\mathbf{k})$ and $B(\mathbf{k})$ diverge and vanish respectively, as $\mathbf{k}\to 0$, according to the scaling laws

$$K(\mathbf{k}) = K(k_\perp \xi_\perp^{\rm NL})^{-\eta_K} f_K[k_z \xi_z^{\rm NL}/(k_\perp \xi_\perp^{\rm NL})^\zeta], \quad (1.2a)$$

$$B(\mathbf{k}) = B(k_\perp \xi_\perp^{\rm NL})^{\eta_B} f_B[k_z \xi_z^{\rm NL}/(k_\perp \xi_\perp^{\rm NL})^\zeta]. \quad (1.2b)$$

Here k_z and $k_\perp$ denote the projections of $\mathbf{k}$ along and perpendicular to the mean normal to the smectic layers, respectively, and K and B without arguments $\mathbf{k}$ will here and hereafter denote the ''bare'' values of the smectic elastic moduli, i.e., their values in the pure (bulk) smectic. The universal exponent ζ is determined by η_B and η_K through Eq. (1.12b).

The ''nonlinear'' length scales $\xi_\perp^{\rm NL}$ and $\xi_z^{\rm NL}$ are the distances in the $\perp$ and z directions, respectively, beyond which the disorder-driven anomalous elasticity is manifest. That is, Eqs. (1.2) only hold for $k_\perp \xi_\perp^{\rm NL} \ll 1$ and $k_z \xi_z^{\rm NL} \ll 1$. If either of these limits is violated, both K and B are $\mathbf{k}$ independent (up to logarithms), as in pure smectics.[24] In three dimensions these nonlinear length scales are given by

$$\xi_\perp^{\rm NL} = \left(\frac{64\pi}{3}\right)^{1/2} \frac{K^{5/4}}{B^{1/4}\Delta_h^{1/2}}, \quad (1.3a)$$

$$\xi_z^{\rm NL} = \frac{64\pi}{3}\frac{K^2}{\Delta_h}. \quad (1.3b)$$

Expressions for these lengths for general dimensionality $d \geqslant 3$ are given in Eqs. (6.6) and (6.8a) of Sec. VI. In the above Δ_h is a measure of the strength of one of two types of disorder, (the other random ''field'' disorder Δ_V being less important in three dimensions) whose relation to various parameters of the aerogel is given in Sec. II. For now, it suffices to say that Δ_h is a monotonically increasing function of the aerogel density, and we expect it to be a smooth, analytic, nonsingular, finite and nonvanishing function of temperature through the bulk NA transition temperature $T_{\rm NA}$. Since K is likewise well behaved through $T_{\rm NA}$, $\xi_z^{\rm NL}$ is as well, a fact that we will make use of shortly.

Although we have been unable to calculate the scaling functions $f_K(x)$ and $f_B(x)$ exactly, we do know that they have the simple property of making $K(\mathbf{k})$ and $B(\mathbf{k})$ independent of k_z when the scaling argument $k_z \xi_z^{\rm NL}/(k_\perp \xi_\perp^{\rm NL})^\zeta$ is $\ll 1$, and independent of $k_\perp$ in the opposite limit. This implies

$$K(\mathbf{k}) \propto \begin{cases} k_\perp^{-\eta_K} & \text{for } k_z \xi_z^{\rm NL} \ll (k_\perp \xi_\perp^{\rm NL})^\zeta, \\ k_z^{-\eta_K/\zeta} & \text{for } k_z \xi_z^{\rm NL} \gg (k_\perp \xi_\perp^{\rm NL})^\zeta, \end{cases} \quad (1.4a)$$

$$B(\mathbf{k}) \propto \begin{cases} k_\perp^{\eta_B} & \text{for } k_z \xi_z^{\rm NL} \ll (k_\perp \xi_\perp^{\rm NL})^\zeta, \\ k_z^{\eta_B/\zeta} & \text{for } k_z \xi_z^{\rm NL} \gg (k_\perp \xi_\perp^{\rm NL})^\zeta. \end{cases} \quad (1.4b)$$

This strong wave-vector dependence of K and B (driven by disorder) is caused by the same mechanism that leads to the much weaker (logarithmic) but still singular divergence and vanishing of K and B (driven by thermal fluctuations) that occurs in disorder-free smectics:[24] the anharmonic elasticity of large fluctuations in the smectic layers.

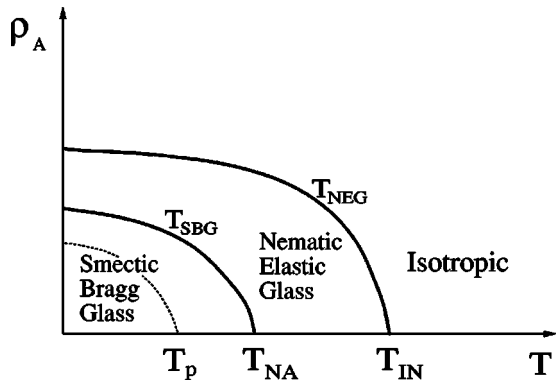

FIG. 2. A schematic (aerogel density ρ_A, temperature T) phase diagram for a smectic liquid crystal, confined inside a low density aerogel, valid *if and only if* all of the bounds, Eq. (1.1) are satisfied. As discussed in the text the ''smectic Bragg glass'' (SBG) and ''nematic elastic glass'' (NEG) phases are, respectively, translationally and orientationally disordered, but are topologically ordered, and are therefore distinct from each other and from the fully disordered ''isotropic'' phase. The dotted line within the SBG phase represents a remnant of the disorder-driven pinning transition of Fig. 6, rounded by the anharmonic elasticity studied in Secs. VI–VIII.

As we will demonstrate in Sec. VI, the effects are larger in the pinned smectic because the disorder induces layer roughness that is much larger than that due to thermal fluctuations, thereby leading to the stronger diverging and vanishing of K and B found here. The stability of the smectic Bragg glass phase depends on η_K and η_B because the elastic moduli K and B determine *both* the size of the orientational fluctuations and the stability of the phase against the unbinding of dislocations. Requiring that real space orientational fluctuations remain finite leads to Eq. (1.1a), while dislocations remain bound only if Eq. (1.1b) is satisfied. If the bounds Eq. (1.1) are satisfied for a real smectic liquid crystal confined inside a low density aerogel, then the resulting ρ_A-T phase diagram will be topologically identical to that displayed in Fig. 2. It is important to note that all of the results quoted here are *equilibrium* results. In contrast, in other pinned elastic media (e.g., Abrikosov flux lattices in dirty superconductors and charge density waves in anisotropic metals), it is often difficult to observe the true equilibrium behavior due to the extremely slow dynamics of those systems, which cause them to drop out of equilibrium, as evidenced by strong hysteretic effects.[3,4]

In this sense, smectics, and liquid crystals in general, appear to be far better systems for investigating the *equilibrium* effects of quenched disorder, since they typically do *not* exhibit hysteresis. This suggests that liquid crystals have intrinsically faster dynamics than, e.g., vortex lattices in superconductors, or charge density wave systems in metals.

That this should be so is hardly surprising: liquid crystals are, after all, ''liquids'' in the sense that, even in the translationally ordered phases like smectics, molecules are quite free to move around. In contrast, the atoms in the conventional crystalline solids are essentially locked into lattice sites, from which they can only escape by thermally activated hopping over substantial energy barriers. Even at room temperature, this hopping is extremely slow; the fact that

even ''high-T_c'' superconductors are much colder than room temperature exacerbates the problem further.

So, because of both their liquidlike *micro*structure and *micro*dynamics, *and* the fact that they are at a higher temperature than, e.g., superconductors, one might have anticipated that it would be far easier to observe the equilibrium effects of quenched pinning in liquid crystals than in other pinned elastic media studied to date. This belief is supported by experiments,[27,19] and so we believe that the *equilibrium* results we obtain here should be directly testable in experiments.

The theoretical analysis that leads to these conclusions is quite interesting and novel. Our first-principles analysis of smectics in aerogel demonstrates that the random pinning induced by the aerogel leads to only two potentially relevant types of random perturbations to the smectic: a random *positional* field disorder (hereafter referred to simply as the ''random field'' and designated by Δ_V), which represents the aerogel's tendency to force the smectic layers to sit at particular *positions*, and a random *orientational* field disorder (hereafter called the ''tilt'' field, with designation Δ_h), reflecting the aerogel's proclivity for particular *orientations* of the nematogens and smectic layers. If we ignore the anharmonic effects that lead to anomalous elasticity, we find that the response of the smectic-*A* phase to the random field disorder in *three* dimensions is in very close mathematical analogy to that of the *XY* model in *two* dimensions (2D).[8] While this suggests that a nontrivial glassy phase might replace the smectic-*A* phase in analogy with the experience with the 2D *XY* model,[8] the presence of *tilt* disorder (which is always generated by the random field) actually destroys this phase. Even in the regime where the random field by itself has no effect at long scales, the tilt disorder leads to short-range smectic order parameter ψ correlations, which fall off exponentially in the direction of the layer normal and as a Gaussian within the smectic layers (within a purely *harmonic* elastic model). This absence of long-ranged order in the elastic model is a strong indication of its limitation and that dislocation defects are likely to proliferate. Focusing on the most important part of randomness, the tilt disorder, we find, in the approximation of ignoring elastic anharmonicities, that disorder *always* creates dislocations. The formalism we use to demonstrate this is new, powerful and potentially applicable to a wide variety of candidate ''Bragg glass'' systems.[25]

Once dislocations are present, the phase is best characterized as a nematic in a random tilt field. However, subsequent examination of orientational fluctuations in this nematic lead to the conclusion that tilt disorder destroys the orientational order of the smectic layers as well. Whether an orientationally and translationally disordered low-temperature phase distinct from the conventionally disordered high temperature isotropic phase is a ''nematic elastic glass'' (NEG), separated from it by a *disclination* unbinding transition, is the subject of an active current investigation.[26]

Including anharmonic elastic effects considerably modifies this picture. Although spatial dimensions d greater than 3 are clearly irrelevant to experiment, it is conceptually extremely useful (and fun) to generalize our model to arbitrary spatial dimensions. We find that for $d>7$, both the random field and the tilt disorder are irrelevant, and a stable, translationally ordered, nonglassy smectic phase exists for sufficiently weak disorder. For $5<d<7$, tilt disorder remains irrelevant, but the random field and the anharmonic terms both become relevant, destroying smectic translational order and leading instead to a stable smectic Bragg glass phase. As in previously studied ''vortex glass'' models of pinned elastic media,[28,9] here too we find that in all spatial dimensions $5<d<7$, real-space positional fluctuations diverge logarithmically with system size:

$$\langle u^2(\mathbf{r})\rangle = c_d \ln[L_\perp f_u(\lambda L_z/L_\perp^2)], \tag{1.5}$$

where $L_{\perp(z)}$ is the linear spatial extent of the system in the $\perp$ (z) direction,

$$\lambda \equiv \sqrt{\frac{K}{B}}, \tag{1.6}$$

$f_u(x)$ is a universal scaling function, and c_d is a universal dimension-dependent constant proportional to $7-d$ for d near 7. As a result, translational correlations throughout the range of spatial dimensions $5<d<7$ decay algebraically,

$$\langle \rho_G^*(\mathbf{r})\rho_G(\mathbf{0})\rangle \propto r_\perp^{-\eta(d)} f_d(\lambda z/r_\perp^2) \tag{1.7}$$

with the exponent $\eta(d)$ a *universal* function of d, and $f_d(x)$ a *universal* (d-dependent) scaling function.[29]

Unlike the vortex glass case, however, here we find anomalous elasticity in this dimension range as well, obtaining

$$K(\mathbf{k}) \propto |\ln[k_\perp f_K(k_z/\lambda k_\perp^2)]|^{\gamma_K(d)}, \tag{1.8a}$$

$$B(\mathbf{k}) \propto |\ln[k_\perp f_B(k_z/\lambda k_\perp^2)]|^{-\gamma_B(d)}, \tag{1.8b}$$

where $f_K(x)$ and $f_B(x)$ are universal scaling functions, and we have, rather remarkably, calculated the universal exponents $\gamma_K(d)$ and $\gamma_B(d)$ *exactly* for all d in this range, $5<d<7$, finding

$$\gamma_K(d) = \frac{1}{2}\left(\frac{-d^2+12d-23}{d^2+6d-13}\right), \tag{1.9a}$$

$$\gamma_B(d) = \frac{3}{2}\left(\frac{d^2-1}{d^2+6d-13}\right). \tag{1.9b}$$

Note that both the fluctuations of $\langle u^2(\mathbf{r})\rangle \propto \ln(L)$ and the anomalous behavior of $K(\mathbf{k})$ and $B(\mathbf{k})$ are very weakly divergent functions of system size L and $\mathbf{k}$ respectively, depending only logarithmically on these quantities. For $d<5$, these weak logarithmic divergences are overwhelmed by power-law divergences caused by the *tilt* disorder, with $K(\mathbf{k})$ and $B(\mathbf{k})$ diverging and vanishing, respectively, according to Eq. (1.2) with,

$$\eta_K = \frac{\epsilon}{5} + O(\epsilon^2), \tag{1.10a}$$

$$\eta_B = \frac{6\epsilon}{5} + 0(\epsilon^2), \tag{1.10b}$$

where $\epsilon \equiv 5-d$, and

$$\langle u^2(\mathbf{r})\rangle = L_\perp^{2\chi}\tilde{f}_u\left(\frac{(L_z/\xi_z^{\rm NL})}{(L_\perp/\xi_\perp^{\rm NL})^\zeta}\right), \tag{1.11a}$$

$$\propto \begin{cases} L_z^{2\chi/\zeta} & \text{for } L_z/\xi_z^{\rm NL} \ll (L_\perp/\xi_\perp^{\rm NL})^\zeta, \\ L_\perp^{2\chi} & \text{for } L_z/\xi_z^{\rm NL} \gg (L_\perp/\xi_\perp^{\rm NL})^\zeta, \end{cases} \tag{1.11b}$$

with the universal roughness χ and anisotropy ζ exponents given by

$$\chi = \frac{\eta_B + \eta_K}{2}, \tag{1.12a}$$

$$\zeta = 2 - \frac{\eta_B + \eta_K}{2}. \tag{1.12b}$$

Here $\tilde{f}_u(x)$ is another universal scaling function.

This result, Eq. (1.11), for $\langle u^2(\mathbf{r})\rangle$ leads to a quantitative, experimentally testable prediction for the x-ray correlation length ξ^X, obtained by equating $\langle u^2(\mathbf{r})\rangle = a^2$, where a is the smectic layer spacing, and solving for L_z, with $L_\perp \to \infty$. This solution for L_z is the x-ray correlation length that will be obtained by scattering off a powder sample (which probably means all samples, since, as discussed earlier, there is probably no long-ranged orientational order in $d=3$, given our best estimates of η_B and η_K). The value of ξ^X so obtained is

$$\xi^X = \begin{cases} \xi_z^{\rm NL}\left(\dfrac{a}{\lambda}\right)^{\zeta/\chi}, & \lambda \ll a, \\ \xi_z^{\rm NL}\left(\dfrac{a}{\lambda}\right)^2, & \lambda \gg a. \end{cases} \tag{1.13}$$

This relation gives a way of experimentally measuring the exponents ζ and χ: since the bare compression modulus $B(T)$ of the pure smectic is a strong function of temperature T near the bulk smectic NA transition temperature $T_{\rm NA}$, vanishing according to $B(T) \propto |T - T_{\rm NA}|^{\tilde{\phi}}$, and since, furthermore, ξ_z^X, a, and K vary smoothly through $T_{\rm NA}$, a plot of $\ln \xi^X$ versus $\ln|T - T_{\rm NA}|$ should yield a straight line for T near $T_{\rm NA}$, with a slope $\tilde{\phi}\zeta/2\chi$, *provided* that we stay far enough below $T_{\rm NA}$ that $\lambda \ll a$, so that the first expression for ξ^X in Eq. (1.13) applies. In deriving this result, we have used the fact that $\lambda = \sqrt{K/B} \propto |T - T_{\rm NA}|^{-\tilde{\phi}/2}$. This prediction for the x-ray correlation length, Eq. (1.13), is schematically displayed in Fig. 3.

A detailed renormalization group analysis[26] of the critical behavior near the pure NA transition (details of which will be presented elsewhere) shows that Eq. (1.13) breaks down altogether at the temperature T_*, which lies below (and for weak disorder very close to) the pure $T_{\rm NA}$. The corresponding reduced temperature $t_* \equiv (T_{\rm NA} - T_*)/T_{\rm NA}$ obeys

$$\xi_{\rm crit}(t_*) = \xi^X, \tag{1.14}$$

where ξ^X is given by the second line of Eq. (1.13), and we have defined a length scale $\xi_{\rm crit}(t)$ derived entirely from properties of the *pure* system

$$\xi_{\rm crit}(t) = \xi_\perp^0\left(\frac{\xi_\perp^{\rm pure}(t)}{\xi_\perp^0}\right)^{2+\tilde{\phi}/2\nu_\perp}, \tag{1.15}$$

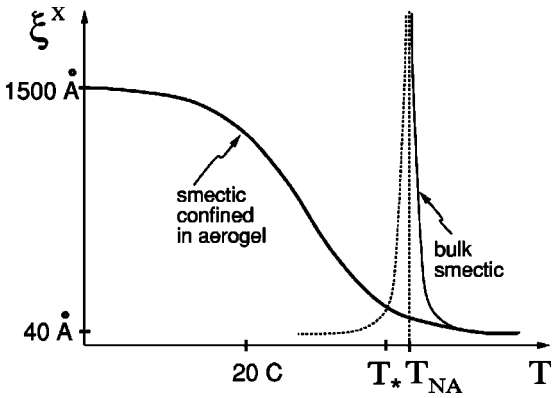

FIG. 3. Finite, temperature-dependent x-ray correlation length $\xi^X(T)$ for a smectic liquid crystal, confined inside a low density aerogel. The essential features of the x-ray correlation length that we predict are (i) saturation as $T \to 0$ of $\xi^X(T)$ at a *finite* ρ_A-dependent value for *arbitrarily low* aerogel density (diverging as $\rho_A \to 0$), (ii) power-law scaling with the *disorder-free* smectic bulk modulus $B(T)$ [see Eq. (1.13)], (iii) crossover from the exponent ζ/χ to 2 in Eq. (1.13) as $T_{\rm NA}$ is approached from below and anomalous elasticity becomes unimportant for ξ^X, (iv) crossover to genuine critical behavior at the temperature T_* at which $\xi_{\rm crit}(T_*)$ (indicated by the dotted curve below $T_{\rm NA}$) $= \xi^X$. In the low density aerogels (e.g., $\rho_A = 0.08$ g/cm^3) we find $T_{\rm NA} - T_* \approx 3$ K (Ref. 20).

where $\xi_\perp^{\rm pure}(t)$ is the x-ray correlation length within the layers of the pure system at a reduced temperature $t \equiv (T - T_{\rm NA})/T_{\rm NA}$ *above* the pure transition temperature $T_{\rm NA}$ and $\xi_\perp^0 \equiv \xi_\perp^{\rm pure}(t \sim 1)$. Here $\nu_\perp$ is the x-ray correlation length exponent for the pure system in the $\perp$ direction, defined via $\xi_\perp^{\rm pure}(t) \propto t^{-\nu_\perp}$.

Theoretically,[16] $\tilde{\phi}$ is expected to be given by $\tilde{\phi} = \nu_{XY} \approx 0.67$. Experimentally,[30] the situation is more complicated, with $\tilde{\phi}$ showing no universality, for reasons that are still unclear (at least to us). Hence, in extracting $\zeta/2\chi$ by the above analysis, the *experimentally* determined value of $\tilde{\phi}$ of the *particular* bulk smectic that is confined in the aerogel should be used. One should also be careful, in such a fit, to treat $T_{\rm NA}$ as a fitting parameter, since the presence of even low density aerogel can presumable shift this temperature slightly. Alternatively, one could avoid these complications by studying the smectic outside the pure system's critical regime, but still reasonably close to the pure $T_{\rm NA}$, since in this regime mean-field theory applies and $\tilde{\phi} = 1$.[31]

Once we know the ratio ζ/χ, we know ζ and χ themselves, since the relation $\zeta = 2 - \chi$, implicit in Eqs. (1.12), provides us with a second equation for the two unknowns ζ and χ. If the roughness exponent χ so obtained is >1, we predict that orientational fluctuations diverge, and hence, long-range orientational order is destroyed. As a result, there will be no ''smectic Bragg glass'' phase, and hence, no thermodynamically sharp transition associated with the pure NA transition that occurs in the absence of disorder. In any event, the prediction Eq. (1.13) for $\lambda \gg a$ will inevitably apply sufficiently close to the pure $T_{\rm NA}$, since $B(T \to T_{\rm NA}) \to 0$, while $K(T \to T_{\rm NA})$ remains nonzero, and, hence $\lambda(T \to T_{\rm NA}) \to \infty \gg a$.

Should it prove that orientational order *is* possible in d

$=3$ [i.e., that the $d=3$ values of the universal exponents η_B and η_K *do* satisfy the inequalities in Eq. (1.1), which would imply that the SBG phase is stable], then single orientational domain scattering *is*, in principle, possible. In such scattering, ξ_z^X which equals ξ^X as given by Eq. (1.13), will be the correlation length along the mean normal to the smectic layers. The correlation length $\xi_\perp^X$ along the smectic layers is then given by

$$\xi_\perp^X = \begin{cases} \xi_\perp^{\rm NL}\left(\dfrac{a}{\lambda}\right)^{1/\chi}, & \lambda \ll a, \\ \xi_\perp^{\rm NL}\left(\dfrac{a}{\lambda}\right), & \lambda \gg a. \end{cases} \tag{1.16}$$

These x-ray correlation lengths should *not*, however, be interpreted as being the length scales beyond which smectic correlations cease. In fact, smectic behavior, by which we mean, e.g., the anomalous smectic elasticity, Eq. (1.2), persists out to much longer lengths. Specifically, in the $\perp$ direction, it persists out to the shorter of the two lengths ξ_O and $\xi_\perp^D$. ξ_O is the distance over which orientational order *would* be well correlated in the absence of dislocations, while $\xi_\perp^D$ is the distance (in the $\perp$ direction) below which dislocations *would* remain bound in the absence of large orientational fluctuations. In actuality, only the prediction of the *smaller* of the two of these lengths is valid. We expect that when $\xi_O \ll \xi_\perp^D$, the large orientational fluctuations also induce dislocation unbinding at about the same length ξ_O. On the other hand, in the opposite ($\xi_O \gg \xi_\perp^D$) limit, the dislocation unbinding occurs first at $\xi_\perp^D$, and on longer scales our system is indistinguishable from a nematic in a random orientational field. Whether the orientational order is stable or not in this case must be reevaluated within an effective random-field nematic model using a Larkin[7] type of analysis.[26]

The ''orientational correlation length'' ξ_O is given by

$$\xi_O = \xi_\perp^{\rm NL}\left(\frac{\xi_z^{\rm NL}}{\lambda}\right)^{1/[2(\chi-1)]}, \tag{1.17}$$

while the ''dislocation length'' scale $\xi_\perp^D$ in the $\perp$ direction is given by

$$\xi_\perp^D = \xi_\perp^{\rm NL}\left(\frac{\lambda T^2 \xi_z^{\rm NL}}{cK^2a^2d^2}\right)^{1/[2(\eta_\kappa-1)]}, \tag{1.18}$$

where d is a microscopic length of order the layer spacing a, and $c(T)$ is a dimensionless constant. $c(T)$ vanishes like $e^{-E_c/T}$ as $T\to 0$ and diverges like $T/E_c(T) \propto |T_{\rm NA}-T|^{-\gamma_\tau}$, where E_c is the ''core energy'' of a dislocation line segment of length d, and γ_τ is the ''line tension'' critical exponent describing how $E_c(T)$ vanishes as $T\to T_{\rm NA}$ from below. We expect the resulting divergence of $c(T)$ to overwhelm the corresponding divergence of $\lambda(T)$, leading to a $\xi_\perp^D$ that gets smaller as $T\to T_{\rm NA}$ from below. Clearly then, sufficiently close to $T_{\rm NA}$, $\xi_\perp^D$ gets to be less than $\xi_\perp^{\rm NL}$, the system enters the strong-disorder regime, and the weak-disorder theory we have presented here no longer applies.

In the z direction, the orientational correlation length is also given by ξ_O, while the dislocation length is given by

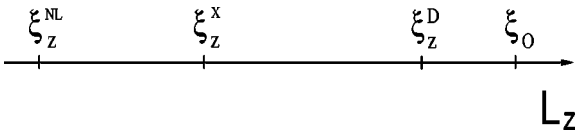

FIG. 4. One possible hierarchy of important length scales in the problem of a three-dimensional weakly disordered smectic, valid for $\lambda \ll a$, $\chi>1$, and $\eta_K>1$. If instead $\chi<1$ and $\eta_K<1$, then the putative SBG phase is stable and ξ_O and $\xi_{\perp,z}^D$ are infinite.

$$\xi_z^D = \xi_z^{\rm NL}\left(\frac{\lambda T^2 \xi_z^{\rm NL}}{cK^2a^2d^2}\right)^{\zeta/[2(\eta_\kappa-1)]}. \tag{1.19}$$

Note that the orientational correlation length given by Eq. (1.17) is always much greater than the nonlinear length $\xi_\perp^{\rm NL}$ in the weak disorder limit, in which $\xi_\perp^{\rm NL}\to\infty$. However, if we hold the disorder strength fixed (i.e., hold Δ_h fixed), and approach the pure NA transition from below, we will always eventually, at some temperature $T_* < T_{\rm NA}$ ($T_* \to T_{\rm NA}$ in weak disorder limit) leave this weak disorder regime (no matter how weak the disorder is), since $\xi_z^{\rm NL}$, as given by Eq. (1.3b), does not change much as $T\to T_{\rm NA}^-$, while $\lambda(T\to T_{\rm NA}^-)\to\infty$. So the ratio $\xi_O/\xi_\perp^{\rm NL}$ decreases without bound as $T\to T_{\rm NA}^-$, and, hence eventually drops below 1, signaling our entry into the strong disorder regime for $T>T_*$ (see Fig. 3). This is not surprising: the smectic's resistance to the perturbing effects of the disorder is provided, in part, by B; so when $B\to 0$, as it does as the pure NA transition is approached, any disorder will eventually look ''strong,'' and our theory will no longer apply. As long as we are at temperatures T far enough below $T_{\rm NA}$, however, our weak disorder results will apply. As we weaken the disorder, by, e.g., reducing aerogel density, we can apply our weak disorder theory closer to $T_{\rm NA}$.

The expression (1.17) for the orientational correlation length ξ_O only holds if the condition for the *stability* of the long-ranged orientational order, Eq. (1.1a), which is equivalent to $\chi<1$, is *violated* [i.e., Eq. (1.17) only holds for $\chi>1$]. If $\chi<1$, ξ_O is infinite. Likewise, Eqs. (1.18) and (1.19) for $\xi_{\perp,z}^D$ only hold if $\eta_K>1$ (i.e., if dislocations *are*, in fact, *unbound*); otherwise, $\xi_{\perp,z}^D$ are infinite. Clearly, if both bounds $\chi<1$ and $\eta_K<1$ are satisfied, *all three* lengths are infinite, smectic behavior holds out to arbitrarily large length scales, and the smectic Bragg glass is a stable phase of weakly disordered smectic liquid crystals. A summary of the many length scales in our theory for three dimensions and their hierarchy, in the weak disorder limit, is illustrated for the case of $\lambda \ll a$ in Fig. 4 and for the case of $\lambda \gg a$ in Fig. 5.

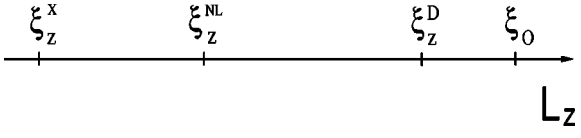

FIG. 5. The hierarchy of important length scales in the problem of a three-dimensional weakly disordered smectic, valid for $\lambda \gg a$, $\chi>1$, and $\eta_K>1$. If instead $\chi<1$ and $\eta_K<1$, then the putative SBG phase is stable and ξ_O and $\xi_{\perp,z}^D$ are infinite.

Just as the proposed ''Bragg glass'' phase of randomly pinned elastic media is distinct from the liquid phase,[9–11] here, too, the absence of long-ranged orientational order does not preclude a ''nematic elastic glass'' phase that would be separated from the high temperature ''isotropic fluid'' phase by a thermodynamically sharp equilibrium phase transition. That such a nematic elastic glass phase may indeed exist is suggested by dynamic light scattering experiments[27,19] that show a dramatic slowing down of director fluctuation relaxations in liquid crystals in aerogel below a temperature $T_{\rm NEG}$ near the bulk NI transition.

We are currently investigating theoretically the possibility of such a nematic elastic glass phase, and initial results indicate that such possibility is, in fact, allowed theoretically. Our results on this subject will appear in a future publication.[26]

The low-temperature analysis presented here is further supported by a complementary approach that investigates the stability of the NA transition to disorder from the high-temperature nematic phase.[32] The results of this latter work substantiate our findings that in three dimensions the NA transition and the smectic-*A* phase are destroyed. The rounded remnant of this transition *can*, however, be studied utilizing self-consistent methods previously successfully applied to pure systems below the lower critical dimension.[33] This method predicts a Lorentzian-squared structure function with a correlation length that monotonically, slowly increases through the bulk NA transition, in good agreement with the x-ray measurements.[20] The calculated specific heat is also in qualitative agreement with the experiments,[20,21] exhibiting a broad, well-defined peak at a temperature slightly lower than the bulk transition temperature $T_{\rm NA}$.

As we hope the introduction made clear, the phenomenology of this system is extremely rich. Our organizational approach to presenting the derivation of these results is the following: starting from the full model, we first throw out all but the simplest effects; only after developing a full understanding of this simplified model do we, one at a time, reintroduce the complicating effects we had initially thrown out, building gradually toward the complexity of the full physical model.

Since many of these complicating effects prove to be important, this approach has the drawback that some of the results derived for the simplified models do *not*, in fact describe our physical system. We have attempted to alert the reader everywhere results of a simplified models differ from our actual predictions for the full theory. Only the latter, of course, should be compared directly with experiments.

The remainder of this paper is organized as follows. In Sec. II, we introduce and motivate our model for smectics in disordered media, and discuss those aspects of the model specific to aerogel. In Sec. III, we derive an elastic low-temperature model of a randomly pinned smectic, at first ignoring anharmonic elasticity and dislocations. Using renormalization group methods, we study the stability of the smectic phase within this model and these approximations. In Sec. IV, we use the results of this RG treatment to calculate various smectic correlation functions, still treating the elastic theory to harmonic order and ignoring dislocations. Section V incorporates the dislocations in the *harmonic* theory, and shows that, in that theory, in three dimensions, dislocations always proliferate, even in the presence of arbitrarily weak disorder, thereby destroying, within the *harmonic* approximation, both the smectic Bragg glass phase and the thermodynamically sharp transition to it.

The effects of the previously ignored anharmonic elastic terms are shown to be very important in Sec. VI, which develops a renormalization group treatment of them in a model that neglects both the random field and dislocations, but includes tilt disorder. The anomalous elasticity Eq. (1.2) and the u-u correlation functions that follow from it are derived in this section. Given the importance of these elastic nonlinearities, we study their effects on dislocation unbinding and orientational order in Sec. VII by incorporating anomalous elasticity into the duality theory developed in Sec. V. We thereby derive the bounds Eq. (1.1a) and Eq. (1.1b) on the anomalous exponents η_B and η_K for the stability of the smectic Bragg glass phase. The irrelevance of the random field in a full, anharmonic theory below $d=5$ is demonstrated by a functional renormalization group treatment in Sec. VIII, in which we also calculate exactly the exponents $\gamma_B(d)$ and $\gamma_K(d)$ that govern the anomalous elasticity for $5<d<7$. In Sec. IX, we demonstrate that in $d=3$, dislocation unbinding is *not* induced by the random field alone. We conclude the main part of the manuscript with Sec. X, where we discuss interpretation of our results in terms of past experiments and their implications for the future experiments, as well as many remaining interesting and important theoretical problems, some under current investigation. The details of the analysis of the random field disorder in three dimensions, the functional renormalization group analysis in higher dimensions, an alternative derivation of the dislocation loop theory, and a fluctuation-corrected mean-field treatment of the dual model of randomly pinned smectic are presented in Appendixes A, B, C, and D, respectively.

II. MODEL

Our theory of the disordered NA transition is based on the de Gennes model. Near the mean-field transition from the nematic to the smectic-*A* phase, the center-of-mass nematogen molecular density $\rho(\mathbf{r})$ (which is liquidlike in the nematic phase) begins to develop strong fluctuations dominated by Fourier components that are integer multiples of the smectic ordering wave vector $\mathbf{q_0}=\hat{\mathbf{n}}2\pi/a$ (a is the layer spacing) parallel to the nematic director $\hat{\mathbf{n}}$. We take the dominant lowest Fourier component $\psi(\mathbf{r})$ as the local (complex scalar) order parameter which distinguishes the smectic-*A* from the nematic phase.[13] It is related to the density $\rho(\mathbf{r})$ by

$$\rho(\mathbf{r})=\mathrm{Re}[\rho_0+e^{i\mathbf{q_0}\cdot\mathbf{r}}\psi(\mathbf{r})], \tag{2.1}$$

where ρ_0 is the mean density of the smectic.

As first discussed by de Gennes,[13] the effective Hamiltonian functional $H_{\rm dG}[\psi,\hat{\mathbf{n}}]$ that describes the NA transition at long length scales, in bulk, pure liquid crystals, is

$$H_{\rm dG}[\psi,\hat{\mathbf{n}}]=\frac{1}{2}\int d^dr\Big[c_\perp|(\boldsymbol{\nabla}_\perp-iq_0\delta\mathbf{n})\psi|^2+c_\parallel|\boldsymbol{\nabla}_\parallel\psi|^2 +t_0|\psi|^2+\frac{1}{2}g_0|\psi|^4\Big]+H_F[\mathbf{n}], \tag{2.2}$$

where $t_0\propto(T-T_{\rm NA}^{\rm pure})/T_{\rm NA}^{\rm pure}$, $T_{\rm NA}^{\rm pure}$ is the NA transition temperature in the pure (bulk) system, $\delta\hat{\mathbf{n}}(\mathbf{r})\equiv\hat{\mathbf{n}}(\mathbf{r})-\hat{\mathbf{n}}_0$ is the fluctuation of the local nematic director $\hat{\mathbf{n}}(\mathbf{r})$ away from its average value $\hat{\mathbf{n}}_0$, which we take to be $\hat{\mathbf{z}}$, subscripts $\parallel$ and $\perp$ denote the directions parallel and transverse to $\hat{\mathbf{n}}_0$, and $H_F[\mathbf{n}]$ is the Frank effective Hamiltonian that describes the elasticity of the nematic order director:

$$H_F[\hat{\mathbf{n}}]=\int d^dr\frac{1}{2}[K_s(\boldsymbol{\nabla}\cdot\hat{\mathbf{n}})^2+K_t(\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{n}})^2 +K_b(\hat{\mathbf{n}}\times\boldsymbol{\nabla}\times\hat{\mathbf{n}})^2], \tag{2.3}$$

where K_s, K_t, and K_b are the bare elastic moduli for splay, twist and bend of the nematic director field, respectively.

The minimal coupling between $\mathbf{n}$ and ψ is enforced by the requirement of global rotational invariance.[13] It is important to emphasize, however, that although the de Gennes Hamiltonian $H_{\rm dG}$ is closely analogous to that of a superconductor, there are essential differences. The physical reality of the nematic $\delta\mathbf{n}$ and the smectic ψ order parameters, in contrast to the gauge ambiguity in the definition of the vector potential and the superconducting order parameter, selects the liquid crystal gauge[34,35] $\delta\mathbf{n}\cdot\hat{\mathbf{n}}_0=0$ (since $|\mathbf{n_0}|^2=1$) as the preferred physical gauge. The strict gauge invariance of $H_{\rm dG}$ is already explicitly broken by the splay term $K_s(\boldsymbol{\nabla}\cdot\hat{\mathbf{n}})^2$ of the Frank Hamiltonian. This is one source of distinction between a smectic-*A* liquid crystal and a superconductor.

In fact, since the only true symmetry of the smectic-*A* liquid crystal is invariance under *global* simultaneous rotation of the smectic layers (equivalent to $\psi\rightarrow\psi e^{i\boldsymbol{\theta}\cdot\mathbf{r}_\perp}$) and the nematic director $\delta\mathbf{n}\rightarrow\delta\mathbf{n}+\boldsymbol{\theta}$ accomplished by a constant (in space) $\boldsymbol{\theta}=$const, and *not* arbitrary *local* gauge transformations, the requirement of gauge invariance severely overrestricts the allowed effective Hamiltonian. That is, the de Gennes model is more symmetric than is required by the physics of the NA transition, and a more general effective Hamiltonian will allow for terms that break nonlinear gauge invariance. All of the extension terms that can be added to de Gennes model are, unfortunately, irrelevant (which is probably why de Gennes did not include them in his original formulation). However, in light of the above remarks, the bending stiffness $\tilde{K}_s$ in the smectic phase [i.e., the coefficient of $(\nabla_\perp^2u)^2$] can differ from the corresponding coefficient K_s in the nematic phase [the coefficient of $(\boldsymbol{\nabla}\cdot\hat{\mathbf{n}})^2$], due to the presence of terms like, e.g., $|\nabla_\perp^2\psi|^2$, in the smectic Hamiltonian, which violate local gauge invariance, but *not* global rotation invariance. This is in contrast to what is usually assumed in all the standard treatments of this problem, which take $\tilde{K}_s$ and K_s to be the same.[14] The importance of this observation is that in mean-field theory it leads to a slope discontinuity in $\tilde{K}_s(T)$ at the $T_{\rm NA}^{\rm mf}$, which is likely to persist in the theory that includes fluctuations.[36]

Guided by these observations, we now proceed to construct the disorder part of the effective Hamiltonian. Because $\psi(\mathbf{r})$ is related to the smectic density $\rho(\mathbf{r})$ via Eq. (2.1), the randomness can couple to it directly. In particular, symmetry allows a coupling of the form

$$H_{d\rho}=\int d^dr\Big[\frac{1}{2}\delta t(\mathbf{r})(\rho-\rho_0)^2+U(\mathbf{r})\rho\Big], \tag{2.4}$$

where both the random $T_c[\delta t(\mathbf{r})]$ and the random potential $U(\mathbf{r})$ are proportional to the local aerogel density $\rho_A(\mathbf{r})$. We will take this aerogel density to be a quenched random variable, and denote averages over it by a horizontal overbar.

This means in particular that the autocorrelations of $\delta t(\mathbf{r})$ and $U(\mathbf{r})$ will be proportional to those of the aerogel, and, hence, long-ranged for fractal aerogel.[37] That is, we expect

$$\overline{U(\mathbf{r})U(\mathbf{r}')}\equiv C_U(\mathbf{r}-\mathbf{r}'), =\Gamma_U\overline{\rho_A(\mathbf{r})\rho_A(\mathbf{r}')}, \tag{2.5}$$

$$\overline{\delta t(\mathbf{r})\,\delta t(\mathbf{r}')}\equiv C_t(\mathbf{r}-\mathbf{r}'), =\Gamma_t\overline{\rho_A(\mathbf{r})\rho_A(\mathbf{r}')}, \tag{2.6}$$

where Γ_U and Γ_t are coupling strengths that depend on the microscopic physics of the aerogel-smectic interaction, but *not*, at low aerogel density, on the aerogel density itself, nor on the smectic order parameter.

The fractal structure of aerogel over a range of length scales can be determined from the aerogel density-density correlation function $\overline{\rho_A(\mathbf{r})\rho_A(0)}$ measured in an x-ray scattering experiment. Writing the aerogel density ρ_A in dimensionless (volume fraction) units, the correlation function $\overline{\rho_A(\mathbf{r})\rho_A(\mathbf{r}')}$ measures the probability that, given that a point $\mathbf{r}$ is occupied, the point $\mathbf{r}'$ is also occupied. The probability that an arbitrary point $\mathbf{r}$ is occupied is clearly proportional to the aerogel density $\overline{\rho_A(L_f)}$, measured at (averaged over) a scale beyond which the aerogel ceases to be fractal. We call this scale L_f. The aerogel is also *not* a fractal for lengths smaller than a microscopic length a_f of the order the aerogel strand diameter. The density of a structure which is a fractal on scales $a_f<r<L_f$, with fractal dimension d_F, is, by definition, given by the ratio of the number of occupied sites $N\sim(L_f/a_f)^{d_F}$ in a volume $V=L_f^d$ to this volume. Hence, the average aerogel density $\rho_A\equiv\overline{\rho_A(L_f)}$ is given by

$$\overline{\rho_A(L_f)}\approx\left(\frac{a_f}{L_f}\right)^{d-d_F}. \tag{2.7}$$

Given that a site $\mathbf{r}$ is occupied, the conditional probability $P(\mathbf{r},\mathbf{r}')$ that a site $\mathbf{r}'$ is also occupied is, roughly, the typical ''mass'' (actually total volume) of material $M(|\mathbf{r}-\mathbf{r}'|)$ contained within a sphere of radius $|\mathbf{r}-\mathbf{r}'|$ centered on a point $\mathbf{r}$ *that is also on the fractal*, divided by the total volume $|\mathbf{r}-\mathbf{r}'|^d$ of that sphere. By definition, $M(|\mathbf{r}-\mathbf{r}'|)\approx a_f^d(|\mathbf{r}-\mathbf{r}'|/a_f)^{d_F}$ (this is what we mean by the fractal dimension d_F). Hence, the conditional probability is $P(\mathbf{r},\mathbf{r}')=M(|\mathbf{r}-\mathbf{r}'|)/|\mathbf{r}-\mathbf{r}'|^d\sim(a_f/|\mathbf{r}-\mathbf{r}'|)^{d-d_F}$. Combining this with Eq. (2.7), we obtain

$$\overline{\rho_A(\mathbf{r})\rho_A(\mathbf{r}')} \approx \left(\frac{a_f}{L_f}\right)^{d-d_F}\left(\frac{a_f}{|\mathbf{r}-\mathbf{r}'|}\right)^{d-d_F}, \tag{2.8a}$$

$$\approx \rho_A\left(\frac{a_f}{|\mathbf{r}-\mathbf{r}'|}\right)^{d-d_F}. \tag{2.8b}$$

We will see in a moment that these long-ranged correlations have no effect on the long distance behavior of the smectic.

Combining the random field energy Eq. (2.4) with the relation Eq. (2.1) between the smectic order parameter ψ and the density ρ, we obtain

$$H_{d\rho}[\psi] = \int d^d r \frac{1}{2}[\delta t(\mathbf{r})|\psi|^2 + V(\mathbf{r})\psi + V^*(\mathbf{r})\psi^*], \tag{2.9}$$

where we have defined the *complex* random potential $V(\mathbf{r})$, which acts on the NA order parameter ψ as a random magnetic field acts on a spin, and is related to the potential disorder $U(\mathbf{r})$ by

$$V(\mathbf{r}) \equiv U(\mathbf{r}) e^{iq_0 z}. \tag{2.10}$$

Note that, despite the long-ranged correlations of $U(\mathbf{r})$, $V(\mathbf{r})$ has only short-ranged correlations. To see this, consider a double Fourier transform of the $\overline{V(\mathbf{r})V^*(\mathbf{r}')}$ correlation function

$$\overline{V(\mathbf{k})V^*(\mathbf{k}')} \equiv \int d^d r d^d r' e^{-i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{k}'\cdot\mathbf{r}'} \overline{V(\mathbf{r})V^*(\mathbf{r}')},$$

$$= (2\pi)^d \delta^d(\mathbf{k}+\mathbf{k}') C_V(\mathbf{k}), \tag{2.11}$$

where

$$C_V(\mathbf{k}) = \int d^d(r-r') e^{i(\mathbf{k}+q_0\hat{\mathbf{z}})\cdot(\mathbf{r}-\mathbf{r}')} \overline{U(\mathbf{r})U(\mathbf{r}')},$$

$$= C_U(\mathbf{k}+q_0\hat{\mathbf{z}}), \tag{2.12}$$

and $C_U(\mathbf{k})$ is the Fourier transform of the $U-U$ correlation, Eq. (2.5). Even though we expect this Fourier transformed correlation function $C_U(\mathbf{k})$ to diverge as $\mathbf{k}\to 0$, due to the power-law spatial correlations in $U(\mathbf{r})$, there is no reason for it to diverge as $\mathbf{k}\to q_0\hat{\mathbf{z}}$, since the aerogel itself has no particular spatial structure at the wavevector of the smectic ordering. Hence, we see from Eq. (2.12) that the V-V autocorrelation function remains finite as $\mathbf{k}\to 0$. Thus, the correlations of $V(\mathbf{r})$ are short ranged, and hence we can accurately capture the long distance physics of the problem by taking those correlations to be zero ranged, and writing

$$\overline{V(\mathbf{r})V^*(\mathbf{r}')} = W\delta^d(\mathbf{r}-\mathbf{r}'), \tag{2.13}$$

where

$$W = C_U(q_0\hat{\mathbf{z}}), \tag{2.14a}$$

$$= \Gamma_U\left(\frac{a_f}{L_f}\right)^{d-d_F}\left(\frac{1}{a_f q_0}\right)^{d_F}, \tag{2.14b}$$

as can be readily seen by Fourier transforming Eq. (2.8).

In addition to the density coupling of Eqs. (2.4) and (2.9), we expect that the long nematogenic molecules that make up the smectic will tend to line up with the randomly oriented aerogel strands,[38] whose orientational correlations we expect to be short ranged. This interaction leads to an additional *orientational* random coupling of the form

$$H_{dn} = \frac{1}{2}\int d^d r[\mathbf{g}(\mathbf{r})\cdot\hat{\mathbf{n}}]^2, \tag{2.15}$$

where the quenched field $\mathbf{g}(\mathbf{r})$ is random and short-range correlated in direction

$$\overline{g_i(\mathbf{r})g_j(\mathbf{r}')} = \Delta_g \delta^d(\mathbf{r}-\mathbf{r}')\delta_{ij}, \tag{2.16}$$

with strength Δ_g proportional to the local aerogel density.

As discussed in the Introduction, motivated by the experimental observations, we assume that near the NA transition, in low-density aerogel samples, the nematic order is well-developed. In this regime, we can consider small disorder-driven deviations from perfect nematic order $\hat{\mathbf{n}}_0 = \hat{\mathbf{z}}$ as small, writing $\hat{\mathbf{n}}(\mathbf{r})$ as

$$\hat{\mathbf{n}}(\mathbf{r}) = \hat{\mathbf{z}}\sqrt{1-|\delta\mathbf{n}|^2} + \delta\mathbf{n}(\mathbf{r}). \tag{2.17}$$

Substituting this representation inside H_{dn}, Eq. (2.15), and, given that $|\delta\mathbf{n}| \ll 1$, keeping only up to quadratic terms in $\delta\mathbf{n}$, and dropping $\delta\mathbf{n}$-independent terms, we obtain

$$H_{dn} \approx \frac{1}{2}\int d^d r\{-g_z(\mathbf{r})^2|\delta\mathbf{n}|^2 + [\mathbf{g}(\mathbf{r})\cdot\delta\mathbf{n}]^2 + 2\mathbf{h}(\mathbf{r})\cdot\delta\mathbf{n}\}, \tag{2.18}$$

where we have defined a quenched random tilt field

$$\mathbf{h}(\mathbf{r}) \equiv g_z(\mathbf{r})\mathbf{g}(\mathbf{r}). \tag{2.19}$$

It is easy to see that for *isotropic* disorder $\mathbf{g}(\mathbf{r})$, the terms quadratic in $\delta\mathbf{n}(\mathbf{r})$, above, cancel each other on average and therefore only make unimportant contributions that are beyond quadratic order in $\delta\mathbf{n}(\mathbf{r})$. As a consequence, for such isotropic disorder, the effective orientational disorder is given by the last term in Eq. (2.18)

$$H_{dn} \approx \int d^d r\, \mathbf{h}(\mathbf{r})\cdot\delta\mathbf{n}. \tag{2.20}$$

We expect that the random field $\mathbf{h}(\mathbf{r})$ is related to the aerogel density structure by the following correlation function:

$$\overline{h_i(\mathbf{r})h_j(\mathbf{r})} = \overline{g_z(\mathbf{r})g_i(\mathbf{r})g_z(\mathbf{r}')g_j(\mathbf{r}')},$$

$$= \Gamma_h \rho_A^2(\mathbf{r})\rho_A^2(\mathbf{r}')t_z(\mathbf{r})t_i(\mathbf{r})t_z(\mathbf{r}')t_j(\mathbf{r}'), \tag{2.21}$$

where $\mathbf{t}(\mathbf{r})$ is the local tangent to the aerogel fibers, and Γ_h is a ρ_A-independent coupling constant that depends on the microscopic aerogel-smectic interaction.

Since we expect this tangent $\mathbf{t}(\mathbf{r})$ to have only short-ranged correlations (with range of order the orientational persistence length of the silica fibers), the above correlation function of the tilt disorder should also be short ranged. Furthermore, it must be isotropic. These considerations, taken

together with the fractal nature of the aerogel, and Eq. (2.21), lead to the following form for the correlation function of the random tilt disorder $\mathbf{h}(\mathbf{r})$:

$$\overline{h_i(\mathbf{r})h_j(\mathbf{r}')} = \Gamma_h \left(\frac{a}{L}\right)^{d-d_F} \delta^d(\mathbf{r}-\mathbf{r}')\,\delta_{ij}\,,$$

$$\equiv \Delta_h \delta^d(\mathbf{r}-\mathbf{r}')\,\delta_{ij}\,, \tag{2.22}$$

which is *short ranged*. This defines the tilt field disorder variance Δ_h.

Thus our model of the NA transition in aerogel is characterized by the effective Hamiltonian functional $H=H_{\rm dG}+H_{d\rho}+H_{dn}$, with $H_{\rm dG}$, $H_{d\rho}$, and H_{dn} given by Eqs. (2.2), (2.9), and (2.20), respectively. This total effective Hamiltonian must be supplemented with correlation functions for the random T_c $(\delta t(\mathbf{r}))$, random field $(V(\mathbf{r}))$, and random tilt $(\mathbf{h}(\mathbf{r}))$ disorders, which are given by Eqs. (2.6)–(2.8), (2.13)–(2.14), and (2.22), respectively. Finally, it is essential to keep in mind that, as discussed above, even for fractal aerogel, although the correlations of the random T_c field $\delta t(\mathbf{r})$ are long ranged (power-law correlated), those of the random field $V(\mathbf{r})$ and the tilt $\mathbf{h}(\mathbf{r})$ are short ranged, with $W \ll \Delta_h$.

III. SMECTIC PHASE AND ITS STABILITY WITHIN THE HARMONIC ELASTIC MODEL

In this section we study the disordered NA model, defined in the preceding section, within the low-temperature phase. While it is tempting to directly analyze the model $H=H_{\rm dG}+H_{d\rho}+H_{dn}$ written in terms of the smectic order parameter ψ, we will not do so here. Our motivation for this is twofold: (i) As discussed in the Introduction, even for the bulk (disorder-free) case the results obtain through such a direct (de Gennes model) approach, have, so far, failed to give predictions that agree with experiments even on the *bulk* NA transition. Having not fully understood the difficulties with the bulk NA transition, we hesitate to use such a high-temperature (''soft-spin'') analysis to study the significantly more complicated NA transition in the presence of disorder (i.e., confined inside the aerogel). (ii) While such a ''soft-spin'' approach is often superior in understanding the critical properties of the *transition* and the high-temperature phase, it is significantly less successful in the analysis of the low-temperature phase. For example, it is *known* to *incorrectly* predict the lower critical dimension in, e.g., the random field Ising model, as well as completely missing the existence of the Kosterlitz-Thouless transition in the XY model. Since one of the main goals of the current initial investigation of the disordered smectics problem is to ascertain the stability and nature of the low-temperature phase, we leave the direct de Gennes model approach to this problem to subsequent publications.[32]

Instead, here, as a first analysis of the problem, we choose the elastic, low-temperature approach in terms of the Goldstone phonon mode u. The predictions of such an approach for the ordered pure smectic phase are free of controversies, and are in agreement with experiments on bulk smectics.

We will not, in either this or the next section (Sec. IV), treat the full elastic model in all its complexity, but rather, begin by simplifying the model by ignoring *elastic anharmonicities* (i.e., terms higher than quadratic in gradients of u). We will also, in both this and the next section, neglect the effects of dislocations, and focus on three spatial dimensions. All of these restrictions will be removed in later sections of the paper, as we build up to the full complexity of the disordered smectic system.

We begin by *assuming* the existence of smectic order and investigate if/when this assumption is violated because of the interaction of the smectic with the random environment of the aerogel. Within the ordered smectic phase, the fluctuations are conveniently described in terms of the fluctuations of the magnitude and phase of ψ. It is easy to show that the fluctuations of the *magnitude* of ψ around the average value $|\psi_0|=\sqrt{t_0/g_0}=$const are ''massive,'' and can therefore be safely integrated out of the partition function, leading to only *finite*, unimportant shifts in the effective elastic moduli. In contrast, the phase of ψ is a U(1) massless Goldstone mode, corresponding to spontaneously broken translational symmetry. It is the essential low energy phonon degree of freedom of the smectic phase, describing the local displacement of the smectic layers from perfect periodic order. In accord with this discussion, deep within the smectic phase, we can represent the smectic order parameter as

$$\psi(\mathbf{r}) = |\psi_0| e^{iq_0 u(\mathbf{r})}, \tag{3.1}$$

safely ignoring (actually integrating out the ''massive'') fluctuations in the magnitude $|\psi_0|$ of ψ. It is important to note that this can be done at any temperature *below* the transition, without any *qualitative* consequences for phenomena occurring on sufficiently long length scales, larger than a well-defined crossover length $\xi_*(T)$. The elastic model is then rigorously valid on length scales larger than $\xi_*(T)$ and breaks down on shorter scales, and therefore, of course can only make predictions about phenomena (e.g., length scales such as ξ^X) larger than $\xi_*(T)$. As $T \to T_{\rm NA}^-$, $\xi_*(T)$ diverges and the range of *length scales* about which the elastic model is able to make predictions shrinks, being pushed out to infinite scales. Consequently, for example, sufficiently close to $T_{\rm NA}^-$, i.e., for $T > T_*$ (see Fig. 3), such that $\xi^X \to \xi_*$ from above, our prediction for ξ^X, based on the weak disorder elastic theory is no longer strictly valid.

Using this low-temperature ansatz (3.1) inside the ψ-dependent part of the effective Hamiltonian, given by Eqs. (2.2) and (2.9) and dropping constant terms, we find

$$H[u,\delta\mathbf{n}] = \int d^d r \left[\frac{B_\perp}{2} |\boldsymbol{\nabla}_\perp u - \delta\mathbf{n}|^2 + \frac{B}{2}(\partial_z u)^2 + \frac{K_s}{2}(\boldsymbol{\nabla}\cdot\delta\mathbf{n})^2 + \frac{K_t}{2}(\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\times\delta\mathbf{n})^2 + \frac{K_b}{2}(\hat{\mathbf{z}}\times\boldsymbol{\nabla}\times\delta\mathbf{n})^2 + \mathbf{h}(\mathbf{r})\cdot\delta\mathbf{n} - |\psi_0| U(\mathbf{r}) \cos\{q_0[u(\mathbf{r})+z]\} \right], \tag{3.2}$$

where $B_\perp = c_\perp |\psi_0|^2 q_0^2$ and $B = c_{||} |\psi_0|^2 q_0^2$. We observe that the fluctuation mode $(\boldsymbol{\nabla}_\perp u - \delta\mathbf{n})$ is ''massive'' and leads to the Anderson-Higgs mechanism, a hallmark of gauge theories. As a consequence, after a simple Gaussian integration

over $\delta\mathbf{n}$, we find that at long length scales, $\delta\mathbf{n}$ fluctuations are constrained to follow $\boldsymbol{\nabla}_\perp u$. The resulting effective elastic Hamiltonian is then obtained by the replacement

$$\delta\mathbf{n}\to\boldsymbol{\nabla}_\perp u, \tag{3.3}$$

everywhere in the Eq. (3.2). This is valid in the long wavelength limit, to quadratic order in gradients of u, and provided dislocations are confined. We obtain

$$H[u]=\int d^dr\left[\frac{B}{2}(\partial_z u)^2+\frac{K}{2}(\nabla_\perp^2 u)^2+\mathbf{h}(\mathbf{r})\cdot\boldsymbol{\nabla}_\perp u - |\psi_0|U(\mathbf{r})\cos\{q_0[u(\mathbf{r})+z]\}\right] \tag{3.4}$$

or, equivalently, in terms of $V(\mathbf{r})$

$$H[u]=\int d^dr\left[\frac{B}{2}(\partial_z u)^2+\frac{K}{2}(\nabla_\perp^2 u)^2+\mathbf{h}(\mathbf{r})\cdot\boldsymbol{\nabla}_\perp u - |\psi_0|[V(\mathbf{r})e^{iq_0u(\mathbf{r})}+V^*(\mathbf{r})e^{-iq_0u(\mathbf{r})}]\right]. \tag{3.5}$$

In the above $K=K_s$, although, as discussed in the beginning of Sec. II, in a model of the NA transition that is more general than the de Gennes model, K and K_s can differ by a *singular* function of the reduced temperature $|T-T_{\rm NA}|$.

To compute self-averaging quantities (e.g., the disorder averaged free energy) we employ the replica ''trick,''[39] which allows us to work with a translationally invariant field theory at the expense of introducing n replica fields (with the $n\to0$ limit to be taken at the end of the calculation). For the free energy this procedure relies on the identity for the $\ln(x)$ function

$$\overline{F}=-T\,\overline{\ln Z}=-T\lim_{n\to0}\frac{\overline{Z^n}-1}{n}. \tag{3.6}$$

After replicating and integrating over the disorder using Eqs. (2.13), (2.14), and (2.22), we obtain

$$\overline{Z^n}=\int[du_\alpha]e^{-H[u_\alpha]/T}. \tag{3.7}$$

The effective translationally invariant replicated Hamiltonian $H[u_\alpha]$ is given by

$$H[u_\alpha]=\int d^dr\left[\sum_{\alpha=1}^n\left(\frac{K}{2}(\nabla_\perp^2u_\alpha)^2+\frac{B}{2}(\partial_z u_\alpha)^2\right)+\frac{1}{T}\sum_{\alpha,\beta=1}^n\left(\frac{\Delta_h}{4}|\boldsymbol{\nabla}_\perp(u_\alpha-u_\beta)|^2 - \Delta_V\cos[q_0(u_\alpha-u_\beta)]\right)\right], \tag{3.8}$$

where $\Delta_V=|\psi_0|^2W$. The statistical symmetry under global rotation forces the disorder generated replica off-diagonal terms to be invariant under $u_\alpha(\mathbf{r})\to u_\alpha(\mathbf{r})+\boldsymbol{\theta}\cdot\mathbf{r}_\perp$. In the replicated effective Hamiltonian Eq. (3.8) the nonlinearities only depend on the difference between different replica fields and therefore do not depend on the ''center of mass'' field $\Sigma_{\alpha=1}^n u_\alpha$, which is therefore a noninteracting field. This implies an exact result that K and B are not renormalized by the disorder in this harmonic approximation.[2] Δ_h, of course, will be renormalized by the random-field nonlinearity.

It is easy to see that the random tilt disorder term with coefficient Δ_h can be rewritten in Fourier space as

$$\frac{\Delta_h}{2}q_\perp^2\sum_{\alpha,\beta}|u_\alpha-u_\beta|^2=\sum_{\alpha,\beta}\Delta_h q_\perp^2[-1+n\,\delta_{\alpha\beta}]u_\alpha u_\beta, \tag{3.9}$$

which, in the $n\to0$ limit leads to the quadratic part of the Hamiltonian

$$H_0[u_\alpha]=\frac{1}{2}\int d^dq\sum_{\alpha,\beta}^n\left[(Kq_\perp^4+Bq_z^2)\,\delta_{\alpha\beta}-\frac{\Delta_h}{T}q_\perp^2\right]u_\alpha u_\beta, \tag{3.10}$$

from which the propagator $G_{\alpha\beta}(\mathbf{q})$ defined through

$$\langle u_\alpha(\mathbf{q})u_\beta(\mathbf{q}')\rangle=G_{\alpha\beta}(\mathbf{q})\,\delta^d(\mathbf{q}+\mathbf{q}') \tag{3.11}$$

can be easily obtained

$$G_{\alpha\beta}(\mathbf{q})=\frac{T\delta_{\alpha\beta}}{Kq_\perp^4+Bq_z^2}+\frac{\Delta_h q_\perp^2}{(Kq_\perp^4+Bq_z^2)^2}, \tag{3.12}$$

using an identity for inverting matrices of the type

$$A_{\alpha\beta}=a\,\delta_{\alpha\beta}+b, \tag{3.13}$$

namely,

$$A^{-1}_{\alpha\beta}=\frac{1}{a}\delta_{\alpha\beta}-\frac{b}{a(a+bn)},$$

$$\underset{n\to0}{=}\frac{1}{a}\delta_{\alpha\beta}-\frac{b}{a^2}. \tag{3.14}$$

In three dimensions, for $\Delta_h=\Delta_V=0$ (disorder-free liquid crystal) at low temperatures ($T<T_{\rm NA}$), the smectic-A phase is described by a fixed plane, defined by the bare values of K and B. Our initial goal is to establish how this fixed plane is destabilized by the disorder, i.e., by the random field and random tilt terms. The calculation is a generalization of that for the 2D random-field XY model[8,2] to the anisotropic elasticity of the smectic-A in three dimensions. We employ the standard momentum shell renormalization group transformation,[40] by writing the displacement field as

$$u_\alpha(\mathbf{r})=u_\alpha^<(\mathbf{r})+u_\alpha^>(\mathbf{r}), \tag{3.15}$$

integrating perturbatively in Δ_V the high wave-vector part $u_\alpha^>(\mathbf{r})$, nonvanishing inside a thin cylindrical momentum shell

$$\Lambda e^{-l}<|\mathbf{q}_\perp|<\Lambda, \tag{3.16a}$$

$$-\infty<q_z<\infty, \tag{3.16b}$$

and rescaling the lengths and long wavelength part of the fields with

$$r_\perp = e^l r'_\perp , \tag{3.17a}$$

$$z = e^{\omega l} z', \tag{3.17b}$$

$$u_\alpha^<(\mathbf{r}) = e^{\phi l} u_\alpha(\mathbf{r}'), \tag{3.17c}$$

so as to restore the ultraviolet cutoff back to Λ. Because the random-field nonlinearity is a periodic function, it is convenient (but not necessary) to take the arbitrary field dimension $\phi = 0$, thereby preserving the period $2\pi/q_0$ under the renormalization group transformation.[41] Under this transformation the resulting effective Hamiltonian functional can be restored into its original form Eq. (3.8) with effective l-dependent couplings. We relegate the details of these calculations to Appendix A, and focus here on the results. These can be succinctly summarized in the renormalization group flow equations

$$\frac{d\Delta_V(l)}{dl} = (2+\omega-\eta)\Delta_V - A_1\Delta_V^2, \tag{3.18a}$$

$$\frac{dK(l)}{dl} = (\omega-2)K, \tag{3.18b}$$

$$\frac{dB(l)}{dl} = (2-\omega)B, \tag{3.18c}$$

$$\frac{d\Delta_h(l)}{dl} = \omega\Delta_h + A_2\Delta_V^2, \tag{3.18d}$$

where we have defined

$$\eta = \frac{q_0^2 T}{4\pi\sqrt{KB}}, \tag{3.19a}$$

$$A_1 = \frac{q_0^4}{8\pi\Lambda^4\sqrt{K^3B}}, \tag{3.19b}$$

$$A_2 = \frac{c\, q_0^6}{4\pi\Lambda^6\sqrt{K^3B}}, \tag{3.19c}$$

and c is a dimensionless number of order 1. Note that $d\eta(l)/dl = 0$, exactly.

As discussed above, the symmetry of the effective Hamiltonian H in Eq. (3.8) guarantees that the flow equations for $K(l)$ and $B(l)$ are *exact* (arising from simple length rescaling with *no diagrammatic* corrections), ignoring (for now) the effects of both anharmonic elastic terms and topological defect loops in u; the latter become important at high temperatures, where they induce the NA transition, by driving the bulk modulus B to zero.[16,42] It is convenient to choose the anisotropy exponent $\omega = 2$, because such a choice keeps $K(l)$ and $B(l)$, Eqs. (3.18b),(3.18c), fixed under the RG. Although it *appears* from Eq. (3.18a) that the glass transition temperature (relevance and irrelevance of Δ_V) depends on the arbitrary choice of ω, it does not. This choice is actually completely arbitrary and does not affect any *physical* quantities. This can be seen by looking at the proper *dimensionless* coupling constant

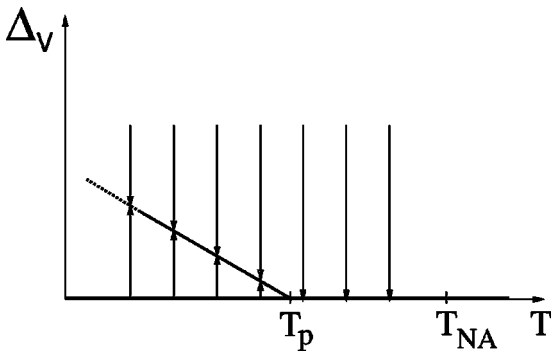

FIG. 6. A fixed line characterizing the pinned glassy phase of a three-dimensional smectic in aerogel. $T_p = 16\pi\sqrt{KB}/q_0^2$ is the pinning transition temperature, taking place *within* the 3D SBG phase. This transition only survives within the harmonic elastic approximation.

$$\tilde{\Delta}_V \equiv A_1\Delta_V, \tag{3.20}$$

whose recursion flow equation can be easily obtained by combining Eqs. (3.18a), (3.18b), and (3.18c)

$$\frac{d\tilde{\Delta}_V(l)}{dl} = (4-\eta)\tilde{\Delta}_V - \tilde{\Delta}_V^2,$$

$$= [4-\eta-\tilde{\Delta}_V]\tilde{\Delta}_V. \tag{3.21}$$

Obviously this flow equation is independent of the arbitrary rescaling exponent ω and has the same form as that for Δ_V, Eq. (3.18a), with $\omega = 2$.

From Eq. (3.21), we then find that for $\eta < 4$ (large elastic moduli and low temperature), or, equivalently, below the pinning transition temperature T_p, given by

$$T_p = \frac{16\pi\sqrt{KB}}{q_0^2}, \tag{3.22}$$

the smectic ($\Delta_V = 0$) fixed plane is unstable to disorder. However, the initial runaway of disorder is halted by the nonlinear terms in $\tilde{\Delta}_V$, which terminate the flow at a new finite disorder fixed line,

$$\tilde{\Delta}_V^*(T) = 4-\eta. \tag{3.23}$$

This new fixed line then describes a randomly pinned, glassy smectic-A phase, analogous to the 2D super-rough phase of crystal surface on a random substrate[2] and the 1+1 vortex glass phase of flux-line vortices (confined to a plane) in type II superconductors.[3] This RG flow structure is summarized in Fig. 6.

It is important to note that the transition at T_p, illustrated in Fig. 6, is taking place *within* the SBG phase. That is, T_p is a transition from the ($T > T_p$) SBG phase, within which the translational random field disorder Δ_V is unimportant at long scales, to the ($T < T_p$) SBG phase, in which Δ_V is relevant

and flows to a nontrivial fixed line. This T_p pinning transition must not be confused with the distinct higher temperature transition, occuring just below $T_{\rm NA}$, in which the putative unpinned ($T>T_p$) SBG ''melts'' into the NEG by unbinding of dislocations. It is most certain that the transition at T_p, derived here within the harmonic elastic approxi mation, will be converted into a crossover by the neglected elastic nonlinearities, analyzed in Secs. VI–VIII. However, it is possible that a rounded ''ghost'' of the transition at T_p will persist in our full theory and will therefore be experimentally observable.

It is enlightening to contrast these RG flows, with their perturbative fixed line, Eq. (3.23), at which the relevant flows of $\tilde{\Delta}_V$ terminate, with those of disorder-free *thermal* sine-Gordon models (e.g., *thermal* roughening transition),[43] where the cosine coupling runs away to strong coupling at low temperatures. In these latter systems the cosine potential becomes relevant upon lowering the temperature, with the system settling down in one of the minima of this *periodic* pinning potential. As the cosine coupling continues to grow, it further reduces thermal fluctuations, suppressing their ability to average away its pinning effects, thereby further increasing its pinning influence. Mathematically this manifests itself in the *positive* nonlinear contribution to the flow equation for the cosine coupling, which results in the absence of a stable *finite* coupling fixed line. In contrast, for disordered problems of the type considered here, as the disorder becomes relevant at low temperatures and begins to grow, it leads to fluctuations (roughening) that are larger than those from purely thermal fluctuations.[2] This disorder-enhanced roughening subsequently leads to a more effective averaging away of the pinning potential, thereby suppressing its effects beyond a certain strength. Mathematically, this is captured by the *negative* sign of the nonlinear $\tilde{\Delta}_V^2$ term in the flow Eq. (3.21), which results in the termination of the flow of $\tilde{\Delta}_V$ and the glassy fixed line described by Eq. (3.23).

The flow Eq. (3.21) also implies that the random-field disorder is irrelevant for $\eta>4$. Since the bulk modulus B vanishes, while K remains finite through $T_{\rm NA}$, η diverges as $T\to T_{\rm NA}^-$, and hence, sufficiently close to the NA transition temperature $T_{\rm NA}$, we are *guaranteed* to have a range of temperatures (within the smectic phase) over which the random-field disorder is irrelevant. However, as we will see below, because tilt disorder Δ_h is a strongly relevant perturbation, the three-dimensional quasi-long-range smectic order for $\eta>4$ will be converted into short-range correlations, even when the random field disorder Δ_V is irrelevant.

It is essential to stress that the renormalization group flow diagram described above (i.e., relevance for $\eta<4$ and irrelevance for $\eta>4$ of the random-field disorder) survives even despite the strong relevance and runaway of the random tilt coupling Δ_h. This occurs because the dimension of the $\cos[q_0(u_\alpha-u_\beta)]$ operator (the random field disorder) is *independent* of Δ_h, to all orders in perturbation theory in Δ_V. This can be easily seen to first order in Δ_V

$$\begin{aligned}\langle -H_{\rm int}\rangle &= \frac{\Delta_V}{T}\int d^3r \sum_{\alpha\beta}\langle \cos[q_0(u_\alpha-u_\beta)]\rangle_>,\\ &= \mathrm{Re}\frac{\Delta_V}{T}\int d^3r\sum_{\alpha\beta} e^{iq_0(u_\alpha^<-u_\beta^<)}\langle e^{iq_0(u_\alpha^>-u_\beta^>)}\rangle_>,\\ &= \frac{\Delta_V}{T}\int d^3r\sum_{\alpha\beta}\cos[q_0(u_\alpha^<-u_\beta^<)]e^{-f_{\alpha\beta}},\\ &= \frac{\Delta_V}{T}\int d^3r\sum_{\alpha\beta}\cos[q_0(u_\alpha^<-u_\beta^<)]e^{-\eta l(1-\delta_{\alpha\beta})},\end{aligned} \tag{3.24}$$

where

$$f_{\alpha\beta}=q_0^2[G^>_{\alpha\alpha}(r=0)-G^>_{\alpha\beta}(r=0)], \tag{3.25}$$

in $G^>_{\alpha\alpha}(0)$ there is no implied sum over α, and

$$G^>_{\alpha\beta}(\mathbf{r})=\int_{\Lambda e^{-l}}^{\Lambda}\frac{d^2q_\perp}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{dq_z}{2\pi}G_{\alpha\beta}(q)e^{i\mathbf{q}\cdot\mathbf{r}}. \tag{3.26}$$

Using

$$G_{\alpha\alpha}(q)-G_{\alpha\beta}(q)=\frac{T}{Kq_\perp^4+Bq_z^2}(1-\delta_{\alpha\beta}), \tag{3.27}$$

which is obviously independent of the tilt-disorder coupling Δ_h, and Fourier transforming, we obtain the Δ_h-independent η given in Eq. (3.19a). Performing length rescalings, Eqs. (3.17), to restore the new cutoff Λe^{-l} back to Λ gives, to first order in Δ_V, the flow equation for Δ_V, Eq. (3.18a).

To see that the recursion relation for Δ_V, Eq. (3.21), is independent of Δ_h to *all* orders in Δ_V, in this otherwise harmonic theory, one can return to the nonreplicated Hamiltonian, Eq. (3.5) and completely eliminate the random tilt field $\mathbf{h}(\mathbf{r})$ via a change of variables

$$u(\mathbf{r})\equiv u'(\mathbf{r})+f(\mathbf{r}), \tag{3.28}$$

with $f(\mathbf{r})$ completely determined by the random tilt field $\mathbf{h}(\mathbf{r})$ via

$$(-B\partial_z^2+K\nabla_\perp^4)f(\mathbf{r})=\boldsymbol{\nabla}_\perp\cdot\mathbf{h}(\mathbf{r}). \tag{3.29}$$

Inserting Eq. (3.28) into Eq. (3.5), and dropping terms that only depend on the random tilt field $\mathbf{h}(\mathbf{r})$, since, as discussed earlier, these have no effect on *any* correlation functions, we have

$$H=\int d^d\mathbf{r}\left[\frac{B}{2}(\partial_z u')^2+\frac{K}{2}(\nabla_\perp^2u')^2+\mathbf{h}(\mathbf{r})\cdot\boldsymbol{\nabla}_\perp u'+B\partial_z f\partial_z u'+K\nabla_\perp^2 f\nabla_\perp^2u'-|\psi_0|[V'(\mathbf{r})e^{iq_0u'(\mathbf{r})}+V'^*(\mathbf{r})e^{-iq_0u'(\mathbf{r})}]\right], \tag{3.30}$$

where we have defined $V'(\mathbf{r}) \equiv e^{iq_0 f(\mathbf{r})} V(\mathbf{r})$. Note that the *statistics* of $V'(\mathbf{r})$ are exactly the same as the *statistics* of $V(\mathbf{r})$, since all we have done is added a new random phase factor to $V(\mathbf{r})$. But since the phase of $V(\mathbf{r})$ was uniformly distributed between 0 and 2π, the phase of $V'(\mathbf{r})$ will also be. Hence, its statistics are the same. Therefore, the model Eq. (3.30) is *only* affected by the presence of the random tilt field $\mathbf{h}(\mathbf{r})$ through the terms

$$\Delta H = \int d^d\mathbf{r}[\mathbf{h}(\mathbf{r})\cdot\boldsymbol{\nabla}_\perp u' + B\partial_z f \partial_z u' + K\nabla_\perp^2 f \nabla_\perp^2 u'], \tag{3.31}$$

which, after integration by parts, vanish by virtue of the choice Eq. (3.29). Thus, the new Hamiltonian is unaffected by the random field $\mathbf{h}(\mathbf{r})$. Since the partition function is obviously invariant under a change of variables, it, and, therefore, the recursion relations for the parameters B, K, and most importantly Δ_V, cannot be affected by $\mathbf{h}(\mathbf{r})$, and hence are independent of Δ_h. The results described above for the disordered three-dimensional smectic-A phase are quite closely analogous to those of Cardy and Ostlund for random symmetry breaking fields in the two-dimensional XY model,[8] except that for pinned smectics, the random-field disorder (the cosine) is guaranteed to become irrelevant as $T \to T_{\mathrm{NA}}^-$ since $\eta(T \to T_{\mathrm{NA}}) \to \infty$.

As can be seen from the detailed renormalization group (RG) analysis of Appendix A and Eq. (3.18d), even if the bare $\Delta_h = 0$ (i.e., no initial tilt disorder), this type of disorder is generated by the random-field disorder (Δ_V) after the high-wave-vector degrees of freedom are integrated out. In contrast to the 2D random-field XY model, where the generated Δ_h disorder is only marginally relevant and only weakly affects the quasi-long-range order (QLRO) found for $\Delta_h = 0$ [converting $\ln(r)$ phase correlations to $\ln^2(r)$], for the 3D smectic-A phase the Δ_h tilt disorder is strongly relevant. As we will see in the next section, the dimensionless coupling that determines the effect of the tilt disorder is

$$\tilde{\Delta}_h \equiv A_1 \Delta_h, \tag{3.32}$$

From the recursion relations Eqs. (3.18a)–(3.18d) we find

$$\frac{d\tilde{\Delta}_h}{dl} = 2\tilde{\Delta}_h + \frac{2cq_0^2}{\Lambda^2}\tilde{\Delta}_V^2. \tag{3.33}$$

For $\eta > 4$, $\tilde{\Delta}_V(l) \to 0$ (as we have seen), and so $d\tilde{\Delta}_h/dl = 2\tilde{\Delta}_h$, which is trivially solved to give

$$\tilde{\Delta}_h(l) = \tilde{\Delta}_h e^{2l}, \tag{3.34}$$

where $\tilde{\Delta}_h$ is the nonuniversal dimensionless ''bare'' coupling, Eq. (3.32). Thus, we see that the tilt disorder is strongly relevant, in contrast to the behavior of the two-dimensional random field XY model,[8] for which the tilt disorder was marginal, in the RG sense, in the phase in which the cosine was irrelevant.

For $\eta < 4$, $\tilde{\Delta}_V \to \tilde{\Delta}_V^* > 0$, with $\tilde{\Delta}_V^*$ given by Eq. (3.23). Now, in the 2D random field XY model, the existence of a nonzero $\tilde{\Delta}_V^*$ in the low-temperature phase implied completely different behavior for the tilt coupling $\tilde{\Delta}_h(l)$ than in the high-temperature phase: a runaway of $\tilde{\Delta}_h(l)$ to infinity as $l \to \infty$ in the low-temperature phase, as opposed to a constant $\tilde{\Delta}_h(l \to \infty)$ in the high-temperature phase.[2] In *our* problem, however, $\tilde{\Delta}_h(l)$ runs away to infinity in *both* phases, and *asymptotically in exactly the same way*. One can see this by solving Eq. (3.33) with $\tilde{\Delta}_V(l)$ replaced by its nonzero fixed point value $\tilde{\Delta}_V^*$ given by Eq. (3.23), finding

$$\tilde{\Delta}_h(l) = \left[\tilde{\Delta}_h + \frac{cq_0^2}{\Lambda^2}(\tilde{\Delta}_V^*)^2\right]e^{2l} - \frac{cq_0^2}{\Lambda^2}(\tilde{\Delta}_V^*)^2, \tag{3.35}$$

which has the same asymptotic behavior as $l \to \infty$ (namely, $\tilde{\Delta}_{h0} e^{2l}$) as in the low-temperature phase. Nonuniversal constants (such as $\tilde{\Delta}_{h0}$) can be different in the low-temperature phase, but the scaling (e^{2l}) is not. We will see in the next section that this implies that *equal time* correlation functions scale in exactly the same way in both the pinned ($\eta < 4$) and nonpinned ($\eta > 4$) phase. The only difference between these two phases is in their dynamics, which are divergently slower in the randomly pinned phase, as we shall show in a future publication.[26]

Physically the strong relevance of Δ_h is expected, because in the smectic-A phase the rotational invariance is *spontaneously* broken. Since each realization of the random tilt disorder *explicitly* breaks the rotational invariance (preserving it only statistically), the smectic layer orientation has a divergent response to Δ_h tilt disorder. At long scales where $\Delta_h \approx 0$ is no longer valid, the *quasi-long-range* order, implied by the nontrivial fixed line, Eq. (3.23), will undergo a rapid crossover to short-range correlations of $\psi(\mathbf{r})$, even in the regime ($\eta > 4$) where the random-field disorder (which always generates random tilt Δ_h) is irrelevant.

IV. RANDOM TILT-ONLY HARMONIC ELASTIC (TOPOLOGICALLY ORDERED) MODEL

For finite Δ_h both the random field (Δ_V) and random tilt (Δ_h) disorders must be treated simultaneously. Since, as described above, our conclusions about the phase diagram for Δ_V are not affected by the tilt disorder Δ_h, we can study the effect of Δ_h on smectic-A order within the regime $\eta > 4$, where the random-field disorder is irrelevant. In this regime, at long enough scales the smectic-A phase is effectively subjected to tilt disorder only. We can therefore safely set $\Delta_V = 0$ and analyze exactly the remaining quadratic theory for any value of Δ_h. As argued in the previous section, and to be shown in the next, the scaling of the results we find here will also apply in the pinned phase where $\eta < 4$ and the cosine disorder is relevant.

Since, within the harmonic-elastic approximation of this section, the effective Hamiltonian is quadratic in the displacement fields u, when $\Delta_V = 0$ all the correlation functions can be computed exactly. In particular the quantity of interest is

$$C(\mathbf{r}_\perp, z) = \overline{\langle [u(\mathbf{r}_\perp, z) - u(\mathbf{0}, 0)]^2 \rangle}. \tag{4.1}$$

Fourier transforming the u fields, we get

$$C(\mathbf{r}_\perp,z)=2\int\frac{d^2q_\perp dq_z}{(2\pi)^3}(1-e^{i\mathbf{q}\cdot\mathbf{r}})\frac{\overline{\langle u(\mathbf{q})u(\mathbf{q}')\rangle}}{\delta^3(\mathbf{q}+\mathbf{q}')}. \quad (4.2)$$

We can write the quenched and thermal averaged $\overline{\langle u(\mathbf{q})u(\mathbf{q}')\rangle}$ in terms of their replicated correlation function:

$$\begin{aligned}\overline{\langle u(\mathbf{q})u(\mathbf{q}')\rangle}&=\langle u_\alpha(\mathbf{q})u_\alpha(\mathbf{q}')\rangle,\\&=\delta^3(\mathbf{q}+\mathbf{q}')G_{\alpha\alpha}(\mathbf{q}),\end{aligned} \quad (4.3)$$

where no sum on α is implied, and the replica propagator $G_{\alpha\beta}$ is given by Eq. (3.12). Using that equation, and Eq. (4.3) for the correlation function, gives

$$C(\mathbf{r}_\perp,z)=C_T(\mathbf{r}_\perp,z)+C_\Delta(\mathbf{r}_\perp,z), \quad (4.4)$$

where we have separated $C(\mathbf{r}_\perp,z)$ into ''thermal'' C_T and ''disorder'' (frozen) C_Δ parts, given by

$$\begin{aligned}C_T&=2T\int\frac{d^2q_\perp dq_z}{(2\pi)^3}\frac{1-e^{i\mathbf{q}\cdot\mathbf{r}}}{Kq_\perp^4+Bq_z^2},\\&=\frac{T}{2\pi\sqrt{KB}}g_T\left(\frac{z\lambda}{r_\perp^2},\frac{r_\perp}{a}\right),\\&=\frac{T}{2\pi\sqrt{KB}}\left[\ln\left(\frac{r_\perp}{a}\right)-\frac{1}{2}Ei\left(\frac{-r_\perp^2}{4\lambda|z|}\right)\right],\end{aligned} \quad (4.5)$$

and

$$\begin{aligned}C_\Delta&=2\Delta_h\int\frac{d^2q_\perp dq_z}{(2\pi)^3}\frac{q_\perp^2(1-e^{i\mathbf{q}\cdot\mathbf{r}})}{(Kq_\perp^4+Bq_z^2)^2},\\&=\frac{\Delta_h}{32\pi B^2\lambda^3}\Bigg\{4\lambda|z|e^{-r_\perp^2/4\lambda|z|}\\&\quad+r_\perp^2\left[Ei\left(\frac{-4\lambda L_z}{L_\perp^2}\right)+Ei\left(\frac{-r_\perp^2}{4\lambda|z|}\right)\right.\\&\quad\left.+2\ln\left(\frac{L_\perp}{r_\perp}\right)\right]\Bigg\},\end{aligned} \quad (4.6)$$

where $Ei(x)$ is the exponential integral function, and, as promised, the effective dimensionless coupling $\tilde{\Delta}_h$, defined in Eq. (3.32), naturally appears above. In the above, we have considered a finite system whose shape is a rectangular parallelepiped of linear dimensions $L_\perp\times L_\perp\times L_z$, L_z being the length of the system along the ordering (z) direction. Unless it has a huge aspect ratio, such that $L_z\sim L_\perp^2/\lambda\gg L_\perp$, any large system ($L_\perp,L_z\gg\lambda$) will have $\lambda L_z\ll L_\perp^2$. In this limit, the asymptotic behaviors of C_Δ are

$$C_\Delta\approx\begin{cases}\dfrac{\Delta_h}{32\pi B^2\lambda^3}\left[4\lambda|z|+r_\perp^2\ln\left|\dfrac{L_z}{z}\right|\right], & \lambda|z|\gg r_\perp^2,\\ & \lambda L_z\ll L_\perp^2,\\ \dfrac{\Delta_h}{16\pi B^2\lambda^3}r_\perp^2\ln\left(\dfrac{2\sqrt{\lambda L_z}}{r_\perp}\right), & \lambda|z|\ll r_\perp^2,\\ & \lambda L_z\ll L_\perp^2.\end{cases} \quad (4.7)$$

Note that, although in principle the second ($r_\perp^2\ln|L_z/z|$) term in the $\lambda|z|\gg r_\perp^2$ expansion of C_Δ dominates the $\lambda|z|$ term in the thermodynamic limit $L_z\to\infty$ (taking that limit at fixed $r_\perp$ and z), in practice, for any reasonable system size L_z, the first ($\lambda|z|$) term actually dominates if $\lambda|z|/r_\perp^2$ gets appreciably bigger than 1.

An unusual feature of this result is that not only do the mean squared fluctuations of u at a given point in space diverge as a function of system size, but even the *relative* displacement of two points with *finite* separations $(r_\perp,z)$ diverge as the system sizes $(L_\perp,L_z)$ go to infinity. This is because the mean squared real space *orientational* fluctuations $\overline{\langle|\delta\mathbf{n}(\mathbf{r})|^2\rangle}$ also diverge as $L_{\perp,z}\to\infty$:

$$\begin{aligned}\langle|\delta\mathbf{n}(\mathbf{r})|^2\rangle&=\langle|\boldsymbol{\nabla}_\perp u(\mathbf{r})|^2\rangle,\\&=\Delta_h\int\frac{d^2q_\perp dq_z}{(2\pi)^3}\frac{q_\perp^4}{(Kq_\perp^4+Bq_z^2)^2},\\&=\frac{\Delta_h}{16\pi^2B^2\lambda^3}\int_{q_\perp>\max[(\lambda L_z)^{-1/2},L_\perp^{-1}]}\frac{d^2q_\perp}{q_\perp^2},\\&=\frac{\Delta_h}{8\pi B^2\lambda^3}\ln(\min[\sqrt{\lambda L_z},L_\perp]).\end{aligned} \quad (4.8)$$

Although not experimentally relevant, it is instructive to generalize this calculation to spatial dimensions $d>3$. Keeping only the tilt disorder (we will return to justify this later), and repeating the calculation just presented, we find

$$C_\Delta(\mathbf{r}_\perp,z)\propto\begin{cases}\Delta_h|z|^{(5-d)/2}, & \lambda|z|\gg r_\perp^2,\\ \Delta_h r_\perp^{5-d}, & \lambda|z|\ll r_\perp^2,\end{cases} \quad (4.9)$$

which diverges at large distances for all $d<5$. This divergence signals the destruction by tilt disorder of the (quasi-) long-ranged smectic translational order for $d<5$. Note that this divergence occurs even for arbitrarily weak disorder (i.e., arbitrarily small Δ_h). This agrees with the experimental observation that, even when the aerogel density becomes very low, smectic translational order is still destroyed, as manifested in the nonzero width of the x-ray scattering peaks associated with the smectic layering.

How are these results, Eq. (4.1), for the equal time correlation functions modified by the presence of the cosine (random field) disorder term in Eqs. (3.4),(3.8)? Aside from a temperature-dependent modification of the prefactor $\Delta_h\to\Delta_h+\text{const}(T_p-T)^2$, below $d=5$, they are not modified at all. We will now demonstrate this fact in $d=3$, and defer to Sec. VIII the demonstration for $3<d<5$.

For $\eta>4$, the cosine clearly does not affect the long distance behavior, since its coefficient Δ_V is irrelevant (i.e., flows to zero under renormalization). Furthermore, we need not worry that it might be *dangerously* irrelevant, since we have just calculated correlation functions with Δ_V set equal to zero, and encountered no difficulties.

What happens with $\eta<4$? For η near 4 $(4-\eta\ll 1)$, the fixed point value $\tilde{\Delta}_V^*=4-\eta$ of $\tilde{\Delta}_V$ is also $\ll 1$, and our perturbative (in $\tilde{\Delta}_V$) renormalization group remains valid. We can therefore use this RG to calculate the equal time correlations for $\eta<4$ and $4-\eta\ll 1$. This calculation must itself be done perturbatively in $\tilde{\Delta}_V^*$, since the full Hamiltonian with $\tilde{\Delta}_V\neq 0$ is not quadratic. However, such a perturbation theory diverges for small wave vectors when $\eta<4$. Indeed, the growth under the RG of small $\tilde{\Delta}_V$ for $\eta<4$ is a signal and a consequence of this divergence. Fortunately, we can use the renormalization group transformation to relate the correlation functions at small wave vector to those at large wave vector, where perturbation theory should be reliable. Indeed, simply repeating for correlation functions the rescalings done earlier for the partition function, we can show that

$$\begin{aligned} &C_{\alpha\beta}[\mathbf{q};B(0),K(0),\Delta_h(0),\Delta_V(0)] \\ &\quad\equiv\frac{\langle u_\alpha(\mathbf{q})u_\beta(\mathbf{q}')\rangle}{\delta^3(\mathbf{q}+\mathbf{q}')} \\ &\quad= e^{(2+\omega)l}C_{\alpha\beta}[e^l\mathbf{q}_\perp,e^{\omega l}q_z;B(l),K(l),\Delta_h(l),\Delta_V(l)]. \end{aligned} \tag{4.10}$$

Now, in order to insure that the rescaled correlation function on the right-hand side of this expression can be safely evaluated using perturbation theory in $\tilde{\Delta}_V(l)$, we will choose l such that $e^l q_\perp=\Lambda$, the ultraviolet cutoff. With this choice, we have

$$C_{\alpha\beta}[\mathbf{q};B(0),K(0),\Delta_h(0),\Delta_V(0)]=\left(\frac{\Lambda}{q_\perp}\right)^{2+\omega}C_{\alpha\beta}[\Lambda,(\Lambda/q_\perp)^\omega q_z;B(l^*),K(l^*),\Delta_h(l^*),\Delta_V(l^*)], \tag{4.11}$$

where $l^*\equiv\ln(\Lambda/q_\perp)$, and $B\equiv B(0)$, $K\equiv K(0)$, $\Delta_h\equiv\Delta_h(0)$, $\Delta_V\equiv\Delta_V(0)$. Note that $l^*\to\infty$ as $q_\perp\to 0$.

To calculate the right-hand side of Eq. (4.11) in perturbation theory in $\Delta_V(l^*)$, we expand the cosine in the full Hamiltonian Eq. (3.8) to quadratic order, obtaining, in Fourier space

$$\begin{aligned} H[u_\alpha]=\frac{1}{2}\int_{\mathbf{q}}\Bigg[&(Bq_z^2+Kq_\perp^4)\sum_{\alpha=1}^{n}|u_\alpha(\mathbf{q})|^2 \\ &+\frac{1}{2}(\Delta_h q_\perp^2+2\Delta_V q_0^2)\sum_{\alpha,\beta=1}^{n}|u_\alpha(\mathbf{q})-u_\beta(\mathbf{q})|^2\Bigg], \end{aligned} \tag{4.12}$$

which we immediately recognize as *identical* to the tilt-only Hamiltonian Eq. (3.10), *except* for the replacement $\Delta_h\to\Delta_h+2\Delta_V q_0^2/q_\perp^2$. Thus, we can immediately calculate the correlation function on the right-hand side of Eq. (4.11) by simply making this replacement in the tilt only propagator Eq. (3.12), obtaining

$$\begin{aligned} &C_{\alpha\beta}[\Lambda,(\Lambda/q_\perp)^\omega q_z;B(l^*),K(l^*),\Delta_h(l^*),\Delta_V(l^*)] \\ &\quad=\frac{T\delta_{\alpha\beta}}{K(l^*)\Lambda^4+B(l^*)(\Lambda/q_\perp)^{2\omega}q_z^2} \\ &\qquad+\frac{\Delta_h(l^*)\Lambda^2+2\Delta_V(l^*)q_0^2}{[K(l^*)\Lambda^4+B(l^*)(\Lambda/q_\perp)^{2\omega}q_z^2]^2}. \end{aligned} \tag{4.13}$$

To finish writing this expression entirely in terms of $\mathbf{q}$, we need to calculate the Hamiltonian parameters $B(l)$, $K(l)$, $\Delta_h(l)$, and $\Delta_V(l)$ from the recursion relations Eqs. (3.18a)–(3.18d) and evaluate them at $l^*=\ln(\Lambda/q_\perp)$. We find

$$B(l^*)=\left(\frac{\Lambda}{q_\perp}\right)^{2-\omega}B, \tag{4.14a}$$

$$K(l^*)=\left(\frac{\Lambda}{q_\perp}\right)^{\omega-2}K, \tag{4.14b}$$

$$\begin{aligned} \Delta_h(l^*)&=\frac{\tilde{\Delta}_h(l^*)}{A_1(l^*)}, \\ &=\left(\frac{\Lambda}{q_\perp}\right)^{\omega}\left[\Delta_h+\frac{c\,q_0^2}{\Lambda^2A_1}\tilde{\Delta}_V^{*2}\right], \end{aligned} \tag{4.14c}$$

$$\begin{aligned} \Delta_V(l^*)&=\frac{\tilde{\Delta}_V(l^*)}{A_1(l^*)}, \\ &=\left(\frac{\Lambda}{q_\perp}\right)^{\omega-2}\frac{\tilde{\Delta}_V^*}{A_1}, \end{aligned} \tag{4.14d}$$

where $A_1\equiv A_1(l=0)$, and in this last equation we have assumed that $q_\perp$ is sufficiently small that $l^*=\ln(\Lambda/q_\perp)$ is sufficiently *large* that $\tilde{\Delta}_V(l^*)$ will have flown to very near its η-dependent fixed point value $\tilde{\Delta}_V^*$, as given by Eq. (3.23). Likewise, in the expression for $\tilde{\Delta}_h(l)$, we have used the solution Eq. (3.35) for $\tilde{\Delta}_h(l)$, the effective tilt coupling constant, and again assumed that $l^*=\ln(\Lambda/q_\perp)$ is very large, so that the second (l-independent) term in Eq. (3.35) is negligible compared to the first, exponentially growing term. Both of these approximations become asymptotically *exact* as $q_\perp\to 0$, and l^*, as a result, $\to\infty$. Inserting these results for the renormalized elastic and coupling constants into the expression Eq. (4.13) for the rescaled correlation function, and us-

ing *that* result in the matching formula Eq. (4.11) for the original correlation function at small wave-vector yields, after a bit of algebra

$$C_{\alpha\beta}(\mathbf{q};B,K,\Delta_h,\Delta_V) = \frac{T\delta_{\alpha\beta}}{Kq_\perp^4+Bq_z^2} + \frac{(\Delta_h+cq_0^2\tilde{\Delta}_V^{*2}/\Lambda^2A_1)q_\perp^2+(2q_0^2\tilde{\Delta}_V^*/A_1)(q_\perp/\Lambda)^4}{(Kq_\perp^4+Bq_z^2)^2}, \tag{4.15}$$

where all of the parameters (K,B,Δ_h,A_1) in this expression are *bare* parameters (i.e., evaluated at $l=0$). This expression is *identical* to the result in the Δ_V irrelevant phase ($\eta>4$) except for: (i) the $\tilde{\Delta}_V^*(q_\perp/\Lambda)^4$ term, which is clearly negligible (as $q_\perp\to 0$) relative to the $(\Delta_h+c\,q_0^2\tilde{\Delta}_V^{*2}/\Lambda^2A_1)q_\perp^2$ term, and (ii) the enhancement of the strength of the tilt disorder according to $\Delta_h\to\Delta_h+c\,q_0^2\tilde{\Delta}_V^{*2}/\Lambda^2A_1$. The heretofore neglected effects of the *anharmonic* terms coming from the cosine on the *rescaled* correlation function (and hence on the original one, since they are related by the matching formula), should be even smaller, since $\tilde{\Delta}_V^*$ is $\ll 1$ for $4-\eta \ll 1$. As discussed earlier, this argument *cannot* be invalidated by possible infrared divergences in the perturbation theory, since we are calculating the *rescaled* correlation function at *large* wave vector. The effects of these divergences on the short wavelength correlation functions are *implicitly* included in the matching expression Eq. (4.11) through our use of the renormalized parameters on the right-hand side, since these renormalizations include anharmonic effects.

Hence, at long wavelengths, aside from the modification of the prefactor $\Delta_h\to\Delta_h+\text{const}(T_p-T)^2$ (for $T<T_p$), we recover the same asymptotic form for the *equal-time* correlation functions in the randomly pinned ($\eta<4$) and nonpinned ($\eta>4$) phases. Although, strictly speaking, we have only derived this result near $\eta=4$, where our perturbation theory is valid, it must apply throughout the entire low-temperature phase, since *only* a phase transition can change the asymptotic scaling of the correlation functions. Barring the existence of such a phase transition controlled by some strong coupling fixed point, which we of course cannot rule out with the perturbative analysis performed here, our results for the scaling behavior of the equal-time u-u correlation functions should persist all the way down to $\eta=0$; i.e., throughout the smectic phase.

In any case, in this elastically harmonic phonon (no-dislocations) model, there is *guaranteed* to be a pinned phase below $\eta=4$ whose *static* correlation functions have $r(q)$ dependence *identical*, at long wavelengths, with those in the nonpinned phase above $\eta=4$. The distinction between these phases lies in the temperature dependence of the parameters in these static correlation functions (as we discuss below) and in their dynamical properties.

That is, it is important to note that in spite of the above finding that static correlation functions *scale* (with position or wave vector) identically in both the $\eta>4$ and $\eta<4$ phases, the existence of the transition at T_p to the low-temperature translationally pinned ($\eta<4$) phase *can* still in principle be detected in a static experiment. This can be seen by noting that in the presence of such a transition, all Δ_h-dependent physical quantities, which would otherwise be smooth functions of temperature, are *nonanalytic* in T at T_p. This is because of the additional contribution to Δ_h^{eff} in Eq. (4.15), proportional to $(\tilde{\Delta}_V^*)^2\propto|T_p-T|^2$, that arises below T_p, $\Delta_h^{\text{eff}}(T)=\Delta_h+\text{const}(T_p-T)^2$. As a result, other physical quantities derived from Δ_h will have a discontinuous second derivative. Although this is a quite subtle effect, it is an unambiguous static experimental signature of the transition to the pinned smectic glass phase at $T=T_p$. However, as briefly discussed just below Fig. 6, we demonstrate in Secs. VI–VIII that this transition, derived here within the harmonic elastic theory, will be rounded once nonlinear elastic effects are taken into account. It is, however possible, that rounded remnants of the nonanalyticities at T_p, discussed above, will still be experimentally observable.

The static correlation function derived in Eq. (4.15) shows that long-ranged smectic translational order is destroyed in $d=3$. To make quantitative comparisons with experiments, it is useful to calculate translational correlation lengths $\xi_\perp^X$ and ξ_z^X, whose inverses will give the width of the broadened x-ray diffraction peaks. These correlation lengths are the distances $r_\perp$ and z at which the mean squared relative displacement correlation function $C(\mathbf{r}_\perp,z)$ is of order a^2, where $a=2\pi/q_0$ is a lattice constant.

When there is no disorder ($\Delta_h=0$), the above criterion leads to

$$\xi_\perp^T = a e^{\pi a^2\sqrt{KB}/T}, \tag{4.16a}$$

$$= \sqrt{\lambda\xi_z^T}. \tag{4.16b}$$

As discussed in the extensive literature on the subject,[14] however, the slow (logarithmic) divergence of the thermal part C_T of C implies that, for $\eta<2$, the x-ray scattering peaks do *not*, in fact, become broad for $|\mathbf{q}_\perp|<1/\xi^T$; instead, they become power-law divergences, rather than the Lorentzians one might otherwise expect.

The more strongly divergent part C_Δ of C *does* lead to genuine broadening, for $|\mathbf{q}|<1/\xi_{\perp,z}^X$, where $\xi_{\perp,z}^X$ are defined as those length scales (i.e., values of $r_\perp$ and z) at which $C_\Delta(r_\perp,z)=a^2$. Having computed C_Δ within the elastically harmonic theory, we thereby find

$$\xi_z^X = a^2\frac{8\pi KB}{\Delta_h}, \tag{4.17a}$$

$$\xi_\perp^X = 4a\left(\frac{\pi B^2\lambda^3}{\Delta_h\ln(2\sqrt{\lambda L_z}/\xi_\perp^X)}\right)^{1/2}. \tag{4.17b}$$

In fact, we powder average the x-ray scattering, since, as we will show in a moment, the smectic in aerogel lacks long-ranged orientational order, as well as translational order. The broad *ring* of x-ray scattering that results from this averaging will have its width determined entirely by ξ_z^X:

$$\kappa_{\text{powder}} \cong (\xi_z^X)^{-1} = \frac{\Delta_h}{8\pi KBa^2}, \tag{4.18}$$

a gratifyingly simple prediction.

Although it is more difficult to measure experimentally, it is nonetheless of interest to calculate the harmonic orientational correlation lengths $\xi_{\perp,z}^{O,h}$, which are defined as the values of $L_{\perp,z}$ beyond which the mean squared orientational fluctuations $\overline{\langle|\delta\mathbf{n}|^2\rangle}$ get to be of order 1. If we ignore dislocations and anharmonic elasticity, we can obtain these lengths by simply equating the ''pure phonon'' result Eq. (4.8) for $\overline{\langle|\delta\mathbf{n}|^2\rangle}$ to 1. This gives

$$\xi_\perp^{O,h} = a e^{4\pi B^2 \lambda^3/\Delta_h} = a e^{\lambda \xi_z^X/2a^2}, \tag{4.19a}$$

$$\xi_z^{O,h} = \frac{a^2}{\lambda} e^{8\pi B^2 \lambda^3/\Delta_h} = \frac{a^2}{\lambda} e^{\lambda \xi_z^X/a^2}, \tag{4.19b}$$

In the next section, we will show that, unsurprisingly, these lengths give the distance beyond which dislocations unbind, invalidating the purely elastic phonon theory studied in this section.

Thus, orientational order persists out to *much* larger distances than translational order, in the limit of weak disorder where *all* the correlation lengths get large. Indeed, in this limit the orientational correlation lengths grow exponentially with the translational ones. This qualitatively agrees with experimental determinations of the orientational correlation length, as indirectly inferred from specific heat data and related measurements.[20]

In Sec. VI, we will show that anharmonic elastic effects, which we have ignored up to now, change the relation between the x-ray correlation length ξ_z^X and the orientational correlation lengths $\xi_{\perp,z}^O$ from exponential to power law. However, the fact that $\xi_{\perp,z}^O$ both remain $\gg \xi_z^X$ continues to hold, validating our use of elastic theory (which was predicated on the assumption that orientational fluctuations are small), to calculate the x-ray correlation lengths. In any case, it would be very interesting to measure the relation between translational and orientational correlation lengths as a function of temperature.

All of the above results apply subject to our two initial assumptions: (1) that dislocations were *not* generated by the disorder, and (2) that anharmonic terms in the *elastic* Hamiltonian could be neglected. In the next section, we will show that, if we continue to assume (2), assumption (1) is wrong: in the *harmonic* elastic approximation, in 3*d* dislocations *are* created even by arbitrarily weak disorder.

However, the effects of these dislocations turn out, in the weak disorder limit, to be felt only on length scales longer than the smaller of the orientational length $\xi_{\perp,z}^O$ and the dislocation unbinding length $\xi_{\perp,z}^D$, both of which are *much* longer than the translational correlation lengths. Thus, our above calculations of the correlation lengths remain valid. However, in the presence of these unbound dislocation loops, the smectic glass phase is destroyed. It may be replaced by a nematic glass phase, however, as discussed in the Introduction. The static properties of our system in the presence of dislocations are analytically accessible, and we analyze them in the next section. Furthermore, the behavior of the correlation functions found above is modified by elastic nonlinearities (which lead to anomalous elasticity), which also have a nontrivial effect on dislocations, as we demonstrate in Sec. VII.

V. DISLOCATIONS IN THE RANDOM TILT-ONLY HARMONIC MODEL

In the previous sections, we have shown that there is no long-ranged smectic translational order in the presence of the random pinning and tilting fields. However, this by itself is *not* sufficient to prove that there is no phase transition in this model. Indeed, we know of many examples of transitions between two phases which *both* lack long-ranged order. The Kosterlitz-Thouless transition in the $d=2$ XY model[44] is perhaps the most famous example. In that problem, the transition is associated not with the disappearance of long-ranged order, but, rather, with the unbinding of neutral pairs of topological defects (vortices) with increasing temperature.

It is reasonable, therefore, to ask whether the same thing can happen in our model: are topological defects (i.e., smectic dislocation loops) still bound even in the translationally disordered phase we have discussed? If they were, then an equilibrium phase transition would be required to produce the expected high temperature phase, in which dislocations are unbound.

Of course, the divergences that destroy long-ranged order are much stronger (power-law) in our model for $d<5$ than in the 2D XY model, where they are logarithmic. However, there are examples of phases much more strongly disordered than the Kosterlitz-Thouless phase in which topological defects nonetheless remain bound.[45] Furthermore, there has been considerable speculation recently[9–11] that a ''Bragg glass'' phase might exist in pinned superconducting flux-line lattices. This ''Bragg glass'' would be a phase in which the random pinning destroyed the translational order of the flux lattice, but did *not* induce dislocations in the lattice. It seems quite reasonable, therefore, to ask whether an analogous phase occurs in smectics. This section addresses this question, and shows analytically that smectic Bragg glass does not occur in $d=3$ (within a model with *harmonic* elasticity). We will show later that this result maybe invalidated by elastic anharmonic effects.

The starting point of our analytic theory is the ''tilt only'' model, by which we mean Eq. (3.5) with the random potential $V(\mathbf{r})$ set to zero. As discussed earlier, this theory correctly reproduces all of the static correlation functions in *both* the *pinned* ($T<T_p$) and the nonpinned ($T>T_p$) regimes. However, in view of the very strong irrelevance of Δ_V to the static correlation function in *both* phases in the ''phonon only'' approximation, it seems quite plausible that it is also irrelevant when dislocations are included. Indeed we will prove this in Sec. IX.

Setting $V(\mathbf{r})=0$ in the Hamiltonian Eq. (3.5) reduces it to a quadratic theory

$$H[u] = \int d^d\mathbf{r}\left[\frac{B}{2}(\partial_z u)^2 + \frac{K}{2}(\nabla_\perp^2 u)^2 + \mathbf{h}(\mathbf{r})\cdot\boldsymbol{\nabla}_\perp u\right]. \tag{5.1}$$

The effect of dislocations on such a theory can then be treated using the techniques that many authors[44,46–48] have

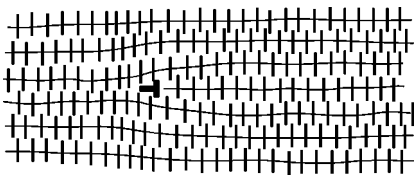

FIG. 7. An *edge* dislocation in the smectic liquid crystal, with the core coming out of the page at the position indicated by the lazy ''**T**'' symbol.

applied to a variety of disorder free systems, including smectics.

Most aspects of this procedure are already described quite well in the literature; therefore, our explanation will be brief until we come to those points that are affected by the presence of the disorder. We begin by recalling the definition of dislocations in smectics.[47] The simplest type is an edge dislocation, illustrated in Fig. 7.

This is the edge of an extra smectic layer inserted into the smectic. Clearly, a pure edge dislocation must form a closed loop lying in the smectic plane, as illustrated in Fig. 8. More generally, dislocations can tip out of the plane of the smectic layers. An extreme case is a screw dislocation, which runs *perpendicular* to the layers.

Mathematically a dislocation is a line, or, more generally, a curve, with the property that when the gradient of the displacement u is integrated around any curve that encloses the dislocation, the result is not zero, as it would be in a dislocation free system, but rather an integral multiple N of the layer spacing a. Mathematically, this means

$$\oint \boldsymbol{\nabla} u \cdot d\mathbf{l} = aN \tag{5.2}$$

or, in differential form:

$$\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{m}, \tag{5.3}$$

where we have defined

$$\mathbf{v} = \boldsymbol{\nabla} u, \tag{5.4}$$

and

$$\mathbf{m}(\mathbf{r}) = \sum_i \int aN_i \mathbf{t}_i(s_i)\, \delta^3[\mathbf{r} - \mathbf{r}_i(s_i)]\, ds_i, \tag{5.5}$$

where s_i parametrizes the ith dislocation loop, $\mathbf{r}_i(s_i)$ is the position of that loop, $\mathbf{t}_i(s_i)$ is its local unit tangent, and N_i the ''charge'' or number of excess layers associated with the dislocation. Note that N_i is independent of s_i, since the charge of a given line is constant along the line defect. Furthermore, Eq. (5.3) implies that

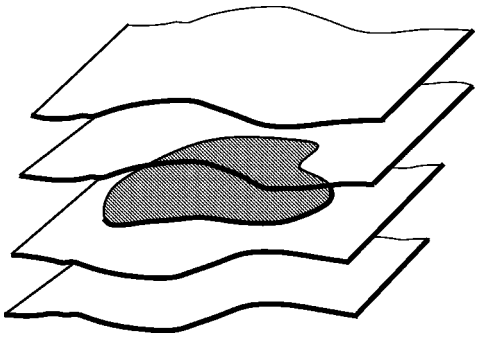

FIG. 8. A three-dimensional illustration of dislocation loop in the smectic liquid crystal.

$$\boldsymbol{\nabla} \cdot \mathbf{m}(\mathbf{r}) = 0, \tag{5.6}$$

which simply means that dislocation lines cannot end in the bulk of the sample; they must either form closed loops or extend entirely through the system.

Now, our procedure for adding dislocation lines to our previous pure phonon model Eq. (5.1) is the following standard one. We separate the field $\mathbf{v} = \boldsymbol{\nabla} u$ into phonon (single-valued) and dislocation (singular) parts

$$\mathbf{v} = \mathbf{v}_p + \mathbf{v}_d, \tag{5.7}$$

where the dislocation part $\mathbf{v}_d$ minimizes[49] the elastic Hamiltonian Eq. (5.1), subject to the constraint of Eq. (5.3) ($\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{m}$). (For an alternative derivation see Appendix C.) This uniquely determines $\mathbf{v}_d(\mathbf{r})$ given the dislocation configuration $\mathbf{m}(\mathbf{r})$. We then insert the decomposition Eq. (5.7) back into the elastic Hamiltonian Eq. (5.1). As a result of the construction that $\mathbf{v}_d(\mathbf{r})$ minimizes Eq. (5.1), all the cross couplings between $\mathbf{v}_d(\mathbf{r})$ and $\mathbf{v}_p(\mathbf{r})$ vanish. We are thus left with a decoupled elastic Hamiltonian for $\mathbf{m}(\mathbf{r})$, which we can use as a basis for studying the statistical mechanics of dislocations.

Let us now implement this procedure. The Euler-Lagrange equation, obtained by minimizing Eq. (5.1), is

$$(B\partial_z^2 - K\nabla_\perp^4) u_d(\mathbf{r}) + \boldsymbol{\nabla}_\perp \cdot \mathbf{h}(\mathbf{r}) = 0. \tag{5.8}$$

Rewriting this in terms of $\mathbf{v}_d(\mathbf{r}) \equiv \boldsymbol{\nabla} u_d(\mathbf{r})$ gives

$$\partial_z v_d^z - \lambda^2 \nabla_\perp^2 \boldsymbol{\nabla}_\perp \cdot \mathbf{v}_d^\perp + \frac{1}{B} \boldsymbol{\nabla}_\perp \cdot \mathbf{h}(\mathbf{r}) = 0, \tag{5.9}$$

where $\lambda^2 \equiv K/B$.

In Fourier space, this becomes

$$q_z v_d^z + \lambda^2 q_\perp^2 \mathbf{q}_\perp \cdot \mathbf{v}_d^\perp + \frac{1}{B} \mathbf{q}_\perp \cdot \mathbf{h}(\mathbf{q}) = 0. \tag{5.10}$$

The constraint Eq. (5.3) becomes, in Fourier space,

$$i\mathbf{q} \times \mathbf{v}_d = \mathbf{m}, \tag{5.11}$$

which has the general solution

$$\mathbf{v}_d = \frac{i\mathbf{q} \times \mathbf{m}}{q^2} + \mathbf{q}\phi, \tag{5.12}$$

where ϕ is the smooth elastic distortion around the dislocation line, to be determined by the Euler-Lagrange equation (5.8). Inserting the above expression for $\mathbf{v}_d$ into Eq. (5.10) and solving for ϕ gives

$$\phi = -\frac{iq_z(1 - \lambda^2 q_\perp^2)\epsilon_{zij} q_i m_j}{\Gamma_q q^2} - \frac{\mathbf{q}_\perp \cdot \mathbf{h}}{B\Gamma_q}, \tag{5.13}$$

where we have defined the inverse of the smectic propagator

$$\Gamma_q \equiv q_z^2 + \lambda^2 q_\perp^4. \tag{5.14}$$

Inserting this into Eq. (5.12) and then substituting this final expression for $\mathbf{v}_d$ into the original elastic Hamiltonian Eq. (5.1) gives the defect interaction Hamiltonian

$$H_d = \int_{\mathbf{q}} \left[\frac{Kq_\perp^2}{2\Gamma_q} P_{ij}^\perp m_i(\mathbf{q}) m_j(-\mathbf{q}) + \mathbf{m}(\mathbf{q}) \cdot \mathbf{a}(-\mathbf{q}) \right], \tag{5.15}$$

where $P_{ij}^\perp(\mathbf{q}) = \delta_{ij}^\perp - q_i^\perp q_j^\perp / q_\perp^2$, $\mathbf{a}(\mathbf{q})$ (not to be confused with the lattice spacing a) is a Fourier transform of the quenched field related to the original random tilt field $\mathbf{h}(\mathbf{q})$ via

$$\mathbf{a}(\mathbf{q}) = i \left[\frac{\mathbf{q} \times \mathbf{h}}{q^2} - \frac{(\hat{\mathbf{z}} \times \mathbf{q}) \mathbf{q} \cdot \mathbf{h}}{\Gamma_q q^2} q_z (1 - \lambda^2 q_\perp^2) \right], \tag{5.16}$$

and we have dropped unimportant terms that depend only on the quenched random variables (and not on $\mathbf{m}$). The first ($\mathbf{h}$-independent) term in this expression Eq. (5.15) is just the usual smectic dislocation energy for a pure system.[48]

To treat this model, we perform a duality transformation. We begin by putting the model on a simple cubic lattice (to make the model well defined at short distances); now, $\mathbf{m}(\mathbf{r})$ is defined on the sites $\mathbf{r}$ of the lattice, and takes on values

$$\mathbf{m}(\mathbf{r}) = \frac{a}{d^2} [n_x(\mathbf{r}), n_y(\mathbf{r}), n_z(\mathbf{r})], \tag{5.17}$$

where the n_i's are integers, and d is the cubic lattice constant used in the discretization. The partition function for this model is then

$$Z[\mathbf{h}(\mathbf{q})] = \sum_{\{\mathbf{m}(\mathbf{r})\}}{}' \, e^{-S[\mathbf{m}]}, \tag{5.18}$$

where

$$S[\mathbf{m}] \equiv \frac{1}{T} \left[H_d[\mathbf{m}] + E_c \frac{d^4}{a^2} \sum_{\mathbf{r}} |\mathbf{m}(\mathbf{r})|^2 \right], \tag{5.19}$$

and the sum is over all discrete configurations of $\mathbf{m}$'s given by Eq. (5.17), satisfying the dislocation line continuity constraint

$$\nabla \cdot \mathbf{m} = 0, \tag{5.20}$$

where the divergence now represents a *lattice* divergence, H_d is given by Eq. (5.15), and we have added a core energy term $E_c \Sigma_{\mathbf{r}} |\mathbf{m}(\mathbf{r})|^2$, to account for energies near the core of the defect line that are not accurately treated by our continuum elastic theory. We call the reader's attention to the fact that the partition function still depends implicitly on the configuration of the random tilt-disorder fields $\{\mathbf{h}\}$ through $\mathbf{a}$ in Eq. (5.15).

To proceed, we enforce the constraint $\nabla \cdot \mathbf{m} = 0$ by introducing a new auxiliary field $\theta(\mathbf{r})$, rewriting the partition function Eq. (5.18) as

$$Z = \prod_{\mathbf{r}} \int d\theta(\mathbf{r}) \sum_{\{\mathbf{m}(\mathbf{r})\}} e^{-S[\mathbf{m}] + i \Sigma_{\mathbf{r}} \theta(\mathbf{r}) \nabla \cdot \mathbf{m}(\mathbf{r}) d^2/a}, \tag{5.21}$$

where the sum over $\{\mathbf{m}\}$ is now unconstrained. The constraint $\nabla \cdot \mathbf{m}(\mathbf{r}) = 0$ is enforced by integration over $\theta(\mathbf{r})$, since

$$\delta[\nabla \cdot \mathbf{m}(\mathbf{r})] = \int_0^{2\pi} \frac{d\theta(\mathbf{r})}{2\pi} e^{i\theta(\mathbf{r}) \nabla \cdot \mathbf{m}(\mathbf{r}) d^2/a}, \tag{5.22}$$

where the δ is a Kronecker delta, since $\mathbf{m}\, d^2/a$, and, hence, $\nabla \cdot \mathbf{m}\, d^2/a$, are integer valued.

Now we can ''integrate'' (actually sum) by parts, and rewrite

$$\sum_{\mathbf{r}} \theta(\mathbf{r}) \nabla \cdot \mathbf{m}(\mathbf{r}) = -\sum_{\mathbf{r}} \mathbf{m}(\mathbf{r}) \cdot \nabla \theta(\mathbf{r}) + \text{surface terms}. \tag{5.23}$$

Our next step is to introduce a dummy gauge field $\mathbf{A}$ to mediate the long-ranged interaction between defect loops in the Hamiltonian Eq. (5.15). This is accomplished by rewriting the partition function as

$$Z = \prod_{\mathbf{r}} \int d\theta(\mathbf{r}) d\mathbf{A}(\mathbf{r}) \sum_{\{\mathbf{m}(\mathbf{r})\}} e^{-S[\mathbf{m}, \theta, \mathbf{A}]} \delta(\nabla \cdot \mathbf{A}) \delta(A_z), \tag{5.24}$$

with

$$S = \frac{1}{T} \sum_{\mathbf{r}} \left[\mathbf{m}(\mathbf{r}) \cdot \left(-i \frac{Td^2}{a} \nabla \theta(\mathbf{r}) + d^3 [i\mathbf{A}(\mathbf{r}) + \mathbf{a}(\mathbf{r})] \right) + E_c \frac{d^4}{a^2} |\mathbf{m}|^2 \right] + \frac{1}{2T} \sum_{\mathbf{q}} \frac{\Gamma_q}{Kq_\perp^2} |\mathbf{A}|^2, \tag{5.25}$$

where $\mathbf{a}$ is the quenched gauge field defined in Eq. (5.16). It is straightforward to check that, upon performing the Gaussian integral over $\mathbf{A}$, subject to the indicated constraints $\nabla \cdot \mathbf{A} = A_z = 0$, we recover the original long-ranged interaction between dislocation lines in Eq. (5.15).

The two goals of all of these manipulations have now been achieved: the sum on $\{\mathbf{m}(\mathbf{r})\}$ is now unconstrained, and the sum on each site over $\mathbf{m}(\mathbf{r})$ is now decoupled from that on every other site. Furthermore, this sum is readily recognized to be nothing more than the ''periodic Gaussian'' made famous by Villain.[50] The partition function Eq. (5.24) can thus be rewritten

$$Z = \prod_{\mathbf{r}} \int d\theta(\mathbf{r}) d\mathbf{A}(\mathbf{r}) \delta[\nabla \cdot \mathbf{A}(\mathbf{r})] \delta(A_z) \times \exp\left[-\sum_{\mathbf{r},i} V_p \left[\theta(\mathbf{r} + \hat{\mathbf{x}}_\mathbf{i}) - \theta(\mathbf{r}) - \frac{ad}{T} [A_i(\mathbf{r}) - i a_i(\mathbf{r})] \right] - \frac{1}{2T} \sum_{\mathbf{q}} \frac{\Gamma_q}{Kq_\perp^2} |\mathbf{A}|^2 \right], \tag{5.26}$$

where the well-known 2π-periodic Villain potential $V_p(x)$, defined by

$$e^{-V_p(x)} \equiv \sum_{n=-\infty}^{\infty} e^{-n^2 E_c/T + ixn} \tag{5.27}$$

has the usual property that the *smaller* E_c/T is (i.e., the *higher* the temperature in the original random-tilt smectic

model), the sharper the potential minima. Thus *raising* the temperature in the original model is equivalent to *lowering* the temperature in the dual model Eq. (5.26). It is precisely this familiar temperature inversion associated with duality that leads to an *inverted XY* transition for three-dimensional disorder-free superconductors[15] and bulk smectics.[16] It also plays an important role here, as we shall see in a moment.

Standard universality arguments imply that replacing the periodic potential $V_p(x)$ in Eq. (5.26) by *any* other nonsingular periodic function should not change the universality class of the transition. In particular, we could replace $V_p(x)$ by $\cos(x)$. The resultant model would be precisely the ''fixed length'' version of the ''soft spin,'' or Landau-Ginsburg-Wilson model, with the *complex* ''action''

$$S=\sum_{\mathbf{r}}\left[\frac{c}{2}\left(\nabla+\frac{ad}{T}(i\mathbf{A}+\mathbf{a})\right)\psi^{*}\cdot\left(\nabla-\frac{ad}{T}(i\mathbf{A}+\mathbf{a})\right)\psi+t|\psi|^{2}+u|\psi|^{4}\right]+\sum_{\mathbf{q}}\frac{\Gamma_q}{2TKq_{\perp}^{2}}|\mathbf{A}(\mathbf{q})|^{2},\tag{5.28}$$

where $\psi(\mathbf{r})$ is a complex ''disorder'' parameter field whose phase is $\theta(\mathbf{r})$; the reduced temperature t, quartic coupling u, and $c(T)$ are parameters of the model with

$$c(T)=\left.\frac{d^{2}V_p(x)}{d^{2}x}\right|_{x=0}\times O(1),\tag{5.29a}$$

$$\approx\begin{cases}2e^{-E_c/T}, & T\ll E_c,\\ T/2E_c, & T\gg E_c.\end{cases}\tag{5.29b}$$

Because of the duality transformation's inversion of the temperature axis, the reduced temperature t is a monotonically *decreasing* function of the temperature T (of the original dislocation loop model), which vanishes at the mean-field transition temperature T_{MF} of the fixed length model Eq. (5.26).

Universality also implies that this ''soft-spin'' model should be in the same universality class as the fixed length model Eq. (5.26). We shall, therefore, henceforth work with model Eq. (5.28), because it is more straightforward to analyze perturbatively.

As we undertake that analysis, it is important to keep in mind that, as a consequence of the duality inversion of the temperature axis, the *ordered* phase of the dual model Eq. (5.28) corresponds to the *disordered* (i.e., dislocation loops unbound) phase of the original dislocation loop gas model. That is, the low dual-temperature phase described by

$$\langle\psi(\mathbf{r})\rangle\neq 0,\tag{5.30}$$

corresponds to the *disordered* dislocation-unbound phase of the smectic liquid crystal. In the absence of disorder ($\mathbf{a}=0$) this model is exactly the dual version of the NA dislocation loop model derived and studied by one of us.[16]

Disorder is included in Eq. (5.28) through the quenched gauge-field $\mathbf{a}(\mathbf{r})$, which is related to the random tilt field $\mathbf{h}(\mathbf{r})$ by Eq. (5.16). The partition function

$$Z[\mathbf{h}]=\int[d\psi][d\mathbf{A}]e^{-S[\psi,\mathbf{A},\mathbf{h}]}\delta(\nabla\cdot\mathbf{A})\,\delta(A_z),\tag{5.31}$$

with S given by Eq. (5.28), is thus an implicit function of the random tilt field configuration $\mathbf{h}(\mathbf{r})$.

As in the previous section, we will cope with this dependence of Z on the quenched field $\mathbf{h}(\mathbf{r})$ using the replica trick. Doing so leads us to calculate

$$\overline{Z^{n}}=\int[d\mathbf{a}]\prod_{\alpha=1}^{n}[d\psi_\alpha][d\mathbf{A}_\alpha]\times e^{-S_r[\psi_\alpha,\mathbf{A}_\alpha,\mathbf{a}]}P[\mathbf{a}]\,\delta(\nabla\cdot\mathbf{A}_\alpha)\,\delta(A_z^{\alpha}),\tag{5.32}$$

with

$$S_r=\sum_{\mathbf{r},\alpha}\left[\frac{c}{2}\left(\nabla+\frac{ad}{T}(i\mathbf{A}_\alpha+\mathbf{a})\right)\psi_\alpha^{*}\cdot\left(\nabla-\frac{ad}{T}(i\mathbf{A}_\alpha+\mathbf{a})\right)\psi_\alpha+t|\psi_\alpha|^{2}+u|\psi_\alpha|^{4}\right]+\sum_{\mathbf{q},\alpha}\frac{\Gamma_q}{2TKq_{\perp}^{2}}|\mathbf{A}_\alpha(\mathbf{q})|^{2},\tag{5.33}$$

The probability distribution $P[\mathbf{a}]$ of the field $\mathbf{a}$ in Eq. (5.32) is Gaussian, since $\mathbf{a}$ is linear in $\mathbf{h}$ and the distribution of $\mathbf{h}$ is Gaussian, defined by Eq. (2.22). Thus, the distribution $P[\mathbf{a}]$ is completely specified by the average $\overline{a_i(\mathbf{q})a_j(-\mathbf{q})}$. This is easily evaluated, using the relation Eq. (5.16) between $\mathbf{a}(\mathbf{q})$ and $\mathbf{h}(\mathbf{q})$. We find

$$\overline{a_i(\mathbf{q})a_j(-\mathbf{q})}=\overline{h_l(\mathbf{q})h_n(-\mathbf{q})}\tag{5.34a}$$

$$\times\left(\frac{\epsilon_{ikl}q_k}{q^{2}}-\frac{\epsilon_{zki}q_k}{\Gamma_q q^{2}}q_l q_z(1-\lambda^{2}q_{\perp}^{2})\right)\times\left(\frac{\epsilon_{jmn}q_m}{q^{2}}-\frac{\epsilon_{zmj}q_m}{\Gamma_q q^{2}}q_n q_z(1-\lambda^{2}q_{\perp}^{2})\right),$$

$$=\Delta_h\left(\frac{P_{ij}}{q^{2}}+\frac{q_z^{2}q_{\perp}^{2}(1-\lambda^{2}q_{\perp}^{2})^{2}P_{ij}^{\perp}}{\Gamma_q^{2}q^{2}}\right),\tag{5.34b}$$

where ϵ_{ijk} is the usual fully antisymmetric third rank unit tensor. To obtain the second equality we have used $\overline{h_l(\mathbf{q})h_n(-\mathbf{q})}=\Delta_h\delta_{ln}$, and defined the standard transverse projection operators

$$P_{ij}=\delta_{ij}-q_iq_j/q^{2},\tag{5.35a}$$

$$P_{ij}^{\perp}=(1-\delta_{iz})(1-\delta_{jz})(\delta_{ij}-q_i^{\perp}q_j^{\perp}/q_{\perp}^{2}).\tag{5.35b}$$

We now need merely to consider the statistical mechanics of the model defined by Eqs. (5.32),(5.33), in the limit $n\to 0$.

From previous classic works[15,51,16] we have considerable experience analyzing the critical properties of gauge theories of the type defined by the action in Eq. (5.33). As we know from work on the zero field normal-to-superconductor transition,[15] and the analogous analysis of the bulk, clean nematic-to-smectic-A transitions,[16] there are two possible regimes: (i) the extreme type I regime, in which $u\ll 1$, and, as a result, the ψ fluctuations are subdominant to those of the gauge field $\mathbf{A}$, allowing a mean-field treatment of ψ. In this case the remaining gauge field fluctuations can be treated

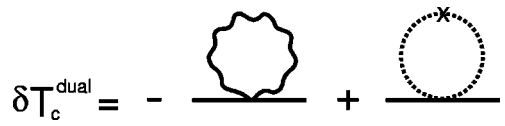

FIG. 9. Feynman diagrams that dominate the renormalization of the reduced *dual* temperature *t*, leading to an *upward* shift in the *dual* T_c. In $d \leqslant 3$ these drive the transition temperature in the original smectic model to 0.

exactly. (ii) The type II regime, in which $u>1$ and *both* ψ and **A** fluctuations must be treated within a considerably more involved RG analysis.

A complete analysis of the critical properties of this smectic dislocation loop unbinding transition, described by the action in Eq. (5.33), is beyond the scope of the present paper.[26] Here our goal is to determine the stability of the topologically ordered smectic Bragg glass phase to disorder-induced dislocation loop unbinding. To answer this question, it is sufficient to study the effect of diagrammatic corrections on the reduced dual temperature *t*, as we demonstrate in detail in Appendix D for the type I limit, delaying the analysis of the type II limit to a future publication.[26] By computing the disorder-averaged free energy, we find that in *either* regime the lowest order contribution to the renormalized *dual* temperature t_R comes from the average of the ''diamagnetic'' terms

$$\delta S = \frac{ca^2d^2}{2T^2} \sum_{\mathbf{r}} \left(\langle |\mathbf{A}|^2 \rangle - \overline{|\mathbf{a}|^2} \right) |\psi|^2, \tag{5.36}$$

graphically illustrated in Fig. 9. Generalizing to *d* dimensions these give

$$t_R = t_0 + \frac{(d-2)ca^2d^2}{T^2} \int \frac{d^dq}{(2\pi)^d} \left[\frac{KTq_\perp^2}{\Gamma_q} - \frac{\Delta_h q_z^2 q_\perp^2}{q^2 \Gamma_q^2} \right], \tag{5.37}$$

where the first term in the square brackets comes from the first graph in Fig. 9, with the internal wiggly line representing $\mathbf{A}_\alpha$ fluctuations, while the second comes from the second graph with the internal dotted line representing the quenched gauge field **a**. The negative sign of the disorder contribution leads to an *increase* in the *dual* T_c, and can be traced back to the fact that the action S_r Eq. (5.33), is complex.

The second, disorder, term in this integral dominates the first as $\mathbf{q} \to 0$. Indeed, this integral diverges in the infrared for $d \leqslant 3$

$$\int d^dq \frac{q_z^2 q_\perp^2}{q^2 (q_z^2 + \lambda^2 q_\perp^4)^2} \propto \int d^{d-1}q_\perp \frac{q_\perp^8}{q_\perp^{10}} \propto \int \frac{d^{d-1}}{q_\perp^2}, \tag{5.38}$$

where the first proportionality follows from the fact that the dominant regime of the integral over q_z is $q_z \sim \lambda q_\perp^2$, as can be easily seen by contour integration. This divergence implies that the renormalized *dual* temperature t_R is driven to $-\infty$ (note the sign) by the disorder in $d \leqslant 3$. Indeed, we find in $d=3$

$$t_R = t_0 - \frac{ca^2d^2\Delta_h}{\lambda T^2} \ln(L/a) \times O(1), \tag{5.39}$$

where L is an infrared cutoff (e.g., the lateral extent of the smectic layers) and a is the ultraviolet cutoff (e.g., the size of the liquid crystal molecules ≈ 10 Å).

The unusual *negative* ($d=3$) divergent correction to t in Eq. (5.39) implies that the *dual* order parameter is always in its *ordered* phase, which, in turn, implies that the dislocation loops of the original smectic model are always *unbound*, thereby destroying the topological order of the smectic Bragg glass, even at $T=0$, for *any* amount of disorder, no matter how small. This conclusion, of course, only holds within the *harmonic* elastic approximation made in this section, with the treatment that approximately includes these nonlinear elastic effects given in Sec. VII.

It is important to note that although the dual disordered gauge model, Eq. (5.28), can be extended to any dimension, it rigorously only describes the original disordered smectic in $d=3$, the dimension in which the duality correspondence was established between the two models. However, if one can entertain the idea that the relation between the two models extends to $d>3$, then we have found a quite nontrivial result for this higher-dimensional disordered smectic. For $3 < d \leqslant 5$ the disordered smectic will then have strongly divergent displacement fluctuations Eq. (4.9), but will maintain topological order, i.e., no condensation of dislocation loops takes place in this range of dimension (for $d>3$). From this point of view this higher-dimensional disordered smectic is in fact a true ''Bragg'' glass.[9–11]

VI. ANOMALOUS ELASTICITY IN THE TILT-ONLY MODEL OF DISORDERED SMECTICS

Up to now we have analyzed disordered smectics ignoring *elastic anharmonic* effects. These effects are important even in pure smectics, as was discovered some time ago by Grinstein and Pelcovits (GP).[24] As reported in a recent paper,[18] we have discovered that in *disordered* smectics these elastic anharmonicities play an even more important role; specifically, we find disorder-driven *power law* divergences, in contrast to the weak, thermally-driven *logarithmic* divergences that occur in pure smectics.[24] Here we provide details of these calculations, describe their effect on the analysis of previous sections, and discuss them in the context of recent experiments.[12,19–22]

The anomalous elasticity of *pure* (bulk) smectics[24] is characterized by layer compressional and tilt moduli $B(\mathbf{k})$ and $K(\mathbf{k})$ which vanish and diverge, respectively, at long wavelengths ($\mathbf{k} \to 0$). This beautiful result is actually a general property of all one-dimensional crystals, in which the direction of the 1D ordering wave vector is chosen *spontaneously*. As a consequence of this *spontaneous* breaking of rotational symmetry (a property possessed by smectics but not, e.g., charge density waves), in the presence of fluctuations a compression can be relieved by smoothing out these fluctuations, thereby leading to an effective compressional modulus $B(\mathbf{k})$ that vanishes at long wavelengths. This is very similar to the ability of free dislocations in a solid to relieve an imposed shear. Similarly, in the presence of fluctuations, a bending of smectic layers necessarily leads to a compression, which implies an effective tilt modulus $K(\mathbf{k})$ which diverges at long wavelengths. This phenomenon is also analogous to the thermally driven anomalous elasticity

of polymerized membranes, in which the bending rigidity modulus diverges, and the shear and bulk moduli vanish at long length scales.[52–54]

Although quite fascinating, the anomalous elasticity of GP for pure 3D smectics is somewhat academic, because for $d=3$ the nonlinearities responsible for the anomalous elasticity are marginally irrelevant, and therefore lead to weakly (logarithmically) k-dependent elastic moduli. These, in practice, are quite difficult to detect experimentally, and have never been observed in bulk (pure) smectics.

As mentioned above, the main ingredients necessary for the existence of anomalous elasticity are *spontaneously* broken rotational invariance, and the presence of fluctuations, i.e., wrinkles in the smectic layers—both of which exist (even at zero temperature) in smectics in random environments. In this section, we demonstrate the existence of anomalous elasticity that is significantly stronger, in all spatial dimensions $d<5$, than the marginal anomalous elasticity of thermal smectics. This stronger anomalous elasticity is driven by quenched disorder, and controlled by a new, zero-temperature fixed point that is perturbatively accessible in $d=5-\epsilon$. One can physically appreciate why this elastic anomaly is so much stronger in smectics with quenched disorder by realizing that the quenched disordered fluctuations are more divergent (at long wavelengths) than their thermal counterparts [see e.g., Eqs. (4.5) and (4.6)].

As discussed at the end of Sec. III, in dimensions $d<5$ we expect that the random field disorder to be significantly less important than the random tilt disorder for the properties of smectics with quenched disorder. We therefore expect the effective elastic Hamiltonian in Eq. (3.4) with $U(\mathbf{r})=0$ to be a good starting point for the description of smectics confined inside low-density aerogels. However, such a model misses an important ingredient: nonlinear elasticity which takes into account the underlying rotational invariance of the smectic phase,[24] hidden by the *spontaneous* choice of the layers to stack along the $\hat{\mathbf{z}}$ direction. A careful analysis, starting with the de Gennes model, that keeps track of such elastic nonlinearities, leads to[24]

$$H[u]=\int_{\mathbf{r}}\left[\frac{K}{2}(\nabla_\perp^2 u)^2+\frac{B}{2}\left(\partial_z u-\frac{1}{2}(\boldsymbol{\nabla}_\perp u)^2\right)^2+\mathbf{h}(\mathbf{r})\cdot\boldsymbol{\nabla}_\perp u\right], \tag{6.1}$$

as the proper rotationally invariant elastic Hamiltonian for the disordered smectic. This form guarantees that a uniform rotation of smectic layers costs zero energy in the absence of the random tilt field.[55] The field $\mathbf{h}(\mathbf{r})\equiv g_z(\mathbf{r})\mathbf{g}(\mathbf{r})$ is the quenched random tilt disorder, defined in Sec. II. As discussed in that section, it has a vanishing average and Gaussian statistics, completely characterized by the two-point correlation function

$$\overline{h_i(\mathbf{r})h_j(\mathbf{r}')}=\Delta_h\delta^d(\mathbf{r}-\mathbf{r}')\,\delta_{ij}, \tag{6.2}$$

which we take to be short-ranged for the reasons discussed in detail in Sec. II.

To compute self-averaging quantities (e.g., the disorder averaged free energy) and to assess the importance of the nonlinearities at long scales it is convenient (but not necessary) to employ the replica ''trick''[39] that relies on the identity $\overline{\ln Z}=\lim_{n\to 0}(\overline{Z^n}-1)/n$. As discussed in Sec. III, this allows us to work with a translationally invariant field theory at the expense of introducing n replica fields (with the $n\to 0$ limit to be taken at the end of the calculation). After replicating and integrating over the disorder $\mathbf{h}(\mathbf{r})$, and using Eq. (6.2), we obtain an effective translationally invariant replicated Hamiltonian

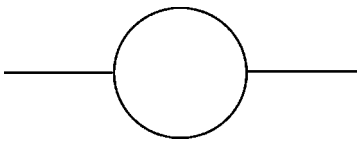

FIG. 10. Feynman graph that renormalizes the elastic moduli K, B and the tilt-disorder variance Δ_h.

$$H[u_\alpha]=\frac{1}{2}\int_{\mathbf{r}}\sum_{\alpha=1}^{n}\left[K(\nabla_\perp^2 u_\alpha)^2+B\left(\partial_z u_\alpha-\frac{1}{2}(\boldsymbol{\nabla}_\perp u_\alpha)^2\right)^2\right]-\frac{\Delta_h}{2T}\int_{\mathbf{r}}\sum_{\alpha,\beta=1}^{n}\boldsymbol{\nabla}_\perp u_\alpha\cdot\boldsymbol{\nabla}_\perp u_\beta \tag{6.3}$$

from which the noninteracting propagator $G_{\alpha\beta}(\mathbf{q})\equiv V^{-1}\langle u_\alpha(\mathbf{q})u_\beta(-\mathbf{q})\rangle_0$ (where V is the system volume) can be easily obtained (see Sec. III),

$$G_{\alpha\beta}(q)=TG(\mathbf{q})\,\delta_{\alpha\beta}+\Delta_h q_\perp^2 G(\mathbf{q})^2, \tag{6.4}$$

with $G(\mathbf{q})=1/(Kq_\perp^4+Bq_z^2)$.

We first attempt to assess the effects of the anharmonicities, disorder and thermal fluctuations by performing a simple perturbation expansion in the nonlinearities of $H[u_\alpha]$. The lowest order correction δB to the bare elastic compressional modulus B comes from a part of the diagram in Fig. 10. A standard analysis gives

$$\delta B=-\frac{B^2}{2}\int_{\mathbf{q}}^{>}[TG(\mathbf{q})^2+2\Delta_h q_\perp^2 G(\mathbf{q})^3]q_\perp^4, \tag{6.5a}$$

$$\approx -B^2\Delta_h\int_{-\infty}^{\infty}\frac{dq_z}{2\pi}\int^{>}\frac{d^{d-1}q_\perp}{(2\pi)^{d-1}}\,\frac{q_\perp^6}{(Kq_\perp^4+Bq_z^2)^3}, \tag{6.5b}$$

$$\approx -\frac{3}{16}\,\frac{C_{d-1}}{5-d}B\Delta_h\left(\frac{B}{K^5}\right)^{1/2}L^{5-d}, \tag{6.5c}$$

where we have dropped the subdominant thermal part, focused on $d<5$, which allows us to drop the uv-cutoff (Λ) dependent part which vanishes for $\Lambda\to\infty$, and cutoff the divergent contribution of the long wavelength modes via the infrared cutoff restriction $q_\perp>1/L$, where L is the linear extent of the system. The constant $C_d=2\pi^{d/2}/[(2\pi)^d\Gamma(d/2)]$ is the surface area of a d-dimensional sphere divided by $(2\pi)^d$. Clearly the anhamonicities become important when the fluctuation corrections to the elastic constants (e.g., δB above) become comparable to the bare values. The divergence of this correction as $L\to\infty$ signals the breakdown of conventional harmonic elastic theory on length scales longer than a crossover scale $\xi_\perp^{\mathrm{NL}}$, which we define as the value of L at which $|\delta B(\xi_\perp^{\mathrm{NL}})|=B$. In d dimensions, this definition gives

$$\xi_\perp^{\rm NL}=\left(\frac{16(5-d)K^{5/2}}{3C_{d-1}B^{1/2}\Delta_h}\right)^{1/(5-d)}, \tag{6.6}$$

which for physical 3D smectics is given by

$$\xi_\perp^{\rm NL}=\left[\frac{64\pi}{3}\frac{K^{5/2}}{B^{1/2}\Delta_h}\right]^{1/2}, \quad \text{for } d=3. \tag{6.7}$$

We can also obtain a nonlinear crossover length in the z direction

$$\xi_z^{\rm NL}=(\xi_\perp^{\rm NL})^2/\lambda, \tag{6.8a}$$

$$=\frac{64\pi K^2}{3\Delta_h} \quad \text{for } d=3, \tag{6.8b}$$

where $\lambda\equiv(K/B)^{1/2}$, by imposing the infrared cutoff in the z direction.

To understand the physics beyond this crossover scale—i.e., to make sense of the apparent infrared divergences found in Eq. (6.5c)—we turn to the renormalization group. As discussed in Sec. III, we employ the standard momentum shell renormalization group transformation,[40] by writing the displacement field as $u_\alpha(\mathbf{r})=u_\alpha^<(\mathbf{r})+u_\alpha^>(\mathbf{r})$, integrating perturbatively in the nonlinear coupling B the high wave-vector part $u_\alpha^>(\mathbf{r})$ with support in Fourier space in an infinitesimal cylindrical shell ($\Lambda e^{-l}<q_\perp<\Lambda$ and $-\infty<q_z<\infty$), and rescaling the lengths and long wavelength part of the fields as in Eqs. (3.17) with $r_\perp=r_\perp' e^l$, $z=z' e^{\omega l}$, and $u_\alpha^<(\mathbf{r})=e^{\phi l}u_\alpha'(\mathbf{r}')$, so as to restore the ultraviolet cutoff back to Λ. The underlying rotational invariance insures that the graphical corrections preserve the rotationally invariant operator $[\partial_z u-\frac{1}{2}(\boldsymbol{\nabla}_\perp u)^2]$, renormalizing it as a whole. It is therefore convenient (but not necessary) to choose the dimensional rescaling that also preserves this operator. It is easy to see that this choice leads to

$$\phi=2-\omega. \tag{6.9}$$

This rescaling then leads to the zeroth order RG flow of the effective couplings

$$K(l)=Ke^{(d-1-\omega)l}, \tag{6.10a}$$

$$B(l)=Be^{(d+3-3\omega)l}, \tag{6.10b}$$

$$\left(\frac{\Delta_h}{T}\right)(l)=\left(\frac{\Delta_h}{T}\right)e^{(d+1-\omega)l}. \tag{6.10c}$$

From these dimensional couplings one can construct two other effective dimensional couplings

$$\tilde g_1\equiv\left(\frac{B}{K^3}\right)^{1/2}, \tag{6.11a}$$

$$\tilde g_2\equiv\Delta_h\left(\frac{B}{K^5}\right)^{1/2}, \tag{6.11b}$$

whose flows are independent of the arbitrary rescaling exponents ω and ϕ, and are given by

$$\tilde g_1(l)=\tilde g_1 e^{(3-d)l}, \tag{6.12a}$$

$$\tilde g_2(l)=\tilde g_2 e^{(5-d)l}. \tag{6.12b}$$

$\tilde g_1$ is just the coupling that becomes relevant in $d<3$ and was discovered by Grinstein and Pelcovits to lead to anomalous elasticity in pure thermal smectics. It is, however, only marginally irrelevant in $d=3$ and therefore only leads to weak anomalous elasticity in physical 3D smectics.[24] In contrast, the upper critical dimension d_{uc} below which $\tilde g_2$ becomes relevant is $d_{uc}=5$, and we therefore expect a significantly stronger anomalous elasticity, that should be experimentally observable, in *disordered* 3D smectics. These observations imply that temperature is a strongly irrelevant variable near the disorder dominated fixed point and does not feed back in a dangerous way. We will therefore set $T=0$ in all subsequent calculations.

We now turn to the RG computation of the one-loop graphical corrections to the flow of the couplings defined above. The required integration over the high wave-vector components of u_α can only be accomplished perturbatively in the nonlinearities of $H[u]$. Since the most relevant coupling $\tilde g_2$ becomes important for $d<5$, we will control the infrared divergences by performing an expansion in $\epsilon=5-d$.

The change in the Hamiltonian due to integrating out these short length modes is

$$\delta H[u_\alpha^<]=\langle H_i[u_\alpha^<+u_\alpha^>]\rangle_>-\frac{1}{2T}\langle H_i^2[u_\alpha^<+u_\alpha^>]\rangle_>^c\cdots, \tag{6.13}$$

where $H_i[u_\alpha^<+u_\alpha^>]$ is the nonlinear part of the Hamiltonian H, Eq. (3.8), which contains three- and four-point vertices. The averages above are performed with the quadratic part of H, Eq. (3.8), using the propagator $G_{\alpha\beta}(\mathbf{q})$, Eq. (6.4), with only an infinitesimal cylindrical shell (i.e., no cutoff on q_z) of modes $u_\alpha^>$ integrated out. The superscript c denotes a cumulant average. It is easy to verify that the first term in δH (first order in H_i) does not lead to corrections of the elastic constants. Aside from correcting the free energy, it generates an operator *linear* in $[\partial_z u-\frac{1}{2}(\boldsymbol{\nabla}_\perp u)^2]$ which corresponds to a renormalization of the smectic wave vector q_0. These corrections turn out to be finite (irrelevant) near $d=5$, and we will therefore not keep track of them here.

The renormalization of K, B and Δ_h comes from the second term in Eq. (6.13), for example, from parts which are second order in the three-point vertex, with four of the six fields contracted. The generic diagram that corrects the elastic moduli and the disorder variance is illustrated in Fig. 10, with the part diagonal in the replica indices α,β (i.e., part proportional to $\delta_{\alpha\beta}$) renormalizing K and B, and the part independent of α,β correcting Δ_h.

For the calculation of δB, the loop integrals can be performed at vanishing external momentum, and after a short calculation one finds

$$\delta B = -\frac{B^2}{2}\int_{\mathbf{q}}^{>}[TG(\mathbf{q})^2 + 2\Delta_h q_\perp^2 G(\mathbf{q})^3]q_\perp^4,$$

$$\approx -B^2\Delta_h\int_{-\infty}^{\infty}\frac{dq_z}{2\pi}\int^{>}\frac{d^{d-1}q_\perp}{(2\pi)^{d-1}}\frac{q_\perp^6}{(Kq_\perp^4+Bq_z^2)^3},$$

$$\approx -\frac{3}{16}g_2 B dl, \tag{6.14}$$

where going from the first line to second line above we dropped the irrelevant finite temperature part, and defined a dimensionless coupling constant

$$g_2 \equiv \Delta_h\left(\frac{B}{K^5}\right)^{1/2} C_{d-1}\Lambda^{d-5}. \tag{6.15}$$

Similar, but more involved calculations for K and Δ_h give

$$\delta K \approx \frac{1}{32}g_2 K dl, \tag{6.16a}$$

$$\delta\left(\frac{\Delta_h}{T}\right) \approx \frac{1}{64}g_2\left(\frac{\Delta_h}{T}\right)dl, \tag{6.16b}$$

which lead to the following RG flow equations:

$$\frac{dB(l)}{dl} = \left(d+3-3\omega-\frac{3}{16}g_2(l)\right)B(l), \tag{6.17a}$$

$$\frac{dK(l)}{dl} = \left(d-1-\omega+\frac{1}{32}g_2(l)\right)K(l), \tag{6.17b}$$

$$\frac{d\Delta_h(l)}{dl} = \left(d+1-\omega+\frac{1}{64}g_2(l)\right)\Delta_h(l). \tag{6.17c}$$

These, together, lead to the RG flow of the dimensionless coupling g_2 defined in Eq. (6.15)

$$\frac{dg_2(l)}{dl} = \epsilon g_2(l) - \frac{5}{32}g_2(l)^2, \tag{6.18}$$

where we remind the reader that in this section $\epsilon = 5-d$. As required, the flow of g_2 is independent of the arbitrary choice of the anisotropy rescaling exponent ω. The RG flow Eq. (6.18) shows that the Gaussian $g_2^* = 0$ fixed point becomes unstable for $d<5$, and the low-temperature phase is controlled by a stable, nontrivial, glassy $T=0$ fixed point at

$$g_2^* = \frac{32}{5}\epsilon. \tag{6.19}$$

The existence of this nontrivial fixed point leads to anomalous elasticity. To see this it is convenient to use our RG results to evaluate the connected disordered averaged two-point $u(\mathbf{k})$ correlation function $G(\mathbf{k}) \propto \overline{\langle|u(\mathbf{k})|^2\rangle} - \overline{\langle u(\mathbf{k})\rangle\langle u(-\mathbf{k})\rangle}$. As discussed (and utilized) in Sec. III, the power of the renormalization group is that it establishes a connection between a correlation function at a small wave vector (which is impossible to calculate in perturbation theory due to the infrared divergences) to the same correlation function at large wavevectors, which can be easily calculated in a controlled perturbation theory. This relation for $G(\mathbf{k})$ is

$$G(\mathbf{k}_\perp, k_z, K, B, g_2) = e^{(3+d-\omega)l}G(\mathbf{k}_\perp e^l, k_z e^{\omega l}, K(l), B(l), g_2(l)), \tag{6.20}$$

where the prefactor on the right-hand side comes from the dimensional rescaling (remembering the momentum conserving δ function), after using the exact rotational Ward identity $\phi = 2-\omega$, and we have ''traded-in'' the disorder variable Δ_h for the dimensionless coupling g_2. To establish the anomalous behavior of K, we look at $k_z = 0$. We then choose the rescaling variable l^* such that

$$k_\perp e^{l^*} = \Lambda. \tag{6.21}$$

We also choose $k_\perp$ sufficiently small such that $g_2(l^*)$ has reached our nontrivial fixed point g_2^*. Eliminating l^* in favor of $k_\perp$, we then obtain

$$G(\mathbf{k}_\perp, 0, K, B, g_2) = \left(\frac{\Lambda}{k_\perp}\right)^{3+d-\omega} G(\Lambda, 0, K(l^*), B(l^*), g_2^*). \tag{6.22}$$

Since the right-hand side is evaluated with the transverse wave vector at the Brillouin zone boundary, it is easily evaluated perturbatively for a small fixed point coupling g_2^*. To lowest order we obtain

$$G(\mathbf{k}_\perp, 0, K, B, g_2) \approx \frac{(\Lambda/k_\perp)^{3+d-\omega}}{\Lambda^4 K(\Lambda/k_\perp)^{(d-1-\omega+g_2^*/32)}}, \tag{6.23a}$$

$$\equiv \frac{1}{K(k_\perp)k_\perp^4}, \tag{6.23b}$$

where we integrated Eq. (6.17b) to obtain $K(l^*)$, and defined the anomalous tilt modulus which diverges at long length scales

$$K(k_\perp) = K(k_\perp/\Lambda)^{-\eta_K}, \tag{6.24}$$

with an anomalous exponent

$$\eta_K = \frac{1}{32}g_2^*, \tag{6.25a}$$

$$= \frac{1}{5}\epsilon, \tag{6.25b}$$

$$= \frac{2}{5}, \quad \text{for } d=3. \tag{6.25c}$$

Similar calculations for the other coupling constants and other directions of $\mathbf{k}$ show that, in general,

$$K(\mathbf{k}) = K(k_\perp\xi_\perp^{\mathrm{NL}})^{-\eta_K} f_K[k_z\xi_z^{\mathrm{NL}}/(k_\perp\xi_\perp^{\mathrm{NL}})^\zeta], \tag{6.26a}$$

$$B(\mathbf{k}) = B(k_\perp\xi_\perp^{\mathrm{NL}})^{\eta_B} f_B[k_z\xi_z^{\mathrm{NL}}/(k_\perp\xi_\perp^{\mathrm{NL}})^\zeta], \tag{6.26b}$$

$$\Delta_h(\mathbf{k}) = \Delta_h (k_\perp \xi_\perp^{\rm NL})^{-\eta_\Delta} f_\Delta[k_z \xi_z^{\rm NL}/(k_\perp \xi_\perp^{\rm NL})^\zeta], \tag{6.26c}$$

with the anisotropy exponent

$$\zeta \equiv 2 - (\eta_B + \eta_K)/2, \tag{6.27}$$

which would $=2$ in the absence of anharmonic effects,

$$\eta_B = \frac{3}{16} g_2^*, \tag{6.28a}$$

$$= \frac{6}{5}\epsilon, \tag{6.28b}$$

$$= \frac{12}{5} \quad \text{for } d=3, \tag{6.28c}$$

and

$$\eta_\Delta = \frac{1}{64} g_2^*, \tag{6.29a}$$

$$= \frac{1}{10}\epsilon, \tag{6.29b}$$

$$= \frac{1}{5} \quad \text{for } d=3. \tag{6.29c}$$

Of course, we do not completely trust the extrapolation of these small ϵ results down to $\epsilon=2$ $(d=3)$. However, since by definition $dg_2/dl=0$ at the nontrivial fixed point, this condition implies an *exact* relation between the anomalous exponents

$$5 - d + \eta_\Delta = \frac{\eta_B}{2} + \frac{5}{2}\eta_K, \tag{6.30}$$

which is obviously satisfied by the anomalous exponents, Eqs. (6.25b),(6.28b),(6.29b), computed here to first order in ϵ. This Ward identity between the anomalous exponents can be equally easily obtained from a self-consistent integral equation for the u-u correlations functions, using renormalized wave-vector-dependent elastic moduli and disorder variance.[26]

At length scales beyond $\xi_\perp^{\rm NL}$ and $\xi_z^{\rm NL}$, the elasticity and fluctuations of the disordered smectic are controlled by our glassy fixed point. One of the important consequences can be seen in the layer fluctuations, which can be inferred from x-ray scattering experiments. For instance, layer displacement fluctuations along z are described by

$$C(z) \equiv \overline{\langle [u(0_\perp, z) - u(0_\perp, 0)]^2 \rangle},$$

$$\approx \int \frac{d^d k}{(2\pi)^d} \frac{2[1-\cos(k_z z)]\Delta_h(\mathbf{k}) k_\perp^2}{[K(\mathbf{k}) k_\perp^4 + B(\mathbf{k}) k_z^2]^2}, \tag{6.31}$$

One can then naturally define the x-ray translational correlation length ξ_z^X as the length along z at which $C(z=\xi_z^X) \equiv a^2$, where a is the smectic layer spacing. A simple calculation, using Eqs. (6.26a)–(6.26c) leads in 3D to

$$\xi_z^X = (a/\lambda)^{\zeta/\chi} K^2/\Delta_h, \tag{6.32a}$$

$$= (a/\lambda)^{\zeta/\chi} \xi_z^{\rm NL}, \tag{6.32b}$$

where

$$\chi \equiv (\eta_B + \eta_K)/2, \tag{6.33}$$

and this result should be contrasted with the expression for the x-ray correlation length ξ_z^X, Eq. (4.17a), calculated within the *harmonic* theory, i.e., ignoring the anomalous elasticity. Note that this x-ray correlation length is finite even as $T \to 0$, as illustrated in Fig. 3. This result is consistent with the experimental observation[12,20] that the x-ray correlation length for smectics in aerogel saturates at some finite value at low temperatures. Note also that this length should be different for different smectics in the same aerogel, since B, K, and Δ_h will change from smectic to smectic. Since, as discussed in Sec. II, we expect Δ_h to be a monotonically increasing function of the aerogel density ρ_A, with a simplest aerogel model giving $\Delta_h \propto \rho_A$, Eq. (2.8b), the aerogel density dependence of ξ_z^X could test the prediction of Eq. (6.32). Likewise, the *temperature* dependence of ξ_z^X could be used to determine the ratio ζ/χ, since the *bulk* $K(T)$ and $B(T)$ that implicitly appear in Eq. (6.32) have temperature dependences that can be extracted from measurements on *bulk* materials.

Note also that this correlation length is longer than the nonlinear crossover length for $\lambda < a$ (i.e., for large B). For $\lambda \gg a$ (small B), $C(z)$ reaches a^2 before z reaches $\xi_z^{\rm NL}$, and hence anharmonic effects are unimportant. In this case, the correlation length ξ_z^X can be determined in the harmonic theory [which amounts to evaluating the integral in Eq. (6.31) with $K(\mathbf{k})$, $B(\mathbf{k})$, and $\Delta_h(\mathbf{k})$ replaced by their constant (bare) values K, B, and Δ_h]. As shown in Sec. IV, this gives $\xi_z^X = a^2 BK/\Delta_h = (a/\lambda)^2 \xi_z^{\rm NL}$, which is, reassuringly, much less than $\xi_z^{\rm NL}$ in the limit $a \ll \lambda$ in which we have asserted it applies.

In this section, in our treatment of elastic nonlinearities we have ignored dislocations, which in Sec. V we have shown to unbind in a *harmonic* theory, for arbitrarily weak disorder. However, as we have shown in that section, their effects are only felt on length scales greater than $\xi_z^D = (a^2/\lambda) e^{\lambda \xi_z^X / a^2}$, where t_0 is a parameter of order BK, which is much longer, in the weak disorder $(\Delta_h \to 0,\ \xi_z^X \to \infty)$ limit, than the translational correlation length ξ_z^X found here. However, it is nevertheless important to study the simultaneous effects of anomalous elasticity and dislocations. We do this in Sec. VII by reconsidering the criterion for dislocation unbinding, studied in Sec. V, now in the presence of anomalous elasticity. We will find that anomalous elasticity leads to shortening of ξ_z^D, which nonetheless remains much longer than ξ_z^X. Hence, for low density aerogels, our predictions for ξ_z^X and the anomalous elasticity studied here remain valid.

Finally, our entire discussion so far has focused only on the effects of *orientational* tilt disorder. In Sec. VIII we will consider the *translational* disorder (i.e., random pinning of the positions of the layers) and will demonstrate that it is *less* important, at long wavelengths (in $d<5$), than the orienta-

tional disorder. Thus, the results described herein are directly applicable to real smectics, where both kinds of disorder are present.

VII. EFFECTS OF ANOMALOUS ELASTICITY ON DISLOCATION UNBINDING AND SMECTIC CORRELATIONS

In Sec. VI we established that, at least at the ''phonon-only'' level, the anharmonic terms in the elastic energy are extremely important below $d=5$. We therefore need to also consider the effects of anharmonic elasticity (and the resulting anomalous elasticity) on dislocations and smectic correlation functions.

A. Dislocations and orientational order

In this subsection we incorporate anomalous elasticity into our duality model and use it to derive a criterion, in terms of the anomalous exponents η_B and η_K, for the dislocation unbinding to take place in the presence of anomalous elasticity. Combining these results with an analysis of orientational order we establish bounds on η_B and η_K for the stability of the smectic Bragg glass phase. However, since we do not know the numerical values of η_B and η_K in $d=3$, we cannot say whether or not, in the full, anharmonic theory, dislocations are, in fact, unbound.

Unfortunately, a full *anharmonic* theory of dislocations is simply intractable. In particular, the fact that, in an anharmonic theory, the interaction energy between dislocations *cannot* be written as a sum of pairwise interactions (since their fields do not simply add) makes it impossible to even write down a general expression for the energy of an arbitrary dislocation configuration. At best, one might hope to be able to write down the energy for a few simple, high symmetry configurations (e.g., a single, straight dislocation line). Such specialized results would be of no use in a full statistical theory of defect unbinding, which requires consideration of very complicated, tangled configurations of dislocations, which, for entropic reasons, dominate the free energy near the dislocation binding transition. Furthermore, even *if* one *could* write down the anharmonic energy for an arbitrary dislocation configuration, it would presumably be anharmonic in the dislocation fields $\mathbf{m}(\mathbf{r})$, and hence, those fields could *not* be decoupled by a simple Hubbard-Stratonovich trick, as they can in the harmonic approximation.

For all of these reasons, a completely honest treatment of dislocations in the full, anharmonic model is impossible, at least for mere mortals like us. Rather than despair, however, we will use a slightly dishonest (i.e., uncontrolled) approximation: we will simply replace the elastic *constants* K and B, and tilt disorder variance Δ_h in our expression Eq. (5.37) for the renormalized dislocation unbinding transition temperature, with the renormalized, wave-vector-dependent moduli $K(\mathbf{q})$, $B(\mathbf{q})$, and $\Delta_h(\mathbf{q})$, derived from the anharmonic, tilt-only elastic theory of Sec. VI.

Doing so, we obtain for the downward renormalization of the *dual* transition temperature

$$\delta t = -\mathrm{O}(1)\times\frac{ca^2d^2}{T^2}\int\frac{\Delta_h(\mathbf{q})q_z^2 d^dq}{[q_z^2+\lambda^2(\mathbf{q})q_\perp^4]^2}, \tag{7.1}$$

where $\Delta_h(\mathbf{q})$ is given by Eq. (6.26c), c is given by Eq. (5.29), and $\lambda^2(\mathbf{q})\equiv K(\mathbf{q})/B(\mathbf{q})$, with $K(\mathbf{q})$ and $B(\mathbf{q})$ given by Eqs. (6.26a) and (6.26b), respectively. In writing Eq. (7.1) we have anticipated that the integral is dominated by region $q_z \ll q_\perp$, as can be seen from Eq. (7.3), below.

Imposing an infrared cutoff $q_\perp > L^{-1}$ on the wave-vector integral in Eq. (7.1), where L is the lateral ($\perp$) real-space linear extent of the system, and changing variables in that integral as follows:

$$\mathbf{q}_\perp \equiv \frac{\mathbf{Q}_\perp}{L}, \tag{7.2a}$$

$$q_z \equiv \frac{Q_z}{L^\zeta}\frac{\lambda}{(\xi_\perp^{\mathrm{NL}})^\chi}, \tag{7.2b}$$

with $\chi\equiv(\eta_B+\eta_K)/2$, we find that

$$\delta t \approx -\left(\frac{L}{\xi_\perp^{\mathrm{NL}}}\right)^\gamma \frac{ca^2d^2\Delta_h}{\lambda T^2} \times\int_{L\Lambda>Q_\perp>1}\frac{Q_\perp^{-\eta_\Delta}Q_z^2 f_\Delta(Q_z/Q_\perp^\zeta)d^dQ}{[Q_z^2+Q_\perp^{2\zeta}f_K(Q_z/Q_\perp^\zeta)/f_B(Q_z/Q_\perp^\zeta)]^2}, \tag{7.3}$$

with

$$\gamma=\zeta+\eta_\Delta+1-d. \tag{7.4}$$

In deriving Eq. (7.4), we have used the scaling relation $\zeta=2-(\eta_B+\eta_K)/2$, and Eq. (6.30) defined in Sec. VI.

Now, it is equally straightforward to show, by anisotropic power counting, that the integral over $\mathbf{Q}$ in Eq. (7.3) is convergent in the ultraviolet when $\gamma>0$, and is therefore independent of its upper cutoff $L\Lambda$, for $L\to\infty$. As a result, for $\gamma>0$, the correction to the reduced *dual* temperature t depends on L only through the overall prefactor L^γ, i.e.,

$$\delta t \approx -\left(\frac{L}{\xi_\perp^{\mathrm{NL}}}\right)^\gamma \frac{ca^2d^2\Delta_h}{\lambda T^2}\times\mathrm{O}(1), \tag{7.5}$$

and therefore, for $\gamma>0$, diverges with system size.

As discussed in our original treatment of duality, such an infinite *downward* renormalization of t in the *dual* model implies that dislocations are *always* unbound, even at zero temperature, even for arbitrarily weak disorder. Thus, our criterion for determining whether or not dislocations are always unbound in $d=3$ is very simple: if γ, as given by Eq. (7.4), is >0, dislocations *are* always unbound, and *no* stable Bragg glass phase exists, while if $\gamma<0$, dislocations will be bound, and a stable Bragg glass phase *will* exist, at least at sufficiently low temperatures and weak disorder.

So it behooves us to determine γ in $d=3$. Using $\zeta=2-(\eta_B+\eta_K)/2$, and the exact scaling relation Eq. (6.30) between η_Δ, η_B, and η_K, we can rewrite Eq. (7.4) in terms of η_Δ and η_K as

$$\gamma=2(\eta_K-1). \tag{7.6}$$

Using our ϵ-expansion result Eq. (1.10a) for η_K, we obtain

$$\gamma=2\left(\frac{\epsilon}{5}-1\right)+O(\epsilon^2), \tag{7.7a}$$

$$=-\frac{6}{5}<0, \tag{7.7b}$$

the last equality holding in $d=3$ ($\epsilon=2$), if we drop the unknown $O(\epsilon^2)$ term. This result *appears* to imply that dislocations are bound, and the smectic Bragg glass is *stable* in three dimensions. Of course, it is always risky to quantitatively extrapolate an expansion for small ϵ out to $\epsilon=2$, but this is the best we can do in $d=3$. And it seems to suggest that the smectic Bragg glass is *stable* in $d=3$.

Independent of the ϵ-expansion, our result implies the strict *necessary, but not sufficient* condition for the stability of the smectic Bragg glass phase

$$\eta_K<1. \tag{7.8}$$

If, on the other hand, this condition is violated, then dislocations will unbind on length scales longer than $\xi_\perp^D$, where $\xi_\perp^D$ is the value of L in Eq. (7.5) such that the correction δt becomes of $O(t)$. This gives the value of $\xi_\perp^D$ quoted in the Introduction, with $c(T)$ given by Eq. (5.29). We note that, although $\lambda(T)\to\infty$ as $T\to T_{\rm NA}$ from below, so does $c(T)$, since the renormalized dislocation core energy vanishes at $T_{\rm NA}$. We expect the divergence of $c(T)$ to overwhelm that of $\lambda(T)$, leading to a dislocation length that gets ever shorter as $T\to T_{\rm NA}$ from below. Eventually, as we approach $T_{\rm NA}$ from below, $\xi_\perp^D$, as given by Eq. (1.18), will get smaller than $\xi_\perp^{\rm NL}$; at this point, we are in the strong coupling regime, and our weak-disorder results no longer apply.

There is another criterion, in addition to Eq. (7.8), that must be satisfied for the smectic Bragg glass to be stable in $d=3$: *long-ranged orientational* order must exist. This is clear, since we have implicitly assumed the existence of such order throughout our calculation (e.g., in writing $\hat{\mathbf{n}}=\hat{\mathbf{z}}+\delta\mathbf{n}$ and *assuming* $|\delta\mathbf{n}|\ll 1$). Surely, if the layer orientations are not ordered, the layers themselves cannot retain their integrity (i.e., remain undislocated). So long-ranged orientational order must be preserved for the smectic Bragg glass phase to be stable. An obvious corollary of this conclusion is that orientational *fluctuations* $\overline{\langle|\delta\mathbf{n}(\mathbf{r})|^2\rangle}$ must remain *finite* (as the system size $L\to\infty$) for the smectic Bragg glass to be stable.

We will now show that within a $5-d=\epsilon$ expansion, this condition is definitely *not* satisfied: orientational fluctuations *do* diverge in $d=3$. Hence, the ϵ expansion implies that [notwithstanding the result $\gamma<0$, Eq. (7.7)] the smectic Bragg glass phase is *not* stable in three dimensions.

To show this, we use the fact that $\delta\mathbf{n}(\mathbf{r})=\boldsymbol{\nabla}_\perp u(\mathbf{r})$ to write

$$\overline{\langle|\delta\mathbf{n}(\mathbf{r})|^2\rangle}=\int\frac{d^dq}{(2\pi)^d}\overline{\langle|\delta\mathbf{n}(\mathbf{q})|^2\rangle}, \tag{7.9a}$$

$$=\int\frac{d^dq}{(2\pi)^d}q_\perp^2\overline{\langle|\mathbf{u}(\mathbf{q})|^2\rangle}, \tag{7.9b}$$

$$=\int\frac{d^dq}{(2\pi)^d}\frac{\Delta_h(\mathbf{q})q_\perp^4}{[B(\mathbf{q})q_z^2+K(\mathbf{q})q_\perp^4]^2}, \tag{7.9c}$$

where we have kept only the dominant, disorder induced term in $\overline{\langle|u(\mathbf{q})|^2\rangle}$, Eq. (3.12), in writing the last equality. Imposing an infrared cutoff $q_\perp>L^{-1}$, and power counting on this integral exactly as we just did in evaluating Eq. (7.3), we find that

$$\overline{\langle|\delta\mathbf{n}(\mathbf{r})|^2\rangle}\propto L^\delta, \tag{7.10}$$

with

$$\delta=-d-3+\eta_\Delta+2\eta_B+3\zeta. \tag{7.11}$$

Using $\zeta=2-(\eta_B+\eta_K)/2$ and the exact scaling relation Eq. (6.30), we can rewrite this as

$$\delta=\eta_B+\eta_K-2, \tag{7.12a}$$

$$=2(\chi-1). \tag{7.12b}$$

Thus, stability of long-ranged *orientational* order, and hence, stability of the smectic Bragg glass phase itself, requires $\delta<0$, or

$$\eta_B+\eta_K<2, \tag{7.13}$$

which, *not* coincidentally, is equivalent to the condition that the anisotropy exponent $\zeta>1$. The reason that these two conditions are equivalent is clear. As we go down in dimension, η_B and η_K grow, and the anisotropy exponent ζ decreases below 2, with the system becoming more isotropic. The dimension at which the orientational order disappears must be *exactly* the dimension at which the system becomes completely isotropic, i.e., $\zeta=1$, since, without orientational order, there can be no distinction between different directions. Thus, the lower critical dimension for orientational order is given by $\zeta=2-(\eta_B+\eta_K)/2=1$, which implies $\eta_B+\eta_K=2$, the borderline of the inequality Eq. (7.13).

If the condition Eq. (7.13) is violated, then for length scales beyond the orientational correlation length, ξ_O,

$$\xi_O=\xi_\perp^{\rm NL}\left(\frac{\xi_z^{\rm NL}}{\lambda}\right)^{1/[2(\chi-1)]}, \tag{7.14}$$

obtained by equating $\overline{\langle|\delta\mathbf{n}(\mathbf{r})|^2\rangle}$ to 1, the orientational order is destroyed, our smectic description breaks down, no stable SBG exists, and the system is isotropic. On the other hand, if condition in Eq. (7.13) holds, i.e., $\chi<1$, ξ_O is infinite and SBG is stable.

The length scales $\xi_\perp^{\rm NL}$ and ξ_O, derived in Sec. VI and in this section, respectively, lead to nontrivial, and in principle experimentally measurable, crossover behavior. At the shortest length scales, $q_\perp\gg 1/\xi_\perp^{\rm NL}$, nonlinear elasticity is unimportant and the dominant modes correspond to $q_z\sim\lambda q_\perp^2$. At longer scales, for $q_\perp<1/\xi_\perp^{\rm NL}$ this quadratic behavior crosses over to $q_z\xi_z^{\rm NL}\sim(q_\perp\xi_\perp^{\rm NL})^\zeta$, controlled by the nontrivial anomalous elasticity fixed point studied in Sec. VI. The anisotropy exponent ζ satisfies $1<\zeta<2$ if Eq. (7.13) is satisfied and orientational order is stable, but is less than 1 if this condition is violated and ξ_O is finite. In this latter case, the

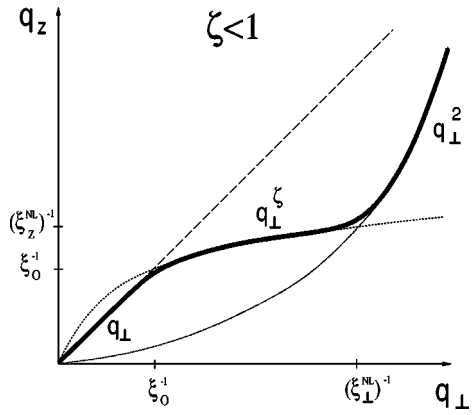

FIG. 11. Crossover in the dominant wave-vector regimes in $q_\perp, q_z$ plane, for $\zeta<1$.

anomalous elasticity scaling $q_z\xi_z^{\rm NL}\sim(q_\perp\xi_\perp^{\rm NL})^\zeta$ crosses over to the isotropic scaling $q_z\sim q_\perp$ for lengths scales longer than ξ_O. This orientational crossover length scale is, reassuringly, precisely the orientational correlation length given in Eq. (7.14), obtained from the condition of $\overline{\langle|\delta\mathbf{n}(\mathbf{r})|^2\rangle}=1$. It is important to note that, even for $\zeta<1$, in the dominant regime of wave vectors for $|\mathbf{q}|>1/\xi_O$, the condition $q_z\ll q_\perp$ is always satisfied. Hence, even in the absence of long-ranged orientational order beyond ξ_O, this justifies our use, throughout the smectic regime ($|\mathbf{q}|>1/\zeta_O$), of the approximation $q_z\ll q_\perp$, which is indeed the usual approximation one makes in treating smectic elasticity. These dominant wave-vector regimes in $q_\perp, q_z$ plane, various important length scales, and the corresponding crossovers are illustrated in Fig. 11.

Combining the stability conditions Eq. (7.8) and Eq. (7.13) with the rigorous bound

$$\eta_B+5\,\eta_K>4 \tag{7.15}$$

that follows from requiring $\eta_\Delta>0$ in the exact scaling relation Eq. (6.30) in $d=3$, we find that the three-dimensional smectic Bragg glass phase can only be stable when η_B and η_K lie in the shaded quadrilateral in Fig. 1.

Our ϵ-expansion results from Eqs. (6.25c) and (6.28c), extrapolated to $d=3$

$$\eta_K=\frac{2}{5}, \tag{7.16a}$$

$$\eta_B=\frac{12}{5}, \tag{7.16b}$$

lie well outside this stable region. Whether this extrapolation can be trusted in $d=3$ is, of course, an open question. *We are therefore left, despite our best efforts as displayed in this manuscript, unable to decide whether or not the smectic Bragg glass phase actually exists in* $d=3$. This question can therefore only be answered by experiments.

B. Smectic correlation functions

In addition to having important effects on dislocation unbinding, as demonstrated in the previous subsection, anharmonic elasticity also changes the smectic correlation functions from those derived for the elastically harmonic model of Sec. IV. We study these effects in this subsection.

We focus on the smectic layer displacement correlation function defined by Eq. (4.1)

$$C(\mathbf{r}_\perp,z)=\overline{\langle[u(\mathbf{r}_\perp,z)-u(\mathbf{0},0)]^2\rangle}. \tag{7.17}$$

To compute $C(\mathbf{r}_\perp,z)$, we proceed as in Sec. IV, except that instead of using the bare values of elastic moduli K, B, and tilt-disorder variance Δ_h, we will use their wavevector-dependent counterparts $K(\mathbf{k})$, $B(\mathbf{k})$, and $\Delta_h(\mathbf{k})$, given in Eqs. (6.26). We expect this procedure to take into account the long length scale effects of anharmonic elasticity. We obtain

$$C(\mathbf{r}_\perp,z)\approx\int\frac{d^dk}{(2\pi)^d}\,\frac{2[1-\cos(\mathbf{k}\cdot\mathbf{r})]\Delta_h(\mathbf{k})k_\perp^2}{[K(\mathbf{k})k_\perp^4+B(\mathbf{k})k_z^2]^2}, \tag{7.18}$$

where we have only kept the dominant disorder-induced contribution. Performing an asymptotic analysis similar to that of the previous subsection, we find that the harmonic elasticity result, Eq. (4.9) is replaced by

$$C(\mathbf{r}_\perp,z)\approx\lambda^2\begin{cases}\left(\dfrac{z}{\xi_z^{\rm NL}}\right)^{(\eta_B+\eta_K)/\zeta}, & z/\xi_z^{\rm NL}\gg(r_\perp/\xi_\perp^{\rm NL})^\zeta,\\ \left(\dfrac{r_\perp}{\xi_\perp^{\rm NL}}\right)^{\eta_B+\eta_K}, & z/\xi_z^{\rm NL}\ll(r_\perp/\xi_\perp^{\rm NL})^\zeta,\end{cases} \tag{7.19}$$

valid for $d<5$ and on scales beyond the crossover scales $\xi_{z,\perp}^{\rm NL}$, when the smectic elasticity is controlled by the nontrivial anomalous fixed point with nonvanishing η_B,η_K, and the anisotropy exponent $\zeta\equiv2-(\eta_B+\eta_K)/2<2$.

The above result for smectic layer undulations has obvious implications for the smectic order parameter $[\psi(\mathbf{r})]$ correlations. Within the Debye-Waller approximation, which amounts to ignoring higher order cumulants, the smectic correlation function is given by

$$S(\mathbf{r}_\perp,z)\equiv\overline{\langle\psi^*(\mathbf{r})\psi(\mathbf{0})\rangle}, \tag{7.20a}$$

$$\approx e^{-(q_0^2/2)C(\mathbf{r})}. \tag{7.20b}$$

It is short-ranged, confirming our earlier claims about the destruction (in $d<5$) of the long-range smectic order by an arbitrarily small amount of tilt disorder, even in the presence of anharmonic elasticity.

Although the anomalous elasticity does not prevent the destruction of the long-ranged smectic order, it does lead to some physically appealing features in the above expression for $S(\mathbf{r}_\perp,z)$, and $C(\mathbf{r})$, Eq. (7.19), which are absent when these elastic anharmonic effects are not taken into account. In particular, we observe that at the lower-critical dimension in which the orientational order becomes short ranged (i.e., $\eta_B+\eta_K\to2$ and $\zeta\to1$), the smectic correlation function $S(\mathbf{r})$ is isotropic and decays as a Gaussian along both z and $r_\perp$. In this case of short-ranged orientational order this faster than exponential decay of smectic correlations of course only persists up to the orientational correlation length, beyond which we expect the order to decay simply exponentially.

VIII. ANOMALOUS ELASTICITY AND TRANSLATIONAL PERIODIC DISORDER IN $3<D<7$

The analysis of nonlinear elasticity of the preceding two sections was confined to the tilt-only model in which the random *field* disorder is neglected. In this section we study the effect of the periodic field disorder in the presence of anharmonic elasticity and show that, for all $d<5$, and, in particular in $d=3$, such random-field disorder

$$H_1^{\rm rf}=-\frac{1}{T}\int d^d\mathbf{r}|\psi_0|U(\mathbf{r})\cos[q_0(u(\mathbf{r})+z)] \tag{8.1}$$

has no effect on the long-wavelength ''pure phonon'' tilt-only disorder theory; that is, as long as dislocations are bound, the anomalous elastic theory of the previous section is correct in $d<5$. In the next section (Sec. IX), we show that periodic disorder does not induce dislocation unbinding.

We begin by showing that the periodic disorder itself induces anomalous elasticity in all spatial dimensions $d<7$. Fortunately, the induced corrections to B and K, and to the u fluctuations themselves, are so weak (only logarithmic) in *all* spatial dimensions $d<7$ that, as soon as $d<5$, they are completely overwhelmed by the *power-law* divergent fluctuations and renormalizations induced by the *tilt* disorder that we studied in Sec. VI. These latter fluctuations and renormalizations therefore dominate, and are unaffected by the presence of the periodic random field disorder. Therefore, once again, the static correlations that we derived in Sec. VI from the 5-ϵ expansion on the tilt-only disorder model are correct, in the long-wavelength limit, even for the full model.

To derive these results, we consider the full disordered smectic phonon model. In addition to the two types of disorder (random tilt and periodic random field disorders), described by the effective Hamiltonian in Eq. (3.8), this model also includes the anharmonic elasticity (considered in the previous section in the tilt-only phonon model), but still ignores unbound dislocations. Generalizing Eq. (3.8) to anharmonic elasticity, as we did in Sec. VI for the tilt-only model, we obtain the effective Hamiltonian for such a phonon model of disordered smectic

$$\begin{aligned}H=\int d^dr\Bigg\{&\sum_{\alpha=1}^{n}\left[\frac{K}{2}(\nabla_\perp^2 u_\alpha)^2+\frac{B}{2}\left(\partial_z u_\alpha-\frac{1}{2}(\boldsymbol{\nabla}_\perp u_\alpha)^2\right)^2\right]\\&+\frac{1}{T}\sum_{\alpha,\beta=1}^{n}\left(\frac{\Delta_h}{4}|\boldsymbol{\nabla}_\perp(u_\alpha-u_\beta)|^2\right.\\&\left.-\Delta_V\cos[q_0(u_\alpha-u_\beta)]\right)\Bigg\}.\end{aligned} \tag{8.2}$$

To study the long distance behavior of this model in general dimensions d, for reasons that we now explain, we must first generalize it further.

First, recall that in the Sec. III derivation of the model, the periodicity of the smectic phase required the translational disorder (i.e., the random field Δ_V) to be a function that is invariant under discrete translations by smectic lattice constant a (i.e., periodic with period a). In writing down Eq. (8.2), we took this function to be the *lowest* cosine harmonic. Our implicit justification for this was that in $d=3$, in harmonic elastic theory, the eigenvalue λ_n for the nth harmonic (with wave vector nq_0)

$$H_n^{rf}=-\frac{1}{T}\int d^d\mathbf{r}|\psi_0|U(\mathbf{r})\cos[nq_0(u(\mathbf{r})+z)] \tag{8.3}$$

is given by a simple generalization of

$$\lambda_1=4-\eta, \tag{8.4}$$

[see Eq. (3.21)] to be

$$\lambda_n=4-\eta n^2. \tag{8.5}$$

As long as η is fixed under the RG flow, this shows that the *higher* harmonics of the random pinning potential are less relevant, i.e., they are irrelevant at the point at which the lowest $n=1$ harmonic becomes relevant. However, here we argue that in d dimensions $\eta(l)$ flows according to

$$\frac{d\eta(l)}{dl}=-\theta(l)\eta(l), \tag{8.6}$$

with the ''thermal eigenvalue'' $\theta>0$ for $d>d_c$ with $d_c>3$. We can demonstrate this explicitly in three limiting cases described below.

The first is the truncated model Eq. (3.8) analyzed in Sec. III, which ignores anharmonic elasticity. Using the exact expressions, Eqs. (3.18b),(3.18c) for the flow of K and B (see discussion in Sec. III), and the constancy of q_0 under the RG, inside η, Eq. (3.19a), we obtain

$$\theta_{\rm harmonic}=d-3, \tag{8.7}$$

exactly, in this elastically *harmonic* model.

We can also compute θ near $d=5$ in a model defined by the Hamiltonian in Eq. (6.3) that includes anharmonic elasticity but leaves out the random field disorder (Δ_V), which is, as we show below, subdominant for $d<5$. In this case, combining Eqs. (6.17a),(6.17b) with the flow of q_0 (governed by the eigenvalue $\phi=2-\omega$) and the fixed point value for g_2, Eq. (6.19), inside Eq. (3.19a), we obtain

$$\theta_{d\to 5^-}=d-3-\frac{\epsilon_5}{2} \tag{8.8}$$

exact to order $\epsilon_5\equiv 5-d$.

Finally, as we will show in this section (with details given in Appendix B) for $5<d<7$,

$$\theta_{d\to 7^-}(l)=d-3+O(1/l), \tag{8.9}$$

In these three regimes for $d>d_c\approx 3$, $\theta>0$, which leads to $\eta(l\to\infty)\to 0$. Based on Eq. (8.5), the implication of this is that the spectrum of random field harmonics is *degenerate*, i.e., an infinite class of operators $H_n^{\rm rf}$, Eq. (8.3), are all equally relevant and must be treated simultaneously. Since these higher harmonics are generated by the RG, we are forced to consider a more general model

$$H=\int d^dr\Bigg\{\sum_{\alpha=1}^{n}\left[\frac{K}{2}(\nabla_\perp^2 u_\alpha)^2+\frac{B}{2}\left(\partial_z u_\alpha-\frac{1}{2}(\boldsymbol{\nabla}_\perp u)^2\right)^2\right]+\frac{1}{T}\sum_{\alpha,\beta=1}^{n}\Bigg(\frac{1}{4}\Delta_h(u_\alpha-u_\beta)|\boldsymbol{\nabla}_\perp(u_\alpha-u_\beta)|^2-\Delta_V(u_\alpha-u_\beta)\Bigg)\Bigg\},\tag{8.10}$$

where $\Delta_h(u_\alpha-u_\beta)$ and $\Delta_V(u_\alpha-u_\beta)$ are arbitrary *periodic* functions (with period a, where a is the smectic layer spacing), describing the field-dependent variances of the tilt and translational disorders. Rather than following the flow for an infinite number of couplings (one for each Fourier component of the random potentials) we apply functional RG (FRG) methods developed by Fisher in the context of the random field Ising and XY models[56] to our study of disordered smectics.

A. Functional RG for $5<d<7$: irrelevance of random-tilt disorder

We integrate out the short scale phonon modes within an infinitesimal momentum shell near the uv cutoff and study how the elastic moduli $B(l)$, $K(l)$ and the disorder variance *function* $\Delta_V(u,l)$ evolve under this RG transformation. As discussed in detail in Sec. VI, the tilt disorder Δ_h [characterized by the dimensionless coupling g_2, Eq. (6.15)] is irrelevant by simple power counting for $d>5$. We will show at the end of Appendix B that this conclusion persists, even in the presence of random-field disorder, although the argument is subtler. Hence for $5<d<7$ we can focus on the random-field-only model of a randomly pinned smectic. We relegate the details of the FRG procedure to Appendix B. For $5<d<7$, the results are summarized by the FRG flow equations

$$\frac{dK}{dl}=\left[d-3+\frac{(d^2-12d+23)\tilde{\Delta}_V''(0,l)}{2(d^2-1)\lambda^2(l)}\right]K,\tag{8.11}$$

$$\frac{dB}{dl}=\left[d-3+\frac{3}{2}\frac{\tilde{\Delta}_V''(0,l)}{\lambda^2(l)}\right]B,\tag{8.12}$$

and

$$\partial_l\tilde{\Delta}_V(u,l)=\epsilon\tilde{\Delta}_V(u,l)+\frac{\eta(l)}{q_0^2}\tilde{\Delta}_V''(u,l)-\frac{3(d^2-6d+11)}{2(d^2-1)\lambda^2(l)}\tilde{\Delta}_V''(0,l)\tilde{\Delta}_V(u,l)+\frac{1}{2}[\tilde{\Delta}_V''(u,l)]^2-\tilde{\Delta}_V''(u,l)\tilde{\Delta}_V''(0,l),\tag{8.13}$$

where

$$\tilde{\Delta}_V(u,l)\equiv\frac{C_{d-1}\Lambda^{d-7}}{4\sqrt{K^3(l)B(l)}}\Delta_V(u,l)\tag{8.14}$$

is the appropriate measure of random field disorder, $\epsilon\equiv 7-d$,

$$\lambda(l)=\sqrt{\frac{K(l)}{B(l)}},\tag{8.15a}$$

$$\eta(l)=\frac{q_0^2T}{4\pi\sqrt{K(l)B(l)}},\tag{8.15b}$$

and primes in Eq. (8.13) indicate a partial derivative with respect to u.

Note that we have written the flow equations for $K(l)$ and $B(l)$ in general d, rather than specializing to $d=7-\epsilon$. The reason for this is that for *all* $d>5$ (i.e., wherever we can ignore *tilt* disorder), the recursion relations Eqs. (8.11) and (8.12) become asymptotically *exact* as $\tilde{\Delta}''(0,l)/\lambda^2(l)\to 0$, which, as we will show in a moment, it always does for $5<d<7$.

To demonstrate the above assertion we first assume (and verify *a posteriori)* that near $d=7$, the flow equations (8.11)–(8.13) lead to a perturbative fixed point at which $\tilde{\Delta}_V(u,l\to\infty)\equiv\tilde{\Delta}_V^*(u)$ is a universal function of order ϵ with a finite (negative) second derivative $\tilde{\Delta}_{V*}''(0)\approx -q_0^2O(\epsilon)$. Then, combining Eqs. (8.11),(8.12) into a recursion relation for $\lambda(l)$, as defined by Eq. (8.15a), we obtain

$$\frac{d\lambda}{dl}=-\frac{\tilde{\Delta}_V''(0,l)}{\lambda(l)}\left(\frac{d^2+6d-13}{2(d^2-1)}\right)\tag{8.16}$$

which, assuming that $\tilde{\Delta}_V''(0,l\to\infty)\to\tilde{\Delta}_{V*}''(0)<0$, has the solution

$$\lambda(l)=\lambda_0\sqrt{1+l/l_0},\tag{8.17}$$

with

$$\lambda_0\equiv\lambda(l=0),\tag{8.18a}$$

$$l_0=\frac{\lambda_0^2}{|\tilde{\Delta}_{V*}''(0)|}\frac{d^2-1}{d^2+6d-13}.\tag{8.18b}$$

The main result of this analysis is that at large length scales, $l\gg l_0$, $\lambda^2(l)$ *grows* as a *universal* function of l

$$\lambda^2(l\to\infty)=\frac{d^2+6d-13}{d^2-1}|\tilde{\Delta}_{V*}''(0)|l.\tag{8.19}$$

Hence, as asserted above, as $\lambda^2(l)$ grows, the graphical corrections to the flow of $K(l)$ and $B(l)$ diminish, Eqs. (8.11),(8.12) become asymptotically *exact*, and the third term in Eq. (8.13) vanishes as $l\to\infty$. All this is still subject to the assumption that $\tilde{\Delta}_{V*}''(0)\neq 0$.

We can begin demonstrating this latter assumption by combining Eqs. (8.11),(8.12) into a recursion relation for $\eta(l)$, given by Eq. (8.15b). Comparing the resulting equation with the generic flow equation for $\eta(l)$, Eq. (8.6), we find that $\theta(l)$ (for $l\gg l_0$) is given by a *universal* function

$$\theta(l)=d-3-\left(\frac{d^2-3d+5}{d^2+6d-13}\right)\frac{1}{l}. \tag{8.20}$$

We note that besides being universal, $\theta(l)$ is furthermore independent of the fixed point value $\tilde{\Delta}''_{V*}(0)$, as long as it is finite. As a result of Eq. (8.20) we conclude that for $d>3$, $\eta(l)$ is strongly irrelevant. Physically, this reflects the strong irrelevance of thermal fluctuations for $d>3$, with the properties of the pinned smectic determined by a competition between elastic and random pinning energies (both infinitely larger than the thermal energy k_BT). As a result, the second term on the right hand side of Eq. (8.13) flows to zero exponentially fast, and can therefore be dropped. Likewise, the $1/\lambda(l)^2$ term in Eq. (8.13) also vanishes as $l\to\infty$ (albeit more slowly), and can also be neglected. This leads to a closed recursion relation for $\tilde{\Delta}_V(u,l)$, which has been extensively studied in the context of the random field XY model[56] and other pinned periodic media.[57,9]

The fixed point solution $\tilde{\Delta}_*(u)$ is readily found by setting $\partial_l\tilde{\Delta}_V(u,l)=0$, which leads to an ordinary differential equation for $\tilde{\Delta}_{V*}(u)$:

$$\epsilon\tilde{\Delta}_{V*}(u)+\frac{1}{2}[\tilde{\Delta}''_{V*}(u)]^2-\tilde{\Delta}''_{V*}(u)\tilde{\Delta}''_{V*}(0)=0. \tag{8.21}$$

The exact form of $\tilde{\Delta}_{V*}(u)$ is not important for our conclusions (except for one exponent); all that matters is that a fixed point solution $\tilde{\Delta}_{V*}(u)$ exists (which it does for $5<d<7$). However, for completeness we summarize the fixed-point results for $\tilde{\Delta}_{V*}(u)$. Focusing, for convenience, on the random *field* correlator $\tilde{\Delta}_{F*}(u)\equiv\tilde{\Delta}''_{V*}(u)$, which satisfies

$$\epsilon\tilde{\Delta}_{F*}(u)+[\tilde{\Delta}'_{F*}(u)]^2+\tilde{\Delta}''_{F*}(u)[\tilde{\Delta}_{F*}(u)-\tilde{\Delta}_{F*}(0)]=0, \tag{8.22}$$

we obtain

$$\tilde{\Delta}_{F*}(u)=-\min_{n\in\mathcal{Z}}\frac{\epsilon}{q_0^2}\left\{\frac{1}{6}[q_0u-(2n+1)\pi]^2-\frac{\pi^2}{18}\right\}, \tag{8.23}$$

which has cusps at $u=n2\pi/q_0=na$, n integer. It is easy to show[56] that at $T=0$ ($\eta=0$) the cusp nonanalyticity in $\tilde{\Delta}_{F*}(u)$ develops at *finite* ''time'' l, while for finite $T(\eta\neq0)$ this singularity is cutoff within a boundary layer of thickness $\sim\eta(l)/\epsilon q_0$ at the value $\tilde{\Delta}''_{F*}(0,l)\approx\epsilon^2\pi^2/9\eta(l)$. We also see from the solution for $\tilde{\Delta}_{F*}(u)=\tilde{\Delta}''_{V*}(u)$, Eq. (8.23), that $\tilde{\Delta}''_{V*}(0)=-\epsilon\pi^2/9q_0^2<0$, thereby, together with Eq. (8.19), demonstrating that $\lambda^2(l\to\infty)\to\infty$, as assumed earlier. Having established the existence of the zero-temperature fixed point for $5<d<7$, characterized by the universal form of the disorder correlator, Eq. (8.23), we now turn to the implications of these results for the elastic properties of the randomly pinned smectic.

Using Eqs. (8.15a)–(8.18b) inside the flow equations for $K(l)$ and $B(l)$, Eqs. (8.11),(8.12), and solving for these elastic moduli, we obtain

$$K(l)=e^{(d-3)l}K_p(l), \tag{8.24a}$$

$$B(l)=e^{(d-3)l}B_p(l), \tag{8.24b}$$

where the physically measurable moduli $B_p(l)$ and $K_p(l)$ are

$$K_p(l)=K_0\left(1+\frac{l}{l_0}\right)^{\gamma_K}, \tag{8.25a}$$

$$B_p(l)=B_0\left(1+\frac{l}{l_0}\right)^{-\gamma_B}, \tag{8.25b}$$

with

$$\gamma_K(d)=\frac{1}{2}\left(\frac{-d^2+12d-23}{d^2+6d-13}\right) \tag{8.26}$$

and

$$\gamma_B(d)=\frac{3}{2}\left(\frac{d^2-1}{d^2+6d-13}\right). \tag{8.27}$$

In the range of interest the exponents $\gamma_K(d)$ and $\gamma_B(d)$ are monotonically decreasing and increasing functions of d, respectively, and do not change significantly for $5<d<7$: $\gamma_K(5)=1/7$, $\gamma_K(7)=1/13$, $\gamma_B(5)=6/7$, $\gamma_B(7)=12/13$. As advertised, we therefore find (for $5<d<7$) that the randomly pinned smectic exhibits logarithmically weak anomalous elasticity, with the bend modulus K_p diverging and the compression modulus B_p vanishing logarithmically at large length scales $e^l\to\infty$.

With these results in hand, we can calculate the corresponding smectic time-persistent correlation function

$$C(\mathbf{k})\equiv\frac{\overline{\langle u(\mathbf{k})\rangle\langle u(\mathbf{k}')\rangle}}{\delta^d(\mathbf{k}+\mathbf{k}')}, \tag{8.28}$$

which determines the long-range spatial smectic correlations. Following the RG matching method, described in detail in Sec. VI, we relate $C(\mathbf{k})$ at the small wave vectors of interest to us, to the same quantity computed in the effective theory with rescaled l-dependent couplings and evaluated at larger wave vectors $e^lk_\perp, e^{\omega l}k_z$:

$$C(\mathbf{k})=e^{(d-1+\omega)l}C[e^lk_\perp, e^{\omega l}k_z, B(l), K(l), \Delta_V(u,l)]. \tag{8.29}$$

Here let us consider the limit $k_z\ll\lambda k_\perp^2$. Now, we set (for convenience) $\omega=2$ and choose $e^{l*}=\Lambda/k_\perp$. The latter condition guarantees that no infrared divergences appear in the perturbative calculation of the right-hand side of Eq. (8.29) (since it is done at large wave vector $e^{l*}k_\perp=\Lambda$ at the Brillouin zone boundary). At these short length scales (Λ^{-1}), phonon displacements u are genuinely small and the random potential variance $\Delta_V(u_\alpha-u_\beta)$ [Eq. (8.10)] can be safely expanded in a convergent power series in $u_\alpha-u_\beta$. Expanding the Hamiltonian Eq. (8.10) to the lowest (quadratic) nontrivial order maps the full nonlinear FRG problem at these high wave vectors onto the random tilt-only model Eq. (6.3) with effective k-dependent couplings $K(l^*)$, $B(l^*)$, and $\Delta_h^{\rm eff}(l^*)$ given by

$$\Delta_h^{\rm eff}(l^*)=-2\Delta''_V(0,l^*)/\Lambda^2, \tag{8.30}$$

$[\Delta_V''(0,l^*)<0$, as we show below]. Using this effective theory and Eqs. (3.11),(3.12) to compute the right-hand side of Eq. (8.29), and taking $k_\perp$ sufficiently small, so that l^* is sufficiently large to guarantee that the fixed point Eq. (8.23) is reached, we readily obtain

$$C(\mathbf{k}) \approx \frac{-2e^{(d+1)l^*}\Delta_V''(0,l^*)}{[K_p(l^*)e^{(d-3)l^*}\Lambda^4 + B_p(l^*)e^{(d-3)l^*}e^{4l^*}k_z^2]^2}, \tag{8.31a}$$

$$\approx \frac{-8\tilde{\Delta}_{V*}''(0)\sqrt{B_p(l^*)K_p(l^*)^3}\,k_\perp^{7-d}}{[K_p(l^*)k_\perp^4 + B_p(l^*)k_z^2]^2 C_{d-1}}. \tag{8.31b}$$

To obtain Eq. (8.31b) above, we used Eq. (8.14) to relate Δ_V to $\tilde{\Delta}_V$ and Eq. (8.25) to express our results in terms of the physical elastic moduli K_p and B_p. Comparing our final expression for $C(\mathbf{k})$ with the expression for this quantity in Gaussian theory Eq. (3.12) justifies our calling $K_p(l^*)$ and $B_p(l^*)$ the ''physical'' K and B of the nonlinear theory of the randomly pinned smectic. Using Eqs. (8.25) for these, in the limit $l^* \gg l_0$, with $l^* = \ln(\Lambda/k_\perp)$, and $k_z \ll \lambda k_\perp^2$ we find the *anomalous* moduli

$$K(\mathbf{k}) \propto |\ln(k_\perp/\Lambda)|^{\gamma_K(d)}, \tag{8.32a}$$

$$B(\mathbf{k}) \propto |\ln(k_\perp/\Lambda)|^{-\gamma_B(d)}. \tag{8.32b}$$

Similar analysis for an arbitrary direction in $k_\perp$-k_z space leads to Eqs. (1.8), advertised in the Introduction.

The Fourier transform of $C(\mathbf{k})$ measures the spatial phonon correlations

$$C(\mathbf{r}) \equiv \overline{\langle [u(\mathbf{r}_\perp,z) - u(\mathbf{0}_\perp,0)]^2 \rangle}, \approx \int \frac{d^d k}{(2\pi)^d} 2[1-\cos(\mathbf{k}\cdot\mathbf{r})]C(\mathbf{k}). \tag{8.33}$$

Using Eq. (8.31b), but ignoring the subdominant $[\ln(\ln r)]$ corrections arising from weak anomalous elasticity, we obtain for $z=0$

$$C(\mathbf{r}_\perp,0) \approx \frac{-8\tilde{\Delta}_{V*}''(0)}{C_{d-1}} \int_{k_\perp < r_\perp^{-1}} \frac{d^d k}{(2\pi)^d} \frac{\sqrt{B(\mathbf{k})K(\mathbf{k})^3}\,k_\perp^{7-d}}{[K(\mathbf{k})k_\perp^4 + B(\mathbf{k})k_z^2]^2}, \approx \frac{-4\tilde{\Delta}_{V*}''(0)}{C_{d-1}} \int_{k_\perp < r_\perp^{-1}} \frac{d^{d-1}k_\perp}{(2\pi)^{d-1}} \frac{1}{k_\perp^{d-1}}, = \frac{4\epsilon\pi^2}{9q_0^2}\ln(r_\perp/a). \tag{8.34}$$

We therefore find that throughout the range of dimensions $5<d<7$, the smectic displacements u grow logarithmically (up to $\ln\ln$ corrections due to anomalous elasticity), with a universal amplitude determined by Eq. (8.23). This superuniversal logarithmic roughness is analogous to that previously found for the random field XY model[28] and for the Abrikosov vortex lattice in a dirty superconductor.[9]

The superuniversal smectic Bragg glass phase found in this section is quite exotic. However, since it only exists for $5<d<7$, it unfortunately has little experimental relevance for physical three-dimensional disordered smectics. Furthermore, because the *periodic* disorder-driven layer fluctuations and the related anomalous elasticity diverge only *logarithmically*, while the *tilt* disorder, which becomes relevant for $d<5$, leads to power-law roughness, Eq. (4.9), the former is completely swamped by the latter in $d<5$.

B. Functional RG for $3<d<5$: irrelevance of random-field disorder

The above claim can readily be demonstrated by extending the functional renormalization group analysis presented above and in Appendix B to $d<5$. The resulting set of flow equations for $K(l)$, $B(l)$, and $\Delta_h(u,l)$, valid for $d<5$ [assuming the validity of the *elastic* model, Eq. (8.10), i.e., that dislocations are bound] is

$$\frac{dK(l)}{dl} = \left[d-3+\frac{1}{32}g(0,l) + \frac{(d^2-12d+23)\tilde{\Delta}_V''(0,l)}{2(d^2-1)\lambda^2(l)} \right] K, \tag{8.35}$$

$$\frac{dB(l)}{dl} = \left[d-3-\frac{3}{16}g(0,l) + \frac{3}{2}\frac{\tilde{\Delta}_V''(0,l)}{\lambda^2(l)} \right] B, \tag{8.36}$$

$$\partial_l \Delta_h(u,l) = \left(d-1+\frac{1}{64}g(u,l) \right)\Delta_h(u,l) + \left(\frac{\eta(l)}{q_0^2} + D(u) \right)\Delta_h''(u,l) + c[\tilde{\Delta}_{V*}'''(u)]^2 \sqrt{K(l)^3 B(l)}\,\Lambda^{5-d}/C_{d-1}, \tag{8.37}$$

where we have defined

$$g(u,l) \equiv \Delta_h(u,l)\left(\frac{B(l)}{K^5(l)} \right)^{1/2} C_{d-1}\Lambda^{d-5}, \tag{8.38}$$

the ''position-dependent diffusivity''

$$D(u) \equiv c_2[\tilde{\Delta}_{V*}''(u) - \tilde{\Delta}_{V*}''(0)], \tag{8.39}$$

and c and c_2 are dimensionless constants of order unity. Note that these recursion relations reduce to those of the tilt-only model considered in Sec. VI, if we take the tilt disorder $\Delta_h(u,l)$ to be a constant independent of u, and drop the random field disorder. In these relations, for convenience, we have chosen the anisotropy exponent $\omega=2$, so that, as discussed earlier, q_0 does not renormalize.

The recursion relation for the dimensionless random field disorder $\tilde{\Delta}_V$ is given by Eq. (8.13). Note that since we are now interested in $d<5$, $\tilde{\Delta}_V$ may grow quite large (since its linear eigenvalue $\epsilon = 7-d$ is no longer small), so we cannot neglect the $O(\tilde{\Delta}_V^3)$ nonlinearities in Eq. (8.13). Because we

do not know these, we cannot actually calculate the fixed point function $\tilde{\Delta}_{V*}(u)$ that $\tilde{\Delta}_V(u,l)$ flows to upon renormalization. However, as in $d>5$, we do not need to. Rather, we only need to argue that it exists, is finite, and that its second derivative at the origin, $\tilde{\Delta}''_{V*}(0)$ is finite and negative.

To make this argument, we first note that if we drop the second and third terms on the right-hand side of Eq. (8.13) and ignore higher order nonlinear terms, the resulting equation certainly has a fixed point solution; namely, our explicit solution Eq. (8.23). Now let us consider the effects of the other terms. The $\eta(l)$ term can only *reduce* $\tilde{\Delta}_V$, being a ''diffusive'' type of term. So, while it may drive $\tilde{\Delta}_V(u,l)$ to zero as $l\to\infty$, it certainly *cannot* have the opposite effect of driving $\tilde{\Delta}_V$ to infinity. Hence $\tilde{\Delta}_{V*}(u)$, and therefore, $\tilde{\Delta}''_{V*}(0)$, remain finite in the presence of this term.

If we assume that $\tilde{\Delta}_{V*}(u)$ *is* finite, we can show *a posteriori* that the third term renormalizes exponentially (in renormalization group ''time'' l) to zero, since, as we will show in a moment, $\lambda(l)\to\infty$ exponentially in l. This should be contrasted with the behavior of this term in $5<d<7$, where it vanished much more slowly (as $1/l$) as $l\to\infty$. The more rapid vanishing of this term in $d<5$ is a consequence of the stronger renormalization of B and K due to the *tilt* disorder field (Δ_h), which becomes relevant for $d<5$.

The crucial point here is that this much stronger renormalization of B and K due to this relevance of the tilt field (Δ_h) does *not* destabilize the $\tilde{\Delta}_V$ fixed point. Since this relevance of the tilt field is the only qualitative difference between $d>5$ and $d<5$, showing that it does not destabilize the $\tilde{\Delta}_V$ fixed point by itself shows that there is no more reason to doubt the stability of this fixed point for $d<5$ than for $5<d<7$. Indeed, if anything, the argument is *strengthened*, since the $1/\lambda(l)$ term vanishes even more rapidly for $d<5$.

Hence, the only possible problem can come from the unknown higher (than second) order terms in $\tilde{\Delta}_V$, in Eq. (8.13). Not knowing these terms, we cannot make a very compelling argument that they do not destabilize the fixed point. However, the same objection could have been raised for $d>5$ but well below 7. Indeed, an analogous objection can be raised for *any* fixed point that is found in an ϵ expansion once one goes well below the upper-critical dimension. We simply assume (as is done in all standard ϵ expansions) that the stability of the FRG fixed point, rigorously derived for $\epsilon\to 0$, persists well below the upper-critical dimension, i.e., for $d<5$.

So it seems plausible to assume that a finite fixed point function $\tilde{\Delta}_{V*}(u)$, with finite $\tilde{\Delta}''_{V*}(0)\leqslant 0$, exists for $d<5$. Let us now investigate the consequences of this assumption. We will see in a moment that they are virtually nonexistent: the asymptotic RG flows of $B(l)$, $K(l)$, and $\Delta_h(l)$, and long distance behavior of the static smectic correlation functions that they imply, are *exactly* those we found in Sec. VI, where we neglected the periodic potential altogether.

To see this, let us take $\tilde{\Delta}_V(u,l)$ to have flown to its fixed point, and combine the recursion relations for $B(l)$, $K(l)$, and $\Delta_h(u,l)$ into recursion relations for the dimensionless coupling function $g(u,l)$, defined in Eq. (8.38), which evolves under the RG transformation according to

$$\partial_l g(u,l)=\left(5-d-\frac{5}{32}g(u,l)+c_3\frac{\tilde{\Delta}''_{V*}(0)}{\lambda^2(l)}\right)g(u,l)+\left(\frac{\eta(l)}{q_0^2}+D(u)\right)g''(u,l)+c\frac{(\tilde{\Delta}'''_{V*}(u))^2}{\lambda^2(l)}, \tag{8.40}$$

where we have defined $c_3\equiv 3/4+(d^2-12d+23)/[2(d^2-1)]$. Similarly, from Eqs. (8.35),(8.36), we obtain the recursion equation for $1/\lambda^2(l)\equiv B(l)/K(l)$

$$\frac{d\lambda^{-2}(l)}{dl}=-\frac{7}{32}g(0,l)\lambda^{-2}(l)+a_1\tilde{\Delta}''_{V*}(0)\lambda^{-4}(l), \tag{8.41}$$

where a_1 is a constant whose numerical values is unimportant for our argument.

We will now prove that for $d<5$, at long scales (large l), $g(u,l)$ flows to $g_2^*=32(5-d)/5$, the constant, u-independent fixed point value of the dimensionless coupling g_2, Eq. (6.15), that we found in our earlier, tilt-only treatment of the disordered smectic, described in Sec. VI.

To see this, we write

$$g(u,l)=g_2^*+\delta g(u,l), \tag{8.42a}$$

$$=\frac{32}{5}(5-d)+\delta g(u,l), \tag{8.42b}$$

and expand the recursion relation Eq. (8.40) to linear order in $\delta g(u,l)$, obtaining

$$\partial_l\delta g(u,l)=[d-5+D(u)\partial_u^2]\delta g(u,l)+\frac{S(u)}{\lambda^2(l)}, \tag{8.43}$$

where we have defined the ''source term''

$$S(u)\equiv c_3 g_2^*\tilde{\Delta}''_{V*}(0)+c[\tilde{\Delta}'''_{V*}(u)]^2, \tag{8.44}$$

and have dropped the $\eta(l)$ term which decays away exponentially to 0 as $l\to\infty$.

To demonstrate the stability of the tilt-only u-independent fixed point, we now only need to show that Eq. (8.43) implies that $\delta g(u,l)$ decays to 0 as $l\to\infty$. To do this we expand $\delta g(u,l)$ in eigenfunctions of the operator $D(u)\partial_u^2$; that is, we write

$$\delta g(u,l)\equiv\sum_{n=0}^{\infty} g_n(l)\phi_n(u), \tag{8.45}$$

where $\phi_n(u)$ are periodic functions of u with period a satisfying the eigenvalue equation

$$D(u)\frac{d^2\phi_n(u)}{du^2}=\Gamma_n\phi_n(u). \tag{8.46}$$

It is straightforward to see that all of the eigenvalues Γ_n are $\leqslant 0$, provided that $D(u)\geqslant 0$ for all u, as it is in $7-\epsilon$ dimensions, where we can explicitly calculate it. We will

assume that $D(u)$ continues to be $\geqslant 0$ for $d<5$, and, in fact, all the way down to the physical dimension $d=3$.

Using the orthogonality of the $\phi_n(u)$ basis, with

$$\int_0^a \frac{\phi_n(u)\phi_{n'}(u)}{D(u)}du = I_n \delta_{nn'}, \tag{8.47}$$

and

$$I_n \equiv \int_0^a \frac{\phi_n^2(u)}{D(u)}du, \tag{8.48}$$

standard projection analysis leads to a set of decoupled recursion relations for the l-dependent expansion coefficients $g_n(l)$ in Eq. (8.45)

$$\frac{dg_n(l)}{dl} = (d-5+\Gamma_n)g_n(l) + \frac{S_n}{\lambda^2(l)}, \tag{8.49}$$

where the source term is given by

$$S_n \equiv \int_0^a \frac{S(u)\phi_n(u)}{I_n D(u)}du. \tag{8.50}$$

Now, since all of the Γ_n are $\leqslant 0$, and we are considering $d<5$, and since, as we show momentarily, $\lambda(l)\to\infty$ as $l\to\infty$, we see that all of the $g_n(l)$ flow to zero as $l\to\infty$, provided only that all of the S_n, Eq. (8.50) are finite. We leave the demonstration of the latter fact[58] as a straightforward homework exercise. Hence as asserted, $g(u,l)$ flows to a stable fixed point $g^*(u)=g_2^*=32(5-d)/5$, identical to that found in the tilt-only model of Sec. VI.

It is now easy to complete the proof, by demonstrating via Eq. (8.19) that near this fixed point $\lambda^{-2}(l)$ flows rapidly to zero:

$$\lambda^{-2}(l) = \lambda^{-2}(0)e^{-(7g_2^*/32)l}, \tag{8.51a}$$

$$= \lambda^{-2}(0)e^{-[7(5-d)/5]l}. \tag{8.51b}$$

Although this result holds only near $d=5$, it is straightforward to generalize the above analysis to arbitrary $d<5$, by relating the graphical corrections to B and K to the exact anomalous exponents η_B and η_K of the tilt-only model. We find that $g_2(l\to\infty)$ again flows to the same fixed point value it would have in the absence of the periodic potential, while $\lambda^{-2}(l)$ now flows to zero as

$$\lambda^{-2}(l) = \lambda^{-2}(0)e^{-(\eta_B+\eta_K)l}. \tag{8.52}$$

So in all dimensions $d<5$, $\lambda^{-2}(l)$ flows exponentially to zero, in contrast to $5<d<7$, where $\lambda^{-2}(l)$ vanishes slowly, as $1/l$. This completes the *a posteriori* argument in our earlier demonstration that $\tilde{\Delta}_V(u,l)$ flows to a finite fixed point and that the tilt disorder $g(u)$ flows to a stable, u-independent fixed point identical to that found for the tilt-only model.

All that remains to complete our argument that the random periodic potential has no effect on the static properties of disordered smectics in $d<5$ is to consider the correlation functions themselves. Using yet again the trajectory integral matching formalism, we can show that

$$C(\mathbf{q}) \equiv \frac{\overline{\langle u(\mathbf{q})u(\mathbf{q}')\rangle}}{\delta^d(\mathbf{q}+\mathbf{q}')}, \tag{8.53}$$

$$= \left(\frac{\Lambda}{q_\perp}\right)^{d+1}\left[\frac{T}{K(l^*)\Lambda^4 + B(l^*)(\Lambda/q_\perp)^4 q_z^2} + \frac{\Delta_h(l^*)\Lambda^2 - 2\Delta_V''(0,l^*)}{[K(l^*)\Lambda^4 + B(l^*)(\Lambda/q_\perp)^4 q_z^2]^2}\right], \tag{8.54}$$

with $l^* \equiv \ln(\Lambda/q_\perp)$. This only differs from our expression for this correlation function in the tilt-only model, Eq. (6.31), by the presence of the constant term $\Delta_V''(0,l^*)$ in the numerator of the second term. Using

$$\Delta_h(l) = g_2(l)\frac{K^{5/2}(l)}{B^{1/2}(l)C_{d-1}\Lambda^{d-5}}, \tag{8.55a}$$

$$\to g_2^* \frac{K^{5/2}(l)}{B^{1/2}(l)C_{d-1}\Lambda^{d-5}}, \tag{8.55b}$$

and

$$\Delta_V(u,l) \underset{l\to\infty}{\to} \frac{4\Lambda^{7-d}}{C_{d-1}} K^{3/2}(l)B^{1/2}(l)\tilde{\Delta}_{V*}(u), \tag{8.56}$$

it is easy to show that the ratio of this $\Delta_V''(0,l^*)$ term to $\Delta_h(l_*)\Lambda^2$ is

$$\frac{\tilde{\Delta}_{V*}''(0)}{g_2^*}\frac{B(l_*)}{K(l_*)} = \frac{\text{const}}{\lambda(l_*)^2}, \tag{8.57}$$

which vanishes as $q_\perp \to 0$ and $l_* \to \infty$, since $\lambda^{-2}(l_*)$ does.

To summarize: *every* possible *static* effect of the periodic potential Δ_V vanishes, as $l\to\infty$, like $\lambda^{-2}(l)$, which vanishes exponentially as $l\to\infty$. Hence, the periodic potential (the random field disorder) has no effect on any *static* properties of the disordered smectic in $d<5$. Our tilt-only model is sufficient to treat those.

All of the above arguments, of course, only apply if we ignore dislocations. In the next section we include these in our analysis.

IX. FAILURE OF PERIODIC POTENTIAL TO INDUCE DISLOCATION UNBINDING

In Secs. III, IV, and VIII we have demonstrated that in three-dimensional (or more generally for $d<5$) randomly pinned smectics, the tilt disorder (Δ_h) asymptotically leads to significantly larger layer disordering than the random field displacement disorder (Δ_V). We thereby argued that static correlation functions for a general disordered smectic can be accurately computed within a tilt-disorder-only model, studied in Secs. IV–VII. However, the validity of these arguments and the conclusions that follow from them critically rely on the stability of the *elastic* model within which these assessments are made. That is, it is in principle possible that although the periodic disorder is less important than the random tilt pinning in disordering smectic layers in the *elastic* model, it could nevertheless drive dislocation unbinding,

thereby invalidating the elastic model and the conclusions drawn from it.

Therefore, to make the arguments of the previous sections complete, it is essential to assess whether the random-field disorder (Δ_V) *alone* can drive proliferation of dislocation loops, and this is the subject of current section. To this end we adapt Fisher's[59] real space renormalization group analysis to disordered smectics and show that the random periodic disorder *alone* does *not* induce a proliferation of dislocation loops in three-dimensional smectics, thereby concluding our justification for the tilt-only model of randomly pinned smectics.

We first review Fisher's analysis as it applies to the three-dimensional random-field XY model. Recall that within a defect-free model the elastic energy is balanced against the pinning energy, both much larger than the thermal energy k_BT. The compromise in this competition leads to phase variations that grow logarithmically with distance and typical ground-state energies that vary as L^{θ}, where L is system size and θ is (minus) of the thermal eigenvalue exponent, which, due to statistical symmetry, is given exactly by

$$\theta_{xy}=d-2. \tag{9.1}$$

We wish to know whether it energetically pays for the system to insert a vortex loop. Confining our discussion to $d=3$, a vortex loop of length L will cost the system elastic energy $E_{\rm el}\sim JL\ln L$, where the multiplicative logarithm is associated with the long-range XY phase deformation around the vortex line and J is the spin-wave stiffness. For such a randomly placed vortex loop, which relieves strains by allowing spin rotations of order 2π, the system can typically gain disorder energy $E_{\rm disorder}\sim L^{\theta}\sim L$. Hence, this crude argument would suggest that, because of the additional multiplicative logarithmic factor in $E_{\rm el}$, it never pays to insert arbitrarily large vortex loops into the random-field XY model. However, as argued by Fisher, a judicious optimization of the vortex loop location and conformation can do better energetically, and therefore it is essential to include in the stability analysis the energy minimization over the vortex position and conformation.

To this end we start out with a circular vortex loop and, as illustrated in Fig. 12, try to lower its energy by moving various segments of length Λ_w of it first by a distance w, then by $2w$, etc. We then calculate the renormalization of the line tension $\epsilon(l)$ by pinning at scale $2^l w$. The idea is that each roughly spherical region of size w is statistically independent of all the others; each gains a pinning energy of the order $w^{\theta}=w^{d-2}=w$ (in $d=3$). So the total pinning energy gained from optimizing the conformation of the vortex loop on scale w is given by

$$\delta E_{\rm pin}\sim -w\sqrt{N_{\rm spheres}}, \tag{9.2a}$$

$$\sim -w\sqrt{\frac{\Lambda_w}{w}}, \tag{9.2b}$$

$$\sim -\sqrt{\Lambda_w w}. \tag{9.2c}$$

Balancing this energy against the stretching energy necessary to deform the vortex loop

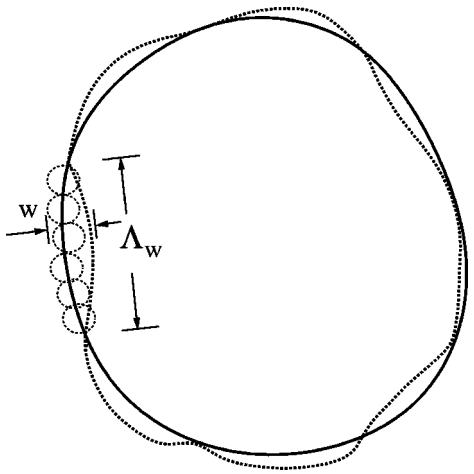

FIG. 12. Illustration of the optimization of a topological defect loop on length scale Λ_w. The full and dashed large loops represent the topological defect line before and after optimization on scale Λ_w, respectively, and the small circles are the isotropic correlated regions corresponding to displacements w.

$$\delta E_{\rm stretch}\sim \epsilon\frac{w^2}{\Lambda_w} \tag{9.3}$$

gives

$$\Lambda_w^{3/2}\sim \epsilon w^{3/2}, \tag{9.4}$$

which predicts the aspect ratio of the energetically most favorable deformation of the vortex loop

$$\Lambda_w\sim \epsilon^{2/3}w. \tag{9.5}$$

Using this in the expression for the pinning energy, we get the best pinning energy gain of

$$\delta E_{\rm pin}\sim -\epsilon^{1/3}w. \tag{9.6}$$

Associating this energetic reduction with the renormalization of the vortex line tension $\epsilon(l)$ gives

$$\delta\epsilon=\frac{\delta E_{\rm el}+\delta E_{\rm pin}}{\Lambda_w}=J\ln 2-\frac{\epsilon^{1/3}w}{\epsilon^{2/3}w}, \tag{9.7}$$

where the first term arises from the elastic energy increase due to the factor of 2 increase in length scale. This result implies the RG recursion relation

$$\frac{d\epsilon(l)}{dl}=J-\frac{1}{\epsilon(l)^{1/3}}. \tag{9.8}$$

Clearly, if the bare vortex line tension $\epsilon(0)\equiv\epsilon$ is sufficiently large, the line tension $\epsilon(l)$ at scale e^l grows as l ($\approx\ln L$), and therefore it is energetically unfavorable for vortex loops to unbind. This therefore argues for the stability of the topologically ordered ''Bragg'' glass phase in a weakly disordered three-dimensional random-field XY model.

We now extend this analysis to smectics pinned by a random periodic potential, with the main difference from the random-field XY model being the anisotropy of the smectic state. We begin by calculating the thermal eigenvalue exponents $-\theta$. As was the case for the random-field XY model,

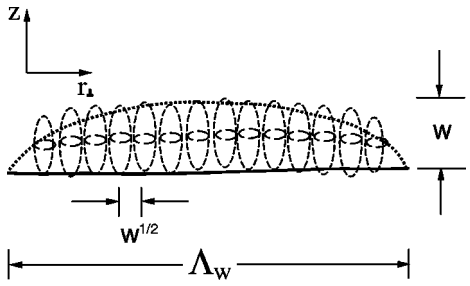

FIG. 13. Illustration of smectic anisotropic cigar-shaped correlation regions inside the dislocation optimization length Λ_w for a displacement w along z of a dislocation running perpendicular to z (case 1).

statistical symmetry guarantees that the smectic elastic moduli B and K are not renormalized by disorder.[60] This therefore implies that the θ exponent can be computed *exactly* by estimating the elastic energy due to layer displacement u of order a lattice constant $O(a)$ inside an anisotropic volume $L_\perp \times L_\perp \times (L_z \sim L_\perp^2)$.

$$E \sim B \int dz d^2 r_\perp (\partial_z u)^2, \tag{9.9a}$$

$$\propto L_\perp^2 L_\perp^2 \frac{1}{L_\perp^4}, \tag{9.9b}$$

$$\sim (L_\perp)^0. \tag{9.9c}$$

In this estimate Eq. (9.9) we have used the fact that, in the *harmonic* smectic elastic theory, distances in the z direction scale like the square of those in the $\perp$ directions. In three dimensions, we therefore find that the ground state of a randomly pinned, elastically *harmonic* smectic is described by

$$\theta_{\text{smectic}} = 0. \tag{9.10}$$

We now need to estimate the pinning energy gain from introducing a dislocation loop and optimizing its configuration, analogously to the analysis for a vortex loop above. The main difference here, however, is that, because of the anisotropy of the smectic phase, there are three dislocation line optimization cases to consider: (1) vertical (along z) w displacement of a horizontal (perpendicular to z) segment (i.e., an edge dislocation), (2) horizontal w displacement of a horizontal segment, (3) horizontal w displacement of a vertical segment (i.e., a screw dislocation).

A. Case 1: Vertical displacement of a horizontal dislocation ($w \| z$)

Because of the elastic anisotropy of the smectic state, in contrast to the spherical correlated volumes of the random-field XY model, here correlated regions are cigar-shaped (directed along z) characterized by dimensions $L_\perp(w) \times L_\perp(w) \times w$, with $L_\perp(w) \sim \sqrt{w}$, as shown in Fig. 13. The corresponding pinning energy change due to optimization of a dislocation segment of length Λ_w is given by

$$\delta E_{\text{pin}} \sim -w^\theta \sqrt{\frac{\Lambda_w}{L_\perp(w)}}, \tag{9.11a}$$

$$\sim -w^0 \sqrt{\frac{\Lambda_w}{\sqrt{w}}}, \tag{9.11b}$$

$$\sim -\frac{\Lambda_w^{1/2}}{w^{1/4}}. \tag{9.11c}$$

Balancing this against the dislocation stretching energy, which, as always, is

$$\delta E_{\text{stretch}} \sim \epsilon \frac{w^2}{\Lambda_w}, \tag{9.12}$$

we get

$$\Lambda_w = \epsilon^{2/3} w^{3/2}. \tag{9.13}$$

We note that $\Lambda_w \gg L_\perp(w) \sim \sqrt{w}$, consistent with the assumption of many ''bubbles'' per segment Λ_w, on which this calculation of δE_{pin} was based. This leads to

$$\delta E_{\text{pin}} \sim \frac{\Lambda_w^{1/2}}{w^{1/4}}, \tag{9.14a}$$

$$\sim -\epsilon^{1/3} w^{1/2}, \tag{9.14b}$$

and predicts a change of the line tension at scale w given by

$$\delta\epsilon = \frac{\delta E_{\text{pin}}}{\Lambda_w}, \tag{9.15a}$$

$$\sim -\frac{1}{\epsilon^{1/3} w}. \tag{9.15b}$$

Note that in the above we have ignored the elastic energy, because we do not need it to prove the stability of smectic ''Bragg'' glass. Even without the *stabilizing* contribution of the elastic energy, Eq. (9.15) implies

$$\frac{d\epsilon(l)}{dl} = -\frac{1}{\epsilon(l)^{1/3}} e^{-l}. \tag{9.16}$$

Clearly, if the bare $\epsilon(l=0) = \epsilon$ is big enough, $\epsilon(l \to \infty)$ will be nonzero. Hence *periodic disorder*, by itself, cannot unbind horizontal loops by displacing them vertically.

B. Case 2: Horizontal displacement of a horizontal dislocation ($w \perp z$)

As illustrated in Fig. 14, here, the volume of the correlated regions is $w \times w \times w^2$, with the long axis along z. The width of the correlated volume *along* Λ_w is now w, so we have

$$\delta E_{\text{pin}} \sim -\sqrt{\frac{\Lambda_w}{w}}. \tag{9.17}$$

Equating this energy to the dislocation stretching energy $\epsilon w^2 / \Lambda_w$, we find

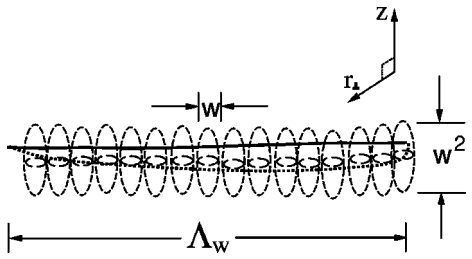

FIG. 14. Illustration of smectic anisotropic cigar-shaped correlation regions inside the dislocation optimization length Λ_w for both the displacement w and the dislocation running perpendicular to z, i.e., along $r_\perp$ (e.g., along x and y) (case 2).

$$\Lambda_w \sim \epsilon^{2/3} w^{5/3}, \tag{9.18}$$

which, upon inserting into $\delta E_{\rm pin}$, Eq. (9.17), gives

$$\delta E_{\rm pin} \sim \epsilon^{1/3} w^{1/3} \tag{9.19}$$

and implies

$$\delta\epsilon = \frac{\delta E_{\rm pin}}{\Lambda_w}, \tag{9.20a}$$

$$\sim \frac{1}{\epsilon^{1/3} w^{4/3}}. \tag{9.20b}$$

This leads to

$$\frac{d\epsilon(l)}{dl} = -\frac{1}{\epsilon(l)^{1/3}} e^{-4l/3}, \tag{9.21}$$

which also demonstrates that there is *no* divergent downward renormalization of ϵ, implying that in the presence of only periodic disorder, horizontal dislocation loops will *not* unbind by displacing horizontally.

C. Case 3: Horizontal displacement of a vertical dislocation ($w \perp z$)

Finally we consider optimizing a z-directed screw dislocation, by displacing it horizontally. This corresponds to a correlated region of volume $w \times w \times w^2$, but, in contrast to case 2, with the axis along Λ_w being the long dimension w^2, as illustrated in Fig. 15. As a result, we have

$$\delta E_{\rm pin} \sim -\sqrt{\frac{\Lambda_w}{w^2}}, \tag{9.22}$$

which when balanced against the dislocation stretching energy gives

$$\Lambda_w \sim \epsilon^{2/3} w^2. \tag{9.23}$$

We note that for $\epsilon \gg 1$ this Λ_w is much greater than w^2, consistent with the assumption that goes into Eq. (9.22). This gives

$$\delta\epsilon \sim \frac{\delta E_p}{\Lambda_w}, \tag{9.24a}$$

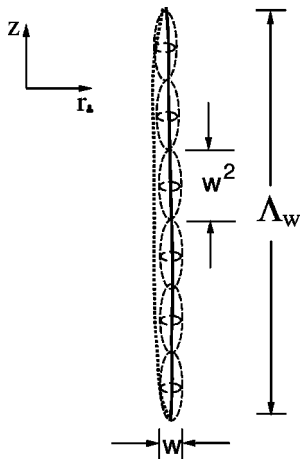

FIG. 15. Illustration of smectic anisotropic cigar-shaped correlation regions inside the dislocation optimization length Λ_w for displacement w along $r_\perp$ and dislocation running along z, i.e. along $r_\perp$ (case 3).

$$= -\frac{1}{\epsilon^{1/3} w^2}, \tag{9.24b}$$

which implies

$$\frac{d\epsilon(l)}{dl} = -\frac{e^{-2l}}{\epsilon(l)^{1/3}} \tag{9.25}$$

whose solution for $\epsilon(l)$, again, is stable as $l \to \infty$.

Since in all three cases we find that the effective dislocation line tension is finite at long scales, we conclude that periodic random-field disorder *alone* cannot unbind dislocation loops in such a randomly pinned smectic.

X. EXPERIMENTAL PREDICTIONS AND FUTURE RESEARCH DIRECTIONS

Our results imply many unequivocal dramatic predictions for experiments. The first and most unequivocal of these is that long-ranged, and even quasi-long-ranged, smectic translational order is destroyed by the presence of even *arbitrarily* weak *quenched* disorder. Furthermore, the decay of smectic translational order induced by such disorder is far more rapid than in previously studied ''Bragg glass'' systems,[28,9–11] being exponential, rather than algebraic. The experimental signature of this is equally unequivocal: broad x-ray scattering peaks with finite peak height, as opposed to the infinitely sharp, divergent peaks associated with quasi-long ranged order in thermal smectics. This qualitative prediction is born out by all experiments on smectics in aerogel to date: all show only broad, finite height x-ray scattering peaks, which can be fit by assuming a finite x-ray correlation length.[12]

In strong contrast to *quenched* (frozen) disorder of this type, it is straightforward to show analytically that weak short-ranged ''*annealed*'' disorder, in which the random environment can rearrange to accomodate the smectic's preference for an ordered state, has no *qualitative* effect on the long scale properties of liquid crystal phases. Such weak annealed disorder only leads to *finite* renormalizations of the effective parameters in our model.[26] Consequently, the smec-

tic phase, its quasi-long-ranged smectic translational order, and the transition into it are *stable to weak annealed* disorder, which might arise, for example, due to a low concentration of microscopic inclusions.

Of course, strictly speaking, all physical random confining structure are elastic, even aerogel, and therefore are able to deform to some extent in response to the smectic, which resists distortions. It is easy to see that this additional ''elastic compliance'' ingredient can be described as an annealed component of the disorder, in addition to the quenched part that is the main subject of our work. In light of the qualitative unimportance of weak annealed disorder, discussed above, our theory for purely quenched random structures immediately extends to deformable structures, which are quenched for large deformations, i.e., described by a finite shear modulus.

A notable example of liquid crystals confined in random structures in which these elastic annealed features appear are the recent experiments on smectics in *aerosil*.[61,62] Since *typical* weak hydrogen bonds that link the aerosil network together have bond-breaking energies of order k_BT, a large fraction of the aerosil network can rearrange itself in response to stresses imposed on it by the smectic and thereby come to thermal equilibrium with it. However, measurements[62] show that even these tenuous, weakly bonded structures are characterized by a small but a *finite* shear modulus. Hence even aerosils are unable to perform arbitrary rearrangements to equilibrate with the ordered smectic. Therefore, while they have a large annealed component, aerosils contain a small quenched component, which dominates the qualitative physics of the confined smectic. We therefore expect a destruction of the quasi-long-ranged translational order for smectics confined in aerosils, with the x-ray correlation length obeying our predictions for low density aerogels.

However, there are three important consequences of the large annealed disorder component present in aerosils. First, while *weak annealed* disorder should only lead to a finite renormalization of effective parameters, *strong* annealed disorder can destroy the quasi-long-ranged smectic order by driving some of the effective elastic moduli negative. Secondly, given the empirical fact that the *bulk* NA transition appears to display nonuniversal behavior,[30] we would expect this nonuniversality to also manifest itself for smectics ''confined'' in low-density aerosils, where the effective model parameters are functions of aerosil density due to the presence of the annealed disorder component. Thirdly, because of this large annealed disorder component in aerosils and a correspondingly small quenched component, we expect aerosils to be described by significantly smaller values of our effective quenched disorder parameters Δ_V and Δ_h than for aerogels of the same solid volume fraction. Consequently, experiments in aerosils are more appropriate for exploring our weak quenched disorder theory and to search for the delicate and ''fragile'' SBG and NEG phases.

These nontrivial qualitative predictions are clearly in agreement with the experimental observations of smectics confined in aerosils:[62] (i) for low aerosil densities these experiments observe significantly longer (than in equivalent volume fractions aerogels), finite smectic x-ray correlation lengths, (ii) they measure heat capacities with aerosil density dependent (and therefore nonuniversal) exponents, and (iii) in aerosils beyond a critical density they find a behavior that is qualitatively similar to smectics confined in aerogels.

It is important to note that in aerogels and aerosils, in contrast to what one might have naively expected, the smectic correlation length is *not* limited by some sort of ''finite size'' effect associated with the ''pore size'' of the aerogel. That is, the smectic confined in aerogel does *not* act as if it has simply been broken up into many small volumes which do not interact with each other.

If this ''finite size'' scenario *were* the case, one would expect to see the smectic translational correlation length ξ^X grow with decreasing temperature as T approached the pure NA transition temperature $T_{\rm NA}^{\rm pure}$ from above exactly as it did in the pure (bulk) smectic. Indeed, one would expect it to track its temperature-dependent value $\xi_{\rm pure}^X(T)$ in the pure system until T got close enough to $T_{\rm NA}^{\rm pure}$ that $\xi_{\rm pure}^X(T)=\xi_{\rm aerogel}^X(T)=L_P$, the ''pore size'' of the aerogel. At this point $\xi^X(T)$ would abruptly saturate, and remain constant at L_P for all lower temperatures.

This is emphatically *not* what is seen in experiments on aerogels.[12,19,20] Rather the growth of ξ^X continues right through $T_{\rm NA}^{\rm pure}$, and only saturates at some temperature well below $T_{\rm NA}^{\rm pure}$. This is completely consistent with our picture, in which ξ^X is determined by a competition between the smectic elasticity, which tries to keep the system ordered, and the random forces exerted by the aerogel strands, which disorder it. In this picture, the saturation of ξ^X with decreasing temperature occurs not *above* $T_{\rm NA}^{\rm pure}$, but, rather, well *below* $T_{\rm NA}^{\rm pure}$; specifically, at the temperature at which the smectic elastic compression modulus $B(T)$ saturates.

Liquid crystals have also been confined in other random but significantly more ordered structures. Some examples are Anopore[63] and Vycor,[64] which in contrast to the tenuous aerogel strand structure present on all scales, consist of a random distribution of empty pores, with fairly regular wall structure and much narrower pore size distribution. Even these systems can in principle be understood within the general model presented here. However, they fall into the strong disorder regime of our model, and we are therefore unable to utilize our weak disorder approach to make quantitative predictions, aside from the observation that the quasi-long-ranged smectic order should be destroyed in these systems as well, consistent with experiments.

The qualitative agreement between our theory and experiment can be made quantitative through our expression Eq. (1.13), which relates ξ^X to the smectic elastic constant $B(T)$. As discussed in the Introduction, this expression implies that sufficiently far below $T_{\rm NA}^{\rm pure}$ (where our weak disorder theory applies), a simple power-law relation between $\xi^X(T)$ and the ''bare'' layer compressional modulus $B_{\rm bare}(T)$ of the pure system holds:

$$\xi^X(T)\propto[B_{\rm bare}(T)]^{\zeta/2\chi}, \tag{10.1a}$$

$$\propto[B_{\rm bare}(T)]^{1/\chi-1/2}. \tag{10.1b}$$

The bare $B_{\rm bare}(T)$ can be determined in a number of ways. One way is simply to measure the layer compressional

modulus $B_{\rm pure}(T)$ of a pure (bulk, aerogel-free) sample of the same smectic liquid crystal material, and pray that this is not changed when the smectic is placed in aerogel. However, since prayers are not always answered, a direct, simultaneous *in situ* determination of $B_{\rm bare}(T)$ in the smectic inside the aerogel is clearly desirable.

This could be done by determining the squared magnitude of the smectic order parameter $|\psi|^2$ from, e.g., the integrated intensity under the (broadened) smectic x-ray peak. In mean-field theory, which should hold for bare quantities sufficiently far below $T_{\rm NA}$, $B_{\rm bare}(T) \propto |\psi|^2$. Thus, we should, in this mean-field regime, observe a universal power-law relation between the x-ray correlation length ξ^X and $|\psi|^2$: $\xi^X \propto |\psi|^{2/\chi-1}$. Note that in the absence of anomalous elasticity, the second regime of Eq. (1.13) [$\xi^X = \xi_z^{\rm NL}(a/\lambda)^2$] would apply, and we would get $\xi^X \propto B \propto |\psi|^2$, the last proportionality holding only in the mean-field regime. Thus, *any* departure of the ξ^X-B relation from linearity is evidence for anomalous elasticity. Furthermore, observing such a power-law relation determines χ.

Preliminary analysis of ξ^X versus B data,[65] as determined above, supports the relation Eq. (10.1), with $\chi = 0.8 \pm 0.1$. Note that $\chi < 1$, favoring *stability* of the smectic Bragg glass phase, although we have, as yet, no experimental way of determining if the other necessary relation for the stability of the SBG, namely, $\eta_K < 1$, is satisfied.

While our prediction for $\xi^X(T)$ is not as quantitative as we would like, since we do not have quantitatively reliable predictions for ζ and χ in $d=3$ (our $5-d=\epsilon$-expansion results not being quantitatively trustworthy when $\epsilon = 2$), it nonetheless makes it possible, by experimentally determining ζ/χ for *one* smectic in *one* aerogel, to predict the temperature dependence of ξ^X for *any* smectic in *any* low density aerogel. Furthermore, once ζ/χ is known, ζ and χ separately are also known, since $\zeta = 2-\chi$. Knowledge of these exponents then makes other predictions possible.

In particular, it leads to our second prediction which concerns whether or not the transition (which replaces the NA transition in bulk smectics) from the nematic to the low-temperature phase (which replaces the ordered smectic) is destroyed by disorder. Surprisingly, this question can, in principle, have a different answer than the question of whether or not smectic translational order is destroyed. This is because our analysis shows that, even though smectic translational order *is always* destroyed, a distinct ''smectic Bragg glass'' phase (SBG), in which dislocations remain bound and long-ranged *orientational* (nematic) order persists, maybe possible in the presence of sufficiently weak disorder. If it is, then a thermodynamically sharp transition into it (with decreasing temperature) replaces the NA transition of the pure system, as this phase itself replaces the smectic-A phase. Consequently, an exotic and quite unique feature of this transition is that, while thermodynamic quantities (e.g., heat capacity) display singularities at the transition to the putative smectic Bragg glass, the x-ray scattering peak remains broad through the transition. Unfortunately, whether or not this (SBG) phase exists, in principle, depends on whether or not the imprecisely known anomalous exponents η_B and η_K satisfy the bounds Eq. (1.1).

Fortunately, the above-described experimental determination of χ could answer this question, since one of the bounds Eq. (1.1a) is simply $\chi < 1$. Thus, if the x-ray measurements, described above, find $\chi < 1$, a SBG phase can exist, and a thermodynamically sharp remnant of the pure system's NA transition could be seen in sufficiently low density aerogels. On the other hand, if χ proves to be >1, no sharp transition should be seen. This is a definite, falsifiable prediction that could be tested experimentally: if $\chi > 1$, no sharp transition should be seen. The converse conclusion, that if $\chi < 1$, a sharp transition should be seen, need not hold, since it is possible that $\chi < 1$, but the bound $\eta_K < 1$ is violated. For the same reason, even if the current experimental evidence that there is *no* transition even for arbitrarily low density aerogel holds up, we can *not* conclude that $\chi > 1$.

It is also important to note, that despite the absence of the long-ranged ordered smectic phase, it *is* possible, as always, to have a thermodynamically sharp *first-order* transition *within* the disordered (nematic) phase. At such a transition, which would be from a nematic with a short smectic correlation length to a nematic with longer (but still finite) smectic correlation length, thermodynamic quantities (such as the heat capacity) could displaying sharp nonanalytic features (such as latent heat). We believe that the recent calorimetry experiments on 7O.4/aerogel samples by Haga and Garland are examples of such a first order transition.[66] As a consequence of our main result that only short range smectic order is possible for smectics confined in quenched random environments, we unambiguously predict a broad, finite x-ray scattering peak for the system studied by Haga and Garland.

If the SBG phase *does* prove to be possible, then we can make far more detailed tests of our predictions. In particular, the simple fact that the SBG phase has long-ranged orientational order is, in itself, a striking prediction that could be tested very easily by looking at optical birefringence in smectics in aerogel. The existence of such long-ranged order would make possible even more quantitative tests for our predictions of anomalous elasticity, by making it possible to measure director fluctuations $\overline{\langle \delta n_i(\mathbf{q}) \delta n_j(-\mathbf{q})\rangle}$ by light scattering. Since these are related to smectic layer fluctuations by the usual smectic relation $\delta \mathbf{n}(\mathbf{r}) = \boldsymbol{\nabla} u(\mathbf{r})$, we can immediately write

$$\overline{\langle \delta n_i(\mathbf{q}) \delta n_j(-\mathbf{q})\rangle} = q_i q_j \overline{\langle |u(\mathbf{q})|^2\rangle}, \tag{10.2a}$$

$$= \frac{\Delta_h(\mathbf{q}) q_i q_j q_\perp^2 V}{[B(\mathbf{q}) q_z^2 + K(\mathbf{q}) q_\perp^4]^2} + \frac{k_B T q_i q_j V}{B(\mathbf{q}) q_z^2 + K(\mathbf{q}) q_\perp^4}, \tag{10.2b}$$

where i and j run over all indices but z, and V is the volume of the system. The first, quenched, term in this expression, Eq. (10.2b), dominates as $\mathbf{q} \to 0$; thus fitting this piece to the observed light scattering would enable a direct experimental measurement of $K(\mathbf{q})$, $B(\mathbf{q})$, and $\Delta_h(\mathbf{q})$, and, hence, a direct test of the scaling predictions Eq. (6.26) for these quantities,

as well as a determination of the anomalous exponents η_K, η_B, and η_Δ. For instance, if we take $\mathbf{q}$ in the $x-z$ plane, Eq. (10.2) implies

$$\overline{\langle|\delta n_x(\mathbf{q})|^2\rangle}=q_x^{-4-\eta_\Delta+2\eta_K}f_n\left(\frac{q_z\xi_z^{\rm NL}}{(q_\perp\xi_\perp^{\rm NL})^\zeta}\right). \quad (10.3)$$

Thus, a log-log plot of $\overline{\langle|\delta n_x(\mathbf{q})|^2\rangle}$ for $q_z=0$ has slope $-4-\eta_\Delta+2\eta_K$. The simple observation that this slope is not -4 would confirm the existence of anomalous elasticity, by showing that η_Δ and η_K were $\neq 0$. The determination of the anisotropy exponent ζ by collapsing the plots of $\overline{\langle|\delta n_x(\mathbf{q})|^2\rangle}$ versus q_x for a variety of q_z's would then fix $\eta_B+\eta_K$. Our exact scaling relation between η_Δ, η_B and η_K, Eq. (6.30) would then provide a third equation to fix all three exponents. That the results satisfied our bounds Eq. (1.1) for the stability of the SBG phase would then provide a nontrivial check on the consistency of our theory.

A more stringent test could be provided by measuring the second, k_BT term in Eq. (10.2). Although, as mentioned earlier, this term is subdominant to the first, disorder term, it could nonetheless be resolved by *time-dependent* light scattering. The reason for this is that the first, disorder term of Eq. (10.2) represents the *static*, time-persistent response of the smectic to the random tilt field of the aerogel, while the k_BT term represents thermal fluctuations about this disorder-determined smectic layer configuration. That is, if we consider the unequal time correlation function $\overline{\langle\delta n_i(\mathbf{q},t)\,\delta n_j(-\mathbf{q},0)\rangle}$, we expect

$$\overline{\langle\delta n_i(\mathbf{q},t)\,\delta n_j(-\mathbf{q},0)\rangle}=\frac{\Delta_h(\mathbf{q})q_iq_jq_\perp^2\,V}{[B(\mathbf{q})q_z^2+K(\mathbf{q})q_\perp^4]^2}+\frac{k_BTq_iq_j\,f(\mathbf{q},t)V}{B(\mathbf{q})q_z^2+K(\mathbf{q})q_\perp^4}, \quad (10.4)$$

where $f(\mathbf{q},t)$ is a function we do not know, but which presumably will have some sort of slow, ''glassy'' decay, vanishing as $t\to\infty$, and going to 1 as $t\to 0$, thereby recovering the equal-time correlation function, Eq. (10.2). Using these two limits immediately implies

$$\lim_{t\to\infty}\overline{\langle\delta n_i(\mathbf{q},t)\,\delta n_j(-\mathbf{q},0)\rangle}-\overline{\langle\delta n_i(\mathbf{q},0)\,\delta n_j(-\mathbf{q},0)\rangle}=\frac{k_BTq_iq_jV}{B(\mathbf{q})q_z^2+K(\mathbf{q})q_\perp^4}, \quad (10.5)$$

which allows a completely independent determination of η_K and η_B from the approach based strictly on the equal-time correlation function described earlier. Consistency of the two approaches would demonstrate the validity of the exact scaling relation between η_K, η_B, and η_Δ.

We remind the avid experimentalist that the entire above discussion *only* applies if the SBG phase is stable. If the SBG phase does *not* exist, then our experimental predictions are reduced to Eq. (1.13) for the x-ray correlation length, the anomalous elasticity lengths given by Eqs. (1.3), and the ''ghost'' of nonanalyticity in T in these lengths, associated with the remnant of T_p below $T_{\rm NA}$ (see Fig. 6), defined by the temperature at which the pure smectic order parameter correlation exponent η passes 4.

Presumably, there are experiments that could test our predictions for the orientational correlation length ξ_O, but we will be unable to make such predictions until we have developed a theory of the nematic elastic glass behavior that holds on longer length scales. This is a topic for future research.[26]

There are a number of other promising areas for future research. One is the subject of smectic order inside *stretched* aerogels, which we are currently studying.[67] In that system we have shown that uniaxially stretching or compressing the aerogel matrix can stabilize the smectic Bragg glass phase. The universality class of the resultant SBG now depends upon whether the aerogel is stretched or compressed: in the former case, it lies in the universality class of the so-called ''vortex glass'' phase, which has recently been much discussed in the context of the random field XY model[10,11] and randomly pinned Abrikosov vortex lattices,[9] while compression leads to a totally novel kind of ''$m=1$ Lifshitz SBG.'' A detailed discussion of this problem is in preparation.[67]

We expect a new glass phase similar in many respects to the ''$m=1$'' glass described above, but in a different universality class, to occur when ''discotics'' (i.e. liquid crystals with *two solidlike* directions and *one liquidlike*) are absorbed into *unstressed* aerogel. Work on this interesting problem is also currently underway.[67]

It is also interesting to consider what happens to a *cholesteric* liquid crystal confined in aerogel.[68] Although the symmetry of such a system is *identical* to that of a smectic studied here, the addition of another long length scale (namely, the cholesteric pitch) may make it possible to access a novel type of ''random manifold'' regime of pinned elastic media.

Probably the most interesting and challenging problem is the theory of dynamics of these smectic Bragg glass systems, which would enable a detailed understanding of recent dynamic light scattering experiments.[19] If the smectic Bragg glass phase exists, which we are quite certain it does in smectics in anisotropic aerogels and discotics in isotropic aerogels,[67] it should have slow glassy dynamics similar to those observed by Bellini *et al.*[19] Given its unusual *static* properties (e.g., anomalous elasticity), its dynamics may be quite unusual, even for glasses.

As noted in the Introduction, here we have studied smectics in aerogel by perturbing around the low temperature translationally and orientational ordered smectic phase. However, another approach to this problem, more convenient to understanding the effects of disorder near and above the bulk NA transition, is based on the de Gennes model.[32] This complementary high-temperature approach makes it possible to predict, e.g., the rounded specific heat and x-ray scattering peaks observed in many experiments on smectics in aerogel.[12,19–22]

Finally, we note that the formalism we developed in Sec. V for disordered smectics can be applied to a variety of other disordered problems, including gauge glasses, randomly pinned Abrikosov vortex lattices and other pinned periodic media. These applications will be discussed in a future publication.[25]

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APPENDIX A: DETAILS OF THE $d=3$ SINGLE HARMONIC RENORMALIZATION GROUP ANALYSIS

Here we present the details of the derivation of the renormalization group flow equations Eqs. (3.18a),(3.18b) for the three-dimensional disordered smectic model, in the harmonic elastic approximation defined by the replicated Hamiltonian Eq. (3.8)

$$H[u_\alpha]=H_0[u_\alpha]+H_\Delta[u_\alpha], \tag{A1}$$

where the quadratic $H_0[u_\alpha]$ and the interaction $H_\Delta[u_\alpha]$ parts of the Hamiltonian are respectively given by

$$H_0=\frac{1}{2}\int d^dq\sum_{\alpha,\beta}^{n}[(Kq_\perp^4+Bq_z^2)\delta_{\alpha\beta}-\Delta_h q_\perp^2]u_\alpha u_\beta, \tag{A2a}$$

$$H_\Delta=-\int d^dr\sum_{\alpha\neq\beta}^{n}{}'\ \Delta_V\cos[q_0(u_\alpha(\mathbf{r})-u_\beta(\mathbf{r}))]. \tag{A2b}$$

In the above and throughout, in the interaction term H_Δ, we have for convenience excluded the $\alpha=\beta$ term inside the summation. This exclusion is indicated by a prime, and is justified since the $\alpha=\beta$ term is a constant. Also for simplicity, throughout this appendix we have set $T=1$; the factors of T can be easily restored by the replacement $K\to K/T$, $B\to B/T$ $\Delta_h\to\Delta_h/T^2$, and $\Delta_V\to\Delta_V/T^2$. From H_0, the propagator $G_{\alpha\beta}(\mathbf{q})$ can be easily obtained,

$$G_{\alpha\beta}(\mathbf{q})=G_T(\mathbf{q})\delta_{\alpha\beta}+G_{\Delta_h}(\mathbf{q}), \tag{A3a}$$

$$=\frac{\delta_{\alpha\beta}}{Kq_\perp^4+Bq_z^2}+\frac{\Delta_h q_\perp^2}{(Kq_\perp^4+Bq_z^2)^2}. \tag{A3b}$$

In three dimensions, a direct perturbative calculation in Δ_V is divergent in the thermodynamic limit even for an arbitrarily small Δ_V. Nevertheless, physical observables can be computed utilizing standard methods of the renormalization group,[40] in which one avoids infrared divergences by integrating out degrees of freedom a momentum ''shell'' at a time. In this procedure, the goal is to establish how the effective Hamiltonian functional changes after the renormalization group transformation, which can be neatly summarized by the renormalization group flow equations for the effective parameters.

We focus on $\overline{Z^n}$ as the physical quantity to keep fixed under the infinitesimal renormalization group transformation. Following the standard RG procedure we write the Fourier-transformed displacement field as $u_\alpha(\mathbf{q})=u_\alpha^<(\mathbf{q})+u_\alpha^>(\mathbf{q})$, where $u_\alpha^<(\mathbf{q})$ is the long wavelength set of modes, nonzero for $0<|\mathbf{q}_\perp|<\Lambda e^{-l}$, with q_z unrestricted, and $u_\alpha^>(\mathbf{q})$ are the short wave-vector degrees of freedom that have support only within a thin cylindrical momentum shell

$$\Lambda e^{-l}<|\mathbf{q}_\perp|<\Lambda, \tag{A4a}$$

$$-\infty<q_z<\infty, \tag{A4b}$$

We first integrate, perturbatively in Δ_V, the high wave-vector part $u_\alpha^>$ out of $\overline{Z^n}$:

$$\overline{Z^n}=\int[du_\alpha^<][du_\alpha^>]e^{-H_0[u_\alpha^<+u_\alpha^>]-H_\Delta[u_\alpha^<+u_\alpha^>]}, \tag{A5a}$$

$$=\int[du_\alpha^<]e^{-H_0[u_\alpha^<]-\delta H[u_\alpha^<]}, \tag{A5b}$$

where $\delta H[u_\alpha^<]$ is obtained by integrating out the short wavevector degrees of freedom

$$e^{-\delta H[u_\alpha^<]}\equiv\int[du_\alpha^>]e^{-H_0[u_\alpha^>]-H_\Delta[u_\alpha^<+u_\alpha^>]}, \tag{A6a}$$

$$=Z_0^>\langle e^{-H_\Delta[u_\alpha^<+u_\alpha^>]}\rangle_>, \tag{A6b}$$

$$=Z_0^>\left[1-\langle H_\Delta[u_\alpha^<+u_\alpha^>]\rangle_>+\frac{1}{2}\langle H_\Delta[u_\alpha^<+u_\alpha^>]^2\rangle_>+\cdots\right], \tag{A6c}$$

The perturbative correction to the nonconstant part of the effective Hamiltonian can therefore be expressed in terms of a cumulant expansion

$$\delta H[u_\alpha^<]=\langle H_\Delta[u_\alpha^<+u_\alpha^>]\rangle_>^c-\frac{1}{2}\langle H_\Delta[u_\alpha^<+u_\alpha^>]^2\rangle_>^c+\cdots, \tag{A7}$$

where

$$\langle H_\Delta[u_\alpha^<+u_\alpha^>]^2\rangle_>^c=\langle H_\Delta[u_\alpha^<+u_\alpha^>]^2\rangle_>-\langle H_\Delta[u_\alpha^<+u_\alpha^>]\rangle_>^2. \tag{A8}$$

To first order in the interaction Δ_V we obtain

$$\langle H_\Delta\rangle_>=-\Delta_V\int d^3r\sum_{\alpha\neq\beta}{}'\ \langle\cos[q_0(u_\alpha-u_\beta)]\rangle_>,$$

$$=-\Delta_V\mathrm{Re}\int d^3r\sum_{\alpha\neq\beta}{}'\ e^{iq_0(u_\alpha^<-u_\beta^<)}\langle e^{iq_0(u_\alpha^>-u_\beta^>)}\rangle_>,$$

$$=-\Delta_V\int d^3r\sum_{\alpha\neq\beta}{}'\ \cos[q_0(u_\alpha^<-u_\beta^<)]e^{-q_0^2f_{\alpha\beta}},$$

$$= -\Delta_V \int d^3r \sum_{\alpha\neq\beta}{}' \cos[q_0(u^<_\alpha - u^<_\beta)] e^{-\eta l(1-\delta_{\alpha\beta})}, \tag{A9}$$

with η as given in Eq. (3.19a), where

$$f_{\alpha\beta} = G^>_{\alpha\alpha}(\mathbf{r}=0) - G^>_{\alpha\beta}(\mathbf{r}=0), \tag{A10}$$

in $G^>_{\alpha\alpha}(0)$ there is *no* implied sum over α and

$$G^>_{\alpha\beta}(\mathbf{r}) = \int_{\Lambda e^{-l}}^{\Lambda} \frac{d^2q_\perp}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} G_{\alpha\beta}(q) e^{i\mathbf{q}\cdot\mathbf{r}}. \tag{A11}$$

Using

$$G_{\alpha\alpha}(\mathbf{q}) - G_{\alpha\beta}(\mathbf{q}) = \frac{1}{Kq_\perp^4 + Bq_z^2}(1-\delta_{\alpha\beta}), \tag{A12}$$

which is obviously independent of the tilt-disorder variance Δ_h, and Fourier transforming the last equality in Eq. (A12), we obtain

$$\eta = \frac{q_0^2}{4\pi\sqrt{KB}}. \tag{A13}$$

Hence, for an infinitesimal δl the ''graphical'' correction to Δ_V to first order in Δ_V is

$$\delta\Delta_V^{(1)} = -\eta\Delta_V \delta l. \tag{A14}$$

We now proceed to compute the corrections to $\delta H[u^<_\alpha]$ to second order in Δ_V, which we anticipate will result in higher order corrections to the random field disorder $\delta\Delta_V$, as well as a correction to tilt disorder variance Δ_h. We compute

$$\begin{aligned}\langle H_\Delta^2\rangle^c_> &\equiv \langle H_\Delta[u^<_\alpha + u^>_\alpha]^2\rangle^c_>, \\ &= \Delta_V^2 \int_{\mathbf{r}\mathbf{r}'} \sum_{\alpha\neq\beta,\alpha'\neq\beta'}{}' T^{\alpha'\beta'}_{\alpha\beta}(\mathbf{r}-\mathbf{r}'),\end{aligned} \tag{A15}$$

where $\int_{\mathbf{r}} \equiv \int d^3r$, the prime on the sum now excludes $\alpha=\beta$ and/or $\alpha'=\beta'$ terms, and

$$\begin{aligned}T^{\alpha'\beta'}_{\alpha\beta}(\mathbf{r}-\mathbf{r}') &\equiv \langle \cos[q_0(u_\alpha(\mathbf{r}) - u_\beta(\mathbf{r}))] \\ &\quad \times \cos[q_0(u_{\alpha'}(\mathbf{r}') - u_{\beta'}(\mathbf{r}'))]\rangle^c_>, \\ &= \frac{1}{4}\sum_{q,q'} \langle e^{iq[u_\alpha(\mathbf{r}) - u_\beta(\mathbf{r})] + iq'[u_{\alpha'}(\mathbf{r}') - u_{\beta'}(\mathbf{r}')]}\rangle^c_>, \\ &= \frac{1}{4}\sum_{q,q'} e^{iq[u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r})] + iq'[u^<_{\alpha'}(\mathbf{r}') - u^<_{\beta'}(\mathbf{r}')]} \\ &\quad \times S^{\alpha'\beta'}_{\alpha\beta}(\mathbf{r}-\mathbf{r}').\end{aligned} \tag{A16}$$

In the above expression the q and q' variables independently range over two values $\pm q_0$, and $S^{\alpha'\beta'}_{\alpha\beta}(\delta\mathbf{r})$ (with $\delta\mathbf{r}\equiv\mathbf{r}-\mathbf{r}'$) is the result of the cumulant average over $u^>_\alpha$ fields given by

$$\begin{aligned}S^{\alpha'\beta'}_{\alpha\beta}(\delta\mathbf{r}) &\equiv \langle e^{iq(u^>_\alpha(\mathbf{r}) - u^>_\beta(\mathbf{r})) + iq'(u^>_{\alpha'}(\mathbf{r}') - u^>_{\beta'}(\mathbf{r}'))}\rangle^c_>, \\ &= e^{-q_0^2(f_{\alpha\beta} + f_{\alpha'\beta'})} \\ &\quad \times [e^{-qq'(G^>_{\alpha\alpha'}(\delta\mathbf{r}) + G^>_{\beta\beta'}(\delta\mathbf{r}) - G^>_{\alpha\beta'}(\delta\mathbf{r}) - G^>_{\beta\alpha'}(\delta\mathbf{r}))} \\ &\quad - 1], \\ &= e^{-q_0^2(f_{\alpha\beta} + f_{\alpha'\beta'})} \\ &\quad \times [e^{-qq'G^>_T(\delta\mathbf{r})(\delta_{\alpha\alpha'} + \delta_{\beta\beta'} - \delta_{\alpha\beta'} - \delta_{\beta\alpha'})} - 1],\end{aligned} \tag{A17}$$

where to obtain the last equation above we used Eq. (A3a). The real space propagator of the short wavelength modes $G^>_T(\delta\mathbf{r})$ is by definition a partial Fourier transform of the propagator in momentum space, Eqs. (A3a),(A3b), which for infinitesimal renormalization group transformation is integrated only over an infinitesimal shell of wave vectors $\Lambda e^{-\delta l} < q_\perp < \Lambda$. This implies that the exponential in the last equation can be expanded in powers of an infinitesimal $G^>_T(\delta\mathbf{r})$ proportional to δl

$$\begin{aligned}S^{\alpha'\beta'}_{\alpha\beta}(\delta\mathbf{r}) &= e^{-q_0^2(f_{\alpha\beta} + f_{\alpha'\beta'})}\Big[-4qq' G^>_T(\delta\mathbf{r})\delta_{\alpha\alpha'} \\ &\quad + \frac{q_0^4}{2}[G^>_T(\delta\mathbf{r})]^2 \\ &\quad \times (\delta_{\alpha\alpha'} + \delta_{\beta\beta'} - \delta_{\alpha\beta'} - \delta_{\beta\alpha'})^2 + \cdots\Big],\end{aligned} \tag{A18a}$$

$$\begin{aligned}&= e^{-q_0^2(f_{\alpha\beta} + f_{\alpha'\beta'})}[-4qq' G^>_T(\delta\mathbf{r})\delta_{\alpha\alpha'} \\ &\quad + 2q_0^4[G^>_T(\delta\mathbf{r})]^2 \\ &\quad \times (\delta_{\alpha\alpha'} + \delta_{\alpha\alpha'}\delta_{\beta\beta'} - 2\delta_{\alpha\beta'}\delta_{\beta\beta'}) + \ldots],\end{aligned} \tag{A18b}$$

where to obtain the first terms in the respective square brackets of Eqs. (A18a),(A18b), above, we anticipated that $S^{\alpha'\beta'}_{\alpha\beta}$ will be summed over q and q', which allowed us to make simplifying changes in summation variables $\alpha,\alpha',\beta,\beta'$. Inserting this last expression Eq. (A18b) into $T^{\alpha'\beta'}_{\alpha\beta}(\delta\mathbf{r})$, Eq. (A16), and performing the sum over q,q', we obtain

$$\begin{aligned}T^{\alpha'\beta'}_{\alpha\beta}(\delta\mathbf{r}) &= q_0^2 e^{-q_0^2 G^>_T(0)(2-\delta_{\alpha\beta}-\delta_{\alpha'\beta'})} \\ &\quad \times [2G^>_T(\delta\mathbf{r})\delta_{\alpha\alpha'} A^{\alpha'\beta'}_{-\alpha\beta}(\mathbf{r},\mathbf{r}') \\ &\quad + q_0^2[G^>_T(\delta\mathbf{r})]^2(\delta_{\alpha\alpha'} + \delta_{\alpha\alpha'}\delta_{\beta\beta'} \\ &\quad - 2\delta_{\alpha\beta'}\delta_{\beta\beta'}) A^{\alpha'\beta'}_{+\alpha\beta}(\mathbf{r},\mathbf{r}')],\end{aligned} \tag{A19}$$

where

$$A^{\alpha'\beta'}_{\pm\alpha\beta}(\mathbf{r},\mathbf{r}') = \cos[q_0(u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r}) - u^<_{\alpha'}(\mathbf{r}') + u^<_{\beta'}(\mathbf{r}'))] \pm \cos[q_0(u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r}) + u^<_{\alpha'}(\mathbf{r}') - u^<_{\beta'}(\mathbf{r}'))]. \quad \text{(A20)}$$

Naively, it appears that the $\delta_{\alpha\beta}$, $\delta_{\alpha'\beta'}$, and $-2\delta_{\alpha\beta'}\delta_{\beta\beta'}$ terms above contribute to the renormalization of Δ_V to second order in Δ_V. However, upon summation over $\alpha,\beta,\alpha',\beta'$, these terms, in fact, vanish, because in the sum over replica indices we exclude the diagonal terms $\alpha=\beta$ and $\alpha'=\beta'$. If we had kept these diagonal contributions in our definition of H_Δ, a (somewhat less obvious) cancellation would have taken place between various terms with the final results, of course, unchanged. Inserting the resulting expression Eq. (A19) into Eq. (A15), we obtain

$$\langle H^2_\Delta\rangle^c_> = I_1 + I_2 + I_3, \quad \text{(A21)}$$

where we have defined three contributions to $\langle H^2_\Delta\rangle^c_>$,

$$I_1 = \Delta_V^2 \int_{\mathbf{r},\delta\mathbf{r}} {\sum_{\alpha\neq\beta,\alpha\neq\beta}}' q_0^2 e^{-2q_0^2 G^>_T(0)} \times [2G^>_T(\delta\mathbf{r}) + q_0^2[G^>_T(\delta\mathbf{r})]^2] \times \cos[q_0(u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r}) - u^<_\alpha(\mathbf{r}') + u^<_{\beta'}(\mathbf{r}'))], \quad \text{(A22a)}$$

$$I_2 = \Delta_V^2 \int_{\mathbf{r},\delta\mathbf{r}} {\sum_{\alpha\neq\beta,\alpha\neq\beta'}}' q_0^2 e^{-2q_0^2 G^>_T(0)} \times [-2G^>_T(\delta\mathbf{r}) + q_0^2[G^>_T(\delta\mathbf{r})]^2] \times \cos[q_0(u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r}) + u^<_\alpha(\mathbf{r}') - u^<_{\beta'}(\mathbf{r}'))]\}, \quad \text{(A22b)}$$

$$I_3 = \Delta_V^2 \int_{\mathbf{r},\delta\mathbf{r}} {\sum_{\alpha\neq\beta}}' q_0^4 e^{-2q_0^2 G^>_T(0)}[G^>_T(\delta\mathbf{r})]^2 \times \{\cos[q_0(u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r}) - u^<_\alpha(\mathbf{r}') + u^<_\beta(\mathbf{r}'))] + \cos[q_0(u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r}) + u^<_\alpha(\mathbf{r}') - u^<_\beta(\mathbf{r}'))]\}, \quad \text{(A22c)}$$

We now carefully analyze each of the above terms. Although naively I_1 and I_2 appear as three-replica terms, in fact, as we will see below, I_1 renormalizes both Δ_V and Δ_h, and I_2 is irrelevant.

I_1 naturally splits up into a $\beta=\beta'$ term and everything else:

$$I_1 = \Delta_V^2 \int_{\mathbf{r},\delta\mathbf{r}} q_0^2 e^{-2q_0^2 G^>_T(0)}\{2G^>_T(\delta\mathbf{r}) + q_0^2[G^>_T(\delta\mathbf{r})]^2\} \times (I_{1a} + I_{1b}), \quad \text{(A23)}$$

where

$$I_{1a} = {\sum_{\alpha\neq\beta',\beta\neq\beta'}}' \cos[q_0(u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r}) - u^<_\alpha(\mathbf{r}') + u^<_{\beta'}(\mathbf{r}'))], \quad \text{(A24a)}$$

$$I_{1b} = {\sum_{\alpha\neq\beta}}' \cos[q_0(u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r}) - u^<_\alpha(\mathbf{r}') + u^<_\beta(\mathbf{r}'))]. \quad \text{(A24b)}$$

Since the function $G^>_T(\delta\mathbf{r})$ multiplying the above expressions is a short-ranged function of $\delta\mathbf{r}$ (with range on the order Λ^{-1}, the inverse of the ultra-violet cutoff Λ), we can safely expand the above expressions in powers of $\delta\mathbf{r}$:

$$\begin{aligned} I_{1a} &= {\sum_{\alpha\neq\beta',\beta\neq\beta'}}' \cos[q_0(u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r}) - u^<_\alpha(\mathbf{r}+\delta\mathbf{r}) + u^<_{\beta'}(\mathbf{r}+\delta\mathbf{r}))], \\ &\approx {\sum_{\alpha\neq\beta',\beta\neq\beta'}}' \cos[q_0(u^<_\beta(\mathbf{r}) - u^<_{\beta'}(\mathbf{r}) + \delta\mathbf{r}\cdot\nabla[u^<_\alpha(\mathbf{r}) - u^<_{\beta'}(\mathbf{r})])], \\ &\approx {\sum_{\beta\neq\beta'}}' (n-2)\cos[q_0(u^<_\beta(\mathbf{r}) - u^<_{\beta'}(\mathbf{r}))] + \text{irrelevant terms}, \end{aligned} \quad \text{(A25)}$$

where the ''irrelevant terms'' are of the form of a product of $\cos[q_0(u^<_\beta(\mathbf{r}) - u^<_{\beta'}(\mathbf{r}))]$ and derivatives of $u^<_\alpha(\mathbf{r}) - u^<_{\beta'}(\mathbf{r})$, and are clearly less important at long length scales than the term that we have kept. Similarly for I_{1b}, we obtain

$$\begin{aligned} I_{1b} &= {\sum_{\alpha\neq\beta}}' \cos[q_0(u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r}) - u^<_\alpha(\mathbf{r}+\delta\mathbf{r}) + u^<_\beta(\mathbf{r}+\delta\mathbf{r}))], \\ &\approx {\sum_{\alpha\neq\beta}}' \cos[q_0(\delta\mathbf{r}\cdot\nabla[u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r})])], \\ &\approx {\sum_{\alpha\neq\beta}}' \left(1 - \frac{q_0^2}{2}|\delta\mathbf{r}\cdot\nabla[u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r})]|^2\right) + \text{irrelevant terms}. \end{aligned} \quad \text{(A26)}$$

Performing similar local expansion in powers of $\delta\mathbf{r}$ of the term I_2, Eq. (A22b), it is easy to see that I_2 contributes terms of the form of a product $\cos[q_0(2u^<_\alpha(\mathbf{r}) - u^<_\beta(\mathbf{r}) - u^<_{\beta'}(\mathbf{r}))]$ and derivatives of $u^<_\alpha(\mathbf{r}) - u^<_{\beta'}(\mathbf{r})$, which are *all irrelevant* at long scales, near the glass transition temperature, and we therefore drop them.

Finally, the analysis of I_3 is also very similar. The *second* cosine in Eq. (A22c) generates a higher harmonic and is therefore irrelevant as in the analysis of I_2, while the *first* cosine gives

$$I_3=\Delta_V^2\int_{\mathbf{r},\delta\mathbf{r}}\sum_{\alpha\neq\beta}{}'\; q_0^4 e^{-2q_0^2 G_T^>(0)}[G_T^>(\delta\mathbf{r})]^2$$
$$\times\cos[q_0(u_\alpha^<(\mathbf{r})-u_\beta^<(\mathbf{r})-u_\alpha^<(\mathbf{r}+\delta\mathbf{r})+u_\beta^<(\mathbf{r}+\delta\mathbf{r}))]$$
$$\approx\Delta_V^2\int_{\mathbf{r},\delta\mathbf{r}}\sum_{\alpha\neq\beta}{}'\; q_0^4 e^{-2q_0^2 G_T^>(0)}[G_T^>(\delta\mathbf{r})]^2$$
$$\times\left(1-\frac{q_0^2}{2}|\delta\mathbf{r}\cdot\boldsymbol{\nabla}[u_\alpha^<(\mathbf{r})-u_\beta^<(\mathbf{r})]|^2\right)+\text{irrelevant terms.} \tag{A27}$$

Putting all this together into Eq. (A21) and dropping irrelevant and constant terms (which renormalize the constant part of the free energy), we obtain

$$\langle H_\Delta^2\rangle_>^c\approx-\Delta_V^2 e^{-2q_0^2 G_T^>(0)}$$
$$\times\int\sum_{\mathbf{r}\alpha\neq\beta}{}'\left(A(2-n)\cos[q_0(u_\alpha^<(\mathbf{r})-u_\beta^<(\mathbf{r}))]\right.$$
$$\left.+\frac{1}{2}C_{ij}\partial_i[u_\alpha^<(\mathbf{r})-u_\beta^<(\mathbf{r})]\partial_j[u_\alpha^<(\mathbf{r})-u_\beta^<(\mathbf{r})]\right), \tag{A28}$$

where we have defined

$$A\equiv q_0^2\int_{\delta\mathbf{r}}\{2G_T^>(\delta\mathbf{r})+q_0^2[G_T^>(\delta\mathbf{r})]^2\}, \tag{A29a}$$

$$C_{ij}\equiv 2q_0^4\int_{\delta\mathbf{r}}\{G_T^>(\delta\mathbf{r})+q_0^2[G_T^>(\delta\mathbf{r})]^2\}\delta r_i\delta r_j. \tag{A29b}$$

The first term in A and C_{ij} is just a Fourier transform of $G_T^>(\mathbf{r})$ evaluated at $\mathbf{q}=0$. Because the propagator $G_T^>(\mathbf{q})$ of the high wave-vector modes by definition only has support near the lattice cutoff $q_\perp=\Lambda$, with $G_T^>(\mathbf{q}=0)=0$, the first terms in A and C_{ij} vanish. With this simplification A and C_{ij} can be evaluated via contour integration.

For A we easily find

$$A=q_0^4\int_{\delta\mathbf{r}}[G_T^>(\delta\mathbf{r})]^2, \tag{A30a}$$
$$=q_0^4\int_{\Lambda e^{-\delta l}}^{\Lambda}\frac{d^2q_\perp}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{dq_z}{2\pi}[G_T^>(\mathbf{q})]^2, \tag{A30b}$$
$$=q_0^4\int_{\Lambda e^{-\delta l}}^{\Lambda}\frac{d^2q_\perp}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{dq_z}{2\pi}\frac{1}{(Kq_\perp^4+Bq_z^2)^2}, \tag{A30c}$$
$$=\frac{q_0^4}{8\pi\Lambda^4\sqrt{K^3B}}\delta l \tag{A30d}$$
$$\equiv A_1\delta l. \tag{A30e}$$

The evaluation of C_{ij} is similar but a little more involved. Because of the anisotropic scaling $q_z^2\sim q_\perp^4\ll q_\perp^2$ enforced by the quadratic part of the elastic Hamiltonian, Eq. (3.8), it is clear that we are only interested in C_{ij} with i,j taking on values x,y, for which

$$C_{ij}^\perp\equiv 2q_0^6\int_{\delta\mathbf{r}}[G_T^>(\delta\mathbf{r})]^2\delta r_i^\perp\delta r_j^\perp, \tag{A31a}$$
$$\equiv C_\perp\delta_{ij}^\perp, \tag{A31b}$$

where the second equation follows by rotational invariance in the xy plane, which enforces $C_{xx}=C_{yy}\equiv C_\perp$ and $C_{xy}=C_{yx}=0$, with

$$C_\perp=q_0^6\int d^2r\,dz\,r_\perp^2[G_T^>(r_\perp,z)]^2, \tag{A32a}$$
$$=\frac{c\,q_0^6}{4\pi\Lambda^6\sqrt{K^3B}}\delta l, \tag{A32b}$$
$$\equiv A_2\delta l. \tag{A32c}$$

In the above c is a dimensionless constant of order 1. Now comparing the expression for $\langle H_\Delta^2\rangle_>^c$, Eq. (A28), with the ''bare'' Hamiltonian, Eq. (3.8), we find

$$\delta\Delta_V^{(2)}=-\frac{1}{2}(2-n)A_1\Delta_V^2\delta l, \tag{A33a}$$
$$\underset{n\to 0}{=}-A_1\Delta_V^2\delta l,$$
$$\delta\Delta_h=A_2\Delta_V^2\delta l. \tag{A33b}$$

We note that the graphical correction to the random field disorder Δ_V is *stabilizing* (negative), leading to a stable Cardy-Ostlund–like glassy fixed line (Fig. 6), *only* for $n\to 0$ $(n<2)$, which can be physically understood from the discussion given in Sec. III.

After the above integration of the short wavelength modes $u_\alpha^>$, the effective Hamiltonian is a functional of the long wavelength modes $u_\alpha^<$, which have an ultraviolet cutoff Λe^{-l} that is different from the cutoff Λ of the original theory. Therefore, in order to identify the new effective coupling constants $\Delta_V(l),\Delta_h(l),\ldots,$ we need to perform the second part of the renormalization group transformation, that involves the rescaling of variables, which restores the cutoff in the effective theory back to Λ. The rescaling that accomplishes this is

$$r_\perp=r'_\perp e^l,$$
$$z=z'e^{\omega l},$$
$$u_\alpha^<(\mathbf{r})=e^{\phi l}u_\alpha(\mathbf{r}'). \tag{A34}$$

Because the random-field nonlinearity is a periodic function, it is convenient (but not necessary) to take the arbitrary field dimension $\phi=0$, thereby preserving the period $2\pi/q_0$ under the renormalization group transformation.[41] Under this transformation the resulting effective Hamiltonian functional can

be restored into its original form, Eq. (3.8), with effective l-dependent couplings that satisfy differential recursion relations, Eqs. (3.18).

APPENDIX B: DETAILS OF THE $3<D<7$ FUNCTIONAL RENORMALIZATION GROUP ANALYSIS, INCLUDING ANHARMONIC ELASTICITY

In this appendix we present the details of the functional renormalization group (FRG) calculation for the effective replicated Hamiltonian, $H=H_0+H_{\rm int}$, given by Eq. (8.10), with the quadratic part H_0

$$H_0[u_\alpha]=\frac{1}{2}\int d^dr\sum_{\alpha=1}^{n}\left[K(\nabla_\perp^2 u_\alpha)^2+B(\partial_z u_\alpha)^2\right], \quad \text{(B1)}$$

and the anharmonic part of the effective Hamiltonian $H_{\rm int}=H_{\rm anh}+H_\Delta$ given by

$$H_{\rm anh}=\frac{1}{2}\int d^dr\sum_{\alpha=1}^{n}\left[-B\partial_z u_\alpha(\boldsymbol{\nabla}_\perp u_\alpha)^2+\frac{B}{4}(\boldsymbol{\nabla}_\perp u_\alpha)^4\right], \quad \text{(B2a)}$$

$$H_\Delta=\int d^dr\sum_{\alpha,\beta=1}^{n}\left[\frac{1}{4}\Delta_h(u_\alpha-u_\beta)|\boldsymbol{\nabla}_\perp(u_\alpha-u_\beta)|^2-\Delta_V(u_\alpha-u_\beta)\right], \quad \text{(B2b)}$$

which incorporates both the elastic anharmonicities, previously studied in Sec. VI, and the random field and tilt disorders of a randomly pinned smectic liquid crystal. The result of the analysis presented below is the set of the FRG flow equations given in Eqs. (8.11)–(8.13) of Sec. VIII.

Our RG analysis will closely follow the approach presented in the previous RG sections, in particular paralleling the calculation and notation for a single harmonic disorder in three dimensions, presented in Appendix A. The difference here is that we are interested in the effect of disorder in dimensions higher than 3. As a result, as discussed in detail in Sec. VIII, in these higher dimensions all harmonics of the random pinning potential are equally relevant and need to be kept track of by studying the evolution of arbitrary periodic functions $\Delta_V(u_\alpha-u_\beta)$ and $\Delta_h(u_\alpha-u_\beta)$ under the RG transformation. Furthermore, in contrast to Appendix A, here, in addition, we are including anharmonic nonlinearities, which lead to the weak logarithmic anomalous elasticity discussed in Sec. VIII for $5<d<7$, and to the power-law anomalous elasticity for $d<5$ discussed in Sec. VI.

1. Assuming tilt disorder is irrelevant for $d>5$

We have argued in Secs. III and IV, that, although in a full model of a randomly pinned smectic both random tilt and field disorders are present, for $3<d<5$, the random tilt disorder always dominates over the random field (periodic) disorder. However, as the power-counting suggests, for $d>5$, the random tilt disorder becomes irrelevant and smectic correlations are dominated by the random field disorder. In the second subsection of this appendix we will explicitly demonstrate that this conclusion remains valid even when potentially singular diagrammatic corrections to $\Delta_h(u_\alpha-u_\beta)$ are taken into account. Hence, for the remainder of this subsection, confining our discussion to dimensions $d>5$, where random tilt disorder is irrelevant, we leave it out of our analysis.

The logic of the renormalization group analysis here is the same as that used in Appendix A: we separate the fields u_α into high and low wave-vector components $u_\alpha^>$ and $u_\alpha^<$, respectively, and perturbatively integrate the $u_\alpha^>$ out of the replicated partition function. At leading order, the correction to the quadratic Hamiltonian H_0 is formally given by a cumulant expansion

$$\delta H[u_\alpha^<]=\langle H_{\rm int}[u_\alpha^<+u_\alpha^>]\rangle_>^c-\frac{1}{2}\langle H_{\rm int}[u_\alpha^<+u_\alpha^>]^2\rangle_>^c+\cdots, \quad \text{(B3)}$$

where, as in Appendix A, we have set $T=1$, but, in contrast to the calculations there, have not excluded diagonal terms in the replica sum.

We first focus on the contributions to the random field disorder $\Delta_V(u)$. We begin by noting that the renormalization of $\Delta_V(u)$ gets *no* contributions from the elastic anharmonicities $\partial_z u_\alpha(\boldsymbol{\nabla}_\perp u_\alpha)^2$ and $(\boldsymbol{\nabla}_\perp u_\alpha)^4$, i.e., from the vertices in $H_{\rm anh}$, Eq. (B2a). This is because the graphs that look like they might renormalize $\Delta_V(u_\alpha-u_\beta)$ have q's on the external legs and hence renormalize *only* the tilt disorder (and other less relevant operators), which, as we discussed above, is irrelevant for $d>5$, which is the case we are considering here.

To first order in the disorder $\Delta_V(u_\alpha-u_\beta)$, keeping only the relevant, nonconstant terms, we obtain

$$\begin{aligned}\delta H_\Delta^{(1)}&=-\sum_{\alpha,\beta}\int_{\mathbf{r}}\langle\Delta_V(u_\alpha-u_\beta)\rangle_>,\\ &\approx-\frac{1}{2}\sum_{\alpha,\beta}\int_{\mathbf{r}}\Delta_V''(u_\alpha^<-u_\beta^<)\langle(u_\alpha^>-u_\beta^>)^2\rangle_>,\\ &\approx-dl\frac{C_{d-1}\Lambda^{d-3}}{2\sqrt{BK}}\sum_{\alpha,\beta}\int_{\mathbf{r}}\Delta_V''(u_\alpha^<-u_\beta^<),\end{aligned} \quad \text{(B4)}$$

which when compared to the definition of H_Δ gives

$$\delta\Delta_V^{(1)}(u)\approx dl\frac{C_{d-1}\Lambda^{d-3}}{2\sqrt{BK}}\Delta_V''(u), \quad \text{(B5)}$$

with the primes indicating a partial derivative with respect to u. To obtain the final result above, we used the Gaussian part of the total effective Hamiltonian

$$H_{\rm Gauss}=\frac{1}{2}\int_{\mathbf{r}}\left\{\sum_\alpha\left[B(\partial_z u_\alpha)^2+K(\nabla_\perp^2 u_\alpha)^2\right]-\sum_{\alpha\beta}\Delta_V''(0)(u_\alpha-u_\beta)^2\right\} \quad \text{(B6)}$$

to calculate

$$\langle(u_\alpha^>-u_\beta^>)^2\rangle_>=2\int_{\mathbf{q}}^{>}\left[\langle|u_\alpha(\mathbf{q})|^2\rangle-\langle u_\alpha(\mathbf{q})u_\beta(-\mathbf{q})\rangle\right] \quad \text{(B7)}$$

using the corresponding propagator

$$\langle u_\alpha(\mathbf{q})u_\beta(-\mathbf{q})\rangle=\frac{\delta_{\alpha\beta}}{Bq_z^2+Kq_\perp^4}-\frac{2\Delta_V''(0)}{[Bq_z^2+Kq_\perp^4]^2}, \quad \text{(B8)}$$

obtained from $H_{\rm Gauss}$ by manipulations virtually identical to those described earlier for the tilt-only model, Sec. VI.

The contribution to second order in $\Delta_V(u)$ is given by

$$\delta H_\Delta^{(2)}\approx-\frac{1}{4}\sum_{\alpha_1,\beta_1,\alpha_2,\beta_2}\int_{\mathbf{r_1},\mathbf{r_2}}\Delta_V''[u_{\alpha_1}^<(\mathbf{r}_1)-u_{\beta_1}^<(\mathbf{r}_1)]\times\Delta_V''[u_{\alpha_2}^<(\mathbf{r}_2)-u_{\beta_2}^<(\mathbf{r}_2)]I_{\alpha_1\beta_1}^{\alpha_2\beta_2}(\mathbf{r}_1,\mathbf{r}_2), \quad \text{(B9)}$$

where

$$I_{\alpha_1\beta_1}^{\alpha_2\beta_2}=\langle[u_{\alpha_1}^>(\mathbf{r}_1)-u_{\beta_1}^>(\mathbf{r}_1)]^2[u_{\alpha_2}^>(\mathbf{r}_2)-u_{\beta_2}^>(\mathbf{r}_2)]^2\rangle_>^c, \quad \text{(B10a)}$$

$$=\langle[u_{\alpha_1}^>(\mathbf{r}_1)-u_{\beta_1}^>(\mathbf{r}_1)][u_{\alpha_2}^>(\mathbf{r}_2)-u_{\beta_2}^>(\mathbf{r}_2)]\rangle_>^2, \quad \text{(B10b)}$$

$$=(\delta_{\alpha_1\alpha_2}+\delta_{\beta_1\beta_2}-\delta_{\alpha_1\beta_2}-\delta_{\beta_1\alpha_2})^2[G_T^>(\mathbf{r}_1-\mathbf{r}_2)]^2, \quad \text{(B10c)}$$

$$=4(\delta_{\alpha_1\alpha_2}+\delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}-2\delta_{\alpha_1\beta_2}\delta_{\beta_1\beta_2})[G_T^>(\delta\mathbf{r})]^2, \quad \text{(B10d)}$$

$\delta\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2$, we used the Wick decomposition theorem for Gaussian random variables (a,b,c,d)

$$\langle abcd\rangle=\langle ab\rangle\langle cd\rangle+\langle ac\rangle\langle bd\rangle+\langle ad\rangle\langle bc\rangle, \quad \text{(B11)}$$

to go from Eq. (B10a) to Eq. (B10b), and used the symmetry of the summand in $\delta H_\Delta^{(2)}$, Eq. (B9) under α,β interchange to get Eq. (B10d).

Substituting the last expression for $I_{\alpha_1\beta_1}^{\alpha_2\beta_2}$, Eq. (B10d), into $\delta H_\Delta^{(2)}$ above and using the short-range property of $G_T^>(\delta\mathbf{r})$ to expand $\mathbf{r}_1$ as $\mathbf{r}_1=\mathbf{r}_2+\delta\mathbf{r}$, keeping only the most relevant terms, we obtain

$$\delta H_\Delta^{(2)}\approx-\delta l\,G_2\int_{\mathbf{r}}\sum_{\alpha,\beta}\Big\{\Delta_V''[u_\alpha^<(\mathbf{r})-u_\beta^<(\mathbf{r})]^2-2\Delta_V''[u_\alpha^<(\mathbf{r})-u_\beta^<(\mathbf{r})]\Delta_V''(0)+\sum_\gamma\Delta_V''[u_\alpha^<(\mathbf{r})-u_\beta^<(\mathbf{r})]\Delta_V''[u_\alpha^<(\mathbf{r})-u_\gamma^<(\mathbf{r})]\Big\}, \quad \text{(B12)}$$

where the constant G_2 is defined by

$$G_2\delta l=\int_{\delta\mathbf{r}}[G_T^>(\delta\mathbf{r})]^2, \quad \text{(B13a)}$$

$$=\int_{\Lambda e^{-\delta l}}^{\Lambda}\frac{d^{d-1}q_\perp}{(2\pi)^{d-1}}\int_{-\infty}^{\infty}\frac{dq_z}{2\pi}[G_T^>(\mathbf{q})]^2, \quad \text{(B13b)}$$

$$=\int_{\Lambda e^{-\delta l}}^{\Lambda}\frac{d^{d-1}q_\perp}{(2\pi)^{d-1}}\int_{-\infty}^{\infty}\frac{dq_z}{2\pi}\frac{1}{(Kq_\perp^4+Bq_z^2)^2}, \quad \text{(B13c)}$$

$$=\frac{C_{d-1}\Lambda^{d-7}}{8\sqrt{K^3B}}\delta l. \quad \text{(B13d)}$$

It is easy to see by power-counting that the three-replica (last) term in $\delta H_\Delta^{(2)}$, Eq. (B12) is irrelevant relative to the two-replica terms and can therefore be neglected. Comparing the resulting expression for $\delta H_\Delta^{(2)}$ with H_Δ, Eq. (B2b) we find

$$\delta\Delta_V^{(2)}(u)\approx\delta l\frac{C_{d-1}\Lambda^{d-7}}{8\sqrt{K^3B}}[\Delta_V''(u)^2-2\Delta_V''(u)\Delta_V''(0)]. \quad \text{(B14)}$$

Combining the first and second order contributions to $\Delta_V(u)$, Eqs. (B5),(B14), with the length and field rescalings, Eq. (A34), necessary to bring the ultraviolet cutoff back to Λ, we obtain the recursion relation for $\Delta_V(u)$ given by

$$\partial_l\Delta_V(u,l)=(d+1)\Delta_V(u,l)+\frac{\eta(l)}{q_0^2}\Delta_V''(u,l)+\left[\frac{1}{2}[\Delta_V''(u,l)]^2-\Delta_V''(u,l)\Delta_V''(0,l)\right]\times\frac{C_{d-1}\Lambda^{d-7}}{4\sqrt{K^3(l)B(l)}}, \quad \text{(B15)}$$

where, for simplicity, in the above we have set the field rescaling $\phi=0$ and used $\omega=2$.

We now turn to the renormalization of the elastic moduli B and K. As discussed in Sec. III, statistical symmetry forbids renormalization of these by disorder alone (the same argument applies to the XY spin stiffness in the random field XY model). However, anharmonic elasticity, special to smectic liquid crystals, conspires with the random field $\Delta_V(u)$ to renormalize B and K.

Calculations very similar to those for the tilt-only model treated in Sec. VI show that the graphical corrections to B and K found in that section can be taken over for the random field disorder, with the identification

$$\Delta_h^{\rm eff}=-2\Delta_V''(0)/\Lambda^2. \quad \text{(B16)}$$

In addition to a detailed perturbative RG calculation, this can also be easily seen by comparing

$$\Delta_V(u_\alpha-u_\beta)\approx\text{const}+\frac{1}{2}\Delta_V''(0)(u_\alpha-u_\beta)^2, \quad \text{(B17)}$$

to the disorder part of the the tilt-only Hamiltonian.

Using this mapping we immediately find

$$\frac{dK}{dl}=\left[d-3+\frac{d^2-12d+23}{8(d^2-1)}\left(\frac{B}{K^5}\right)^{1/2}\Delta_V''(0)C_{d-1}\Lambda^{d-7}\right]K, \quad \text{(B18a)}$$

$$\frac{dB}{dl}=\left[d-3+\frac{3}{8}\left(\frac{B}{K^5}\right)^{1/2}\Delta_V''(0)C_{d-1}\Lambda^{d-7}\right]B. \tag{B18b}$$

Now identifying the dimensionless measure of the random field disorder

$$\tilde{\Delta}_V(u,l)\equiv\frac{C_{d-1}\Lambda^{d-7}}{4\sqrt{K^3(l)B(l)}}\Delta_V(u,l), \tag{B19}$$

and using it inside Eqs. (B15), (B18a), and (B18b), we obtain the FRG flow Eqs. (8.13), (8.11), and (8.12) given in Sec. VIII.

2. Proving that tilt disorder is irrelevant for $d>5$

In the previous subsection we have ignored the tilt disorder for $d>5$, arguing for its irrelevance based on power-counting in these higher dimensions. However, the (supernaturally) alert reader might be concerned (as we were at first) that the singular behavior of $\tilde{\Delta}_V^*(u)$, Eq. (8.23) [in particular, the divergence of its fourth derivative with respect to u, $\tilde{\Delta}_V^{(iv)}(u,l)$ as $l\to\infty$] could potentially invalidate the simple power-counting argument that tilt disorder is irrelevant for $d>5$. Here, we demonstrate that this does not happen, by deriving a *functional* renormalization group recursion relation for $\Delta_h(u,l)$ to *linear* order in $\Delta_h(u,l)$ itself at the (FRG) fixed point $\tilde{\Delta}_V^*(u)$, Eq. (8.23) found in Sec. VIII. We obtain

$$\partial_l\Delta_h(u,l)=(d-1)\Delta_h(u,l)+\left(\frac{\eta(l)}{q_0^2}+D(u)\right)\Delta_h''(u,l)+c(\tilde{\Delta}_{V*}'''(u))^2\sqrt{K^3(l)B(l)}\Lambda^{5-d}/C_{d-1}, \tag{B20}$$

where we have defined the ''position-dependent diffusivity''

$$D(u)\equiv c_2[\tilde{\Delta}_{V*}''(u)-\tilde{\Delta}_{V*}''(0)], \tag{B21}$$

and c and c_2 are dimensionless constants of order unity.

As in the tilt-only model, the important quantity proves not to be $\Delta_h(u,l)$, but rather the dimensionless combination

$$g(u,l)\equiv\Delta_h(u,l)\left(\frac{B(l)}{K^5(l)}\right)^{1/2}C_{d-1}\Lambda^{d-5}, \tag{B22}$$

whose recursion relation readily follows from those for $K(l)$, $B(l)$, and $\Delta_h(u,l)$, Eqs. (B18a), (B18b), and (B20), respectively,

$$\partial_l g(u,l)=\left(5-d+c_3\frac{\tilde{\Delta}_{V*}''(0)}{\lambda^2(l)}\right)g(u,l)+\left(\frac{\eta(l)}{q_0^2}+D(u)\right)g''(u,l)+c\frac{[\tilde{\Delta}_{V*}'''(u)]^2}{\lambda^2(l)}, \tag{B23}$$

where $c_3\equiv 3/4+(d^2-12d+23)/[2(d^2-1)]$.

The solution to this inhomogeneous linear partial differential equation can, as usual, be written as a sum of the solution to the *homogeneous* equation obtained by dropping the last term, plus any solution of the inhomogeneous equation. The solution to the homogeneous equation can be written as an expansion in a complete basis

$$g_{\rm hom}(u,l)=\sum_{n=0}^{\infty}g_n(l)\phi_n(u), \tag{B24}$$

where the set $\phi_n(u)$ are the eigenfunctions of the operator $D(u)\partial_u^2$ defined and discussed in Sec. VIII.

The expansion coefficients $g_n(l)$ can be readily found from Eq. (B23) to be

$$g_n(l)=g_n(0)e^{-\lambda_n l}\exp\left(-\int_0^l dl'\frac{c_3|\tilde{\Delta}_{V*}''(0)|}{\lambda^2(l')}\right), \tag{B25}$$

with

$$\lambda_n\equiv d-5+|\Gamma_n|, \tag{B26}$$

which is positive for all n for $d>5$, and we have dropped the $\eta(l)$ term, which vanishes exponentially fast as $l\to\infty$. Since the second exponential in Eq. (B25) is a monotonically decreasing function of l, every $g_n(l)$, and, hence, the entire homogeneous solution Eq. (B24) for $g(u,l)$, vanishes as $l\to\infty$.

Thus, the only possible problem with neglecting tilt disorder (for $d>5$) must come from the *inhomogeneous* solution for $g(u,l)$. We will now show that this vanishes as well, thereby proving that tilt disorder is irrelevant for $d>5$. We do this by finding an asymptotic (large l) solution $g_I(u,l)$ to the full, inhomogeneous equation of the form

$$g_I(u,l)=\sum_{n=1}^{\infty}\frac{f_n(u)}{l^n}, \tag{B27}$$

which clearly vanishes as $l\to\infty$.

Inserting Eq. (B27) into Eq. (B23), and keeping in mind that $1/\lambda^2(l)$ is itself $\propto 1/l$, as $l\to\infty$ [see Eq. (8.17)], we find an equation for $f_1(u)$ by equating coefficients of the leading order $1/l$ terms on both sides of Eq. (B23). The resulting equation reads

$$D(u)f_1''(u)-(d-5)f_1(u)=-\frac{cA(d)|\tilde{\Delta}_{V*}'''(0)|^2}{|\tilde{\Delta}_{V*}''(0)|}, \tag{B28}$$

where we have defined $A(d)\equiv(d^2-1)/(d^2+6d-13)$ and used Eq. (8.17) for $\lambda(l)$. Once $f_1(u)$ is known, the remaining f_n's for $n>1$ can be determined by equating coefficients of $1/l^n$ on both sides of Eq. (B23) after inserting the ansatz Eq. (B27). This gives

$$D(u)f_n''(u)-(d-5)f_n(u)=(c_3A(d)+1-n)f_{n-1}(u). \tag{B29}$$

Now, to complete the proof that $g_I(u,l)$ vanishes as $l\to\infty$, we need merely show that all f_n's obtained from Eq.

(B28) and Eq. (B29) are finite for all n. If $f_1(u)$ is finite, then clearly $f_2(u)$ is also finite, since no derivatives of $f_1(u)$ [which could potentially diverge if $f_1(u)$ is cusped] appear in Eq. (B29). This finiteness argument then obviously recursively carries through via Eq. (B29) to *all* of other $f_n(u)$'s.

Although $f_1(u)$ can be found explicitly applying standard integration methods to Eq. (B28), our only goal here is to demonstrate that $f_1(u)$ is finite. To do this we simply expand $f_1(u)$ in eigenfunctions $\phi_n(u)$ of the operator $D(u)\partial_u^2$:

$$f_1(u)=\sum_{n=0}^{\infty} a_n\phi_n(u). \tag{B30}$$

Inserting this expansion into Eq. (B28), multiplying both sides by $\phi_m(u)$, integrating from $u=0$ to $u=a$, and using the eigenvalue equation, Eq. (8.46) and the orthogonality relation Eq. (8.47), we obtain

$$a_m=\frac{cA(d)}{(d-5+|\Gamma_m|)|\tilde{\Delta}''_{V*}(0)|}S_m, \tag{B31}$$

where the source

$$S_m\equiv\int_0^a\frac{|\tilde{\Delta}'''_{V*}(u)|^2\phi_m(u)}{I_mD(u)}du \tag{B32}$$

was shown to be finite for all m (including $m=0$) in Sec. VIII.

Hence, $f_1(u)$ will be finite for all u, *provided* that the sum

$$\sum_{n=0}^{\infty}\frac{S_n\phi_n(u)}{d-5+|\Gamma_n|}=\frac{|\tilde{\Delta}''_{V*}(0)|}{cA(d)}f_1(u) \tag{B33}$$

converges for all u.

To prove that it does, let us follow the obvious convention of ordering the eigenmodes n such that $|\Gamma_n|$ is a monotonically increasing function of n. Then, for large n, we can use the WKB solution for the eigenfunctions $\phi_n(u)$

$$\phi_n(u)=D^{1/4}(u)\sin\left[nq_0\overline{D}^{1/2}\int_0^u\frac{du'}{D^{1/2}(u')}\right] \tag{B34}$$

and eigenvalues Γ_n

$$\Gamma_n=-\overline{D}n^2q_0^2, \tag{B35}$$

where we have defined the suitably averaged diffusion constant

$$\overline{D}\equiv\left[\frac{1}{a}\int_0^a\frac{du'}{D^{1/2}(u')}\right]^{-2}, \tag{B36}$$

to prove the convergence of the sum in Eqs. (B33). Equation (B34) shows that $\phi_n(u)$ is bounded from above for all u (by $[\max_u D(u)]^{1/4}$), which in turn implies that the S_n's are also bounded above [since $\tilde{\Delta}'''_{V*}(u)$ is finite for all u, and so is $\phi_n(u)$] as $n\to\infty$. Then using Eq. (B35) for the Γ_n, we see that the large n behavior of the summand in Eq. (B33) is bounded above by const./n^2. Hence, the sum converges, and so $f_1(u)$ is in fact finite for all u; consequently all $f_n(u)$'s are also finite. This therefore demonstrates that $g(u,l)$ vanishes as $l\to\infty$, proving our assertion that tilt disorder is irrelevant for $d>5$.

APPENDIX C: DERIVATION OF SMECTIC ''COULOMB GAS'' DISLOCATION THEORY, WITHOUT EULER-LAGRANGE EQUATION

In this appendix we derive the Coulomb gas description of smectic dislocation loops characterized by an effective Hamiltonian, Eq. (5.15). However, here, in contrast to the derivation of the main text, we accomplish this *without* minimizing the Hamiltonian with respect to smooth smectic deformations.

Our starting point is the elastic Hamiltonian for a tilt-only model of a randomly pinned smectic, given in Eq. (5.1)

$$H[u]=\int d^d\mathbf{r}\left[\frac{B}{2}(\partial_z u)^2+\frac{K}{2}(\nabla_\perp^2 u)^2+\mathbf{h}(\mathbf{r})\cdot\boldsymbol{\nabla}_\perp u\right]. \tag{C1}$$

We include dislocations in above description by allowing the layer displacement $u(\mathbf{r})$ to be a multi-valued function, such that

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}u=\mathbf{m}. \tag{C2}$$

We then decompose $\boldsymbol{\nabla}u$ into a sum of a singular, *purely transverse* part $\mathbf{v}_d$ satisfying

$$\boldsymbol{\nabla}\times\mathbf{v}_d=\mathbf{m}, \tag{C3}$$

and a nonsingular, purely longitudinal part $\boldsymbol{\nabla}u_p$, according to

$$\boldsymbol{\nabla}u=\mathbf{v}_d+\boldsymbol{\nabla}u_p. \tag{C4}$$

The difference between this approach and our earlier derivation in Sec. V is in our arbitrary choice of the separation of $\boldsymbol{\nabla}u$ into phonon and dislocation parts. In Sec. V, we chose $\mathbf{v}_d$ to minimize H given $\mathbf{m}$; here, we simply choose it to be purely transverse. We will now demonstrate that this difference between these two *arbitrary* choices has no effect on the final answer, as it should not.

Substituting this decomposition into $H[u]$, Eq. (C1), we obtain

$$H_{\rm tot}[u_p,\mathbf{v}_d]=H_{\rm el}[u_p]+H_{\rm v}[\mathbf{v}_d]+H_{\rm int}[u_p,\mathbf{v}_d], \tag{C5}$$

where (dropping the subscript d on $\mathbf{v}$),

$$H_{\rm v}=\int d^d\mathbf{r}\left[\frac{B}{2}v_z^2+\frac{K}{2}(\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp)^2+\mathbf{h}(\mathbf{r})\cdot\mathbf{v}_\perp\right], \tag{C6a}$$

$$=\int d^d\mathbf{r}\left[\frac{B}{2}v_z^2+\frac{K}{2}(\partial_z v_z)^2+\mathbf{h}(\mathbf{r})\cdot\mathbf{v}_\perp\right], \tag{C6b}$$

and

$$H_{\rm int}=\int d^d\mathbf{r}[Bv_z\partial_z u+K(\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp)\nabla_\perp^2 u], \tag{C7a}$$

$$=\int d^d\mathbf{r}[Bv_z\partial_z u-K\partial_z v_z\nabla_\perp^2 u]. \tag{C7b}$$

In going from Eq. (C6a) to Eq. (C6b) and from Eq. (C7a) to Eq. (C7b), we used the purely transverse property of $\mathbf{v}$, $\boldsymbol{\nabla}\cdot\mathbf{v}=0$, or equivalently,

$$\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp+\partial_z v_z=0, \tag{C8}$$

to eliminate $\mathbf{v}_\perp$ in favor of v_z.

We now integrate over the single-valued phonon degree of freedom u_p. After a simple Gaussian integration, some algebra, and Fourier transformation, we obtain an effective Hamiltonian

$$H_d=\int_{\mathbf{q}}\left[\frac{Kq^4}{2\Gamma_q}|v_z(\mathbf{q})|^2+\mathbf{b}(\mathbf{q})\cdot\mathbf{h}(-\mathbf{q})\right], \tag{C9}$$

where, $\Gamma_q=q_z^2+\lambda^2 q_\perp^4$, and $\mathbf{b}(\mathbf{q})$ is given by

$$b_i(\mathbf{q})=\left[P^T_{\perp ij}(\mathbf{q})+\frac{q^2}{\Gamma_q}P^L_{\perp ij}(\mathbf{q})\right]v_{\perp j}(\mathbf{q}), \tag{C10}$$

with

$$P^T_{\perp ij}(\mathbf{q})=\delta^\perp_{ij}-\frac{q^\perp_i q^\perp_j}{q^2_\perp}, \tag{C11a}$$

$$P^L_{\perp ij}(\mathbf{q})=\frac{q^\perp_i q^\perp_j}{q^2_\perp}, \tag{C11b}$$

the transverse and longitudinal projection operators in the space ($\perp$) perpendicular to $\hat{\mathbf{z}}$.

Now Fourier transforming Eq. (C3), and solving it for $\mathbf{v}(\mathbf{q})$, keeping in mind that $\mathbf{v}$ is purely transverse, we find

$$v_i(\mathbf{q})=i\epsilon_{ijk}q_j m_k/q^2. \tag{C12}$$

Using this solution to eliminate $\mathbf{v}$ inside H_d, Eq. (C9), above, in favor of the dislocation density $\mathbf{m}$, we obtain the final expression for the dislocation loop Hamiltonian, Eq. (5.15), quoted in the main text.

APPENDIX D: ANALYSIS OF FLUCTUATIONS IN THE *DUAL* MODEL OF DISORDERED SMECTICS IN TYPE I LIMIT

In this appendix we analyze the type I regime of the dual disordered smectic liquid crystal model derived and studied in Sec. V. In this regime, we will compute *exactly* the effective free energy and as a by-product obtain from it the fluctuation corrections to the reduced dual transition temperature, given in Eq. (5.37).

Our starting point is the replicated dual ''action'' S_r given in Eq. (5.33), together with the quenched ''gauge field'' $\mathbf{a}$ variance, given by Eq. (5.34b). We build on the ideas developed in Ref. 15, generalizing them here to disordered systems. As discussed in these references and in Sec. V, in the type I regime the order parameter ψ is significantly stiffer than the gauge field $\mathbf{A}$ and can therefore be accurately treated in a mean-field approximation. This amounts to taking $\psi(\mathbf{r})$ to be constant in space. The constant ψ is then calculated by minimizing the resultant free energy, which can now be computed *exactly*, without further approximations. As is clear from the discussion in Sec. V, to do so, we must calculate

$$\overline{Z^n}=\frac{1}{N_a}\int'[d\mathbf{a}]\prod_{\alpha=1}^{n}[d\mathbf{A}_\alpha]e^{-S_{mf}[\psi,\mathbf{A}_\alpha,\mathbf{a}]}, \tag{D1}$$

with

$$S_{\rm mf}[\psi,\mathbf{A}_\alpha,\mathbf{a}]=S_r[\psi,\mathbf{A}_\alpha,\mathbf{a}]+\sum_{\mathbf{q}}\frac{\Gamma_q^2q^2}{2\Delta_h q_\perp^2 q_z^2}P^\perp_{ij}a_i(\mathbf{q})a_j(-\mathbf{q}), \tag{D2}$$

$$\begin{aligned}=nV(t|\psi|^2+u|\psi|^4)+\frac{1}{2}\int_{\mathbf{r}}\Bigg[\sum_\alpha|\mathbf{A}_\alpha|^2-n|\mathbf{a}|^2+2i\sum_\alpha\mathbf{A}_\alpha\cdot\mathbf{a}\Bigg]|\psi|^2\\+\frac{1}{2}\sum_{\mathbf{q},\alpha}[G_A^{-1}(\mathbf{q})P^\perp_{ij}(\mathbf{q})A_{\alpha i}(\mathbf{q})A_{\alpha j}(-\mathbf{q})+G_a^{-1}(\mathbf{q})P^\perp_{ij}(\mathbf{q})a_i(\mathbf{q})a_j(-\mathbf{q})].\end{aligned} \tag{D3}$$

The prime on the integral in Eq. (D1) indicates that it is constrained to be taken only over the *transverse* parts of $\mathbf{A}_\alpha$ and $\mathbf{a}$, V is the volume of the system, N_a is the normalization factor coming from the probability distribution of $\mathbf{a}$, and will be chosen such that $\overline{Z^n}\to 1$ as $n\to 0$, and

$$G_A^{-1}=\frac{\Gamma_q}{Kq_\perp^2}, \tag{D4a}$$

$$G_a^{-1}=\frac{\Gamma_q^2q^2}{\Delta_h q_\perp^2 q_z^2}, \tag{D4b}$$

can be read off from Eqs. (5.33) and (5.34b), taking the appropriate long wavelength limit in the latter.

The great virtue of the effective ''action'' $S_{\rm mf}$ is that it is *quadratic* in $\mathbf{A}_\alpha$ and $\mathbf{a}$ and therefore allows exact evaluation of the functional integrals over them in Eq. (D1). Taking these Gaussian integrals, we obtain

$$\ln\overline{Z^n}=-V\Bigg[n(t|\psi|^2+u|\psi|^4)+\frac{n(d-2)}{2}\int_{\mathbf{q}}\ln[G_A^{-1}(\mathbf{q})+|\psi|^2]+\frac{(d-2)}{2}\int_{\mathbf{q}}\ln[G_a^{-1}(\mathbf{q})-n|\psi|^2]\Bigg]-\ln N_a, \tag{D5}$$

where the factors of $d-2$ correspond to the fact that there are $d-2$ transverse ''gauge-field'' components of $\mathbf{A}_\alpha$ and $\mathbf{a}$, due to the transversality constraint on the functional integral discussed above.

We observe that, as expected, the normalization factor N_a for the probability distribution of $\mathbf{a}$, given by

$$\ln N_a=-V\frac{(d-2)}{2}\int_{\mathbf{q}}\ln[G_A^{-1}(\mathbf{q})], \tag{D6}$$

guarantees that $\ln\overline{Z^n}\to 0$ as $n\to 0$, Incorporating N_a into Eq. (D5) gives

$$\ln\overline{Z^n} = -Vn\left[(t|\psi|^2+u|\psi|^4)+\frac{(d-2)}{2}\int_{\mathbf{q}}\ln(G_A^{-1}(\mathbf{q})+|\psi|^2) + \frac{(d-2)}{2n}\int_{\mathbf{q}}\ln(1-nG_a(\mathbf{q})|\psi|^2)\right]. \tag{D7}$$

Expanding this in n, we find

$$\ln\overline{Z^n} = -Vn\left[t|\psi|^2+u|\psi|^4+\frac{d-2}{2}\int_{\mathbf{q}}\ln[G_A^{-1}(\mathbf{q})+|\psi|^2] - \frac{d-2}{2}|\psi|^2\int_{\mathbf{q}}G_a(\mathbf{q})\right]+O(n^2). \tag{D8}$$

Now using the standard replica expression for the average free energy, we finally obtain

$$\overline{F}_{\rm MF} = \lim_{n\to 0}\left(-k_BT\frac{\overline{Z^n}-1}{n}\right) \tag{D9a}$$

$$= \lim_{n\to 0}\left(-k_BT\frac{\ln\overline{Z^n}}{n}\right), \tag{D9b}$$

$$= k_BTV\left[t_R|\psi|^2+u|\psi|^4 + \frac{d-2}{2}\int_{\mathbf{q}}\ln(G_A^{-1}+|\psi|^2)\right], \tag{D9c}$$

where the renormalized reduced temperature t_R is given by

$$t_R = t-\frac{d-2}{2}\int_{\mathbf{q}}G_a(\mathbf{q}), \tag{D10}$$

which, using Eq. (D4b), is precisely the result used in Eq. (5.37) of Sec. V to assess the stability of the disordered smectic to dislocation unbinding. Thus, the result of our more rigorous and systematic perturbation theory is recovered by this fluctuation corrected mean field theory. The annealed term in Eq. (D9c) is analogous to the term that leads to a fluctuation-driven first-order transition in type I superconductors.[15] Here, however, its effects are innocuous: expanding the integrand in $|\psi|^2$ leads only to a finite renormalization of t, plus a non-analytic $|\psi|^{d+1}$ piece, which only leads to a finite renormalization of u in $d=3$.

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[38] We use the expression ''line up'' to designate a fixed *preferred* orientation of the nematogen axis relative to the local aerogel strand axis (or more generally with the local tangent to the surface of the confining structure), which might vary from parallel to perpendicular (homeotropic) alignment, depending on the system. It is easy to show that for isotropic confining structures, with no orientational order, e.g., aerogel, our results are independent of these microscopic details. However, for smectics confined, instead, in orientationally ordered structures, studied in Ref. 67, parallel and homeotropic alignments lie in different universality classes.

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[42] Gauge invariance of the nonlinearities in the de Gennes model requires that even when dislocation loops are included K does not have any diagrammatic renormalization. This strict *non-renormalization* of K can also be seen by studying the transition from the low-temperature side, in terms of the dislocation-loop unbinding picture (Refs. 48,16). In this formulation, K is the coefficient of an operator quadratic in u, that is nonanalytic in wave vector, while the nonlinearities are all analytic in wave vector. Since it is impossible for these analytic nonlinearities to generate terms that are nonanalytic in k, there is strictly no renormalization of K, even in the presence of dislocation loops that drive the bulk modulus B to zero.

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