Electronic transport properties of Sierpinski lattices

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We have studied the electronic transport properties of open Sierpinski gasket systems connected to two electron reservoirs in the presence of a magnetic field. In the framework of a tight-binding model, the systems are composed of one-dimensional ordered chains. A generalized eigenfunction method, which allows one to deal with finite systems consisting of a large number of lattice sites (nodes), is used to calculate the transmission and reflection coefficients of the studied systems. The numerical results show that there are two kinds of symmetries of the transmission coefficient T to magnetic flux Φ, and there are antiresonant state regions ($T=0$) and resonant states ($T=1$). It is different from the open ring systems now the electronic energies of resonant states do not coincide with the eigenenergies of the isolated Sierpinski gasket systems. It is also found that the transmission behavior of the single exit systems is much more complicated than that of two exit systems. [S0163-1829(99)03640-1]

I. INTRODUCTION

In the past decade, rapid progress has been made in the area of mesoscopic physics. Quantum transport in mesoscopic systems has been extensively studied both experimentally and theoretically.[1–19] For mesoscopic systems at very low temperatures, the scattering due to phonons, which is a dephasing scattering, is significantly suppressed and the phase-coherence length of electrons becomes large compared to the system dimension. The scattering in the systems can then be modeled as phase-coherent elastic scatterings. Furthermore, if we consider the electron as a free particle, an idealized sample becomes an electron waveguide, which assumes that the electron transport through the system is perfectly ballistic. In recent years, there have been many works devoted to the study of the electronic properties of mesoscopic systems within the framework of the waveguide theory[9–14] and the tight-binding model.[7,8,15–19] Along these lines, the theoretical work to date has focused largely on the problems related to an isolated ring, or to open ring systems connected via leads to electronic reservoirs together with a magnetic flux Φ through the rings. For an isolated ring, the persistent current has been the focus of attention.[3–6] The idea is based on the possibility that the electron wave function may extend coherently over the whole circumference of the ring, and elastic scatterings, finite temperature, and weak inelastic scatterings do not destroy the coherence. As for the open ring systems, the important problem is to study the relationship among the transmission coefficient T, incident electron energy E, and magnetic flux Φ through the rings. The electron reservoirs in the open ring systems act as the source of energy dissipation or irreversibility, and all scattering processes in the leads and rings are assumed to be elastic. Based on the waveguide theory, Xia[10] has studied the Aharonov-Bohm effect in an open ring by calculating the transmission and reflection amplitudes as functions of the magnetic flux, the arm length, and the wave vector. Singha Deo and Jayannavar[12,13] have studied the quantum transport properties of serial stub or ring structures and the band formation in these geometries. Takai and Ohta[14] have published a series of articles investigating similar problems in the presence of both an electrostatic potential and magnetic flux.

On the other hand, it is well known that the tight-binding model is more flexible in theoretical treatments than the waveguide theory as disorder can be introduced readily and the band-structure effects are included.[20,21] Along these lines, Entin-Wohlman *et al.*[7] and Kowal *et al.*[8] have studied the electronic transport properties of an open single ring. Aldea *et al.*[11] studied the same problems using the Green's-function method. Wu and Mahler[9] have developed the quantum network theory of transport, by which the transmission probability for an open *A*-*B* type ring with an arbitrary form factor has been studied in detail. Liu and co-workers have investigated the persistent current of an isolated disordered ring,[15] the effects of spin interaction on the persistent current,[16] as well as the electronic transport properties of variant ring systems threaded by magnetic flux.[17–19]

Fractals and their properties have been studied by physicists for many years. Lakhtakia *et al.* have studied the construction and the analytic properties of the fractal clusters, and they also investigated the diffusion motion of the Pascal-Sierpinski gaskets by using combinational algebra.[20] For electronic transmission, the fractal lattices are much more complicated in structure compared with the ring systems. One of the main points of interest has been the fact that these self-similar objects are found to serve as a nontrivial model

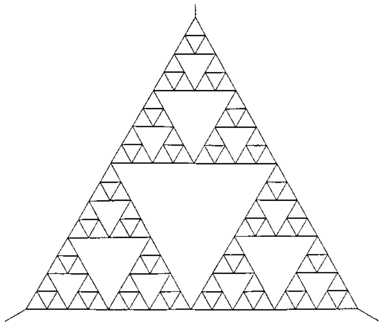

FIG. 1. The fourth-generation Sierpinski gasket lattice, the electronic properties of which are studied in the text.

for the backbone of transport problems. Fractals, in particular deterministic fractals such as the Sierpinski gasket (SG) fractal, possess some special properties, one of which is scale invariance, and do not have any translational order. They in fact bridge the gap between periodic and disordered systems.[22] Therefore, a detailed study on the electronic properties of fractals would lead to new physical results and increase our understanding of nonperiodic systems. Even though there is a large volume in the literature concerned with fractal systems, the study of their electronic properties is not that exhaustive. Along these lines, the energy spectrum and localization of electronic states in an isolated Sierpinski gasket lattice (SGL) have been the subject of many papers.[23–27] Domany *et al.*[23] studied the energy spectrum of the isolated SGL by the use of the recursive technique. Rammal and Toulouse[24] investigated the same problem in the presence of a magnetic field. However, in recent years the belief has been that in a highly correlated self-similar fractal system, such as SGL, localized eigenstates can exist. This should be a kind of structure-induced localization, which is different from Anderson localization due to incoherent scattering.[21] Therefore, the electronic transport properties of this kind of fractal structure would be an interesting problem. Chakrabarti[26] has found that in the absence of magnetic field for the isolated SGL, there are extended electron states. Wang[27] has studied the electronic localization of Sierpinski lattices, and claimed that there exist an infinite number of extended states. He has also studied the magnetic-field effects on the electronic states of the isolated SGL. But to the best of our knowledge, up to now the study on the electronic transport properties of an open SGL has not been reported yet. The reason would be that to deal with an electronic transmission problem of an open SGL, one would have to solve a united equation set, in which the number of equations roughly equals the number of the sites included in the SGL. Therefore, even for a finite SGL, this is a difficult work. Fortunately, we have found an effective approach to solve this problem, in which the transmission and reflection amplitudes are treated together with the electronic wave functions in the sites of the SGL, so that we can deduce a simpler formula to calculate the transmission and reflection coefficients. We have named this approach the generalized eigenfunction method (GEM). By the use of this GEM we have investigated the electronic transport properties of open SGL up to its fourth-generation system containing 123 sites (nodes), which is shown in Fig. 1. By the way, this GEM is formally similar to the fast multipole method (FMM), which is commonly used in electromagnetic scattering problems, but we should point out that they are essentially different from each other.[28] The main purpose of this paper is to investigate the behavior of the transmission coefficient T as the incident electron energy E and the magnetic flux Φ, which penetrates the elementary triangles of the SGL, are varied. Detailed results are given in three-dimensional plots of T against E and Φ, and of which in the two-dimensional cross sections T versus E. It is found that there are two kinds of symmetries of transmission coefficient T to flux Φ. The transmission behavior of single-exit SGL systems is much more complicated than that of two-exit systems. We also found that as the SGL generation increases, the antiresonant regions corresponding to $T=0$ in the E-Φ space progressively increase in both the region number and the region area. This means that in these regions the magnetic flux completely blocks out the electronic transport. This is an interesting quantum phenomenon. On the other hand, we have also calculated the eigenenergy spectrum of the corresponding isolated SGL, and found that in the open SGL case the electron energies of resonant transmission states do not coincide with the eigenenergies of the isolated SGL, which is different from the open ring systems.[18]

This paper is organized as follows. In Sec. II, we introduce the generalized eigenfunction method (GEM) to calculate the transmission and reflection coefficients of an open SGL. The numerical results and discussion of the electronic transport properties are presented in Sec. III. A brief summary is given in Sec. IV.

II. GENERALIZED EIGENFUNCTION METHOD AND ITS APPLICATION IN OPEN SGL SYSTEMS

For the studied open Sierpinski gasket lattices (SGL) which are coupled to two reservoirs via ideal leads, we assume that the leads connected to neighboring sites are composed of one-dimensional ordered chains with on-site energy ε_n and transfer integral t between nearest-neighboring sites. Denoting the incident electron energy by E and the projection of the Wannier wave function on the nth site by ψ_n, in the presence of a magnetic flux Φ the tight-binding equation can be written as[19]

$$(\varepsilon_n - E)\psi_n = \sum_{n'} t_{n,n'}\psi_{n+n'}, \tag{1}$$

where the transfer integral $t_{n,n'}$ equals $te^{\pm i2\pi\Phi/(\Phi_0 S)}$, the $S=3$ is the circumference length of the elementary triangle of the SGL, the magnetic phase $\phi=2\pi\Phi/(\Phi_0 S)$, the $\Phi_0=hc/e$ is the elementary flux quantum, and the sum runs over the nearest neighbors of site n. The wave function ψ_n can be written as the linear combination[18]

$$\psi_n = Ae^{ikn} + Be^{-ikn}, \tag{2}$$

where k is the wave vector, n is the site number, and we take the lattice distance to be unity. In the tight-binding model, the wave vector k is related with the incident electronic en-

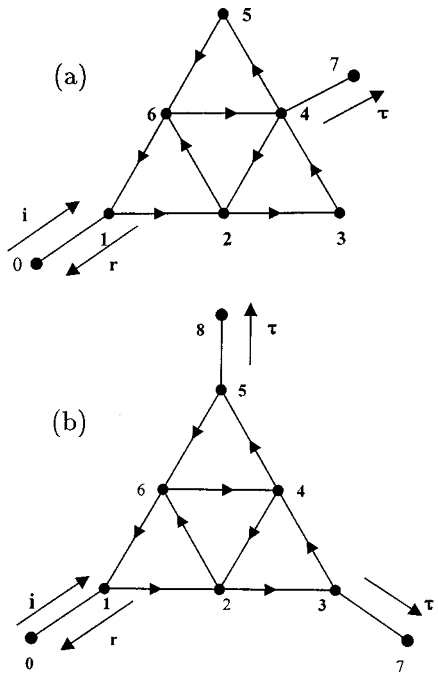

FIG. 2. First-generation Sierpinski lattice coupled to two reservoirs via ideal leads. The magnetic phase is equal to ϕ in the direction of the arrow and $-\phi$ otherwise. (a) One-exit case. (b) Two-exit case; both exits are coupled to the same reservoirs via ideal leads.

ergy E by formula $E=2t\cos k$. We first consider the first-generation SGL with a single exit shown in Fig. 2(a). By the use of Eq. (1) we obtain the tight-binding equations on the sites of the SGL as follows:

$$E\psi_1=te^{-i\phi}\psi_2+te^{-i\phi}\psi_6+t\psi_0,$$

$$\begin{aligned} E\psi_2&=te^{i\phi}\psi_1+te^{-i\phi}\psi_3+te^{i\phi}\psi_4+te^{-i\phi}\psi_6,\\ E\psi_3&=te^{i\phi}\psi_2+te^{-i\phi}\psi_4,\\ E\psi_4&=te^{-i\phi}\psi_2+te^{i\phi}\psi_3+te^{-t\phi}\psi_5+te^{i\phi}\psi_6+t\psi_7,\\ E\psi_5&=te^{i\phi}\psi_4+te^{-i\phi}\psi_6,\\ E\psi_6&=te^{-i\phi}\psi_1+te^{i\phi}\psi_2+te^{-i\phi}\psi_4+te^{i\phi}\psi_5. \end{aligned} \tag{3}$$

On the other hand, for the special sites located in the entry and exit we can write their wave function as[19]

$$\begin{aligned} \psi_0&=1+r \quad (n=0),\\ \psi_1&=e^{ik}+re^{-ik} \quad (n=1),\\ \psi_4&=\tau \quad (n=0),\\ \psi_7&=\tau e^{ik} \quad (n=1), \end{aligned} \tag{4}$$

where r and τ are the reflection and transmission amplitudes of reflecting and outgoing wave functions, respectively. To calculate both of them, we have to solve the above united equation set (3) and (4); evidently this is difficult. If we consider higher-generation SGL, then obtaining an analytic solution seems impossible. To numerically solve this problem, we introduce the following generalized eigenfunction method (GEM). The trick of the GEM is that we treat the amplitudes r and τ the same as the wave functions ψ_i. In this way the united equation set (3) and (4) can be rewritten as the following $(N+2)$-order matrix equation. N is the number of sites in the SGL:

$$\begin{bmatrix} E & e^{-i\phi} & 0 & 0 & 0 & e^{i\phi} & 1 & 0\\ e^{i\phi} & E & e^{-i\phi} & e^{i\phi} & 0 & e^{-i\phi} & 0 & 0\\ 0 & e^{i\phi} & E & e^{-i\phi} & 0 & 0 & 0 & 0\\ 0 & e^{-i\phi} & e^{i\phi} & E & e^{-i\phi} & e^{i\phi} & 0 & e^{ik}\\ 0 & 0 & 0 & e^{i\phi} & E & e^{-i\phi} & 0 & 0\\ e^{-i\phi} & e^{i\phi} & 0 & e^{-i\phi} & e^{i\phi} & E & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & -e^{-ik} & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} \psi_1\\ \psi_2\\ \psi_3\\ \psi_4\\ \psi_5\\ \psi_6\\ r\\ \tau \end{pmatrix} = \begin{pmatrix} -1\\ 0\\ 0\\ 0\\ 0\\ 0\\ e^{ik}\\ 0 \end{pmatrix}. \tag{5}$$

For the sake of simplicity, in the above equation we have chosen $\varepsilon_n=0$ and $t=1$.

If we denote the above matrix equation (5) as

$$M\Psi=C,$$

then the reflection and transmission amplitudes are simply

$$r=(M^{-1}C)_{N+1}, \quad \tau=(M^{-1}C)_{N+2}, \tag{6}$$

and the transmission coefficient $T=|\tau|^2$.

Here we would like to emphasize two points. First, the numerical solution of the above formula (6) is very easy to obtain even with a personal computer. Second, the above generalized eigenfunction method is a very powerful approach to deal with the electronic transport problems in lattice (network) systems, no matter how many entries and exits exist in the studied systems. Even for the quasiperiodic and disordered ones, and for three-dimensional systems, this GEM can also be very efficiently used. For example, the matrix equation of the first-generation SGL with *two exits* shown in Fig. 2(b) can be easily written as follows:

$$\begin{bmatrix} E & e^{-i\phi} & 0 & 0 & 0 & e^{i\phi} & 1 & 0 & 0 \\ e^{i\phi} & E & e^{-i\phi} & e^{i\phi} & 0 & e^{-i\phi} & 0 & 0 & 0 \\ 0 & e^{i\phi} & E & e^{-i\phi} & 0 & 0 & 0 & e^{ik} & 0 \\ 0 & e^{-i\phi} & e^{i\phi} & E & e^{-i\phi} & e^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\phi} & E & {}^{-i\phi} & 0 & 0 & e^{ik} \\ e^{-i\phi} & e^{i\phi} & 0 & e^{-i\phi} & e^{i\phi} & E & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -e^{-ik} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ r \\ \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ e^{ik} \\ 0 \\ 0 \end{pmatrix}, \tag{7}$$

which is now an $(N+3)$-order matrix equation. In deducing the above matrix equation (7) we have used the following relations held in sites 3 and 5 of Fig. 2(b):

$$E\psi_3 = te^{i\phi}\psi_2 + te^{-i\phi}\psi_7 + t\tau_1 e^{ik},$$

$$E\psi_5 = te^{i\phi}\psi_4 + te^{-i\phi}\psi_6 + t\tau_2 e^{ik}.$$

The corresponding reflection and transmission amplitudes of reflecting and outgoing wave functions are, respectively,

$$r = (M^{-1}C)_{N+1}, \quad \tau_1 = (M^{-1}C)_{N+2}, \quad \tau_2 = (M^{-1}C)_{N+3}. \tag{8}$$

From the above example we can see that in the same way we can easily extend the GEM to multientries (and exits) cases. To clarify the name GEM, we should compare the generalized eigenfunction equation (5) with the energy eigenvalue matrix equation of an *isolated* SGL written in the following. If we assume the site energy $\epsilon_n = 0$ for the whole system, then the tight-binding equations in the sites and their corresponding eigenvalue matrix equation are, respectively,

$$\begin{aligned} E\psi_1 &= te^{-i\phi}\psi_2 + te^{-i\phi}\psi_6, \\ E\psi_2 &= te^{i\phi}\psi_1 + te^{-i\phi}\psi_3 + te^{i\phi}\psi_4 + te^{-i\phi}\psi_6, \\ E\psi_3 &= te^{i\phi}\psi_2 + te^{-i\phi}\psi_4, \\ E\psi_4 &= te^{-i\phi}\psi_2 + te^{i\phi}\psi_3 + te^{-t\phi}\psi_5 + te^{i\phi}\psi_6, \\ E\psi_5 &= te^{i\phi}\psi_4 + te^{-i\phi}\psi_6, \\ E\psi_6 &= te^{-i\phi}\psi_1 + te^{i\phi}\psi_2 + te^{-i\phi}\psi_4 + te^{i\phi}\psi_5, \end{aligned} \tag{9}$$

and

$$\begin{bmatrix} E & e^{-i\phi} & 0 & 0 & 0 & e^{i\phi} \\ e^{i\phi} & E & e^{-i\phi} & e^{i\phi} & 0 & {}^{-i\phi} \\ 0 & e^{i\phi} & E & e^{-i\phi} & 0 & 0 \\ 0 & e^{-i\phi} & e^{i\phi} & E & e^{-i\phi} & e^{i\phi} \\ 0 & 0 & 0 & e^{i\phi} & E & e^{-i\phi} \\ e^{-i\phi} & e^{i\phi} & 0 & e^{-i\phi} & e^{i\phi} & E \end{bmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{pmatrix} = 0, \tag{10}$$

where we can see that the above N-order square matrix is the submatrix of matrix $\mathbf{M}$ of the matrix equation (5), and the above eigenwave function vector is the subvector of the corresponding vector of matrix equation (5). That is why we call the method the generalized eigenfunction method.

III. NUMERICAL RESULTS AND DISCUSSION

The formalism mentioned in Sec. II can be easily implemented numerically and the results for both the single- and two-exit cases are obtained up to fourth-generation open SGL with site (node) number $N = 123$. The numerical calculation is easy and quick even with a personal computer. Because the main characters of the transport properties have been revealed in the investigation of the first four generation SGL, it is not necessary to consider the higher-generation systems. In our calculations, the on-site energies are chosen to be $\epsilon_n = 0$ and the transfer integrals $t = -1.0$. To examine the accuracy of our numerical calculations, we check at every intermediate stage of the calculation that the criterion $|\tau|^2 + |r|^2 = 1$ for the transmission and reflection coefficients is satisfied to a tolerance of 10^{-14}. This accuracy enables us to examine with confidence the electronic transport properties of the open SGL.

We consider two basic cases of the open SGL with one and two exits, of which the first-generation systems are

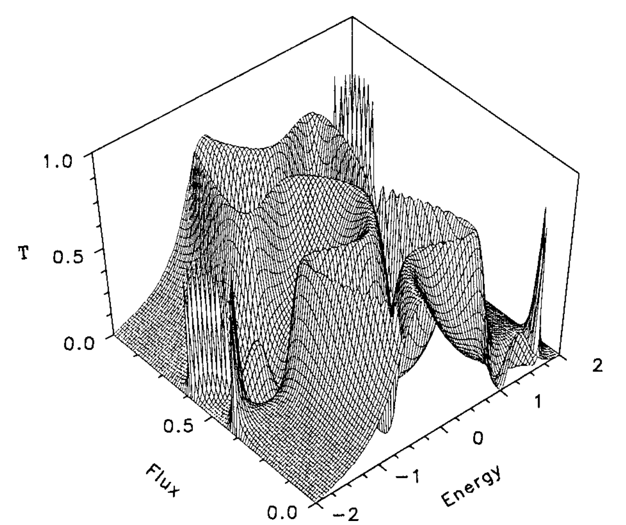

FIG. 3. Transmission coefficient T as a function of magnetic flux Φ and incident electron energy E for the first-generation Sierpinski lattice with a single exit.

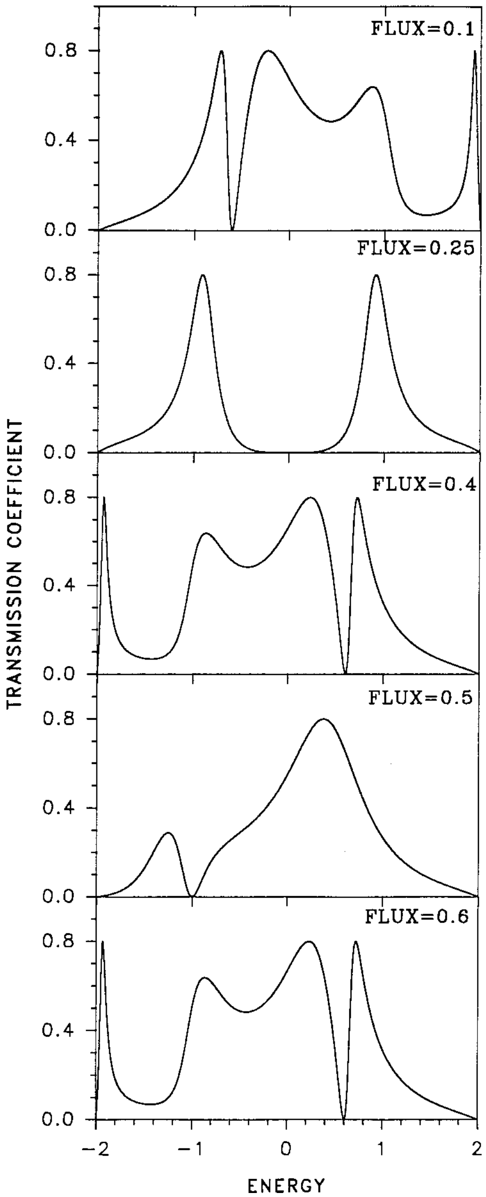

FIG. 4. Transmission coefficient T vs electron energy E as cross sections of Fig. 3. The corresponding flux is marked in the pictures. The two pictures are the same for $\Phi/\Phi_0 = 0.4$ and 0.6, but for $\Phi/\Phi_0 = 0.1$ and 0.4 they are ''antisymmetric'' to energy E (see text).

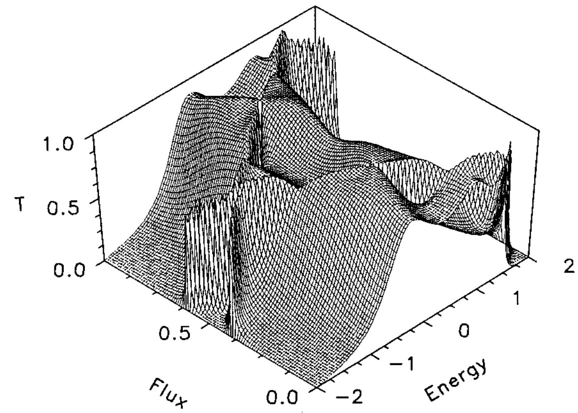

FIG. 5. Transmission coefficient T as a function of magnetic flux Φ and incident electron energy E for first-generation Sierpinski lattice with two exits.

shown in Fig. 2. By the use of the GEM, we have totally calculated the first four generation SGL. The numerical results are shown in Figs. 3–11, in which the typical three-dimensional plots of the transmission coefficient T against the electron energy E and magnetic flux Φ are shown in Figs. 3 and 5 for the first-generation SGL, in Figs. 7 and 8 for the second-generation SGL, in Fig. 10 for the third-generation SGL, and in Fig. 11 for the fourth-generation SGL. For the sake of clear visualization and because of the symmetry of the transmission spectrum, we plotted only a half and a quarter of the whole periodic picture in Figs. 10 and 11, respectively. To display the detail, we have also plotted some pictures of the transmission coefficient T versus energy E, i.e., the cross sections of three-dimensional plots, for the first-generation SGL in Fig. 4 (single exit) and Fig. 6 (two exits), and for the second-generation SGL with two exits in Fig. 9.

From the obtained numerical results, we can see some interesting transport properties, which exist in all studied SGL cases. First, following the enlargement of the SGL, the transmission coefficient T fluctuates more and more, i.e., there exist more and more peaks, valleys, and more and bigger zero-transmission $(T=0)$ regions. This complexity of the transmission behaviors can be understood as the result of the quantum coherence effect among electrons traveling through the SGL. This is due to the fact that the presence of a magnetic flux destroys the time-reversal symmetry and the paths going clockwise and anticlockwise over the systems have different phases. Therefore, when the site number of the systems increases, the variant possibility of quantum coherence also increases, and the transmission coefficient as a function of the electron energy E and magnetic flux Φ becomes increasingly complicated. For the same reason, from the figures we can also see that the antiresonant state region, i.e., the region with $T=0$, enlarges following the increase of the site number. In Figs. 3 and 5 of the first-generation SGL there is no such region, but one does appear in Figs. 7 and 8 of the second-generation SGL and enlarges in the next generations. In the fourth generation SGL several such regions have appeared and their areas have quickly enlarged (see Fig. 11). This global behavior is clearly displayed in the three-dimensional T-E-Φ plots. To show the sophisticated relationship between the incident electron energy and its transmission coefficient, we have plotted the E versus T curves with $\Phi/\Phi_0 = 0.1$, 0.25, 0.4, 0.5, and 0.6, respectively, for the first- and second-generation SGL, and shown them in Figs. 4, 6, and 9, which compliment very well their corresponding 3D plots.

Second, we have noticed the symmetry of transmission behaviors. Because we need to use the eigenvalue matrix equation (10) to discuss the parameter symmetry of the transport property, we investigate in advance the relationship between the resonant electronic states of the *open* SGL and the energy eigenvalues of the *isolated* SGL. An incident electronic state with peak-value transmission coefficient T is called a resonant state. In the open ring systems the electronic energies of resonant states are close to the eigenenergies of the corresponding isolated ring systems.[18] An inter-

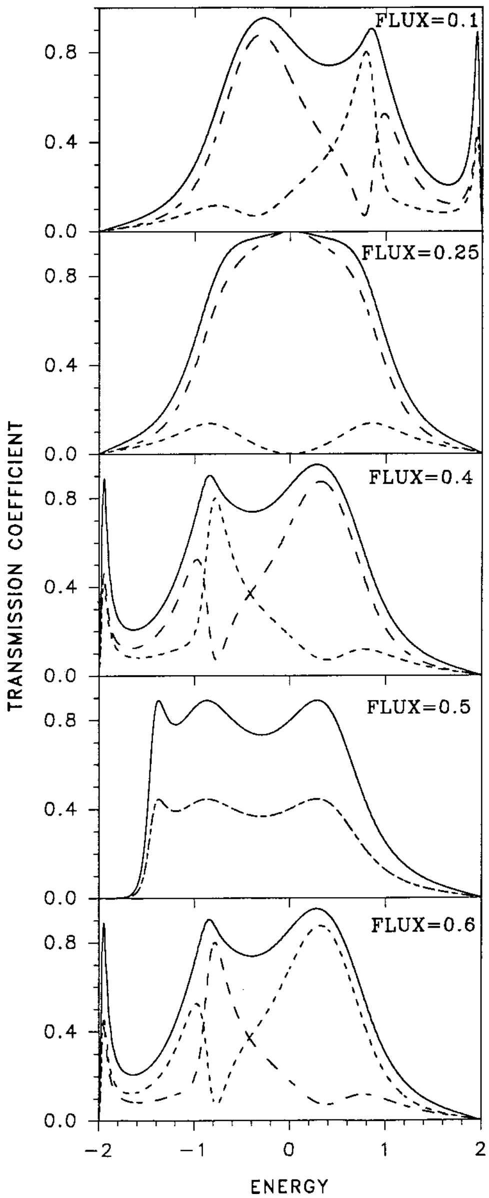

FIG. 6. Transmission coefficient T vs electron energy E as cross sections of Fig. 5. The corresponding flux is marked in the pictures. Long- and short-dashed lines are the transmission coefficients of exits 7 and 8, respectively. Solid lines are the sum of them. The plots show the same symmetry as the single-exit case (see text).

esting question is whether or not there exists the same kind of relationship in the SGL systems. Figure 12 shows the energy eigenvalue spectra of the isolated Sierpinski lattice for the first, second, and fourth generations, which are obtained by calculating Eq. (10). Comparing Fig. 12 with Figs. 3, 5, 7, 8, and 11, which show the T-Φ-E behaviors, we can see that in both the single- and two-exit cases, there is no definite correspondence between the electron energy of the resonant states of the open SGL and the eigenenergy of the isolated SGL. This means that the transport properties of the fractal systems are more complicated than those of the slab systems[13–14] and ring systems.[18] A plausible explanation for this phenomenon should be that the structure of the fractal systems is much more complicated than that of the ring systems, which leads to many more possibilities of variant reflection and transmission, so that quantum coherence effects have a much greater chance to influence the transport properties and finally destroy the correspondence of the two kinds of energies that exist in the open ring systems.

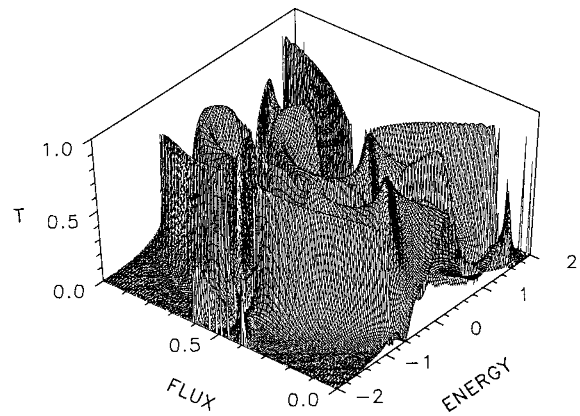

FIG. 7. Transmission coefficient T as a function of magnetic flux Φ and incident electron energy E for second-generation Sierpinski lattice with single exit.

On the other hand, from the energy spectra shown in Fig. 12 we have noticed that there are two kinds of symmetries. First, the energy spectrum is symmetric to $\Phi/\Phi_0=0.5$, i.e., for fluxes Φ/Φ_0 and $1-\Phi/\Phi_0$ two energy bands are exactly the same. Second, to the $\Phi/\Phi_0=0.25$ (or 0.75) the energy spectrum is ''antisymmetric,'' i.e., there is a correspondence between $E(\Phi/\Phi_0)$ and $-E(0.5-\Phi/\Phi_0)$ for $\Phi/\Phi_0\leqslant 0.25$. This symmetrization of the energy spectrum could be understood from the eigenequation (10) of the first-generation SGL. We have obtained the polynomial expression satisfied by eigenenergies E:

$$-2-\cos 6\phi+6E\cos 3\phi-56E^3\cos 3\phi-480E^4+512E^6 =0, \tag{11}$$

from which we can see that the symmetry of the eigenenergy spectrum depends on the symmetry of $\cos 3\phi$, where

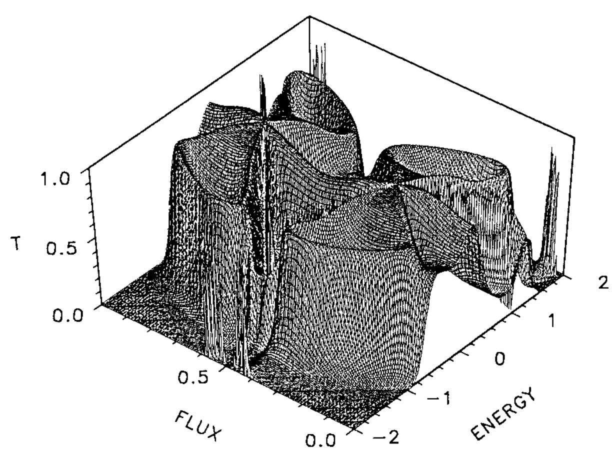

FIG. 8. Transmission coefficient T as a function of magnetic flux Φ and incident electron energy E for second-generation Sierpinski lattice with two exits.

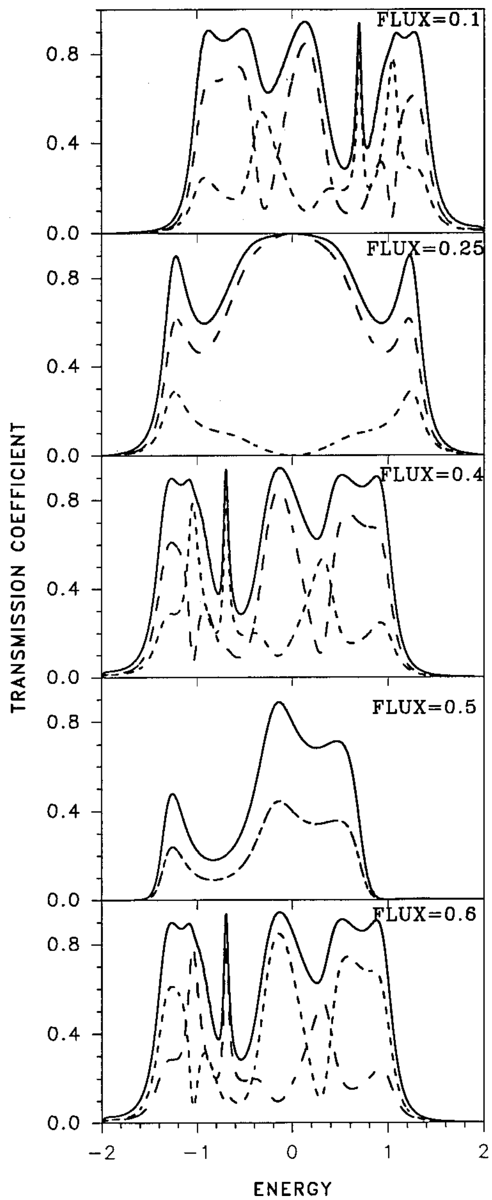

FIG. 9. Transmission coefficient T vs electron energy E as cross sections of Fig. 8. The corresponding flux is marked in the pictures. Long- and short-dashed lines are the transmission coefficients of exits 7 and 8, respectively. Solid lines are the sum of them. The plots show the same symmetry as the first-generation case (see text).

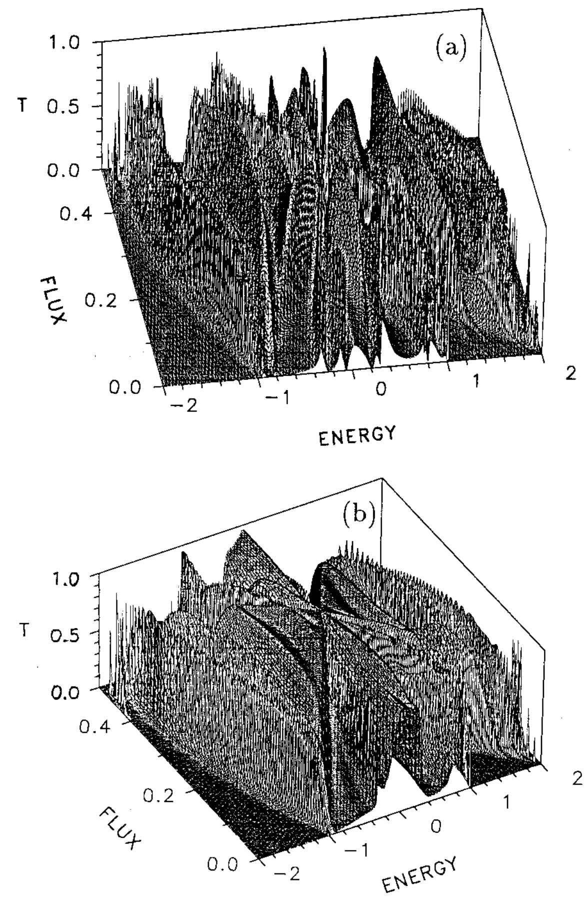

FIG. 10. Transmission coefficient T as a function of magnetic flux Φ and incident electron energy E for third-generation Sierpinski lattice. For the sake of clear visualization only a half-period in flux Φ is shown. (a) Single-exit case. (b) Two-exit case.

$\phi = 2\pi\Phi/3\Phi_0$ so that $\cos 3\phi = \cos 2\pi\Phi/\Phi_0$. This is why there are $\Phi/\Phi_0 = 0.5$ and $\Phi/\Phi_0 = 0.25$ (0.75) kinds of symmetries, because they are merely the symmetries of $\cos 3\phi$. Here we can also see that the energy spectrum is periodic in flux with period $\Phi/\Phi_0 = 1$.

For the same reason, the transmission coefficient T also posseses these two kinds of symmetries. In the three-dimensional plots Figs. 3, 5, 7, and 8, we can find that there is a symmetric plane $(\Phi/\Phi_0 = 0.5, E)$ and two symmetric centers $(\Phi/\Phi_0 = 0.25, E = 0)$ and $(\Phi/\Phi_0 = 0.75, E = 0)$. The symmetric centers are most clearly displayed in Fig. 8, which is a three-dimensional plot for the second-generation SGL with two exits. If we compare the generalized eigenequation (5) with the eigenenergy equation (10), we can see that in the matrix **M** of Eq. (5) the matrix elements related with flux Φ are exactly the same as those of the eigenenergy equation (10). Therefore, they have the same dependence on the flux Φ. For a better view, in Figs. 4 and 6 we show some T-E cross sections of the three-dimensional T-Φ-E plots, which show that the pictures of $\Phi/\Phi_0 = 0.4$ and 0.6 are exactly the same, and those of $\Phi/\Phi_0 = 0.4$ and 0.1 are ''antisymmetric,'' i.e., $T(\Phi/\Phi_0, E) = T(0.5 - \Phi/\Phi_0, -E)$. Therefore, the transmission coefficient T is symmetric for $\pm E$ in the $\Phi/\Phi_0 = 0.25$ case. This point is clearly shown in the two-dimensional plots Figs. 4 and 6. These similarities also originate from the same relationship to flux Φ for both of the matrix equations (5) and (10).

Another interesting phenomenon displayed in the figures is that the single-exit SGL shows more complicated transmission behavior than the two-exit systems, i.e., in the former the transmission spectrum contains more peaks, valleys, and bigger fluctuation. An intuitive explanation for this phenomenon could be that from Fig. 2(b) we can see that the two exits in the open SGL are symmetric, both of which directly connect with the entry site by a straight lead, which serves as a direct ''transport channel.'' This means that for the two-exit systems the multiscattering effect and the quantum-coherent effect of structure are weaker compared with the single-exit systems.

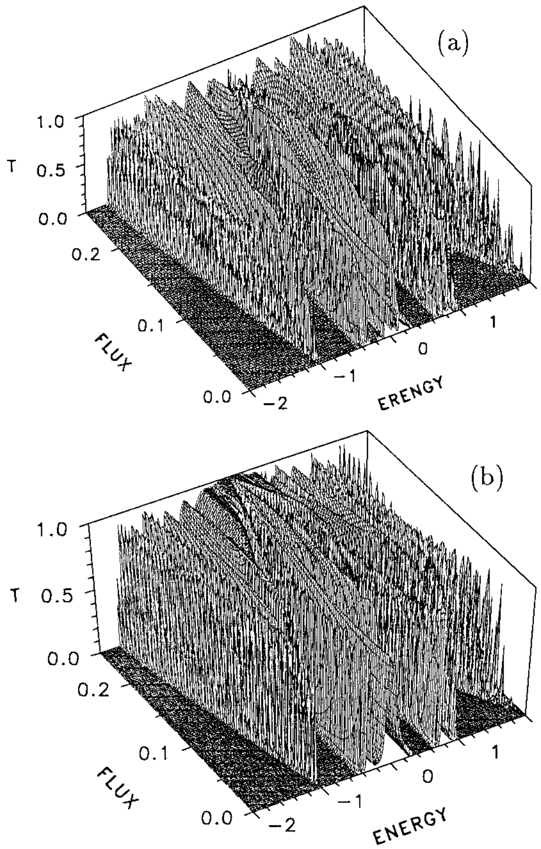

FIG. 11. Transmission coefficient T as a function of magnetic flux Φ and incident electron energy E for fourth-generation Sierpinski lattice. For the sake of clear visualization only a quarter of a period in flux Φ is shown. (a) Single-exit case. (b) Two-exit case.

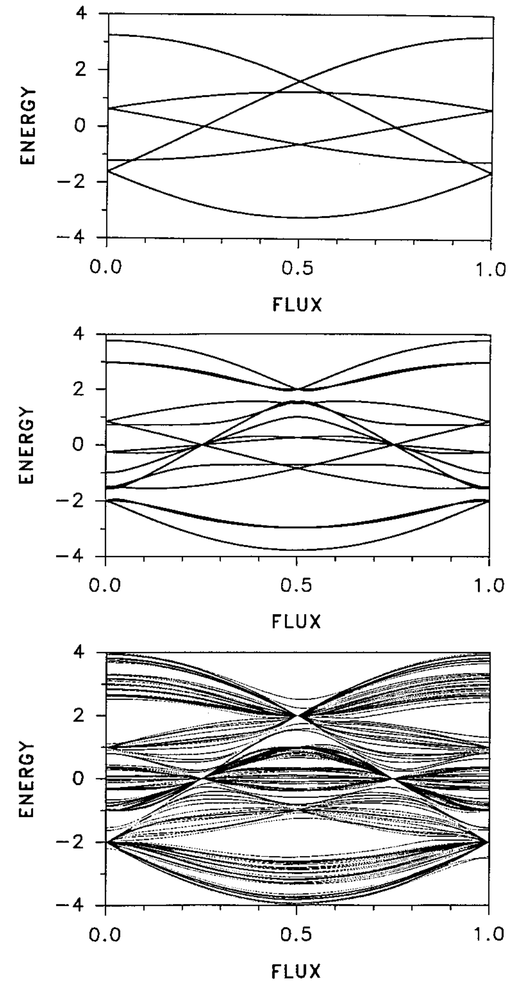

FIG. 12. Energy spectrum of isolated Sierpinski lattice as a function of magnetic flux Φ. From upper to bottom, they correspond to the first-, second-, and fourth-generation Shierpinski lattices, respectively. Readers should notice the symmetries of the spectrum to $\Phi/\Phi_0 = 0.5$ and 0.25.

In Figs. 6 and 9 we show the total and individual transmission coefficients T, T_1, T_2 for the first- and second-generation SGL with two exits. We can see the variation of the transmission coefficients T_1 and T_2 with the change of the magnetic flux Φ. Due to the modulation of the magnetic field, the T_1 and T_2 behaviors are different except in some special cases, such as $\Phi/\Phi_0 = 0$ and 0.5. Generally they periodically exchange the ''position'' following the variance of the flux Φ. This behavior comes from the fact that the two exits are symmetric in the structure of the SGL, therefore in the modulation of the magnetic field the T_1 and T_2 have a phase difference.

IV. BRIEF SUMMARY

We have introduced the generalized eigenfunction method (GEM), which is a very efficient and powerful approach to studying the electronic transport problems of aperiodic systems. By the use of the GEM we have studied the transport properties of open Sierpinski gasket lattices (SGL) coupled to two electron reservoirs via ideal leads. We have investigated the electronic transport properties of an open SGL up to its fourth-generation systems containing site number $N = 123$. The main purpose of this paper is to investigate the behavior of the transmission coefficient T as the incident electron energy E and the magnetic flux Φ, which penetrates the elementary triangles of the SGL, are varied. The detailed numerical results are given in three-dimensional plots of transmission coefficient T against electron energy E and flux Φ, and in which the two-dimensional cross sections are T versus E. It is found that following the enlargement of the SGL, the transmission coefficient T fluctuates more and more, there are more and more resonant peaks, low-transmission valleys, and more and bigger antiresonant states $(T=0)$ regions. In the transmission behaviors there are two kinds of symmetries to flux Φ. In the three-dimensional T-E-Φ plots, the transmission coefficient T has a symmetric plane $(\Phi/\Phi_0 = 0.5, E)$ and two symmetric certers: $(\Phi/\Phi_0 = 0.25, E=0)$ and $(\Phi/\Phi_0 = 0.75, E=0)$. The numerical results show also that the transmission behavior of single-exit SGL systems is much more complicated than that of the two exit systems, because in the former there are direct ''transport channels.'' It is different from the open ring systems now the electronic energies of the resonant states do not coincide with the eigenenergies of the isolated Sierpinski gasket systems. It means that the transport properties of the fractal systems are more complicated than those of the slab systems[13,14] and ring systems.[18] The above results increase our understanding of the transport properties of fractal systems. In the present paper, as an example, we only discussed the SG, which is a simple fractal gasket derived from the Pascal triangle modulo 2, but it is well known that other strictly self-similar gaskets can be derived from Pascal triangle modulo n when n is prime, and even for a nonprime n there also exists self-similarity in the asymptotic sense.[29,30] For these fractal structures, determining what kind of universal property there is in the electronic transport problem would be a very interesting problem, and worth studying.

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