Nonlinear magnetization dynamics of the classical ferromagnet with two single-ion anisotropies in an external magnetic field

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By using a stereographic projection of the unit sphere of magnetization vector onto a complex plane for the equations of motion, the effect of an external magnetic field for integrability of the system is discussed. The properties of the Jost solutions and the scattering data are then investigated through introducing transformations other than the Riemann surface in order to avoid double-valued functions of the usual spectral parameter. The exact multisoliton solutions are investigated by means of the Binet-Cauchy formula. The results showed that under the action of an external magnetic field nonlinear magnetization depends essentially on two parameters: its center moves with a constant velocity, while its shape changes with another constant velocity; its amplitude and width vary periodically with time, while its shape is also dependent on time and is unsymmetric with respect to its center. The orientation of the nonlinear magnetization in the plane orthogonal to the anisotropy axis changes with an external magnetic field. The total magnetic momentum and the integral of the motion coincident with its z component depend on time. The mean number of spins derivated from the ground state in a localized magnetic excitations is dependent on time. The asymptotic behavior of multisoliton solutions, the total displacement of center, and the phase shift of the jth peak are also analyzed.
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I. INTRODUCTION

Nonlinear magnetization dynamics of the classical ferromagnet with two single-ion anisotropies in an external magnetic field can provide an approximative description of various kinds of behavior of magnetic materials as well as the natural starting point for analyzing the anomalous hydrodynamical behavior of low-dimensional magnetic systems. Such fascinating nonlinear dynamic problem exhibits both coherent and chaotic structures depending on the nature of the magnetic interactions, and it is of considerable interest from the point of view of condensed-matter physics, statistical physics, and soliton theory.

Nonlinear magnetization dynamics of the classical ferromagnet can be described by the Landau-Lifschitz equation,[1] special solutions of which have been derived by many authors: Makamura and Sasada[2] found analytic expressions for the permanent profile solitary waves and periodic wave trains; Laksmanan, Ruijgrok, and Thompson[3] discussed the spin-wave spectrum and derived also the solitary wave solution. Tjon and Wright[4] found that a single-solitary wave is stable with respect to small perturbations and that two colliding ones preserve their identity, thus providing evidence that the solitary wave is a bona fide soliton. Kosevich, Ivanov, and Kovalev[5] found a solution by reducing the equation to an appropriate form. Mikeska[6] obtained a solution by reducing the equation of motion to a sine-Gordon equation for a ferromagnet with an easy plane. Long and Bishop[7] proposed another solution which does not tend to the well-known solution of an isotropic ferromagnet when an anisotropy parameter vanishes. Zakharov and Takhtajan[8] found the equivalence of a nonlinear Schrodinger equation and Landau-Lifschitz equation of an anisotropic ferromagnet. Ivanov, Kosevich, and Babich[9] obtained a solution by taking into account only the first-order approximation. Using the Hirota method, Bogdan and Kovalev[10] attempted to construct exact multisoliton solutions of an anisotropic ferromagnet. Svendsen and Fogedby[11] derived the complete spectrum of the Landau-Lifshitz equation by the Hirota method. Using the variation method, Nakumura and Sasada[12] obtained a solution which does not satisfy the equation if it is substituted into the equation of motion.[13] By separating variables in moving coordinates, Quispel and Capel[14] obtained a solution of the Landau-Lifschitz equation of a ferromagnet with an easy plane. Potemina[15] and Kivshar[16] elaborated on the perturbation theory for the Landau-Lifschitz equation describing a biaxial anisotropic ferromagnet.

The general solution of Landau-Lifschitz equation for the special initial condition has been considered by several investigators. Lakshmann[17] shown that the energy and current densities are given by the solutions of a completely integrable nonlinear Schrodinger equation. Takhtajan[18] concluded that the Landau-Lifschitz equation admits a Lax representation and, consequently, falls within the scope of an inverse scattering transformation. Fogedby[19] reviewed the permanent profile solutions of a continuous classical Heisenberg ferromagnet and expounded on the application of an inverse scattering transformation. Sklyanin[20] and Borisov[21] found the Lax pair of Landau-Lifschitz equations for a complete anisotropic ferromagnet, respectively. Mikhailov[22] and Rodin[23] reduced the problem to the Riemann boundary-value problem on a torus, then obtained some results which are expressed by the elliptic functions. Borovik and Kulinich[24,25] derivated the Marchenko equation by an inverse scattering transformation. Pu, Zhou, and Li[26] reported the multisoliton solutions of the Landau-Lifschitz equation in an isotropic ferromagnet in a magnetic field. Chen, Huang, and Liu[27] obtained soliton solutions of the Landau-Lifschitz equation for a spin chain with an easy axis. Yue, Chen, and Huang[28] investigated solitons of the Landau-Lifschitz equation for a spin chain with an easy plane. By means of the Darboux transformation, Huang, Chen, and Liu[29] found the soliton solutions of the Landau-Lifschitz equation for a spin chain with an easy plane. Liu *et al.*[30] studied solitons in a uniaxial Heisenberg spin chain with Gilbert damping in an external magnetic field. Using the method of the Riemann problem with zeros, Yue and Huang[31] investigated solitons for a spin chain with an easy plane.

There are some difficulties in the study of nonlinear magnetization dynamics of a ferromagnet with an anisotropy in an external magnetic field. Its equations of motion, which differ from those of an isotropic ferromagnet, could not be solved by the method of separating variables in moving coordinates.[4] Then, this equation could also not solved by an usual form of inverse scattering transformation since the double-valued function of the spectral parameter is required to introduce a Riemann surface. The reflection coefficient at the edges of cuts in the complex plane could not be neglected even in the case of nonreflection. Thirdly, it is impossible to use Darboux transformation to include the contribution due to the continuous spectrum of the spectral parameter. If we consider the exact solutions of the Landau-Lifschitz equation under various external actions such as an external field in the present paper, a general theory with terms of the continuous spectrum as a starting point is necessary. Finally, an external magnetic field will affect the integrability of the system. This field will change the initial condition of the Landau-Lifschitz equation of a ferromagnet with an anisotropy. It would be instructive if the effect of a magnetic field is discussed. Introducing the coherent-state ansatz, the time-dependent variational principle, and the method of multiple scales, Liu and Zhou investigated the equation of motion and obtained multisolitons in the pure[32] and the biaxial[33] anisotropic antiferromagnets in an external field. Up to date, the effect of an external magnetic field for magnetic systems with anisotropy is treated as various perturbations. The exact solutions of the Landau-Lifschitz equation of the classical ferromagnet with two single-ion anisotropies in an external field have not been obtained yet. On the experimental side,[34,35] a ferromagnet with an easy plane in a symmetry-breaking external transverse field has received continuing interest, though most theoretical treatments have been based on the approximate mapping[6] to a sine-Gordon equation.

This paper focuses on the integrability and nonlinear magnetization dynamics of the classical ferromagnet with two single-ion anisotropies in an external magnetic field. This is an important problem which has been treated to a large extent for the vanishing magnetic field by Sklyanin in a famous, but unpublished preprint, cited in Ref. 20. For magnetic fields with rotational symmetry (an easy-plane or an easy-axis case), the analysis of Ref. 20 can be generalized by transformation to a rotating coordinate frame. For the general direction of the magnetic field the results will be generally investigated in the following content. The plan of this paper is as follows. In Sec. II by using stereographic projection of the unit sphere of the magnetization vector onto a complex plane for the equations of motion, the effect of a magnetic field for integrability of the system will be discussed. Then, though introducing transformations other than the Riemann surface, the properties of the Jost solutions and the scattering of data will be investigated in detail. In Sec. III will be derived the Gel'fand-Levitan-Marchenko equation to construct solutions from the scattering data. The exact multisoliton solutions will be investigated by means of the Binet-Cauchy formula. The total magnetic momentum and its z component will be obtained. Section IV will be devoted to the asymptotic behavior of multisoliton solutions as well as the total displacement of center and the phase shift of the jth peak. Finally, Sec. V will given our concluding remarks.

II. THE EQUATIONS OF MOTION

When we use a macroscopic description, dynamics of the classical ferromagnet is determined by giving at each point of the magnetization vector $\mathbf{M}=(M_x,M_y,M_z)$. The energy of a ferromagnet in this approach called, generally, micromagnetism, is written as the magnetization function. The magnetic energy E of the classical ferromagnet with two single-ion anisotropies in an external magnetic field, including an exchange energy $E_{\rm ex}$, an anisotropic energy $E_{\rm an}$, and a Zeeman energy E_Z can be written as

$$\begin{aligned}E&=E_{\rm ex}+E_{\rm an}+E_Z\\&=\frac{1}{2}\alpha\int\sum_k\frac{\partial\mathbf{M}}{\partial x_k}\frac{\partial\mathbf{M}}{\partial x_k}d^3x-\frac{1}{2}\beta_x\int M_x^2d^3x\\&\quad-\frac{1}{2}\beta_z\int M_z^2d^3x-\mu_B\int\mathbf{M}\cdot\mathbf{B}d^3x,\end{aligned}\tag{1}$$

where μ_B is the Bohn magneton. Equation (1) has an integral of motion $\langle\mathbf{M}^2\rangle\equiv\mathbf{M}_0^2=$const. In the ground state, the quantity M_0 coincides with a so-called spontaneous magnetization $M_0=(2\mu_BS)/a^3$, where S is the atomic spin and a is the interatomic spacing. In the limit $\beta_x=0$, a biaxial anisotropic ferromagnet reduces into an uniaxial anisotropic ferromagnet with an anisotropy axis coincident with the z axis: when $\beta_z>0$, an anisotropy is of an easy-axis type and its magnetization vector in the ground state is directed along the z axis; when $\beta_z<0$ it is of an easy-plane type, its vector $\mathbf{M}$ in the ground state lies in the easy plane in the absence of an external magnetic field and can be directed arbitrarily in this plane. If $E_{\rm an}=0$, a crystal is called an isotropic ferromagnet.

As a function of space coordinates and time, the magnetization vector of the classical ferromagnet $\mathbf{M}(x,t)$ is a solution of the Landau-Lifschitz equation

$$\frac{\partial \mathbf{M}}{\partial t}=\frac{2\mu_B}{\hbar}\mathbf{M}\times\frac{\delta E}{\delta \mathbf{M}}. \quad (2)$$

If we measure the space coordinate x and time t in unit of $l_0=(\alpha/\beta_z)^{1/2}$ and $\omega_0=(2\mu_B\beta_z M_0)/\hbar$, respectively, then according to Eqs. (1) and (2), we can obtain the following equation of motion:

$$\frac{\partial \mathbf{M}}{\partial t}=\mathbf{M}\times\left[\frac{\partial^2 \mathbf{M}}{\partial x^2}+J\mathbf{M}+\mu_B\mathbf{B}\right], \quad (3)$$

where the matrix $J=\mathrm{diag}(J_x,J_y,J_z)$ is related to the anisotropic constants. Equation (3) with $B=0$ is exactly integrable by Sklyanin in a famous, but unpublished paper [20]. The additional terms on the right-hand side of Eq. (3) describes various external actions, e.g., a magnetic field in the present paper, magnetic impurities, dissipative loses, etc. When oscillations of the magnetization vector $\mathbf{M}$ are localized near an easy plane yz, Eq. (3) with $B=0$ could be transformed into a sine-Gordon equation in the limit $J_x \ll J_y < J_z$. Similarly, this equation with $B=0$ also becomes a nonlinear Schrodinger equation in the limit $J_x \approx J_y \ll J_z$ when oscillations of the magnetization vector $\mathbf{M}$ are localized in the vicinity of the vacuum state $\mathbf{M}(x,t)=(0,0,M_0)$. In the special case $\beta_z=0$, an isotropic ferromagnet in an external magnetic field is also completely integrable.[26] When a magnetic field is zero, Eq. (3) is equivalent to a nonlinear Schrodinger equation.[8] Thus Eq. (3) is the most general equation describing the classical ferromagnet with two single-ion anisotropies in an external magnetic field, but its exact solutions have not been obtained so far because the additional terms such as an external magnetic field in the present paper on the right-hand side of Eq. (3) are determined by various perturbations.[33]

We first consider the effect of an external magnetic field on integrability of the system. For magnetic fields with rotational symmetry (an easy-plane or an easy-axis case), the analysis of Ref. 20 can be generalized by going over to a rotating coordinate frame. For the general direction of the magnetic field, we first use a stereographic projection of the unit sphere of magnetization vector onto a complex plane[17,36]

$$P(x,t)=\frac{M_x+iM_y}{1+M_z}. \quad (4)$$

Substituting Eq. (4) into Eq. (3), we can find

$$\begin{aligned}&(1-P^{*2})\Phi_x(P,P^*)-(1-P^2)\Phi_x^*(P,P^*)=0,\\ &-i(1+P^{*2})\Phi_y(P,P^*)-i(1+P^2)\Phi_y^*(P,P^*)=0, \quad (5)\\ &P^*\Phi_z(P,P^*)-P\Phi_z^*(P,P^*)=0,\end{aligned}$$

where Φ_x, Φ_y and Φ_z can be written as

$$\begin{aligned}\Phi_i(P,P^*)=&\,i(1+|P|^2)\frac{\partial P}{\partial t}+(1+|P|^2)\frac{\partial^2 P}{\partial x^2}-2P^*\left(\frac{\partial^2 P}{\partial x^2}\right)^2\\ &+2\Delta J_i P(1-|P|^2)+\mu_B(1+|P|^2)\\ &\times\left[\frac{1}{2}B^x(1-P^2)+\frac{1}{2}iB^y(1+P^2)-B^zP\right], \quad (6)\end{aligned}$$

where $i=x,y,z$, $\Delta J_x=J_z-J_y$, $\Delta J_y=J_x-J_z$, $\Delta J_z=J_y-J_x$, respectively.

The consistency of Eq. (6) implies $\Phi_i(P,P^*)=0$ and $\Phi_i^*(P,P^*)=0$, therefore the evolution equation for the stereographic projection $P(x,t)$ in the presence of the general direction of an external magnetic field becomes

$$\begin{aligned}&i(1+|P|^2)\frac{\partial P}{\partial t}+(1+|P|^2)\frac{\partial^2 P}{\partial x^2}-2P^*\left(\frac{\partial^2 P}{\partial x^2}\right)^2\\ &+2\Delta J_i P(1-|P|^2)+\mu_B(1+|P|^2)\\ &\times\left[\frac{1}{2}B^x(1-P^2)+\frac{1}{2}iB^y(1+P^2)-B^zP\right]=0. \quad (7)\end{aligned}$$

According to Eq. (7), we can analyze the effect of an external magnetic field on the integrability of the system. When an external field is directed along an anisotropic axis, e.g., $\mathbf{B}=[0,0,B^z(t)]$, the magnetic field term in Eq. (7) can be removed by the following gauge transformation $P\rightarrow\tilde{P}=P\exp[i\mu_B\int dtB^z(t)]$, and the system becomes integrable. However, if the magnetic field is transverse, e.g., $\mathbf{B}=[0,B^y(t),0]$, the magnetic field term is not removable by previous gauge transformation and none of the magnetization components remain conserved quantities. Consequently, the combined Galilean plus gauge invariance of the Landau-Lifschitz equation is broken, no Lax pairs seem to exist, and the system appears to be nonintegrable.

The influence of the magnetic field on the classical ferromagnet with an easy axis amounts to a change of the precession frequency of the magnetization vector $\mathbf{M}$ by $\omega_B=\mu_B B$. Therefore, if we can introduce an angular variable $\tilde{\varphi}=\varphi-\omega_B t$ in the polar coordinates (θ,φ), then in terms of the angular variables θ and $\tilde{\varphi}$ Eq. (3) will not depend on B.

However, the magnetization dynamics of the classical ferromagnet with an easy plane is very sensitive to an external magnetic field. Even a weak magnetic field alters the character of the ground state and therefore the form of localized solutions. When an external magnetic field is perpendicular to an easy plane, it does not alter the axial symmetry associated with the z axis, and the form of the ground state depends on the strength of an external field. The critical value is $B_c=[(J_x-J_z)M]/\mu_B$. When an external magnetic field $B^z<B_c$ the magnetization vector $\mathbf{M}$ in the ground state deviates from an easy plane, and it is characterized by an inclination $\theta=\theta_0$ to the z axis, where $\theta_0=\arccos(B^z/B_c)$. The angle φ remains arbitrary. For brevity, such a ground state is referred to as an easy cone. As an external magnetic field increases, the angular opening of the easy cone becomes smaller, especially in the case of $B^z \gg B_c$, and the magnetization vector $\mathbf{M}$ in a nonexcited ferromagnet with an easy plane lies along the z axis.

In the context of the experiments,[34,35] the situation where an external magnetic field lies in an easy plane, e.g., $\mathbf{B}=[B^x(t),0,0]$, or $\mathbf{B}=[0,B^y(t),0]$, seems quite topical. In experiments on samples of a ferromagnet with an easy plane, $CsNiF_3$ and $(C_6H_{11}NH_3)CuBr_3$, an external field is applied as a rule in an easy plane. The presence of an external field, which lies in an easy plane, makes finding soliton solutions of the Landau-Lifschitz equation essentially more difficult.

The magnetic-field term in Eq. (7) is not removable by previous gauge transformation. Thus, we can conclude that a ferromagnet with a uniaxial anisotropy in a transverse magnetic field is, in general, nonintegrable and becomes integrable only in the absence of either an anisotropic interaction or an external field.

Equation (3) may be represented as a compatibility condition $\partial_t L - \partial_x A + [L,A] = 0$ of two equations for 2×2 matrices $\Psi(x,t;\mu,\lambda)$:

$$\frac{\partial \Psi(x,t;\mu,\lambda)}{\partial x} = L(\mu,\lambda)\Psi(x,t;\mu,\lambda),$$

$$\frac{\partial \Psi(x,t;\mu,\lambda)}{\partial t} = A(\mu,\lambda)\Psi(x,t;\mu,\lambda), \tag{8}$$

while

$$L(\mu,\lambda) = -i\rho ns(\mu,\lambda)M_x\sigma_x - i\rho ds(\mu,\lambda)M_y\sigma_y - i\rho cs(\mu,\lambda)M_z\sigma_z,$$

$$\begin{aligned} A(\mu,\lambda) = {} & i2\rho^2 ds(\mu,\lambda)cs(\mu,\lambda)M_x\sigma_x + i2\rho^2 ns(\mu,\lambda)cs(\mu,\lambda)M_y\sigma_y + i2\rho^2 ns(\mu,\lambda)ds(\mu,\lambda)M_z\sigma_z - i\rho ns(\mu,\lambda) \\ & \times\left(M_y\frac{\partial M_z}{\partial x} - M_z\frac{\partial M_y}{\partial x}\right)\sigma_x - i\rho ds(\mu,\lambda)\left(M_z\frac{\partial M_x}{\partial x} - M_x\frac{\partial M_z}{\partial x}\right)\sigma_y - i\rho cs(\mu,\lambda)\left(M_x\frac{\partial M_y}{\partial x} - M_y\frac{\partial M_x}{\partial x}\right)\sigma_z, \end{aligned} \tag{9}$$

where $\sigma_i (i=x,y,z)$ are the Pauli metrics, $ns(\mu,\lambda), ds(\mu,\lambda)$, and $cs(\mu,\lambda)$ are elliptical functions, while μ and ρ are defined as $\mu = (J_yM - J_xM)^{1/2}/2\rho$, $\rho = 1/2(J_zM - J_xM)^{1/2}$. The coefficients in the Lax pairs include two parameters μ and ρ instead of the three $J_i(i=1,2,3)$, because adding a constant to all the J_i does not change Eq. (3). Since the coefficients are double-periodic functions of the parameter λ, it is sufficient to consider λ inside the rectangle $|\mathrm{Re}\,\lambda| \leqslant 2K$, $|\mathrm{Im}\,\lambda| \leqslant 2K'$, where $K(\mu)$ is a complete elliptic integral of the first kind and $K'(\mu) = K[(1-\mu^2)^{1/2}]$.

For an uniaxial anisotropic ferromagnet in an external magnetic field, the Lax pairs can be written as

$$L(\mu,\lambda) = -i\mu M_x\sigma_x - i\mu M_y\sigma_y - i\lambda M_z\sigma_z,$$

$$\begin{aligned} A(\mu,\lambda) = {} & i2\mu\lambda M_x\sigma_x + i2\mu\lambda M_y\sigma_y + i2\mu^2 M_z\sigma_z \\ & - i\mu\left(M_y\frac{\partial M_z}{\partial x} - M_z\frac{\partial M_y}{\partial x}\right)\sigma_x - i\mu\left(M_z\frac{\partial M_x}{\partial x} \right. \\ & \left. - M_z\frac{\partial M_z}{\partial x}\right)\sigma_y - i\lambda\left(M_z\frac{\partial M_y}{\partial x} - M_z\frac{\partial M_x}{\partial x}\right)\sigma_z, \end{aligned} \tag{10}$$

where the spectral parameters λ and μ satisfy the following relation:

$$\lambda^2 = \begin{cases} \mu^2 + 4\rho^2, & \text{for} \quad \beta_z < 0 \ \text{(an easy plane)}; \\ \mu^2 - 4\rho^2, & \text{for} \quad \beta_z > 0 \ \text{(an easy axis)}, \end{cases} \tag{11}$$

and where ρ is defined as

$$\rho = \begin{cases} \frac{1}{4}[(J_x - J_z)M]^{1/2}, & \text{for} \quad \beta_z < 0 \ \text{(an easy plane)}; \\ \frac{1}{4}[(J_z - J_x)M]^{1/2}, & \text{for} \quad \beta_z > 0 \ \text{(an easy axis)}. \end{cases} \tag{12}$$

If one of two parameters in Eq. (11) is taken as an independent parameter, then another is the double-value function of the first, therefore it is necessary to introduce a Riemann surface. In order to avoid the complexity brought about a Riemann surface, introducing another parameter k called the affine parameter, we will consider $\lambda(k)$ and $\mu(k)$ as a single-valued function of k,

$$\lambda = \begin{cases} \dfrac{2\rho(k^2+1)}{k^2-1}, \\ \dfrac{k^2-\rho^2}{k}, \end{cases} \quad \mu = \begin{cases} \dfrac{4\rho k}{k^2-1}, & \text{for an easy plane}, \\ \dfrac{k^2+\rho^2}{k}, & \text{for an easy axis}. \end{cases} \tag{13}$$

There are two different types of physical boundary conditions in Eq. (3). The boundary condition of the first type corresponds to a breatherlike solution, which is usually called a magnetic soliton. For the classical ferromagnet with two single-ion anisotropies in an external magnetic field, in terms of analysis for integrability of Eq. (7), we will study soliton solutions of possessing asymptotes $\mathbf{M} \to \mathbf{M_0} = (0,0,M_0)$, as $x \to \pm\infty$. The corresponding Jost solutions $\Psi_{0\pm}(x,k)$ of Eq. (8) may be chosen as $\Psi_{0\pm}(x,k) \to E(x,k)$ as $x \to \pm\infty$, where $E(x,k) = \exp[-i\rho cs(k)M_0 x\sigma_z]$, while

$$\Psi_0(x,k) = \exp\left\{-i\rho cs(k)M_0\left[x - \frac{2\rho ns(k)ds(k)}{cs(k)}t\right]\sigma_z\right\}$$

for $\mathrm{Im}\,k = 0, 2K'$. There are two independent solutions $E_1(x,k)$ and $E_2(x,k)$ in $E(x,k)$, with every solution having two components,

$$E_1(x,k) = \begin{pmatrix} E_{11}(x,k) \\ E_{21}(x,k) \end{pmatrix}, \quad E_2(x,k) = \begin{pmatrix} E_{12}(x,k) \\ E_{22}(x,k) \end{pmatrix}.$$

$\Psi_{0+}(x,k)$, $\Psi_{0-}(x,k)$, and $\Psi_0(x,k)$ have also two independent solutions $\Psi_{0+1}(x,k)$ and $\Psi_{0+2}(x,k)$, $\Psi_{0-1}(x,k)$ and $\Psi_{0-2}(x,k)$, $\Psi_{01}(x,k)$ and $\Psi_{02}(x,k)$, respectively.

Under an external magnetic field the magnetization vector $\mathbf{M}$ in the ground state of a ferromagnet with an easy plane deviates from an easy plane, and it is characterized by an inclination θ_0 to the z axis and ϕ_0 to the x axis, where the asymptotic magnetization vector $\mathbf{M}$ lies on the surface of an easy cone. The simplest solution of Eq. (3) can be written as $\mathbf{M}\rightarrow\mathbf{M_0} = (M_0 \sin\theta_0 \cos\phi_0, M_0 \sin\theta_0 \sin\phi_0, M_0 \cos\theta_0)$, as $x\rightarrow\pm\infty$, the corresponding Jost solutions $\Psi^p_{0\pm}(x,k)$ of Eq. (8) may be chosen as $\Psi^p_{0\pm}(x,k)\rightarrow E^p(x,k)$ as $x\rightarrow\pm\infty$, where

$$E^p(x,k)=\exp\left[-i\frac{2\rho k}{k^2-1}M_0 \sin\theta_0 x \cos\phi_0\sigma_x - i\frac{2\rho k}{k^2-1}M_0 \sin\theta_0 x \sin\phi_0\sigma_y - i\frac{\rho(k^2+1)}{k^2-1}M_0 \cos\theta_0 x\sigma_z\right],$$

while

$$\Psi^p_0(x,k)=\exp\left\{-i\frac{2\rho k}{k^2-1}M_0 \sin\theta_0 \cos\phi_0 \times\left[x-\frac{2\rho(k^2+1)}{k^2-1}t\right]\sigma_x - i\frac{2\rho k}{k^2-1}M_0 \sin\theta_0 \sin\phi_0\left[x-\frac{2\rho(k^2+1)}{k^2-1}t\right]\sigma_y - i\frac{\rho(k^2+1)}{k^2-1}M_0 \cos\theta_0\left[x-\frac{8\rho k^2}{k^4-1}t\right]\sigma_z\right\}.$$

When an external magnetic field increases, magnetization will be far from an easy plane, and in the case of $B^z \gg B_c$, magnetization will lie along the z axis. When a magnetic fields vanishes, magnetization will lie on an easy plane and can be written as $\mathbf{M_0}=(M_0\cos\phi_0, M_0\sin\phi_0, 0)$. There are two independent solutions $E^p_1(x,k)$ and $E^p_2(x,k)$ in $E^p(x,k)$, with every solution having two components,

$$E^p_1(x,k)=\begin{pmatrix}E^p_{11}(x,k)\\ E^p_{21}(x,k)\end{pmatrix},$$

$$E^p_2(x,k)=\begin{pmatrix}E^p_{12}(x,k)\\ E^p_{22}(x,k)\end{pmatrix}.$$

The solutions $\Psi^p_{0+}(x,k)$, $\Psi^p_{0-}(x,k)$ and $\Psi^p_0(x,k)$ have also two independent solutions $\Psi^p_{0+1}(x,k)$ and $\Psi^p_{0+2}(x,k)$, $\Psi^p_{0-1}(x,k)$ and $\Psi^p_{0-2}(x,k)$, $\Psi^p_{01}(x,k)$ and $\Psi^p_{02}(x,k)$, respectively.

Since the z axis is an easy axis in a ferromagnet, the boundary condition is chosen as $\mathbf{M}\rightarrow\mathbf{M_0}=(0,0,M_0)$ as $x\rightarrow\pm\infty$, and the corresponding Jost solutions $\Psi^a_{0\pm}(x,k)$ of Eq. (8) may be chosen as $\Psi^a_{0\pm}(x,k)\rightarrow E^a(x,k)$ as $x\rightarrow\pm\infty$, where

$$E^a(x,k)=\exp\left[-i\frac{k^2-\rho^2}{2k}M_0 x\sigma_z\right],$$

while

$$\Psi^a_0(x,k)=\exp\left\{-i\frac{k^2-\rho^2}{2k}M_0\left[x-\frac{(k^2+\rho^2)^2}{k(k^2-\rho^2)}t\right]\sigma_z\right\}.$$

Similarly, $E^a(x,k)$ also has two independent solutions $E^a_1(x,k)$ and $E^a_2(x,k)$, with every solution having two components,

$$E^a_1(x,k)=\begin{pmatrix}E^a_{11}(x,k)\\ E^a_{21}(x,k)\end{pmatrix},\quad E^a_2(x,k)=\begin{pmatrix}E^a_{12}(x,k)\\ E^a_{22}(x,k)\end{pmatrix}.$$

$\Psi^a_{0+}(x,k)$, $\Psi^a_{0-}(x,k)$ and $\Psi^a_0(x,k)$ have also two independent solutions $\Psi^a_{0+1}(x,k)$ and $\Psi^a_{0+2}(x,k)$, $\Psi^a_{-1}(x,k)$ and $\Psi^a_{0-2}(x,k)$, $\Psi^a_{01}(x,k)$ and $\Psi^a_{02}(x,k)$, respectively.

By means of the standard procedures of characteristic theory, we can obtain the following integral representation:

$$\Psi_+(x,k)=E(x,k)+\lambda\int_x^\infty dyK_+(x,y)E(y,k),$$
$$\Psi_-(x,k)=E(x,k)+\lambda\int_{-\infty}^x dyK_-(x,y)E(y,k), \tag{14}$$

where the kernels $K_+(x,y)$ and $K_-(x,y)$ depend functionally on magnetization $M(x)$ but are independent of the eigenvalue λ, and $K_\pm(x,\pm\infty)=0$.

For a ferromagnet with an easy plane in an external magnetic field, we can also obtain

$$\Psi^p_+(x,k)=E^p(x,k)+\frac{\rho(k^2+1)}{k^2-1}\int_x^\infty dyK^{p,d}_+(x,y)E^p(y,k) + \frac{2\rho k}{k^2-1}\int_x^\infty dyK^{p,nd}_+(x,y)E^p(y,k),$$

$$\Psi^p_-(x,k)=E^p(x,k)+\frac{\rho(k^2+1)}{k^2-1}\int_{-\infty}^x dyK^{p,d}_-(x,y)E^p(y,k) + \frac{2\rho k}{k^2-1}\int_{-\infty}^x dyK^{p,nd}_-(x,y)E^p(y,k), \tag{15}$$

where $K^p_\pm(x,\pm\infty)=0$, the superscripts d and nd denote the diagonal and nondiagonal parts of the matrix, respectively. While for a ferromagnet with an easy axis in an external magnetic field,

$$\Psi^a_+(x,k)=E^a(x,k)+\frac{k^2-\rho^2}{2k}\int_x^\infty dyK^{a,d}_+(x,y)E^a(y,k) + \frac{k^2+\rho^2}{2k}\int_x^\infty dyK^{a,nd}_+(x,y)E^a(y,k),$$

$$\Psi^a_-(x,k)=E^a(x,k)+\frac{k^2-\rho^2}{2k}\int_{-\infty}^x dyK^{a,d}_-(x,y)E^a(y,k) + \frac{k^2+\rho^2}{2k}\int_{-\infty}^x dyK^{a,nd}_-(x,y)E^a(y,k), \tag{16}$$

where $K^a_\pm(x,\pm\infty)=0$.

III. SOLITONS

By means of the results obtained in the previous section, we will investigate soliton solutions. The reconstruction of magnetization, i.e., the ''potential'' $M(x,t)$, from the time-dependent scattering data is called the ''inverse scattering problem'' and is achieved by means of a linear integral equation, the Gel'fand-Levitan-Marchenko equation. It is well known that the pure soliton solutions correspond to the reflectionless case. In the reflectionless case, the reflectional coefficient $r(k,t)=0$, the Gel'fand-Levitan-Marchenko equation can be written as

$$K_{11}(x,t)+K_{12}(x,t)N''(x,t)=0,$$
$$K_{12}(x,t)-\overline{G(x,t)}-K_{11}(x,t)N'(x,t)=0, \tag{17}$$

where $K_{11}(x,t)$ and $K_{12}(x,t)$ can be expressed by

$$K_{11}(x,t)=i\left\{\frac{\det[I+N''(x,t)M'(x,t)]}{\det[I+N''(x,t)N'(x,t)]}-1\right\},$$
$$K_{12}(x,t)=\frac{\det[I+N''(x,t)N'(x,t)+\overline{H(x)}^{T}\overline{G(x,t)}]}{\det[I+N''(x,t)N'(x,t)]}-1, \tag{18}$$

where $M'(x,t)=N'(x,t)+iH(x)^{T}\overline{G(x,t)}$, while $N'(x,t)$ and $N''(x,t)$ are $N\times N$ matrices.

In order to obtain $K_{11}(x,t)$ and $K_{12}(x,t)$, we will calculate $\det[I+N''(x,t)N'(x,t)]$, $\det[I+N''(x,t)M'(x,t)]$ and $\det[I+N''(x,t)M'(x,t)+\overline{H(x)}^{T}\overline{G(x,t)}]$ by the Binet-Cauchy formula, respectively.

Setting

$$\Gamma_0=\det(I+N''N'), \tag{19}$$

and using the Binet-Cauchy formula, we can obtain

$$\Gamma_0=1+\sum_{r=1}\sum_{1\leq n_1<n_2<\cdots<n_r\leq N}\sum_{1\leq m_1<m_2<\cdots<m_r\leq N}\times\gamma_0(n_1,n_2,\ldots,n_r;m_1,m_2,\ldots,m_r). \tag{20}$$

For a ferromagnet with two single-ion anisotropies in an external magnetic field,

$$\gamma_0(n_1,n_2,\ldots,n_r;m_1,m_2,\ldots,m_r)=(-1)^r\prod_{n}\prod_{m}\prod_{n<n'}\prod_{m<m'}\overline{f_n}f_m\overline{\alpha_n}\alpha_m\frac{\rho^2[\overline{cs(k_n)}-\overline{cs(k_{n'})}]^2[cs(k_m)-cs(k_{m'})]^2}{[\overline{cs(k_m)}-cs(k_m)]^2}, \tag{21}$$

where

$$\alpha_n=\prod_{m\neq n}\frac{cs(k_n)-\overline{cs(k_m)}}{\rho[cs(k_m)-\overline{cs(k_m)}][cs(k_n)-\overline{cs(k_n)}]},$$

$$f_n=\prod_{l=1}^{N}\frac{\overline{cs(k_l)}}{cs(k_l)}b_nH_n^2.$$

For an uniaxial anisotropic ferromagnet in an external magnetic field, we can find

$$\gamma_0(n_1,n_2,\ldots,n_r;m_1,m_2,\ldots,m_r)$$
$$=\begin{cases}(-1)^r\prod_{n}\prod_{m}\prod_{n<n'}\prod_{m<m'}\overline{f_n}f_m\overline{\alpha_n}\alpha_m\dfrac{4\rho^2k_m(\overline{k_n}^4-1)(\overline{k_m}^2-1)(\overline{k_{n'}}^2-\overline{k_n}^2)^2(k_{m'}^2-k_n^2)^2}{\overline{k_m}(k_m^2+1)(\overline{k_n}^2-1)^2(\overline{k_{n'}}^2-1)^2(k_{m'}^2-1)^2(k_m^2-\overline{k_n}^2)}, \\ \text{for an easy plane;} \\ (-1)^r\prod_{n}\prod_{m}\prod_{n<n'}\prod_{m<m'}\overline{f_n}f_m\overline{\alpha_n}\alpha_m\dfrac{4\rho^2\overline{k_n}(k_m^2+1)[\overline{k_n}(\overline{k_{n'}}^2-1)-\overline{k_{n'}}^2(\overline{k_n}^2-1)]^2[k_m(k_{m'}^2-1)-k_{m'}(k_m^2-1)]^2}{k_m(\overline{k_n}^2+1)[\overline{k_n}(k_m^2-1)-k_m(\overline{k_n}^2-1)]^2}, \\ \text{for an easy axis.}\end{cases} \tag{22}$$

where

$$\alpha_n=\begin{cases}\prod_{m\neq n}\frac{(k_n^2-1)(\overline{k_n}^2-1)(\overline{k_m}^2-1)(k_m^2-k_n^2)}{2\rho(k_m^2-1)(\overline{k_n}^2-k_n^2)(\overline{k_m}^2-k_n^2)};\\ \prod_{m\neq n}\frac{(k_n^2-1)(\overline{k_n}^2-1)(\overline{k_m}^2-1)[k_n(k_m^2-1)-k_m(k_n^2-1)]}{4\rho(k_m^2-1)[k_n(\overline{k_m}^2-1)-\overline{k_m}^2(k_n^2-1)][k_n(\overline{k_n}^2-1)-\overline{k_n}(k_n^2-1)]}.\end{cases}$$

$$f_n=\begin{cases}\prod_{l=1}^{N}\frac{(k_l^2+1)(\overline{k_l}^2+1)}{(k_l^2-1)(\overline{k_l}^2-1)}b_nH_n^2, \text{ for an easy plane;}\\ \prod_{l=1}^{N}\frac{\overline{k_l}(k_l^2-1)}{k_l(\overline{k_l}^2-1)}b_nH_n^2, \text{ for an easy axis.}\end{cases} \tag{23}$$

Setting

$$\Gamma_1=\det(I+N''M'), \tag{24}$$

then Γ_1 can be written as

$$\Gamma_1=1+\sum_{r=1}\sum_{1\leqslant n_1<n_2<\cdots<n_r\leqslant N}\sum_{1\leqslant m_1<m_2<\cdots<m_r\leqslant N}\gamma_1(n_1,n_2,\ldots,n_r;m_1,m_2,\ldots,m_r). \tag{25}$$

For the classical ferromagnet with two single-ion anisotropies in an external magnetic field,

$$\gamma_1(n_1,n_2,\cdots,n_r;m_1,m_2,\cdots,m_r)=(-1)^r\prod_n\prod_m\prod_{n<n'}\prod_{m<m'}\overline{f_n}f_m\overline{\alpha_n}\alpha_m\frac{\rho^2cs(k_m)[\overline{cs(k_n)}-\overline{cs(k_{n'})}]^2[cs(k_m)-cs(k_{m'})]^2}{\overline{cs(k_n)}[\overline{cs(k_n)}-cs(k_m)]^2}. \tag{26}$$

For an uniaxial anisotropic ferromagnet in an external magnetic field,

$$\gamma_1(n_1,n_2,\cdots,n_r;m_1,m_2,\cdots,m_r)$$

$$=\begin{cases}(-1)^r\prod_n\prod_m\prod_{n<n'}\prod_{m<m'}\overline{f_n}f_m\overline{\alpha_n}\alpha_m\frac{\rho^2k_m(\overline{k_n}^2-1)(\overline{k_{n'}}^2-\overline{k_n}^2)^2(k_{m'}^2-k_m^2)^2}{\overline{k_n}(k_m^2-1)(\overline{k_{n'}}^2-1)(k_{m'}^2-1)(k_m^2-\overline{k_n}^2)^2},\\ \text{for an easy plane;}\\ (-1)^r\prod_n\prod_m\prod_{n<n'}\prod_{m<m'}\overline{f_n}f_m\overline{\alpha_n}\alpha_m\frac{16\rho^2(k_m^2+1)(\overline{k_n}^2-1)[\overline{k_n}(\overline{k_{n'}}^2-1)-\overline{k_{n'}}(\overline{k_n}^2-1)]^2[k_m(k_{m'}^2-1)-k_{m'}(k_m^2-1)]^2}{(k_m^2-1)(\overline{k_n}^2+1)[\overline{k_n}(k_m^2-1)-k_m(\overline{k_n}^2-1)]^2},\\ \text{for an easy axis.}\end{cases} \tag{27}$$

Third, in order to obtain $\det[I+N''(x,t)N'(x,t)+\overline{H(x)}^T\overline{G(x,t)}]$ in Eq. (18), we will introduce a $N\times(N+1)$ matrix Q'' and a $(N+1)\times N$ matrix Q', $Q''_{nm}=N'_{nm}$, $Q''_{n0}=-i\overline{H_n}$, $Q'_{nm}=N'_{nm}$, $Q''_{n0}=i\overline{G_n}$, $n,m=1,2,\ldots,N$, then $\det(I+Q''Q')$ can be written as

$$\det(I+Q''Q')=1+\sum_{r=1}\sum_{1\leqslant n_1<n_2<\cdots<n_r\leqslant N}\sum_{0\leqslant m_1<m_2<\cdots<m_r\leqslant N}\times Q''(n_1,n_2,\ldots,n_r;m_1,m_2,\ldots,m_r)Q'(m_1,m_2,\ldots,m_r;n_1,n_2,\ldots,n_r), \tag{28}$$

where the sum is decomposed into two parts: one is extended to $m_1=0$, the other to $m_1\geqslant 1$. Except for the same extended to $m_1=0$, Eq. (28) is just Eq. (19), therefore,

$$\det(I+Q''Q')-\det(I+N''N')=\sum_{r=1}\sum_{1\leqslant n_1<n_2<\cdots<n_r\leqslant N}\sum_{1\leqslant m_1<m_2<\cdots<m_r\leqslant N}\times Q''(n_1,n_2,\ldots,n_r;0,m_2,\ldots,m_r)Q'(0,m_2,\ldots,m_r;n_1,n_2,\ldots,n_r). \tag{29}$$

Setting

$$\Gamma_2=\det(I+Q''Q')-\det(I+N''N'), \tag{30}$$

we can obtain

$$\Gamma_2 = \sum_{r=1} \sum_{1 \leq n_1 < n_2 < \cdots < n_r \leq N} \sum_{1 \leq m_1 < m_2 < \cdots < m_r \leq N} \gamma_2(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r). \tag{31}$$

For the classical ferromagnet with two single-ion anisotropies in an external magnetic field,

$$\gamma_2(n_1,n_2,\ldots,n_r;0,m_2,\ldots,m_r) = (-1)^{r+1} \prod_n \prod_m \prod_{n<n'} \prod_{m<m'} \overline{f_n} f_m \overline{\alpha_n} \alpha_m \frac{\rho^2 cs(k_m)[\overline{cs(k_n)} - \overline{cs(k_{n'})}]^2 [cs(k_m) - cs(k_{m'})]^2}{\overline{cs(k_n)}[\overline{cs(k_n)} - cs(k_m)]^2}. \tag{32}$$

For an uniaxial anisotropic ferromagnet in an external magnetic field,

$$\gamma_2(n_1,n_2,\ldots,n_r;0,m_2,\ldots,m_r) = \begin{cases} (-1)^{r+1} \prod_n \prod_m \prod_{n<n'} \prod_{m<m'} \overline{f_n} f_m \overline{\alpha_n} \alpha_m \dfrac{\rho^2 k_m (\overline{k_n}^2 - 1)^2 (\overline{k_{n'}}^2 - \overline{k_n}^2)^2 (k_{m'}^2 - k_m^2)^2}{\overline{k_n}(k_m^2 - 1)(k_{n'}^2 - 1)^2 (k_{m'}^2 - 1)^2 (k_m^2 - \overline{k_n}^2)^2}, & \text{for an easy plane,} \\ (-1)^{r+1} \prod_n \prod_m \prod_{n<n'} \prod_{m<m'} \overline{f_n} f_m \overline{\alpha_n} \alpha_m \\ \times \dfrac{16\rho^2 (k_m^2 + 1)(\overline{k_n}^2 - 1)[\overline{k_n}(\overline{k_{n'}}^2 - 1) - \overline{k_{n'}}(\overline{k_n}^2 - 1)]^2 [k_m(k_{m'}^2 - 1) - k_{m'}(k_m^2 - 1)]^2}{(k_m^2 - 1)(\overline{k_n}^2 + 1)[\overline{k_n}(k_m^2 - 1) - k_m(\overline{k_n}^2 - 1)]^2}, & \text{for an easy axis,} \end{cases} \tag{33}$$

where f_n can also be written as $f_n = \exp(-\Phi_{1n} + i\Phi_{2n})$.

Substituting Eqs. (19), (24), and (30) into Eq. (18), we can obtain K_{11} and K_{12}. Using the following relations:

$$\mathbf{M(x,t)} = [iK(x,x,t) - \sigma_z]\sigma_z[iK(x,x,t) - \sigma_z]^{-1}, \tag{34}$$

we can obtain the multisoliton solutions in the classical ferromagnet with two single-ion anisotropies in an external magnetic field,

$$(\mathbf{M}_n)_x = \mathrm{Re}\left(\frac{2\Gamma_1\Gamma_2}{|\Gamma_1|^2 + |\Gamma_2|^2}\right),$$
$$(\mathbf{M}_n)_y = \mathrm{Im}\left(\frac{2\Gamma_1\Gamma_2}{|\Gamma_1|^2 + |\Gamma_2|^2}\right), \tag{35}$$
$$(\mathbf{M}_n)_z = M_0 - \frac{|\Gamma_1|^2 - |\Gamma_2|^2}{|\Gamma_1|^2 + |\Gamma_2|^2}.$$

For an uniaxial anisotropic ferromagnet in an external magnetic field, the multisoliton solutions can be written as

$$(\mathbf{M}_n)_x = \begin{cases} M_0 \sin\theta_0 \cos\phi_0 - \mathrm{Re}\left(\dfrac{2\Gamma_1\Gamma_2}{|\Gamma_1|^2 + |\Gamma_2|^2}\right), \\ \mathrm{Re}\left(\dfrac{2\Gamma_1\Gamma_2}{|\Gamma_1|^2 + |\Gamma_2|^2}\right), \end{cases}$$

$$(\mathbf{M}_n)_y = \begin{cases} M_0 \sin\theta_0 \sin\phi_0 - \mathrm{Im}\left(\dfrac{2\Gamma_1\Gamma_2}{|\Gamma_1|^2 + |\Gamma_2|^2}\right), \\ \mathrm{Im}\left(\dfrac{2\Gamma_1\Gamma_2}{|\Gamma_1|^2 + |\Gamma_2|^2}\right), \end{cases} \tag{36}$$

$$(\mathbf{M}_n)_z = \begin{cases} M_0 \cos\theta_0 - \dfrac{|\Gamma_1|^2 - |\Gamma_2|^2}{|\Gamma_1|^2 + |\Gamma_2|^2}, & \text{for an easy plane,} \\ M_0 - \dfrac{|\Gamma_1|^2 - |\Gamma_2|^2}{|\Gamma_1|^2 + |\Gamma_2|^2}, & \text{for an easy axis.} \end{cases}$$

Then taking the z axis as the polar axis in the polar coordinates, we can obtain the multisoliton solutions of the classical ferromagnet with two single-ion anisotropies in an external magnetic field,

$$\cos\theta = \frac{2|\Gamma_2|^2}{|\Gamma_1|^2 + |\Gamma_2|^2},$$
$$\varphi = -\arg\Gamma_2 - \arg\Gamma_1. \tag{37}$$

For an uniaxial anisotropic ferromagnet in an external magnetic field, the multisoliton solutions can be written as

$$\cos\theta = \begin{cases} \cos\theta_0 - \dfrac{2|\Gamma_2|^2}{|\Gamma_1|^2 + |\Gamma_2|^2}, \\ 1 - \dfrac{2|\Gamma_2|^2}{|\Gamma_1|^2 + |\Gamma_2|^2}, \end{cases}$$

$$\varphi = \begin{cases} -\arg\Gamma_2 - \arg\Gamma_1, & \text{for an easy plane,} \\ -\arg\Gamma_2 - \arg\Gamma_1, & \text{for an easy axis,} \end{cases} \tag{38}$$

where Γ_1 and Γ_2 are expressed by Eqs. (25) and (31), respectively.

When $n = 1$, the single-soliton solutions of the classical ferromagnet with two single-ion anisotropies in an external magnetic field can be written as

$$(\mathbf{M}_1)_x=\frac{ns(k_1)''^2\sinh\Phi_1\sin\Phi_2+ns(k_1)'^2\cosh\Phi_1\cos\Phi_2}{4ns(k_1)'^2\cosh^2\Phi_1+4ds(k_1)''^2\sinh^2\Phi_1+cs(k_1)''^2},$$

$$(\mathbf{M}_1)_y=\frac{ds(k_1)''^2\sinh\Phi_1\cos\Phi_2-ds(k_1)'^2\cosh\Phi_1\sin\Phi_2}{4ns(k_1)'^2\cosh^2\Phi_1+4ds(k_1)''^2\sinh^2\Phi_1+cs(k_1)''^2}, \tag{39}$$

$$(\mathbf{M}_1)_z=M_0-\frac{2cs(k_1)''^2}{4ns(k_1)'^2\cosh^2\Phi_1+4ds(k_1)''^2\sinh^2\Phi_1+cs(k_1)''^2},$$

where

$$\Phi_1=2\rho cs(k_1)''(x-V_1t-x_{10}),\quad \Phi_2=2\rho cs(k_1)'(x-V_2t-x_{20}), \tag{40}$$

and

$$V_1=4cs(k_1)',\quad V_2=\frac{2(cs(k_1)'^2-cs(k_1)''^2+4\rho^2)}{\rho cs(k_1)'}. \tag{41}$$

The single-soliton solutions of a uniaxial anisotropic ferromagnet in an external magnetic field can be written as

$$(\mathbf{M}_1)_x=M_0\sin\theta_0\cos\phi_0-\frac{2k_1''^2[4k_1'^2+|k_1^2-1|^2\sin^2\Phi_2]}{|k_1^2-1|^2[k_1'^2\cosh^2\Phi_1+k_1''^2\sin^2\Phi_2]},$$

$$(\mathbf{M}_1)_y=M_0\sin\theta_0\sin\phi_0-\frac{2k_1'k_1''[4k_1'k_1''\sinh\Phi_1\cos\Phi_2+(|k_1|^4-1)\cosh\Phi_1\sin\Phi_2]}{|k_1^2-1|^2[k_1'^2\cosh^2\Phi_1+k_1''^2\sin^2\Phi_2]},$$

$$(\mathbf{M}_1)_z=M_0\cos\theta_0-\frac{4k_1'k_1''[k_1''(|k_1|^2+1)\sinh\Phi_1\sin\Phi_2-k_1'(|k_1|^2-1)\cosh\Phi_1\cos\Phi_2]}{|k_1^2-1|^2[k_1'^2\cosh^2\Phi_1+k_1''^2\sin^2\Phi_2]},\quad \text{for an easy plane,} \tag{42}$$

and

$$(\mathbf{M}_1)_x=\frac{16k_1'^2k_1''^2\sinh\Phi_1\sin\Phi_2+(|k_1|^4-1)^2\cosh\Phi_1\cos\Phi_2}{(|k_1|^4-1)^2\cosh^2\Phi_1+16k_1'^2k_1''^2\sinh^2\Phi_1+4k_1''^2(|k_1|^2+1)^2},$$

$$(\mathbf{M}_1)_y=\frac{16k_1'^2k_1''^2\sinh\Phi_1\cos\Phi_2-(|k_1|^4-1)^2\cosh\Phi_1\sin\Phi_2}{(|k_1|^4-1)^2\cosh^2\Phi_1+16k_1'^2k_1''^2\sinh^2\Phi_1+4k_1''^2(|k_1|^2+1)^2},$$

$$(\mathbf{M}_1)_z=M_0-\frac{2k_1''^2(|k_1|^2+1)^2}{(|k_1|^4-1)^2\cosh^2\Phi_1+16k_1'^2k_1''^2\sinh^2\Phi_1+4k_1''^2(|k_1|^2+1)^2},\quad \text{for an easy axis,} \tag{43}$$

where

$$\Phi_1=\begin{cases}\dfrac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}(x-V_1t-x_{10}),\\[2ex] \dfrac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}(x-V_1t-x_{10}),\end{cases}$$

$$\Phi_2=\begin{cases}\dfrac{8\rho k_1'(|k_1|^2-1)}{|k_1^2-1|^2}(x-V_2t-x_{20}), & \text{for an easy plane,}\\[2ex] \dfrac{8\rho k_1'(|k_1|^2-1)}{|k_1^2-1|^2}(x-V_2t-x_{20}), & \text{for an easy axis,}\end{cases} \tag{44}$$

and

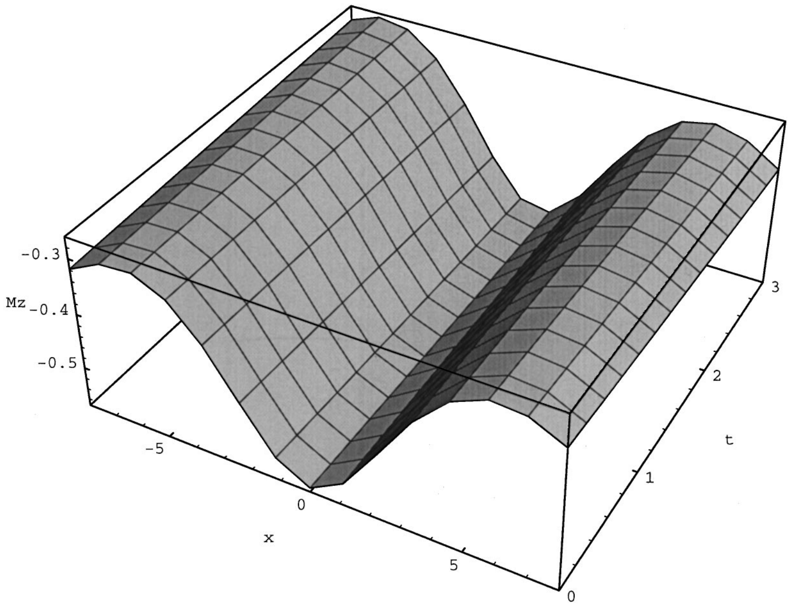

FIG. 1. Some graphical illustrations of the motion of the center and the change of shape of the z component of the nonlinear magnetization $(\mathbf{M}_1)_z$ expressed by Eq. (42) in a ferromagnet with an easy plane, where $\theta_0=30^0$, $\rho=0.1$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, and $x_{20}=0$.

$$V_1=\begin{cases}\dfrac{2\rho[(|k_1|^4-1)+4k_1'^2(|k_1|^2-1)]}{|k_1^2-1|^2}, \\ \dfrac{16\rho k_1'(|k_1|^2-1)}{|k_1^2-1|^2},\end{cases}$$

$$V_2=\begin{cases}\dfrac{2\rho[(|k_1|^2-1)^2-4k_1''^2(|k_1|^2+1)]}{(|k_1|^2-1)|k_1^2-1|^2}, & \text{for an easy plane,} \\ \dfrac{2\rho[4k_1'^2(|k_1|^2-1)^2-4k_1''^2(|k_1|^2+1)^2+|k_1-1|^4]}{k_1'(|k_1|^2-1)|k_1^2-1|^2}, & \text{for an easy axis.}\end{cases} \tag{45}$$

These results show that under the action of an external magnetic field, the nonlinear magnetization of the classical ferromagnet with an anisotropy depends essentially on two parameters, namely, two velocities V_1 and V_2 in Eqs. (41) and (45); the center of nonlinear magnetization moves with a constant velocity V_1, while its shape also changes with another velocity V_2. Figures 1–4 give some graphical illustrations of the motion of the center and the change of shape of the z component of nonlinear magnetization $(\mathbf{M}_1)_z$, expressed by Eq. (42) in a ferromagnet with an easy plane and by Eq. (43) in a ferromagnet with an easy

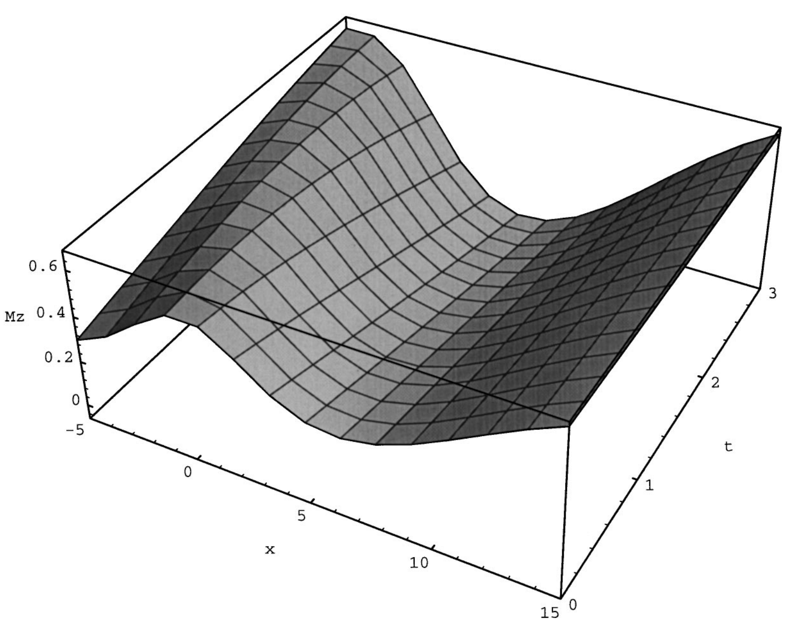

FIG. 2. Some graphical illustrations of the motion of the center and the change of shape of the z component of the nonlinear magnetization $(\mathbf{M}_1)_z$ expressed by Eq. (43) in a ferromagnet with an easy axis, where $\rho=0.1$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, and $x_{20}=0$.

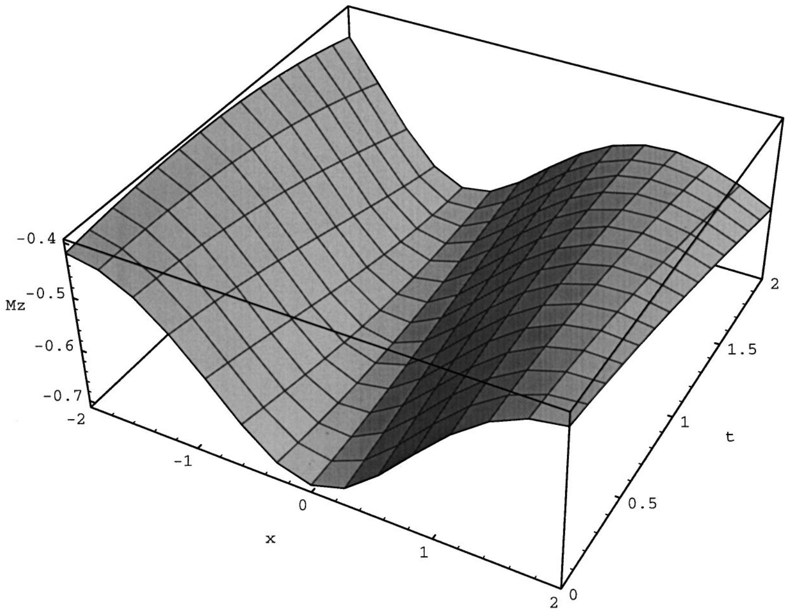

FIG. 3. Some graphical illustrations of the motion of the center and the change of shape of the z component of the nonlinear magnetization $(\mathbf{M}_1)_z$ expressed by Eq. (42) in a ferromagnet with an easy plane, where $\theta_0=30^0$, $\rho=0.3$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, and $x_{20}=0$.

axis, as an anisotropic parameter. Also, there is an external magnetic field increase from $\rho=0.1$ in Figs. 1 and 2 to $\rho=0.3$ in Figs. 3 and 4, where $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, $x_{20}=0$, $\theta_0=30^0$.

If we take the z axis as the polar axis in the polar coordinates, the single-soliton solutions of the classical ferromagnet with two single-ion anisotropies in an external magnetic field can be written as

$$\cos\theta=1-\frac{2cs(k_1)''^2}{4ns(k_1)'^2\cosh^2\Phi_1+4ds(k_1)''^2\sinh^2\Phi_1+cs(k_1)''^2},$$

$$\tan\varphi=\frac{ds(k_1)''^2\sinh\Phi_1\cos\Phi_2-ds(k_1)'^2\cosh\Phi_1\sin\Phi_2}{ns(k_1)''^2\sinh\Phi_1\sin\Phi_2+ns(k_1)'^2\cosh\Phi_1\cos\Phi_2}. \tag{46}$$

The single-soliton solutions of an uniaxial anisotropic ferromagnet in an external magnetic field can be written as

$$\cos\theta=\cos\theta_0-\frac{2[k_1''(|k_1|^2+1)\sinh\Phi_1\sin\Phi_2+k_1'(|k_1|^2-1)\cosh\Phi_1\cos\Phi_2]}{|k_1^2-1|^2[k_1'^2\cosh^2\Phi_1+k_1''^2\sin^2\Phi_2]},$$

$$\tan\varphi=\frac{\sin\theta_0\sin\phi_0-8k_1'k_1''\sinh\Phi_1\cos\Phi_2-2(|k_1|^4-1)\cosh\Phi_1\sin\Phi_2}{\sin\theta_0\cos\phi_0-2k_1''^2[4k_1'^2+|k_1^2-1|^2\sin^2\Phi_2]},\quad \text{for an easy plane,} \tag{47}$$

and

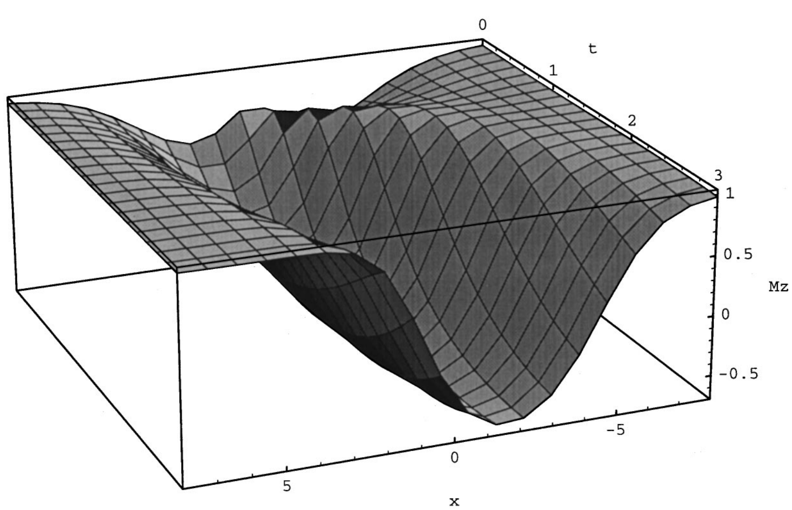

FIG. 4. Some graphical illustrations of the motion of the center and the change of shape of the z component of the nonlinear magnetization $(\mathbf{M}_1)_z$ expressed by Eq. (43) in a ferromagnet with an easy axis, where $\rho=0.3$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, and $x_{20}=0$.

$$\cos\theta=1-\frac{2k_1''^2(|k_1|^2+1)^2}{(|k_1|^4-1)^2\cosh^2\Phi_1+16k_1'^2k_1''^2\sinh^2\Phi_1+4k_1''^2(|k_1|^2+1)^2},$$

$$\tan\varphi=\frac{16k_1'^2k_1''^2\sinh\Phi_1\cos\Phi_2-(|k_1|^4-1)^2\cosh\Phi_1\sin\Phi_2}{16k_1'^2k_1''^2\sinh\Phi_1\sin\Phi_2+(|k_1|^4-1)^2\cosh\Phi_1\cos\Phi_2},\quad \text{for an easy axis.} \tag{48}$$

We can find the following property:

$$\cos(-x,-t)=\cos(x,t). \tag{49}$$

It means that under the action of an external magnetic field the z component of nonlinear magnetization is a symmetric function of space and time, while the orientation of the nonlinear magnetization in the plane orthogonal to the anisotropic axis changes with an external field, and it will be constant when an external field vanishes.

In order to analyze the feature of the previous soliton solutions, setting the preliminary values as zero in the moving coordinates of the soliton, for the classical ferromagnet with two single-ion anisotropies in an external magnetic field, we can obtain

$$\cos\theta=1-\frac{2cs(k_1)''^2}{4ns(k_1)'^2\cosh^2[2\rho cs(k_1)''x]+4ds(k_1)''^2\sinh^2[2\rho cs(k_1)''x]+cs(k_1)''^2},$$

$$\tan\varphi=\frac{ds(k_1)''^2\sinh[2\rho cs(k_1)''x]\cos[2\rho cs(k_1)'(x-V_2t)]-ds(k_1)'^2\cosh[2\rho cs(k_1)''x]\sin[2\rho cs(k_1)'(x-V_2t)]}{ns(k_1)''^2\sinh[2\rho cs(k_1)''x]\sin[2\rho cs(k_1)'(x-V_2t)]+ns(k_1)'^2\cosh[2\rho cs(k_1)''x]\cos[2\rho cs(k_1)'(x-V_2t)]}, \tag{50}$$

and

$$\cos\theta=\cos\theta_0-\frac{\frac{2k_1''(|k_1|^2+1)}{|k_1|^4-1}\sinh\left[\frac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}x\right]\tan\left[\frac{8\rho k_1'(|k_1|^2-1)}{|k_1^2-1|^2}(x-V_2t)\right]+\frac{2k_1'(|k_1|^2-1)}{|k_1|^4-1}\cosh\left[\frac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}x\right]}{|k_1^2-1|^2\left\{k_1'^2\cosh^2\left[\frac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}x\right]+k_1''^2\sin^2\left[\frac{8\rho k_1'(|k_1|^2-1)}{|k_1^2-1|^2}(x-V_2t)\right]\right\}},$$

$$\tan\varphi=\frac{\sin\theta_0\sin\phi_0-\frac{4k_1'k_1''}{|k_1|^4-1}\sinh\left[\frac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}x\right]-\cosh\left[\frac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}x\right]\tan\left[\frac{8\rho k_1'(|k_1|^2-1)}{|k_1^2-1|^2}(x-V_2t)\right]}{\sin\theta_0\cos\phi_0-2k_1''^2\left\{4k_1'^2+\left|k_1^2-1\right|^2\sin^2\left[\frac{8\rho k_1'(|k_1|^2-1)}{|k_1^2-1|^2}(x-V_2t)\right]\right\}},$$

$$\text{for an easy plane,} \tag{51}$$

and

$$\cos\theta=1-\frac{2\frac{4k_1''^2(|k_1|^2+1)^2}{(|k_1|^4-1)^2}}{\cosh^2\left[\frac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}x\right]+\frac{16k_1'^2k_1''^2}{(|k_1|^4-1)^2}\sinh^2\left[\frac{8\rho k_1'(|k_1|^2-1)}{|k_1^2-1|^2}x\right]+\frac{4k_1''^2(|k_1|^2+1)^2}{(|k_1|^4-1)^2}},$$

$$\tan\varphi=\frac{16k_1'^2k_1''^2\sinh\left[\frac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}x\right]-(|k_1|^4-1)^2\cosh\left[\frac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}x\right]\tan\left[\frac{8\rho k_1'(|k_1|^2-1)}{|k_1^2-1|^2}(x-V_2t)\right]}{16k_1'^2k_1''^2\sinh\left[\frac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}x\right]\tan\left[\frac{8\rho k_1'(|k_1|^2-1)}{|k_1^2-1|^2}(x-V_2t)\right]+(|k_1|^4-1)^2\cosh\left[\frac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}x\right]},$$

$$\text{for an easy axis.} \tag{52}$$

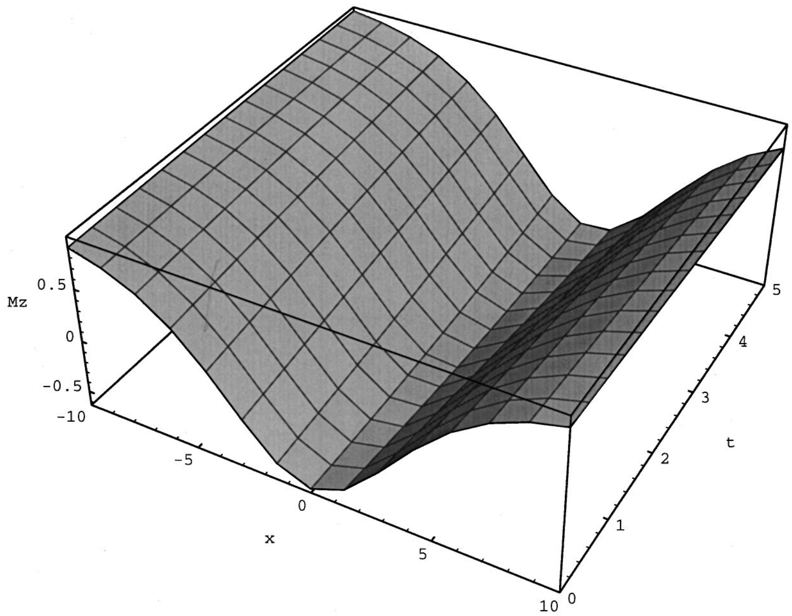

FIG. 5. Some graphical illustrations of the change of amplitude and width of the z component of the nonlinear magnetization $(\mathbf{M}_1)_z$ expressed by Eq. (51) in a ferromagnet with an easy plane, where $\theta_0=30^0$, $\rho=0.2$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, and $x_{20}=0$.

We can also find that under the action of an external magnetic field the amplitudes and widths of the nonlinear magnetization are not constants but vary periodically with time. According to Eqs. (51) and (52), Fig. 5 shows that the amplitude and shape of the z component of the nonlinear magnetization $(\mathbf{M}_1)_z$ in a ferromagnet with an easy plane also changes with a velocity V_2 and it is not symmetrical with respect to the center. Its shape in a ferromagnet with an easy axis is symmetrical with respect to the center by means of Fig. 6, where $\rho=0.2$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\theta_0=30^0$.

Obviously, when an anisotropic parameter $\rho\rightarrow 0$, these soliton solutions in an uniaxial anisotropic ferromagnet reduce to those in an isotropic ferromagnet, for example, the single-soliton solutions (42) and (43) are transformed to

$$(\mathbf{M}_1)_x=\frac{2k_1''}{|k_1|^2}\operatorname{sech}^2[k_1''(x-4k_1't-x_{10})]\left\{k_1''\sinh[k_1''(x-4k_1't-x_{10})]\sin\left[k_1'\left(x-2\left(k_1'-\frac{k_1''^2}{k_1'}\right)t-x_{20}\right)\right]\right.$$
$$\left.+k_1'\cosh[k_1''(x-4k_1't-x_{10})]\cos\left[k_1'\left(x-2\left(k_1'-\frac{k_1''^2}{k_1'}\right)t-x_{20}\right)\right]\right\},$$

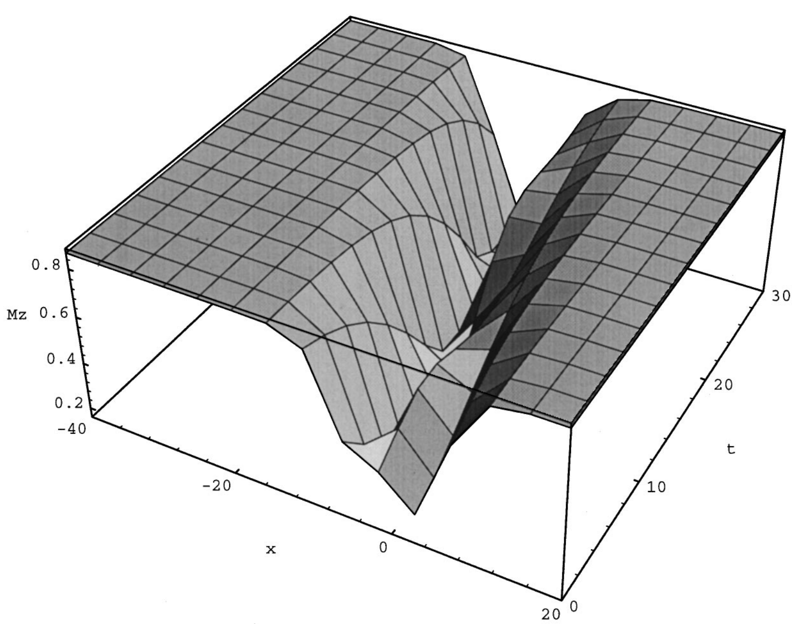

FIG. 6. Some graphical illustrations of the change of amplitude and width of the z component of the nonlinear magnetization $(\mathbf{M}_1)_z$ expressed by Eq. (52) in a ferromagnet with an easy axis, where $\rho=0.2$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, and $x_{20}=0$.

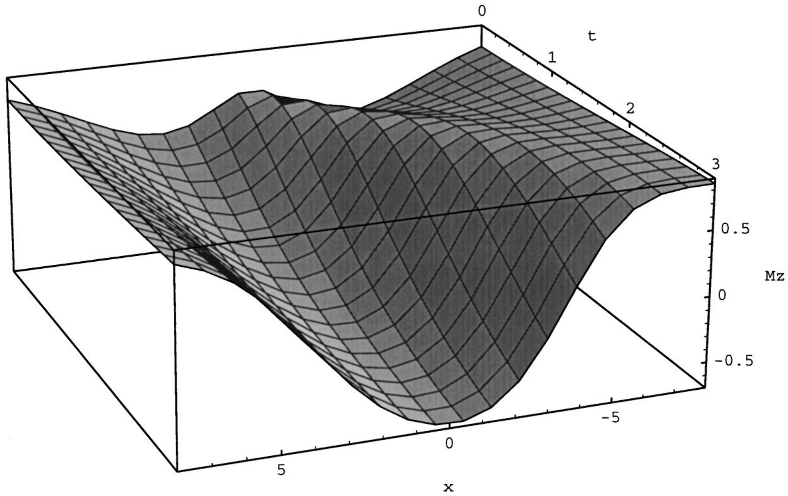

FIG. 7. Some graphical illustrations of the motion of the center and the change of shape of the z component of the nonlinear magnetization $(\mathbf{M}_1)_z$ expressed by Eq. (53) in an isotropic ferromagnet, where $\rho=0$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, and $x_{20}=0$.

$$(\mathbf{M}_1)_y=\frac{2k_1''}{|k_1|^2}\operatorname{sech}^2[k_1''(x-4k_1't-x_{10})]\left\{k_1''\sinh[k_1''(x-4k_1't-x_{10})]\cos\left[k_1'\left(x-2\left(k_1'-\frac{k_1''^2}{k_1'}\right)t-x_{20}\right)\right]\right.$$
$$\left.-k_1'\cosh[k_1''(x-4k_1't-x_{10})]\sin\left[k_1'\left(x-2\left(k_1'-\frac{k_1''^2}{k_1'}\right)t-x_{20}\right)\right]\right\},$$

$$(\mathbf{M}_1)_z=M_0-\frac{2k_1''^2}{|k_1|^2}\operatorname{sech}^2[k_1''(x-4k_1't-x_{10})]. \qquad (53)$$

These results are equal to Eq. (27a) obtained by the method of an inverse scattering transformation in Ref. 26. We also find that under the action of an external magnetic field the center and shape of the z component of nonlinear magnetization do not move with the two velocities V_1 and V_2 as showed by Fig. 7. While taking the z axis as the polar axis in the polar coordinates, we can obtain

$$\cos\theta=1-\frac{2k_1''^2}{|k_1|^2}\operatorname{sech}^2[k_1''(x-4k_1't-x_{10})],$$

$$\varphi=\varphi_0+k_1'\left[x-2k_1'\left(1-\frac{k_1''^2}{k_1'^2}\right)t-x_{20}\right]$$
$$+\tan^{-1}\left\{\frac{k_1''}{k_1'}\tanh[k_1''(x-4k_1't-x_{10})]\right\}. \qquad (54)$$

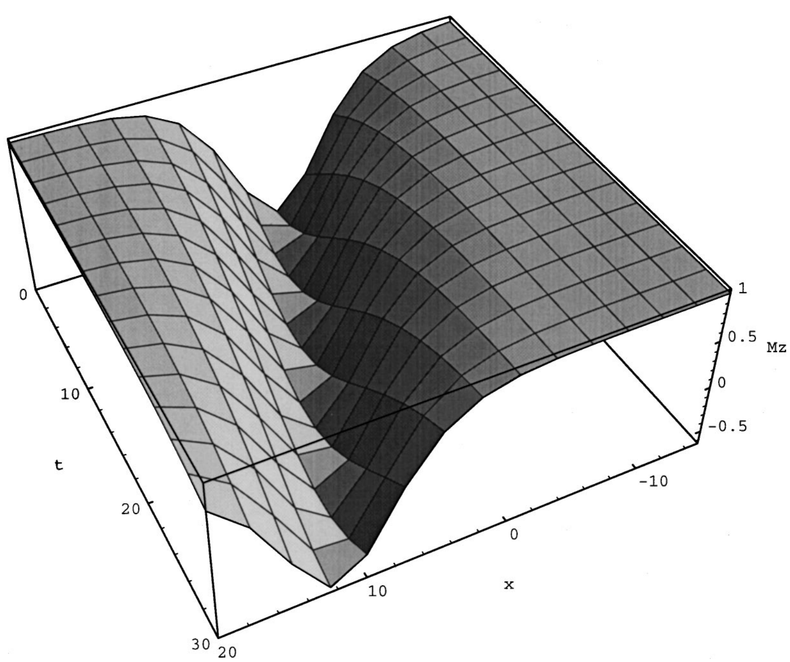

FIG. 8. Some graphical illustrations of the amplitude and width of the z component of the nonlinear magnetization $(\mathbf{M}_1)_z$ expressed by Eq. (54) in an isotropic ferromagnet, which do not change periodically with time, where $\rho=0$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, and $x_{20}=0$.

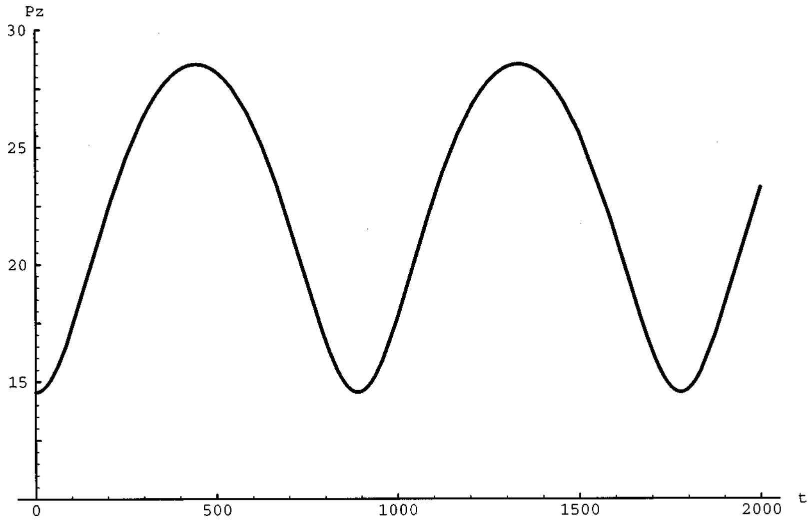

FIG. 9. Some graphical illustrations of the change of the z component of the total magnetic momentum P_z expressed by Eq. (56) in a ferromagnet with an easy plane, where $\theta_0=30^0$, $\rho=0.1$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, and $x_{20}=0$.

It means that the amplitudes and widths of the z component of the nonlinear magnetization do not also vary periodically with time. Figure 8 give some graphical illustrations of the amplitudes and width of the z component of the nonlinear magnetization $(\mathbf{M}_1)_z$ expressed by Eq. (54) in an isotropic ferromagnet, where $\rho=0$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, and $x_{20}=0$. When $t\rightarrow 0$, these results are equivalent to Eq. (22) obtained by means of the method of separating variables in moving coordinates shown in Ref. 4.

The total magnetic momentum

$$\mathbf{P}=M_0\int dx(1-\cos\theta)\nabla\varphi \tag{55}$$

depends on time and it is not a constant under the action of an external magnetic field. The integral of the motion coincident with the z component of the total magnetic momentum

$$P_z=M_0\int dx(1-\cos\theta) \tag{56}$$

is also not a constant. Figures 9 and 10 have given some graphical illustrations of the z component of the total magnetic momentum P_z expressed by Eq. (56) varying periodically with time in an anisotropic ferromagnet with an easy plane and with an easy axis, respectively. In the two figures, we took the following parameters: $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, $x_{20}=0$, $\rho=0.10$, and $\theta_0=30^0$ for an easy plane, respectively. We find that under the action of an external magnetic field, P_z depends periodically on time for a ferromagnet with an easy plane, while P_z in a ferromagnet with an easy axis will decrease as time increases, where P_z has the sense of the mean number of spins deviated from the ground state in localized magnetic excitations. This feature did not appear in the study of all other nonlinear problems in magnetism. When an anisotropic parameter vanishes, the ground state of the isotropic ferromagnet has a constant spin pointing in the z direction and the fixed boundary condition $\mathbf{M}\rightarrow(0,0,M_0)$ when $x\rightarrow\pm\infty$. When an external magnetic field vanishes, the Hamiltonian H, the total magnetic momentum $\mathbf{P}$, and the z component of the total magnetic momentum P_z, i.e., the three constants of motion associated with the global symmetries of the time translation, space translation, and spin rotation, respectively, are in the action angle representation given by the diagonal expressions. In terms of soliton solutions (56), we find that only in the case of an isotropic ferromagnet are the Hamiltonian, the total magnetic momentum $\mathbf{P}$, and the z component of the total magnetic momentum P_z constants of motion, $E=4JM_0^2k_1''+4JM_0B(k_1''/|k_1|^2)$, $P=4M_0\sin^{-1}(k_1''/|k_1|)$, and $P_z=4M_0(k_1''/|k_1|^2)$. Tjon and

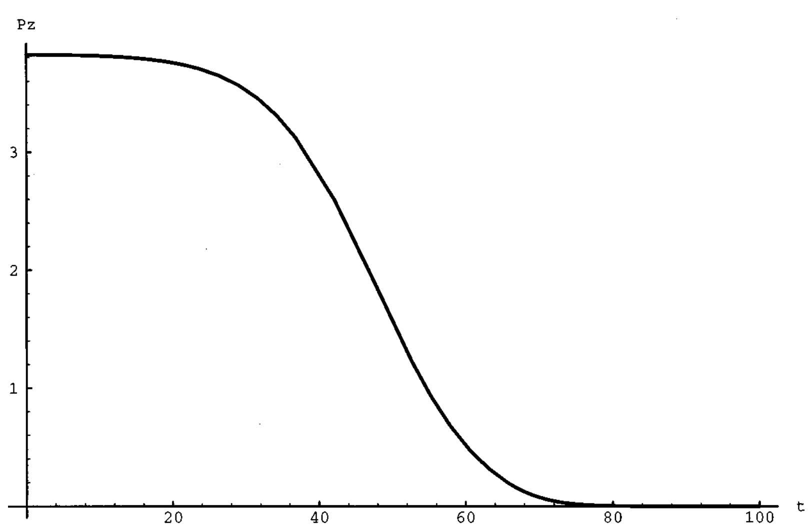

FIG. 10. Some graphical illustrations of the change of the z component of the total magnetic momentum P_z expressed by Eq. (56) in a ferromagnet with an easy axis, where $\rho=0.1$, $k_1'=0.1$, $k_1''=0.2$, $x_{10}=0$, and $x_{20}=0$.

Wright[4] took advantage of this feature in solving the equation of motion. These properties are important for the classical ferromagnet with an anisotropy in an external magnetic field, but they have never been obtained by all the other methods.

IV. THE ASYMPTOTIC BEHAVIOR OF MULTISOLITON SOLUTIONS

Supposing all $k_n''>0$ and $k_1'>k_2'>\cdots>k_N'$, the vicinity of $x=x_{in}+V_{in}t(i=1,2)$ is denoted by Θ_n. In the extreme by large t, these vicinities are separated from left to right as $\Theta_N,\Theta_{N-1},\ldots,\Theta_1$. In the vicinity Θ_j, there are the following limits: $(x-V_{in}t-x_{in0})\to-\infty$, $|f_n|\to\infty$, if $n<j$; $(x-V_{im}t-x_{im0})\to\infty$, $|f_m|\to\infty$, if $m>j$, while

$$\Gamma_0\sim\gamma_0(1,2,\ldots,j-1;1,2,\ldots,j-1)+\gamma_0(1,2,\ldots,j;1,2,\ldots,j),$$

$$\Gamma_1\sim\gamma_1(1,2,\ldots,j-1;1,2,\ldots,j-1)+\gamma_1(1,2,\ldots,j;1,2,\ldots,j),$$

$$\Gamma_2\sim\gamma_2(1,2,\ldots,j;0,1,2,\ldots,j-1).$$

Substituting the explicit expressions into Eqs. (19), (24), and (30), for the classical ferromagnet with two single-ion anisotropies in an external magnetic field, we can obtain the following relations:

$$\Gamma_0\sim1+\frac{\overline{cs(k_j)}ns(k_j)}{cs(k_j)\overline{ns(k_j)}}|F_j|^2,\quad \Gamma_1\sim1+\frac{ns(k_j)}{\overline{ns(k_j)}}|F_j|^2,$$

$$\Gamma_2\sim\frac{2cs(k_j)''}{\overline{ns(k_j)}}\overline{F_j},$$

where

$$F_j=\prod_{n=1}^{j-1}\prod_{m=j+1}^{N}\frac{[cs(k_j)-cs(k_n)][cs(k_j)-\overline{cs(k_m)}]}{[cs(k_j)-\overline{cs(k_n)}][cs(k_j)-cs(k_m)]}f_j.$$

Similarly, for an uniaxial anisotropic ferromagnet in an external magnetic field, we can also find

$$\Gamma_0\sim\begin{cases}1+|F_j|^2\dfrac{k_j(\bar{k}_j^2+1)}{\bar{k}_j(k_j^2+1)};\\[2mm] 1+|F_j|^2\dfrac{\bar{k}_j(k_j^2+1)}{k_j(\bar{k}_j^2+1)};\end{cases}$$

$$\Gamma_1\sim\begin{cases}1+|F_j|^2\dfrac{(k_j^2-1)(\bar{k}_j^2+1)}{(k_j^2+1)(\bar{k}_j^2-1)};\\[2mm] 1+|F_j|^2\dfrac{(k_j^2+1)(\bar{k}_j^2-1)}{(k_j^2-1)(\bar{k}_j^2+1)};\end{cases}$$

$$\Gamma_2\sim\begin{cases}\overline{F_j}\dfrac{4k_j'k_j''(\bar{k}_j^2-1)}{\bar{k}_j|k_j^2-1|^2};\\[2mm] \overline{F_j}\dfrac{4k_j''(\bar{k}_j^2-1)(|k_j|^2+1)}{(\bar{k}_j^2+1)|k_j^2-1|^2};\end{cases}$$

$$F_j=\begin{cases}f_j\displaystyle\prod_{n=1}^{j-1}\prod_{m=j+1}^{N}\frac{(k_m^2-1)(\bar{k}_n^2-1)(k_n^2-k_j^2)(\bar{k}_m^2-k_j^2)}{(k_n^2-1)(\bar{k}_m^2-1)(\bar{k}_n^2-k_j^2)(k_m^2-k_j^2)}, & \text{for an easy axis,}\\[2mm] f_j\displaystyle\prod_{n=1}^{j-1}\prod_{m=j+1}^{N}\frac{(k_m^2-1)(\bar{k}_n^2-1)[k_j(k_n^2-1)-k_n(k_j^2-1)][k_j(\bar{k}_m^2-1)-\bar{k}_m(k_j^2-1)]}{(k_n^2-1)(\bar{k}_m^2-1)[k_j(\bar{k}_n^2-1)-\bar{k}_n(k_j^2-1)][k_j(k_m^2-1)-k_m(k_j^2-1)]}, & \text{for an easy axis.}\end{cases}\tag{57}$$

It can be concluded from the results given above that the classical ferromagnet with two single-ion anisotropies in an external magnetic field has multisoliton solutions in a strict sense. When $t\to\pm\infty$, nonlinear magnetization appear to be the trains of N separating single solitons. The trains at $t\to-\infty$ turn out to be trains at $t\to\infty$ after the collision in the duration of time with the number and shape of solitons unchanged, and the position of center the of mass displaced in the traveling coordinates. The total displacement of the center of the jth peak in the course from $t\to-\infty$ to $t\to\infty$ is determined by

$$X_j=\frac{1}{cs(k_j)''}\left\{\ln\prod_{n=1}^{j-1}\left|\frac{cs(k_j)-cs(k_n)}{cs(k_j)-\overline{cs(k_n)}}\right|-\ln\prod_{m=j+1}^{N}\left|\frac{cs(k_j)-cs(k_m)}{cs(k_j)-\overline{cs(k_m)}}\right|\right\}.\tag{58}$$

However, even in the traveling coordinates the angle $\tan\varphi = \arctan(M_y/M_x)$ contains a linear term in time t. This shows that M_x and M_y manifest themselves as solitons. The total phase shift of the jth peak can be written as

$$\Phi_j = 2\left\{ \arg\left[\prod_{n=1}^{j-1} \frac{cs(k_j)-cs(k_n)}{cs(k_j)-\overline{\overline{cs(k_n)}}} \right] - \arg\left[\prod_{m=j+1}^{N} \frac{cs(k_j)-cs(k_m)}{cs(k_j)-\overline{\overline{cs(k_m)}}} \right] \right\}. \tag{59}$$

For a uniaxial anisotropic ferromagnet in an external magnetic field, the total displacement of the center and the total phase shift of the jth peak in the course from $t \to -\infty$ to $t \to \infty$ are

$$X_j = \begin{cases} \dfrac{|k_j^2-1|^2}{2\rho k_j' k_j''} \left\{ \ln \prod\limits_{n=1}^{j-1} \left| \dfrac{(k_n^2-k_j^2)(\overline{k_n}^2-1)}{(k_n^2-1)(\overline{k_n}^2-k_j^2)} \right| - \ln \prod\limits_{m=j+1}^{N} \left| \dfrac{(\overline{k_m}^2-k_j^2)(k_m^2-1)}{(k_m^2-k_j^2)(\overline{k_m}^2-1)} \right| \right\}; \\ \dfrac{|k_j^2-1|^2}{2\rho k_j''(|k_j|^2+1)} \left[\ln \prod\limits_{n=1}^{j-1} \left| \dfrac{(\overline{k_n}^2-1)[k_j(k_n^2-1)-k_n(k_j^2-1)]}{(k_n^2-1)[k_j(\overline{k_n}^2-1)-\overline{k_n}(k_j^2-1)]} \right| - \ln \prod\limits_{m=j+1}^{N} \left| \dfrac{(\overline{k_m}^2-1)[k_j(k_m^2-1)-k_m(k_j^2-1)]}{(k_m^2-1)[k_j(\overline{k_m}^2-1)-\overline{k_m}(k_j^2-1)]} \right| \right]. \end{cases}$$

$$\Phi_j = \begin{cases} 2\left\{ \arg\left[\prod\limits_{n=1}^{j-1} \dfrac{(k_n^2-k_j^2)(\overline{k_n}^2-1)}{(k_n^2-1)(\overline{k_n}^2-k_j^2)} \right] - \arg\left[\prod\limits_{m=j+1}^{N} \dfrac{(\overline{k_m}^2-k_j^2)(k_m^2-1)}{(k_m^2-k_j^2)(\overline{k_m}^2-1)} \right] \right\}, & \text{for an easy plane;} \\ 2\left[\arg\left(\prod\limits_{n=1}^{j-1} \dfrac{(\overline{k_n}^2-1)[k_j(k_n^2-1)-k_n(k_j^2-1)]}{(k_n^2-1)[k_j(\overline{k_n}^2-1)-\overline{k_n}(k_j^2-1)]} \right) - \arg\left(\prod\limits_{m=j+1}^{N} \dfrac{(\overline{k_m}^2-1)[k_j(k_m^2-1)-k_m(k_j^2-1)]}{(k_m^2-1)[k_j(\overline{k_m}^2-1)-\overline{k_m}(k_j^2-1)]} \right) \right], & \text{for an easy axis.} \end{cases} \tag{60}$$

When an anisotropic parameter $\rho \to 0$, the displacement of the center and the phase shift of the jth peak of an isotropic ferromagnet in an external magnetic field are

$$X_j = \frac{1}{k_j''} \left(\ln \prod_{n=1}^{j-1} \left| \frac{k_n-k_j}{k_n-\overline{\overline{k_j}}} \right| - \ln \prod_{m=j+1}^{N} \left| \frac{k_m-k_j}{k_m-\overline{\overline{k_j}}} \right| \right),$$

$$\Phi_j = 2\left[\arg\left(\prod_{n=1}^{j-1} \frac{k_n(k_n-k_j)}{k_n-\overline{k_j}} \right) - \arg\left(\prod_{m=j+1}^{N} \frac{k_m(k_m-k_j)}{k_m-\overline{k_j}} \right) \right], \tag{61}$$

These results are equal to Eqs. (28a) and (28b) obtained by the method of an inverse scattering transformation in Ref. 26.

V. CONCLUSION

In this section we will compare the present results with those obtained by other methods, then give some concluding remarks. According to Eqs. (39), (42), and (43), we can find that under the action of an external magnetic field nonlinear magnetization in a ferromagnet with an anisotropy depends essentially on two parameters V_1 and V_2 in Eqs. (41) and (45). The center of the nonlinear magnetization moves with a constant velocity V_1, while its shape also changes with another velocity V_2; the depths and widths of a surface of nonlinear magnetization vary periodically with time, and its shape is unsymmetrical with respect to the center. By means of these features, we find that the soliton solutions in a ferromagnet with an anisotropy in the external magnetic field are not expressed in the form of the product of separated variables in moving coordinates.[4] Only when an anisotropic parameter $\rho \to 0$, these soliton solutions in an anisotropic ferromagnet reduce to those in an isotropic ferromagnet, for example, the single-soliton solutions (47) in the polar coordinates are equivalent to Eq. (22) obtained by means of the method of separating variables in the moving coordinates in Ref. 4. Therefore, it is very difficult to investigate the exact soliton solutions in a ferromagnet with an anisotropy in an external magnetic field by means of the method of separating variables.

Reducing the Landau-Lifschitz equations to an appropriate form, Kosevich, Ivanov, and Kovalev[5] found a solution. In terms of Eq. (47) in the polar coordinates, there exist

$$\tan^2\left(\frac{\theta}{2}\right) = \frac{k_1''^2\left\{ |k_1^2-1|^2 \sin^2\left[\dfrac{8\rho k_1''(|k_1|^2+1)}{|k_1^2-1|^2}(x-V_1t-x_{10}) \right] + 4k_1'^2 \right\}}{k_1'^2\left\{ |k_1^2-1|^2 \cosh^2\left[\dfrac{8\rho k_1'(|k_1|^2-1)}{|k_1^2-1|^2}(x-V_2t-x_{10}) \right] - 4k_1''^2 \right\}}. \tag{62}$$

If we compared Eq. (62) with an approximate solution given by Ref. 5, we find that the previous properties of the soliton solutions remain even in the approximation of the order of ρ^2. The solutions of Ref. 5 did not satisfy the Landau-Lifschitz equation for a ferromagnet with an anisotropy even in the first order of anisotropy, and there is no reason to consider it as an approximate solution, since all attempts in this approximation were not successful.

Using the Hirota method, Bogdan and Kovalev[10] sought the soliton solutions of the Landau-Lifschitz equation in a ferromagnet with an anisotropy in the form

$$\mathbf{M}_x + i\mathbf{M}_y = \frac{2fg}{|f|^2+|g|^2}, \quad \mathbf{M}_z = \frac{|f|^2-|g|^2}{|f|^2+|g|^2}, \tag{63}$$

where

$$f = \sum_{n=0}^{[N/2]} \sum_{C_2 n} a(i_1, \ldots, i_{2n}) \exp(\rho_{i_1} + \cdots + \rho_{i_{2n}}),$$

$$g^* = \sum_{m=0}^{[(N-1)/2]} \sum_{C_{2m+1}} a(j_1, \ldots, j_{2m+1}) \times \exp(\rho_{j_1} + \cdots + \rho_{j_{2m+1}}), \tag{64}$$

$$a(i_1, \ldots, i_n) = \begin{cases} \sum_{k<l}^{(n)} a(i_k, i_l), & \text{for } n \geq 2; \\ 1, & \text{for } n = 0,1. \end{cases}$$

where $[N/2]$ is the maximum integer in addition to $N/2$, C_n represents the summation over all combinations of N elements in n, and $\rho_i = (k_i + \omega_i t + \rho_i^0)$. According to the expression of the single-soliton solutions (42) and (43) in this paper, we find that soliton solutions are difficult to express in the form of the Hirota factorization. Obviously, Bogdan and Kovalev[10] did not obtain the desired results.

We have introduced some transformations in Eq. (13), while $k=\infty$ and 0 correspond to $\lambda = \pm 2\rho$ (or $\mu = 0$) and $\mu = \pm\rho$ (or $\lambda = 0$). In the complex μ plane, these two points are the edges of the cuts. This is important to ensure that the Jost solution generated satisfies the corresponding Lax equations. It indicates that the edges of the cuts in the complex plane in an inverse scattering transformation must give a contribution even in the case of nonreflection. Unfortunately, Borovik and Kulinich[24,25] did not apparently consider these effects. Evidently, they did not obtain any expression of the solution.

In the present paper we have used the stereographic projection of the unit sphere of the magnetization vector onto a complex plane for the equations of motion in the classical ferromagnet with two single-ion anisotropies in an external magnetic field, and the effect of a magnetic field for integrability of the system is discussed. Then, introducing some transformations instead of the Riemann surface in order to avoid the double-valued function of the usual spectral parameter, the properties of the Jost solutions and the scattering data in detail are obtained. The Gel'fand-Levitan-Marchenko equation is derived. In the case of no reflection the exact multisoliton solutions are investigated. This method is more effective than the Darboux transformation. The asymptotic behavior of multisoliton solutions in the long-time limit as well as the total displacement of the center and the phase shift of the jth peak are also given. The total magnetic momentum and its z component are obtained. The present inverse scattering transformation method includes the contributions due to the continuous spectrum of the spectral parameter. They may be useful for further theoretical research and practical application.

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