Dynamic renormalization group theory of superfluid helium films

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The statistical mechanics of vortices, which is usually described in terms of the behavior of a two-dimensional Coulomb gas, can be mapped exactly onto a sine-Gordon model. We use this duality to develop a theoretical approach to the description of the response of a superfluid film to an oscillating substrate. Starting from a Hamiltonian for the vortex-gas system that includes a time-dependent superflow, we derive the form of the equivalent sine-Gordon Hamiltonian in terms of a fictitious field. A simple equation of motion is then proposed and we proceed to renormalize it using methods developed by Nozières and Gallet for the roughening transition. The renormalization program allows us to calculate the dynamic response of a superfluid film as measured in torsional oscillator experiments. We find that our method leads to predictions which are closer to experiment than previous phenomenological approaches. [S0163-1829(99)02941-0]

I. INTRODUCTION

The theory of the superfluid behavior of thin helium films is due to Kosterlitz and Thouless[1,2] who showed that thermally activated topological defects (vortex and antivortex pairs) are the dominant fluctuations which mediate the transition from normal to superfluid phases. Kosterlitz[3] used renormalization-group techniques to analyze the phase transition of a dilute gas of vortex antivortex pairs in equilibrium. The vortices in the film behave as charged particles interacting with each other as if they formed a two-dimensional Coulomb-gas.

Attempts to verify the theory proved frustrating as the principal experiments available had to be performed in the linear-response regime and at finite frequency,[4–6] leading to a strong broadening of the transition which could not be described accurately using the equilibrium Kosterlitz-Thouless theory. What was required was an extension of the theory to finite frequencies. Just such an extension was first provided by the theory of Ambegaokar, Halperin, Nelson, and Siggia (AHNS).[7]

The AHNS theory described the vortex dynamics in a phenomenological way, using expressions for the superfluid density and vortex fugacity obtained from the static recursion relations of Kosterlitz. Recently Bowley *et al.*[8,9] showed that the approach of AHNS can be refined, dispensing with the need for fitting parameters and leading to the prediction of a universal property which is readily compared with experiment. An alternative formulation of the dynamic theory was also developed by Minnhagen,[10] but it too relies on a heuristic description of the dynamics very similar to that of AHNS.[11] Furthermore, it leads to predictions very similar to those of the theory of Bowley *et al.*[8]

The driven torsional oscillator is the canonical experiment used to probe the behavior of superfluid films. The period shift and change in the Q-factor of the torsional oscillator are measured and can be related to the vortex response function, known as the dynamic dielectric function, which is calculated in the theory. As we have shown elsewhere[9] previous phenomenological theories based on the Coulomb-gas model alone do not give agreement with experimental results for ^{4}He films on Mylar or Grafoil substrates. It could be that the test fails for films on a Grafoil substrate because the disorder occurs on length scales comparable with other lengths in the theory. However, it is believed that the disorder on Mylar substrates occurs on short length scales; therefore, the discrepancy between theory and experiment for films on Mylar is much more serious. It is this lack of agreement between theory and experiment which caused us to develop a very different description of the dynamics.

Here we describe a new theoretical approach based on the equivalence of the Coulomb-gas model and the sine-Gordon model for a fictitious field ϕ. We propose an equation of motion to describe the relaxation of the fictitious field back to equilibrium. It is the renormalization of this equation of motion in the presence of an oscillating drive which allows us to calculate the dynamic dielectric constant and hence compare our theory with experiment. It must be acknowledged that we are proposing this equation of motion without proof. However, the approach has the important advantage over previous theories in that the dynamic behavior is obtained directly from a renormalization treatment of the equation of motion for the system; it also has the advantage that it leads to agreement with the main features of the experimental results for a wide range of coverages of ^{4}He films on Mylar as measured by McQueeney.[6] However, as we noted elsewhere,[9] the response of helium films on Mylar depends weakly on the thickness of the film, an effect which we are still not able to capture in the theory.

We begin the development of our theory by re-expressing the Coulomb-gas model as a classical field theory. The transformation is performed using the well-known Hubbard-Stratonovich transformation leading to a sine-Gordon model with free energy density $f[\phi(\mathbf{r})]$, given by

$$f[\phi(\mathbf{r})]=\frac{\gamma}{2}[\nabla\phi(\mathbf{r})]^2-V\cos[\phi(\mathbf{r})], \tag{1}$$

expressed in terms of a fictitious field $\phi(\mathbf{r})$.[12–15] The sine-Gordon model has been used with great success by Nozières and Gallet to describe the roughening transition in the weak-coupling limit.[16,17] The duality of the two-dimensional

Coulomb-gas and sine-Gordon models means that an exact translation exists from the language of the roughening transition to that of superfluid films.[12] We can use this duality to obtain the recursion relations for the Coulomb-gas model using the analysis of Nozières and Gallet; this leads to slightly different recursion relations for the superfluid density compared to those found by Kosterlitz.[3]

In this paper we use the sine-Gordon model to discuss the effect of a superflow on the Coulomb gas. Our aim is to generate recursion relations which allow us to calculate the dynamic response of the system. The dynamics of vortices in films is not easy to capture using the conventional picture in which the vortices are treated as particles that make up a two-dimensional Coulomb gas.[7–9] In contrast, the dynamics of the sine-Gordon model at the roughening transition has been analyzed by Nozières and Gallet with considerable success. Here we work the duality transform backwards and use it as the basis of our theory of the dynamics of superfluid films.

Our proposal, in its simplest form, is that the dynamics of the fictitious field ϕ can be found by treating it as if were a nonconserved order parameter which obeys relaxational (model A) dynamics:

$$\eta \frac{\partial \phi}{\partial t} = -\frac{\delta f[\phi]}{\delta \phi} + R, \tag{2}$$

where η is a friction coefficient, R represents the random component of the force, and $f[\phi]$ is the free energy density functional of the resulting sine-Gordon model. This method allows us to develop a renormalization scheme in which the friction coefficient is renormalized as well as the superfluid density and the fugacity of vortices. By extending the theory to include the effect of an oscillating superflow we are able to develop recursion relations that describe the response observed in torsional oscillator experiments. Of course the use of relaxational dynamics to describe the dynamics of superfluid films is not new. Indeed the original theory due to AHNS (Ref. 7) was based on such an approach, and there has been a considerable amount of work recently on the simulation of XY model dynamics using a time dependent Landau-Ginzburg approach.[18,19] However, to our knowledge the idea that model A dynamics could be applied to the fictitious field obtained from a Hubbard-Stratonovich transformation of the Coulomb-gas model is new.

The organization of this work is as follows. In Sec. II we discuss the form of the Hamiltonian for a static superfluid film. First of all, we write down the Hamiltonian in the vortex-gas picture. Then we describe how the Hubbard-Stratonovich transformation can be used to rewrite the Hamiltonian in the form of a sine-Gordon model. Next we discuss the boundary conditions which are required for a film of finite extent and then we consider the effect of a static superflow on the system. In Sec. III we describe the dynamic behavior of superfluid films using the sine-Gordon approach. We begin our discussion of the dynamics by postulating an equation of motion for the system. Then we derive expressions for the dielectric response of the system to an ac superflow. Next we use numerical methods to calculate the dielectric function. Having calculated the dielectric response, we compare the predictions of our theory with the results of experiments and previous phenomenological theories. Finally, in Sec. IV, we discuss the strengths and weaknesses of our model, focusing in particular on the ways in which it might be extended in the future.

II. HAMILTONIAN

The local order parameter for a superfluid film is a complex quantity

$$\Psi = \Psi_0 e^{iS(\mathbf{r})} \tag{3}$$

with both Ψ_0 and S real. If the film is uniform the free energy associated with a gradient in the phase is

$$E = \frac{\hbar^2 \rho_s^0}{2m^2} \int [\nabla S(\mathbf{r})]^2 d^2\mathbf{r} = \frac{1}{2} \rho_s^0 \int \mathbf{v}_s^2(\mathbf{r}) d^2\mathbf{r}, \tag{4}$$

where ρ_s^0 is the superfluid density and $\mathbf{v}_s(\mathbf{r}) = \hbar \nabla S(\mathbf{r})/m$ is the local superfluid velocity. We divide $\mathbf{v}_s$ into two parts,[20] $\mathbf{v}_s = \mathbf{v}_0 + \mathbf{v}_1$: $\mathbf{v}_0$ has zero divergence, $\mathbf{v}_1$ has zero curl. The quantity $\mathbf{v}_1$ is the velocity fluctuation associated with collective modes (third sound); $\mathbf{v}_0$ is the velocity associated with vortices. We can write

$$\nabla \times \mathbf{v}_0 = \frac{2\pi\hbar}{m} n(\mathbf{r}) \hat{\mathbf{k}}, \quad \nabla \cdot \mathbf{v}_0 = 0, \tag{5}$$

where $n(\mathbf{r})$ is the vortex-density per unit area.

We consider a film in which there are N_+ vortices of positive circulation and N_- of negative circulation. Suppose that for a particular configuration there are vortices with positive circulation at sites $\mathbf{r}_\alpha$ and negative circulation at sites $\mathbf{r}_\beta$

$$n(\mathbf{r}) = \sum_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha) - \sum_\beta \delta(\mathbf{r} - \mathbf{r}_\beta). \tag{6}$$

In the Coulomb-gas model positive circulation is equivalent to positive charge on the particles, negative circulation, to negative charge.

We suppose that the film is rectangular, of area $A = L_x L_y$, and that we can impose periodic boundary conditions so that we can define a two-dimensional Fourier transform of $n(\mathbf{r})$ as

$$n(\mathbf{q}) = \int d^2\mathbf{r}\, n(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}}, \tag{7}$$

$$n(\mathbf{r}) = \frac{1}{A} \sum n(\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{r}}. \tag{8}$$

Similarly we have

$$\mathbf{v}_0(\mathbf{q}) = \int d^2\mathbf{r}\, \mathbf{v}_0(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}}, \tag{9}$$

$$\mathbf{v}_0(\mathbf{r}) = \frac{1}{A} \sum \mathbf{v}_0(\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{r}}. \tag{10}$$

Using Eq. (5) we find that

$$\mathbf{v}_0(\mathbf{q}) = i\frac{2\pi\hbar}{mq}n(\mathbf{q}). \tag{11}$$

The kinetic energy associated with vortices can be written as

$$\mathcal{H}_v = \frac{1}{2}\rho_s^0 \int \mathbf{v}_0^2(\mathbf{r})d^2\mathbf{r} = \frac{2\pi^2}{A}\frac{\hbar^2\rho_s^0}{m^2}\sum_q \frac{|n(\mathbf{q})|^2}{q^2}. \tag{12}$$

The total kinetic energy of the system also contains contributions that depend on $\mathbf{v}_1(\mathbf{r})$; these terms arise from nonvortex excitations in the fluid or from a superfluid flow. In what follows we shall assume that the nonvortex contributions have been averaged over and subsumed into the definitions of the microscopic parameters, as have the effects of disorder.

A. Hubbard-Stratonovich transformation

The total energy of a system of N_+ vortices of positive circulation and N_- vortices of negative circulation includes the chemical potential μ of each vortex as well as $\mathcal{H}_v$. Thus the energy is

$$\frac{\mathcal{H}_{\text{total}}}{k_BT} = \frac{2\pi^2K_0}{A}\sum_q \frac{n(\mathbf{q})n(-\mathbf{q})}{q^2} + \frac{\mu(N_+ + N_-)}{k_BT}, \tag{13}$$

where $K_0 = \rho_s^0\hbar^2/m^2k_BT$ represents the unrenormalized superfluid density. The unrenormalized fugacities of the vortices are

$$y_0 = e^{-\mu/k_BT}. \tag{14}$$

The grand partition of the system involves a sum over all configurations of vortices, taking account of the fact that the vortices are identical objects. However, as Samuel points out there is a difference between a three-dimensional and a two-dimensional Coulomb gas:[14] because of infrared divergences, a charge distribution in two-dimensions has an infinite energy unless it is strictly charge neutral. We can either ensure neutrality and work with an infinite area, or we can, as Samuel suggests, ‘‘enclose the system in a grounded conducting casing. If there is excess charge then an equal and opposite charge will be induced on the conductor.’’ We choose to deal with a finite system so that there is no constraint of charge neutrality (N^+ does not have to equal N^- for a finite system); this choice makes the analysis much simpler in the presence of a superflow. Thus

$$\Xi = \sum_{N^+}\frac{{y_0}^{N^+}}{N^+!}\sum_{N^-}\frac{{y_0}^{N^-}}{N^-!}\prod_{\alpha=1}^{N^+}\int\frac{d^2\mathbf{r}_\alpha}{a^2}\prod_{\beta=1}^{N^-}\int\frac{d^2\mathbf{r}_\beta}{a^2}\exp\left(\frac{-\mathcal{H}_v}{k_BT}\right) \tag{15}$$

worked out for all positions and numbers of the vortices. In order to define the model unambiguously we define the positions as the lattice sites of a square lattice of side a. The partition function for a particular number of positive and negative vortices is

$$Z = \sum_{\mathbf{r}_\alpha}^{N_+}\sum_{\mathbf{r}_\beta}^{N_-}\exp\left(-\frac{2\pi^2K_0}{A}\sum_q\frac{n(\mathbf{q})n(-\mathbf{q})}{q^2}\right). \tag{16}$$

The problem is to evaluate Z.

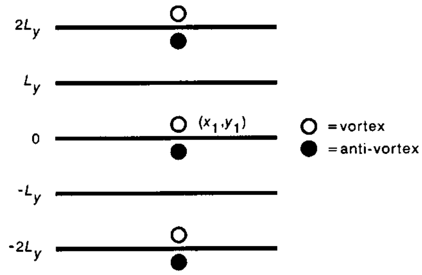

FIG. 1. Set of image charges for a positive vortex at (x_1,y_1) confined between plates at $y=0$ and $y=L_y$.

The Hubbard-Stratonovich transformation turns the sum into a path integral. We write

$$e^{-\mathcal{H}_v/k_BT} = B^{-1}\prod_{\mathbf{q}}\int\mathcal{D}\phi(\mathbf{q})e^{-q^2\phi(\mathbf{q})\phi(-\mathbf{q})/8\pi^2AK_0} \times e^{i[n(\mathbf{q})\phi(-\mathbf{q})+n(-\mathbf{q})\phi(\mathbf{q})]/2A}, \tag{17}$$

with the constant B given by

$$B = \prod_{\mathbf{q}}\int\mathcal{D}\phi(\mathbf{q})e^{-q^2\phi(\mathbf{q})\phi(-\mathbf{q})/8\pi^2AK_0}. \tag{18}$$

We can turn this back into real space to give a path integral,

$$e^{-\mathcal{H}_v/k_BT} = B^{-1}\int\mathcal{D}\phi(\mathbf{r})\exp\left[-\int\left(\frac{(\nabla\phi)^2}{8\pi^2K_0} - in(\mathbf{r})\phi(\mathbf{r})\right)d^2\mathbf{r}\right]. \tag{19}$$

To paraphrase Samuel, if there is excess charge (that is $N^+\neq N^-$) in the bulk then there must be an equal and opposite charge induced in the fictitious conductor which mimics the boundary conditions. We now consider the boundary conditions imposed on the system.

B. Boundary conditions

Suppose the system is confined to the region $0\leq x<L_x$ and $0\leq y<L_y$ in such a way that the normal component of velocity is zero on the $y=0$ and $y=L_y$ edges. To achieve this we can create an infinite series of images of any vortex. To clarify matters let us consider a vortex of positive circulation at (x_1,y_1). The vortex sits between two parallel lines ($y=0$ and $y=L_y$) which act as mirrors, producing an infinite set of images: there are image vortices of negative sign at $(x_1,2mL_y-y_1)$ with m an integer or zero, and image vortices of positive sign at $(x_1,2mL_y+y_1)$. We illustrate this in Fig. 1. It follows that the system is periodic in the y direction with period $2L_y$. To simplify matters let us suppose that the system has periodic boundary conditions in the x direction with period L_x: such boundary conditions are appropriate when the surface is the inner lining of a cylinder covered in a film of helium and the film is laid flat as in Fig. 2.

We can take Fourier transforms in the y direction over an enlarged area with the vortices and their images in the region at $0\leq x<L_x$ and $-L_y\leq y<L_y$. Over the enlarged area the system obeys periodic boundary conditions. When we inte-

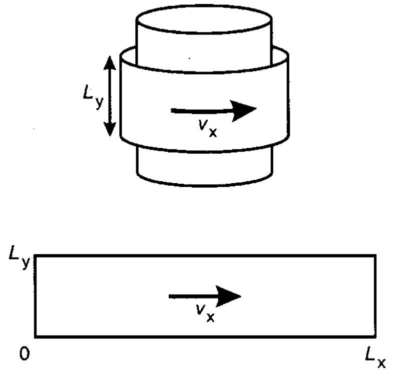

FIG. 2. When it is laid flat the inner lining of a cylinder is analogous to a flat film with periodic boundary conditions along the x direction.

grate over the enlarged area we have a pair of vortices of opposite sign at (x_1,y_1) and $(x_1,-y_1)$, leading to

$$\int_0^{L_x} dx \int_{-L_y}^{L_y} dy\, in(\mathbf{r})\phi(\mathbf{r}) = i[\phi(x_1,y_1) - \phi(x_1,-y_1)]. \tag{20}$$

By taking the integral over the larger area we are doubling the total energy of the system; but this can easily be remedied by dividing by 2. The function $\phi(x_1,y_1)$ can be written as

$$\phi(x_1,y_1) = \phi_1(x_1,y_1) + \phi_2(x_1,y_1), \tag{21}$$

where $\phi_1(x_1,y_1)$ is an even function of y_1 and $\phi_2(x_1,y_1)$ is an odd function. The $\phi_1(x_1,y_1)$ part of the field cancels out, and only the $\phi_2(x_1,y_1)$ part remains. It follows that $\phi(x_1,y_1)$ is an odd function of y_1, a symmetry condition which arises as a direct consequence of the boundary conditions we have imposed.

First let us suppose that there is no superflow. Substituting the expression for the charge density, Eq. (6), into Eq. (19) we obtain a phase factor for each vortex; in this way we get the grand partition function

$$\Xi = B^{-1} \int \mathcal{D}\phi(\mathbf{r}) \exp\left[-\int \left(\frac{(\nabla\phi)^2}{8\pi^2 K_0} - \frac{2y_0 \cos[\phi(\mathbf{r})]}{a^2} \right) d^2\mathbf{r} \right], \tag{22}$$

which involves a path integral over the energy functional

$$\frac{F[\phi(\mathbf{r})]}{k_B T} = \int \left(\frac{(\nabla\phi)^2}{8\pi^2 K_0} - \frac{2y_0 \cos[\phi(\mathbf{r})]}{a^2} \right) d^2\mathbf{r}. \tag{23}$$

The Euler equation which minimizes the free energy is

$$\frac{1}{4\pi^2 K_0} \nabla^2 \phi_0(\mathbf{r}) - \frac{2y_0}{a^2} \sin[\phi_0(\mathbf{r})] = 0. \tag{24}$$

The solution of this equation is $\phi_0(\mathbf{r}) = 0$. Fluctuations about this solution cost an increase in free energy.

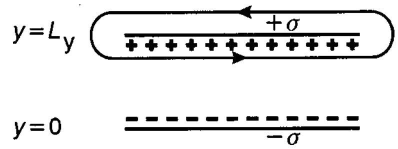

FIG. 3. Contour of integration in the plane of the film for the calculation of the edge charge.

C. Superflow

Now let us turn to the case where there is superflow around the cylinder, that is in the x direction. We can represent superflow by supposing that there is a line of vorticity along $y = L_y$ with charge per unit length σ and another line at $y=0$ with charge per unit length $-\sigma$. There are again an infinite series of images of the lines of vorticity. These lines of vorticity act as sources of the superflow. Using Stokes' theorem for a contour which encircles the edge at $y=L_y$ (see Fig. 3) we get

$$\oint \mathbf{v}_s \cdot d\mathbf{l} = \frac{2\pi\hbar}{m} \int n(\mathbf{r}) \hat{\mathbf{k}} \cdot d\mathbf{S} = \frac{2\pi\hbar}{m} \int \sigma dl. \tag{25}$$

It follows that the superfluid velocity is related to σ as

$$\mathbf{v}_s = \hat{\mathbf{i}} h\sigma/2m. \tag{26}$$

The transformation proceeds as before only now we get the expression

$$e^{-\mathcal{H}_v/k_B T} = B^{-1} \int \mathcal{D}\phi(\mathbf{r}) \exp\left[-\int \left(\frac{(\nabla\phi)^2}{8\pi^2 K_0} - i[n(\mathbf{r}) + \delta n(\mathbf{r})]\phi(\mathbf{r}) \right) d^2\mathbf{r} \right], \tag{27}$$

with $\delta n(\mathbf{r})$ the density of edge charges

$$\delta n(\mathbf{r}) = \frac{\sigma}{2}[\delta(y+L_y) + \delta(y-L_y)] - \sigma\delta(y). \tag{28}$$

The system of charges given by Eq. (28), when replicated in a periodic way in the y direction, produces a charge per unit length σ at $y=(2m+1)L_y$ with m an integer.

The grand partition function now takes the form

$$\Xi = B^{-1} \int \mathcal{D}\phi(\mathbf{r}) \exp\left(-\frac{1}{k_B T} \int f[\phi(\mathbf{r})] d^2\mathbf{r} \right), \tag{29}$$

with the energy functional

$$\frac{f[\phi(\mathbf{r})]}{k_B T} = \left(\frac{1}{8\pi^2 K_0} [\nabla\phi(\mathbf{r})]^2 - \frac{2y_0}{a^2} \cos[\phi(\mathbf{r})] - i\delta n(\mathbf{r})\phi(\mathbf{r}) \right). \tag{30}$$

Notice that in this case the energy functional contains an imaginary term, $-i\delta n(\mathbf{r})\phi(\mathbf{r})$. However, the imaginary term is an odd function of $\phi(\mathbf{r})$ and so we can rewrite the grand partition function in the form

$$\Xi=\left\langle \exp\left(\int i\,\delta n(\mathbf{r})\phi(\mathbf{r})d^2\mathbf{r}\right)\right\rangle=\left\langle \cos\left(\int \delta n(\mathbf{r})\phi(\mathbf{r})d^2\mathbf{r}\right)\right\rangle$$
$$+i\left\langle \sin\left(\int \delta n(\mathbf{r})\phi(\mathbf{r})d^2\mathbf{r}\right)\right\rangle, \tag{31}$$

where the angle brackets denote an average over the field $\phi(\mathbf{r})$ with weight

$$\exp\left[-\int\left(\frac{(\nabla\phi)^2}{8\pi^2K_0}-\frac{2y_0}{a^2}\cos[\phi(\mathbf{r})]\right)d^2\mathbf{r}\right]$$

which is an even function of $\phi(\mathbf{r})$. It follows that odd powers of $\phi(\mathbf{r})$ average to zero and we get

$$\Xi=\left\langle \cos\left(\int \delta n(\mathbf{r})\phi(\mathbf{r})d^2\mathbf{r}\right)\right\rangle. \tag{32}$$

The partition function remains real.[21] In fact imaginary terms in the free energy functional are often generated by the Hubbard-Stratonovich transformation; in particular they arise whenever there is a difference in chemical potential between plus and minus topological charges.[22]

The function which minimizes the free energy density, subject to the boundary conditions, is the solution of

$$\frac{1}{4\pi^2K_0}\nabla^2\phi(\mathbf{r})-\frac{2y_0}{a^2}\sin[\phi(\mathbf{r})]=-i\,\delta n(\mathbf{r}). \tag{33}$$

It is a complex, even function of the coordinate y which we write as $\phi(\mathbf{r})=i\chi(\mathbf{r})$. As emphasized by Samuel,[14] this solution is the equivalent of the Debye-Hückel equation for a coarse-grained Coulomb potential

$$\frac{1}{4\pi^2K_0}\nabla^2\chi(\mathbf{r})-\frac{2y_0}{a^2}\sinh[\chi(\mathbf{r})]=-\delta n(\mathbf{r}). \tag{34}$$

It is this solution which is the minimum of the thermodynamic potential; the system naturally evolves towards this minimum.

If the vortices cost an infinite amount of energy, the fugacity y_0 would be zero and there would be no vortices in the system. The solution when the fugacity y_0 is zero in the region $-L_y<y<L_y$ is

$$\chi_0(\mathbf{r})=2\pi^2K_0\sigma(|y|-L_y/2). \tag{35}$$

The corresponding free energy is

$$\frac{F[\phi(\mathbf{r})]}{k_BT}=\int\frac{f[\phi(\mathbf{r})]}{k_BT}d^2\mathbf{r}=\int\left(\frac{(\nabla\chi)^2}{8\pi^2K_0}\right)d^2\mathbf{r}$$
$$=\frac{\pi^2K_0(L_x\sigma)^2L_y}{2L_x}, \tag{36}$$

which can be thought of as the energy of a charge ''$L_x\sigma$'' stored in a ''capacitor'' of capacitance

$$C=\frac{L_x}{\pi^2K_0L_y} \tag{37}$$

with K_0^{-1} the equivalent of the permittivity of free space ϵ_0. The renormalization program is a systematic way of accounting for the effect of vortices by taking account of terms in powers of the fugacity. The effect of renormalization of the superfluid density (as represented by K) is the equivalent of changing the ''permittivity'' ϵ_0 to $\epsilon\epsilon_0$ with ϵ the dielectric constant

$$\epsilon=\frac{K_0}{K}, \tag{38}$$

where K is the renormalized parameter. In essence we have been describing the Coulomb-gas picture of the system in which the charges form dipoles which act to screen the internal electric field.

III. DYNAMICS

Our aim is to develop a model of the dynamics of the system which describes the motion of the thermally generated vortices. The model must also allow the vortices to fluctuate in all possible ways and to relax back to thermal equilibrium. For example, if the system has too many vortex-antivortex pairs, the model should allow them to decay by mutual annihilation; if there are more positive than negative charges in a region there must be a mechanism in the model which allows them to redistribute themselves so as to relax back to equilibrium. However, rather than use the language of vortices we prefer to express the decay back to equilibrium in terms of the fictitious field $\phi(\mathbf{r})$. The quantity $\phi(\mathbf{r})$ gives a complete specification of the instantaneous state of the system; it is just as powerful in describing the configuration of the system as a description of the positions of all the vortices — including those which were described as ''free'' in the AHNS picture.[7] A description in terms of the fictitious field may not be as physically transparent as a description in terms of the vortex density, but it has the great advantage that it allows us to apply a systematic renormalization scheme even in the dynamic regime.

To obtain the dynamic response we assume that the field $\phi(\mathbf{r})$ is the slowest variable which describes the system. Vortices are assumed to move slowly in a gas of thermal excitations which damp their motion; all other quantities which could describe the system (such as small fluctuations in the phase of the order parameter) relax rapidly towards equilibrium with a much shorter time scale. Under these circumstances the system is at each moment in a thermodynamic state which is in instantaneous equilibrium as far as the faster quantities are concerned. The instantaneous state of the system can be described by the variable $\phi(\mathbf{r})$ alone. When disturbed from equilibrium the value of $\phi(\mathbf{r})$ evolves slowly until it reaches a minimum in the thermodynamic potential.

A. Equation of motion

The simplest proposal is that the equation of motion is of the relaxational form

$$\eta\frac{\partial\varphi(\mathbf{r})}{\partial t}=-\frac{\delta f[\varphi(\mathbf{r})]}{\delta\varphi(\mathbf{r})}+R(\mathbf{r},t), \tag{39}$$

where η is a friction coefficient and R represents a thermal noise term with a Gaussian white spectrum and mean of zero. Equation (39) is the simplest equation of motion we can formulate for a system which is not constrained by a conservation law (φ is not a conserved quantity) and in which there no other relevant quantities which decay on a comparable time scale. In what follows we shall assume that this equation of motion can be used to describe the linear response of the fictitious field $\phi(\mathbf{r})$.

When there is no superflow (no edge charges) we get

$$\eta_0 \frac{\partial \phi(\mathbf{r})}{\partial t} = \frac{k_B T}{4\pi^2 K_0} \nabla^2 \phi(\mathbf{r},t) - \frac{2 y_0 k_B T}{a^2} \sin[\phi(\mathbf{r},t)] + R(\mathbf{r},t). \tag{40}$$

Equation (40) is mathematically equivalent to the equation

$$\eta' \frac{\partial z}{\partial t} = \tilde{\gamma} \nabla^2 z - \frac{2\pi}{b} V \sin\left(\frac{2\pi z(\mathbf{r})}{b}\right) + R \tag{41}$$

used by Nozières and Gallet to describe the dynamics of the position z of the interface for temperatures close to the roughening transition. In their case the free energy density is

$$f[z(\mathbf{r})] = \frac{\tilde{\gamma}}{2} [\nabla z(\mathbf{r})]^2 - V \cos\left(\frac{2\pi z(\mathbf{r})}{b}\right), \tag{42}$$

where $\tilde{\gamma}$ is the surface stiffness, b is the spacing of the planes, and V is the amplitude of the pinning potential.

In Nozières and Gallet's theory the interface is infinite in extent, whereas our treatment has been developed for a rectangular film of finite area. Let the smaller length scale associated with the size of the system be L_y. Suppose we exert a time-varying force on the system which oscillates at a frequency ω and which induces a superflow of maximum amplitude v_s. There is a length scale associated with the dynamics of the system which is

$$r_d = \sqrt{\frac{k_B T}{4\pi^2 K_0 \eta_0 \omega}}. \tag{43}$$

We therefore have two limits: when $r_d \gg L_y$ the system has plenty of time to evolve to the minimum in the thermodynamic potential. A vortex which escapes from its image at one edge of the film has sufficient time to cross the film and be annihilated on the other edge causing the ''charge on the capacitor'' to decay; in more conventional language we are describing the decay of superflow by phase slippage. In the opposite limit, $r_d \ll L_y$ the vortices do not have time to move across the film in the time scale of an oscillation, and so the charge does not decay. In this limit we can treat the system as if it were effectively infinite in extent with an error of order r_d/L_y. In what follows we assume that the system is always in the limit $r_d \ll L_y$.

There is also a length scale associated with the superflow

$$r_F = \frac{\hbar}{m v_s}. \tag{44}$$

We are interested in the linear response of the system so we are concerned with the limit $r_F \gg r_d$. When r_F and r_d are comparable the response is nonlinear.

We can convert the renormalization group analysis of the roughening transition, due to Nozières and Gallet, into that of superfluid films using the dictionary

$$\phi(\mathbf{r}) \rightleftharpoons \frac{2\pi z(\mathbf{r})}{b},$$

$$\frac{2 y_l}{a^2} \rightleftharpoons \frac{V}{k_B T},$$

$$K_l^{-1} \rightleftharpoons \frac{b^2 \tilde{\gamma}}{k_B T},$$

$$\eta_l \rightleftharpoons \frac{b^2}{4\pi^2} \eta',$$

where y_l and K_l are the renormalized fugacity and Kosterlitz parameter, respectively. Nozières and Gallet define $U = V/\Lambda^2$ where Λ is the cutoff in momentum (which has initial value Λ_0), and the scaling parameter $l = \ln(\Lambda_0/\Lambda)$.[23] The recursion relations to order U^2 take the form

$$\frac{dU}{dl} = U(2-n), \tag{45}$$

$$\frac{d \ln \tilde{\gamma}}{dl} = \frac{2\pi^4 U^2}{\tilde{\gamma}^2 b^4} A(n), \tag{46}$$

$$\frac{d \ln \eta'}{dl} = \frac{8\pi^4 U^2}{\tilde{\gamma}^2 b^4} B(n), \tag{47}$$

where the functions $A(n)$ and $B(n)$ are given by

$$A(n) = n \int_0^\infty d\tilde{\rho} \tilde{\rho}^3 \int_0^\infty \frac{d\kappa}{\kappa} e^{-1/4\kappa} e^{-2nh} J_0(\tilde{\rho}) e^{-\tilde{\rho}^2 \kappa}, \tag{48}$$

$$B(n) = n \int_0^\infty d\tilde{\rho} \tilde{\rho}^3 \int_0^\infty d\kappa e^{-1/4\kappa} e^{-2nh} J_0(\tilde{\rho}) e^{-\tilde{\rho}^2 \kappa} \tag{49}$$

with

$$h(\tilde{\rho}, \kappa) = \int_0^1 \frac{dx}{x} [1 - J_0(x\tilde{\rho}) e^{-(\kappa x^2 \tilde{\rho}^2)}], \tag{50}$$

and $n = \pi k_B T/\tilde{\gamma} b^2 = \pi K_l$. The corresponding recursion relations for the superfluid case can be found using our ''dictionary;'' they are

$$\frac{d y_l}{dl} = (2 - \pi K_l) y_l, \tag{51}$$

$$\frac{d K_l^{-1}}{dl} = 8\pi^4 K_l A(\pi K_l) \left(\frac{y_l}{a^2 \Lambda_0^2}\right)^2, \tag{52}$$

$$\frac{d \ln \eta_l}{dl} = 32\pi^4 K_l^2 B(\pi K_l) \left(\frac{y_l}{a^2 \Lambda_0^2}\right)^2. \tag{53}$$

A detailed calculation shows that the value of $A(\pi K_l = 2) = 0.398$. We can make this consistent with the recursion relation of Kosterlitz by choosing a particular value of $a\Lambda_0$. The significant difference between this scheme and that of Kosterlitz lies in the fact that $A(\pi K_l)$ varies as a function of πK_l; for example, it tends to zero as πK_l goes to infinity. If we are interested in the recursion relations away from the fixed point $K_l=2/\pi$, as is the case for the dynamic response, then the difference between the two renormalization schemes becomes important.

B. Dynamic superflow

When the substrate of a superfluid film oscillates, it carries with it the excitations of the normal fluid and leaves the superfluid component behind. In the rest frame of the substrate it is the superfluid which oscillates as if there were a superflow. In this frame a vortex experiences a Magnus force due to the superflow, a force which is balanced by a friction force as the vortex moves through the stationary excitations of the normal fluid. The superflow acts as the driving force that causes the vortices to move and dissipate energy.

We can generate a superflow around the surface mathematically by putting an oscillating charge on the edges of the form

$$\sigma(t)=\sigma_0\sin(\omega t) \tag{54}$$

with the maximum amplitude of the superflow $v_s=h\sigma_0/2m$. These charges generate the field $i\chi_0(\mathbf{r},t)$. The surface charge, and hence $\chi_0(\mathbf{r},t)$, can decay if there are phase-slip processes in which vortices cross the film. The dynamics of ϕ can be separated into two parts: the part $i\chi_0(\mathbf{r},t)$ which decays due to phase slippage, and the remainder $\tilde{\phi}=\phi-i\chi_0(\mathbf{r},t)$ which decays according to the equation

$$\eta_0\frac{\partial\tilde{\phi}(\mathbf{r},t)}{\partial t}=\frac{k_BT}{4\pi^2K_0}\nabla^2\tilde{\phi}(\mathbf{r},t)-\frac{2y_0k_BT}{a^2}\sin[\tilde{\phi}(\mathbf{r},t)+i\chi_0(\mathbf{r},t)]+R(\mathbf{r},t). \tag{55}$$

In principle we could set up an equation describing the behavior of $\chi_0(\mathbf{r},t)$; however, as we are considering the case where $r_d\ll L_y$, the decay of the surface charge during each cycle will be a very small effect and we shall neglect it.

The appearance of the imaginary term in the equation of motion for $\tilde{\phi}$ is a direct consequence of the imaginary term in the Hamiltonian. However, it leads to no mathematical problems: the renormalization scheme can be extended to complex fields and the recursion relations obtained are entirely real. A similar case was considered by Park and Lubensky[22] who showed that static recursion relations could be derived for a system with a Hamiltonian containing imaginary terms.

Equation (55) is almost equivalent to the equation of motion that was used by Giorgini and Bowley[24] to describe the dynamics of driven surfaces near the roughening transition. The equation of motion in that case is

$$\eta'\frac{\partial z_1}{\partial t}=\tilde{\gamma}\nabla^2z_1-\frac{2\pi V}{b}\sin\left(\frac{2\pi}{b}(z_1+z_0)\right)+R \tag{56}$$

with

$$z_0=\left(\frac{F}{b\,\eta'\omega}\right)\sin(\omega t) \tag{57}$$

the term arising from the external driving overpressure $F\cos(\omega t)$. Mathematically we have an analogous equation with two differences. First the corresponding quantity to $2\pi z_0/b$ is now complex

$$\frac{2\pi z_0}{b}\rightleftharpoons i\chi_0(\mathbf{r},t)$$

or

$$\frac{2\pi}{b}\left(\frac{F}{b\,\eta'\omega}\right)\sin(\omega t)\rightleftharpoons i2\pi^2K_0\sigma_0\sin(\omega t)(|y|-L_y/2).$$

We can think of the term $2\pi^2K_0\sigma_0=E_0$ as if it were an ''electric field'' of magnitude E_0 emerging from the charge σ_0.

The second difference from the roughening transition is that the external drive [as represented by $\chi_0(\mathbf{r},t)$] varies with position. In that sense the situation is more closely analogous to the problem of a vicinal surface as treated by Nozières and Gallet,[16] albeit an oscillating vicinal surface. Under renormalization vicinal surfaces exhibit an anisotropy in the surface stiffness. Therefore the model predicts that in the superfluid film the superfluid density should become anisotropic in the nonlinear regime.[25] Here we consider only the linear-response regime.

Giorgini and Bowley evaluate the renormalization of $u_0=\dot{z}_0$ as

$$\frac{d\ln[u_0(\omega)]}{dl}=-\frac{8\pi^4U^2}{\tilde{\gamma}^2b^4}C(n,r_d,r) \tag{58}$$

with $r=r_d/r_F$ the ratio of the diffusion length to the length scale associated with the external force, $r_F=b^2\tilde{\gamma}/F$. In the linear response regime $r\to 0$ and we have

$$C(n,\tilde{r}_d,0)=-in\tilde{r}_d^2\int_0^\infty d\tilde{\rho}\tilde{\rho}\int_0^\infty\left(\frac{d\kappa}{\kappa}\right)e^{-1/4\kappa-2nh}\times J_0(\tilde{\rho})e^{-\tilde{\rho}^2\kappa}(e^{i\kappa\tilde{\rho}^2/\tilde{r}_d^2}-1), \tag{59}$$

where $\tilde{r}_d=r_d\Lambda$. Use of the same technique for the superfluid case allows us to calculate the change in the electric field. As long as the length scale involved is very much smaller than the size of the film we get the recursion relation

$$\frac{d\ln E_0}{dl}=-32\pi^4K_l^2\left(\frac{y_l}{a^2\Lambda_0^2}\right)^2C(\pi K_l,\tilde{r}_d,0). \tag{60}$$

C. Dielectric function

The dynamic ''dielectric function'' $\varepsilon(\omega)$ is defined as the ratio of values of the electric field before and after renormalization

$$\varepsilon(\omega)=\left(\frac{E_0(l=0)}{E_0(l=\infty)}\right). \tag{61}$$

In principle, the dielectric function may be obtained by simply integrating the recursion relation for the external field. To order y_l^2 we get

$$\ln\left(\frac{E_0(l=\infty)}{E_0(l=0)}\right)=-\frac{32\pi^4}{a^2\Lambda_0^2}\int_0^\infty K_l^2 C(\pi K_l,\tilde{r}_d,0)y_l^2\,dl. \quad (62)$$

Hence the inverse of the dielectric constant is

$$\varepsilon(\omega)^{-1}=\exp\left(-\frac{32\pi^4}{a^2\Lambda_0^2}\int_0^\infty K_l^2 C(\pi K_l,\tilde{r}_d,0)y_l^2\,dl\right). \quad (63)$$

This procedure is correct for temperatures below the transition where y_l tends to zero for large l. However, for temperatures above the transition y_l becomes large and a theory based on the small fugacity expansion becomes inaccurate.

This question was considered by Giorgini and Bowley in their analysis of the mathematically identical problem of the roughening transition in the presence of an external harmonic drive.[24] Giorgini and Bowley calculate the dielectric function as a combination of two parts: a term from the short wavelength fluctuations, which is obtained via the renormalization approach, and a second term describing the long wavelength behavior which is dominated by the pinning potential. In terms of the roughening transition this description is valid in the region just below the transition where the pinning potential is finite on a macroscopic scale. This corresponds to the region just above the superfluid transition in thin helium films, which is exactly what is probed by torsional oscillator experiments.

Using the ansatz of Giorgini and Bowley, the expression for the dielectric function given above [Eq. (63)] is modified: the integration is performed up to a finite cutoff l_c and a separate term from the strong-coupling regime (i.e., when $y_l\sim 1$) is then added on. The strong-coupling contribution is obtained from a solution in the long-wavelength limit of the equation of motion for small displacements. The advantage of this method is that it provides a simple calculational scheme; however, the exact value of l_c is not determined by the theory.

The dielectric constant for renormalization up to l_c is defined by the ratio

$$\epsilon_{l_c}(\omega)=\frac{u_0(\omega,l=0)}{u_0(\omega,l=l_c)}. \quad (64)$$

At l_c the equation of motion is

$$\overline{\eta'}(l_c)\frac{\partial\overline{z_1}(l_c)}{\partial t}=\overline{\gamma}(l_c)\nabla^2\overline{z_1}(l_c)-\frac{2\pi}{b}\overline{V}\sin\left(\frac{2\pi(\overline{z_1}(l_c)+\overline{z_0}(l_c))}{b}\right)+\overline{R}(l_c), \quad (65)$$

where all quantities have been renormalized up to the point $l=l_c$, as emphasized by the overlines. This equation can now be written in terms of the quantity $\overline{z}=\overline{z_1}+\overline{z_0}$; hence

$$\overline{\eta'}(l_c)\frac{\partial\overline{z(l_c)}}{\partial t}=\overline{\gamma}(l_c)\nabla^2\overline{z}(l_c)-\frac{2\pi}{b}\overline{V}\sin\left(\frac{2\pi\overline{z(l_c)}}{b}\right)+\overline{R}(l_c)+\frac{\overline{F}\cos(\omega t+\alpha)}{b}, \quad (66)$$

where

$$\frac{\overline{F}\cos(\omega t+\alpha)}{b}=\overline{\eta'}(l_c)\frac{\partial\overline{z_0}(l_c)}{\partial t}. \quad (67)$$

Here $\overline{F}/b$ is the renormalized driving force and α is a phase angle. Giorgini and Bowley neglect the random force, take the long wavelength limit, and solve the equation of motion assuming that the pinning potential is strong enough to allow only small displacements of the interface around the equilibrium position so that

$$\sin\left(\frac{2\pi\overline{z}(l_c)}{b}\right)\simeq\frac{2\pi\overline{z}(l_c)}{b}. \quad (68)$$

The equation of motion for small displacements can be written as a linear equation of the form

$$\overline{\eta'}\frac{\partial\overline{z}}{\partial t}=\overline{\gamma}\nabla^2\overline{z}-\left(\frac{2\pi}{b}\right)^2\overline{Vz}+\overline{R}+\frac{\overline{F}\cos(\omega t+\alpha)}{b}, \quad (69)$$

which is easily be solved if we ignore the random force. In the long wavelength limit, the Fourier component of the velocity due to the renormalized driving force which oscillates as $e^{-i\omega t}$ is

$$u_0(\omega,l=\infty)=\frac{1}{[1+i(4\pi^2\overline{V}/\omega\overline{\eta'}b^2)]}\frac{\overline{F}e^{-i\phi}}{2b\overline{\eta'}}=\frac{u_0(\omega,l=l_c)}{[1+i(4\pi^2\overline{V}/\omega\overline{\eta'}b^2)]}. \quad (70)$$

The final expression for the dielectric function is

$$\ln[\epsilon(\omega)]=\int_0^{l_c}\frac{8\pi^4U^2}{\tilde{\gamma}^2b^4}C(n,\tilde{r}_d,0)\,dl+\ln\left(1+i\frac{4\pi^2\overline{V}}{\omega\overline{\eta'}b^2}\right). \quad (71)$$

Therefore, the procedure for calculating the full dielectric function is to evaluate the recursion relations up to l_c and then to use the renormalized values of $\overline{V}(l_c)$ and $\overline{\eta'}(l_c)$ to calculate the correction to the dielectric constant for large l. Using our dictionary, and the reduced variables

$$X(l)\equiv\frac{2}{\pi K_l}, \quad (72)$$

$$Y(l)\equiv\frac{4\pi U}{k_BT}=\frac{8\pi y_l}{a^2\Lambda_0^2}, \quad (73)$$

the equation for the dielectric function can be written as

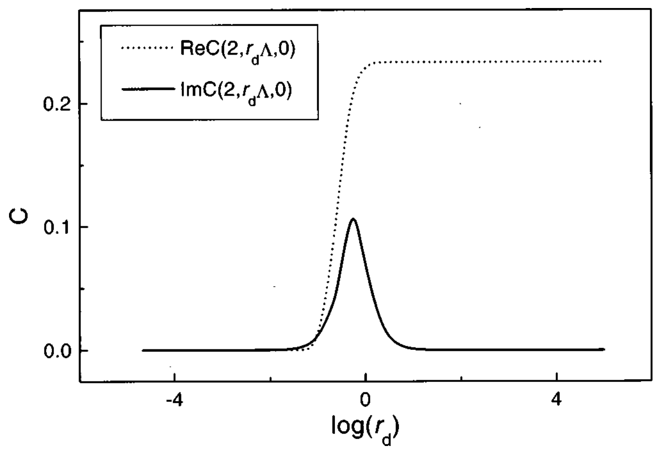

FIG. 4. The variation of functions $\mathrm{Re}\,C(n,\tilde{r}_d,0)$ and $\mathrm{Im}\,C(n,\tilde{r}_d,0)$ with $\tilde{r}_d$ $(=r_d\Lambda)$ for the case $n=2$.

$$\ln[\varepsilon(\omega)]=\int_0^{l_c} dl\,\frac{2Y(l)^2}{X(l)^2}C(2/X,\tilde{r}_d,0)+\ln\left(1+i\,\frac{2Y(l_c)\tilde{r}_d(l_c)^2}{X(l_c)}\right). \tag{74}$$

D. Numerical method

In order to calculate the weak-coupling contribution to the dielectric function the recursion relations must be integrated and consequently the functions A, B, and C have to be evaluated at a whole series of points. The recursion relations were integrated numerically in a straightforward way using a fourth or fifth order Runga-Kutta scheme.[26]

The calculations of the functions A, B, and C proved relatively time consuming. Each of the three functions contains three integrals, two of which run from zero to infinity. We used Rhomberg integration in each case and evaluated the integrals from zero to infinity by splitting the integrals into two parts and mapping them onto a finite range by a change of variables. The behavior of C largely controls the renormalization of the driving field and hence the response function. The variation of $\mathrm{Re}\,C(n,\tilde{r}_d,0)$ with $\tilde{r}_d$ is very close to that of a step function: it increases rapidly from zero for small $\tilde{r}_d$, to a constant value of 0.246 at $\tilde{r}_d\sim 0.5$, for $n=1.5$. However, $\mathrm{Re}\,C(n,\tilde{r}_d,0)$ varies only slightly with n: the height of the step varies from 0.246 for $n=1.5$, to a value of 0.195 when $n=3$. In contrast the function $\mathrm{Im}\,C(n,\tilde{r}_d,0)$ takes the form of a sharp peak around the point $\tilde{r}_d\sim 0.5$, and has a value of zero elsewhere. The effect of variation in n is to modulate the height of the peak: for $n=1.5$, the peak value of $\mathrm{Im}\,C(n,\tilde{r}_d,0)$ is 0.1028, this increases slightly to 0.1035 when $n=1.7$, after which it declines, reaching a value of 0.0897 for $n=3$. The behavior of $\mathrm{Re}\,C(n,\tilde{r}_d,0)$ and $\mathrm{Im}\,C(n,\tilde{r}_d,0)$ for $n=2$ is shown in Fig. 4.

In order to calculate the dielectric function in the transition region we performed a series of integrations of the recursion relations using different initial conditions. The choice of initial values $\tilde{r}_d(0)$ and $Y(0)$ before renormalization, which of course correspond to the microscopic parameters of the model, is not known.

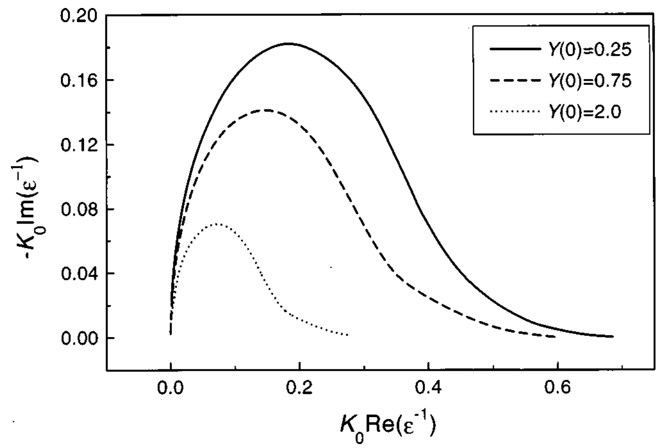

FIG. 5. Parametric plot of the dynamic dielectric function for superfluid films for different values of the parameter $Y(0)$, with $Y(l_c)=1$.

Suppose first of all that $Y(0)$ [and hence $y(0)$] is small. We found that the behavior of $K_0\varepsilon^{-1}(\omega)$ depended only weakly on the exact value of $\tilde{r}_d(0)=r_d(0)\Lambda_0$ as long as it is large. The reason is simple. The effect of $r_d(0)$ is to effectively terminate the recursion relations at this length scale; the transition then occurs at a smaller value of n than for infinite $r_d(0)$ where the transition occurs at $n=2$. As long as length $r_d(0)$ is large the change in n is small and so $A(n)$, $B(n)$, and $C(n,\tilde{r}_d,0)$ are barely affected. We have chosen a value, somewhat arbitrarily, of $\tilde{r}_d(0)=2\times 10^3$.

In contrast, there is a strong dependence on the choice of $Y(0)$. Figure 5 shows the variation in $K_0\varepsilon^{-1}(\omega)$ with $Y(0)$, where in each case the values of $X(0)$ were chosen to sweep through the transition. Although the size of the curve depends on the exact value of $Y(0)$, we find that to a good approximation the shape remains the same.

We also investigated the dependence of the results on the choice of the parameter l_c, as defined by the choice of the value $Y(l_c)$ at which renormalization is stopped. The behavior of $K_0\varepsilon^{-1}(\omega)$ varies quite strongly with $Y(l_c)$, except of course in the low temperature region where y_l remains small, as is shown in Fig. 6. Here $Y(0)=0.75$. The arbitrariness in

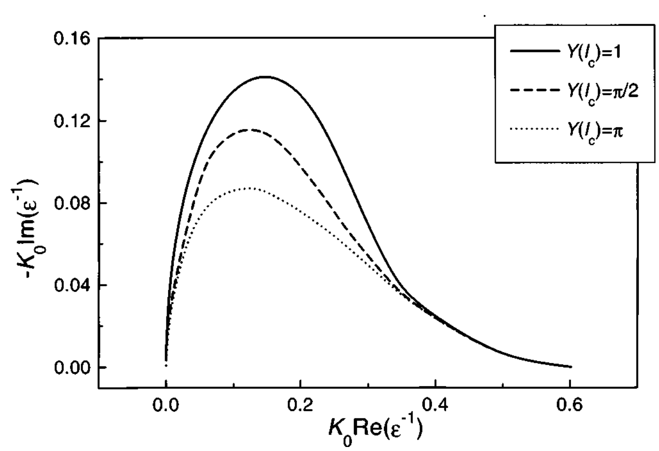

FIG. 6. Parametric plot of the dynamic dielectric function for superfluid films for different values of the cutoff parameter l_c, as defined by the value of $Y(l_c)$, and with $Y(0)=0.75$.

the choice of l_c represents a weakness in the theory, for we have to leave it as an adjustable parameter. We select the value to give the best agreement with the experimental data.

E. Comparison with Experiment

The frequency shift and inverse Q factor measured in torsional oscillator experiments are related to the quantities calculated in the theory by the relations[9]

$$\frac{2\Delta P}{P}=\frac{Am^2k_BT}{M\hbar^2}K_0\,\mathrm{Re}[\varepsilon^{-1}(\omega)], \tag{75}$$

$$\Delta Q^{-1}=\frac{Am^2k_BT}{M\hbar^2}K_0\,\mathrm{Im}[-\varepsilon^{-1}(\omega)], \tag{76}$$

where A is the area of the film, m the mass of an atom of ^{4}He and M is the effective mass of the empty cell. The period shift and change in Q-factor are measured directly, the quantities $K_0\,\mathrm{Re}[\varepsilon^{-1}(\omega)]$ and $K_0\,\mathrm{Im}[-\varepsilon^{-1}(\omega)]$ are predicted from the theory and value of A/M is inferred from the calibration of the oscillator.[5]

In this section the results from torsional oscillator experiments performed by McQueeney,[6] using a Mylar substrate, are compared with the predictions of the dynamic theory.[27] His work is particularly important for two reasons: he measured the response for many helium films of different coverages, and he took great care to reduce the drive level so that the amplitude of oscillation is much less than other experiments—he used a maximum speed of 6 $\mu\,\mathrm{ms}^{-1}$ which means that it should be possible to describe his results within the framework of a linear response theory. McQueeney used the same experimental apparatus as Agnolet *et al.*[5] for which the value of A/M is 266 $\mathrm{m}^2\,\mathrm{kg}^{-1}$.

The results from torsional oscillator experiments are usually presented in the form of curves of $2\Delta P/P$ against T and ΔQ^{-1} against T. However, we choose instead a parametric plot so that the real and imaginary parts of $K_0\varepsilon(\omega)^{-1}$ are plotted against each other. The parametric plot has the advantage that it avoids the need to model the temperature dependence of $K_0\,\mathrm{Re}[\varepsilon(\omega)^{-1}]$ and $K_0\,\mathrm{Im}[\varepsilon(\omega)^{-1}]$ and so it leads to a more direct comparison of theory and experiment.[9]

In Fig. 7 we compare the theoretical curve of $K_0\varepsilon(\omega)^{-1}$, for $Y(l_c)=1.5$ and $Y(0)=0.25$, with McQueeney's data (excluding only the thinnest films in which the temperature of the peak in the dissipation T_p arises at <0.5 K). We have also included for comparison the prediction of the refined phenomenological theory due to Bowley *et al.*,[8] which is based on the work of Ambegaokar *et al.*,[7] and is essentially the same as that made by Wallin[11] on the basis of Minnhagen's theory.[10] In order to obtain the fit we tuned the values of two parameters: we varied $Y(0)$ until there was good agreement at the low temperature end [large $K_0\,\mathrm{Re}\,\varepsilon^{-1}(\omega)$] and then we varied $Y(l_c)$ to fit the average peak in the dissipation $[-K_0\,\mathrm{Im}\,\varepsilon^{-1}(\omega)]$. The theoretical curves are not very sensitive to the value of $Y(0)$; if we reduced $Y(0)$ from 0.25 to half this value (0.125) the curves would shift for large $K_0\,\mathrm{Re}\,\varepsilon^{-1}(\omega)$ by only 5%.

Looking at Fig. 7 it is clear that the method presented here leads to a parametric curve of the dielectric function with a shape very close to those measured by McQueeney in torsional oscillator experiments. There is some deviation between the new theory and experiment at the high temperature end [small $K_0\,\mathrm{Re}\,\varepsilon^{-1}(\omega)$], but this is to be expected: this is the strong coupling regime which we have described with a very crude harmonic approximation to the pinning potential. There is some systematic deviation from the predicted curve near the peak in $K_0\,\mathrm{Im}\,\varepsilon^{-1}(\omega)$ which is not captured by the present theory. Nevertheless, the agreement between our theory and experiment is extremely good; it is substantially better than that obtained using the refined phenomenological theory, which itself is more accurate than AHNS.

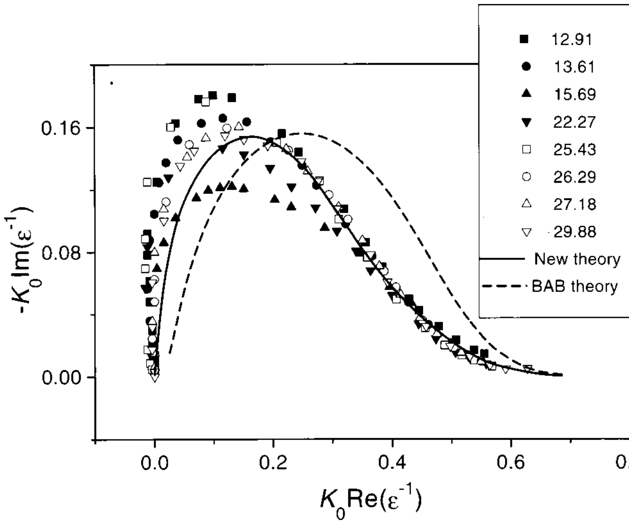

FIG. 7. Comparison of plots of real and imaginary parts of $K_0\varepsilon(\omega)^{-1}$ for a series of films on Mylar, measured by McQueeney, with the new theoretical prediction. The units of film thickness are μmoles. Also shown is the curve predicted by the refined phenomenological theory of Bowley *et al.* (BAB).

It might be thought that the variation in the peak value of $K_0\,\mathrm{Im}\,\varepsilon(\omega)^{-1}$ with film thickness is caused by some systematic variation in l_c with coverage. It is true that there is an increase in the core size as the film thickness is reduced[6,28] but this does not affect the value of l_c. The parameter l_c is simply an artifact of our calculational scheme which has no physical significance: it simply parameterizes the somewhat arbitrary value of the fugacity at which we crossover from the renormalization group calculation in the weak-coupling regime (where y_l is small) to the strong-coupling ansatz of Giorgini and Bowley.

However, the systematic variation in the peak value of $K_0\,\mathrm{Im}\,\varepsilon(\omega)^{-1}$ with film coverage could arise from imperfect coupling between the atomic layers which make up the film,[9] an effect which is not included in our theoretical model. As the coverage increases the first fluid layer of the film fills up and saturates, then the next fluid layer forms and fills up, and so on. There is a coupling between the layers which describes the transfer of particles from one layer to the next. If the coupling under renormalization becomes strong, the layers become locked together: the effect of layering is then negligible for the layers move as a single, uniform film. If the coupling becomes weak under renormalization the layers decouple and move as separate entities. We can expect to see systematic variations in the data, as is observed. Nevertheless, it is not yet clear whether this picture will be able to describe the data correctly.

The value of the energy of a vortex is expected to increase as the coverage increases and hence the fugacity and $Y(0)$ should decrease. One would then expect the curves to evolve as shown in Fig. 5. Once the films become sufficiently thick one would expect the curves to tend to the small $Y(0)$ limit of the theory. Hence we have made our comparison with films where $T_p > 0.5$ K,[6] omitting the data from the thinner films. In fact nothing dramatic happens for the thinner films: the dominant effect is the variation in the peak value of $K_0 \,\mathrm{Im}\, \varepsilon(\omega)^{-1}$ with film thickness and since this is not yet captured by the theory there is no insight to be gained from adding the data from the thinnest plots to Fig. 7. We have also compared our theory with the thinnest films studied by McQueeney ($T_p \simeq 0.18$ K), and the fit is less good. The peak value of $K_0 \,\mathrm{Im}\, \varepsilon(\omega)^{-1}$ is lower (as expected), there appears to be an offset due to additional dissipation, and there is a poorer fit at the high temperature end.

We have not compared our data with the results of torsional oscillator studies carried out using a Grafoil substrate.[29] This is because the data obtained from such studies has a rather different form from that obtained using a Mylar substrate, as is strikingly demonstrated when the data is plotted parametrically.[9,30] The origin of the differences in the data obtained by McQueeney using Mylar and recent work on Grafoil is not yet clear. However, we can suggest three possible reasons. Firstly, the morphology of Grafoil differs from that of Mylar: although Grafoil is atomically smooth, on short length scales it is strongly disordered at a length scale of order the grain size and this may well have a greater effect on the superfluid film than the short-scale inhomogeneities in Mylar, a possibility first suggested in a more general form by AHNS.[7] Secondly, the geometry of the substrate is in the form of a stack of disks in the Grafoil experiment rather than a long sheet wound round itself as is the case with Mylar. Thus edge effects may play a more important role for this geometry. Thirdly, the drive strength used by McQueeney was very much lower than that used either in the Grafoil experiments, or in other experiments using a Mylar substrate: we believe that it is this very low drive level which leads to the characteristic asymmetric shape in the parametric plot of the data, a feature which is not observed either in the Grafoil data[9] or Mylar data obtained using higher drive strengths.[31]

IV. CONCLUSIONS AND DISCUSSION

The theory presented in this paper provides a theoretical framework within which the dynamics of superfluid films can be described. We use the Hubbard-Stratonovich transformation to rewrite the Hamiltonian of the vortex gas as the well known sine-Gordon model which allows us to postulate an equation of motion for the system. The equation of motion is renormalized using the systematic methods of Nozières and Gallet: this leads to a complete set of recursion relations for the parameters of the model so the dielectric function can be calculated directly. The form of the dielectric function predicted by the theory is readily compared with the response of torsional oscillators in the region of the superfluid transition and we find that agreement with experiment is substantially better than that obtained using phenomenological approaches. However, we find that there is some discrepancy between theory and experiment in the high temperature region.

Our theory is also an advance on previous work for reasons apart from the improved agreement with experiment. Because of the analogy with a vicinal surface, we can predict that for a nonlinear drive the superfluid density will become anisotropic, a feature which is to be expected,[25] but is not contained in any of the phenomenological theories.[32] In addition, the theory presented here explicitly describes the response of a finite size system. So far no attempt has been made to describe accurately the geometry of films in real experiments, but this may now be possible using the methods described here.

Despite its successes, there are a number of improvements that can be made in the theory described here. Most importantly, our theory does not describe the crossover between the weak and strong coupling regimes precisely. The advantage of a well-defined calculation of the crossover is that it would both avoid the need to introduce the arbitrary crossover parameter l_c, and hopefully lead to better agreement with experiment in the high temperature region. One way of doing this would be to develop a scheme for renormalizing the equation of motion in the strong coupling regime. An alternative scheme could be based on a numerical simulation of the equation of motion in the strong coupling limit.

Also our model does not yet account for the variation in dynamic response with film thickness and so we cannot use it to understand the systematic variation in the dissipation peak which is observed. In order to describe the effects of layering a far more complex model of the film would be required in which there was imperfect coupling between different atomic layers. However, the approach outlined here should form the basis of such a theory.

In summary, the new theory presented here provides a more accurate approach to the calculation of the dynamic response of superfluid films than the alternative phenomenological theories that have been developed in the past. Not only does it lead to closer agreement with experiment, it also provides a systematic framework within which it should be possible to discuss the subtle effects of nonlinearity and atomic layering on the dynamic response of superfluid films.

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